Combinatorial Solution of Non-diagonalisable Spin Chains

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Based on

- *The One-Loop Spectral Problem of Strongly Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory* by Ipsen, Staudacher, Zippelius [1812.08794]

- *The Integrable (Hyper)eclectic Spin Chain* by Ahn, Staudacher [2010.14515]

- Work in progress by Ahn, Corcoran, Staudacher [2110.soon]

Motivation

- Strongly twisted planar $\mathcal{N} = 4$ super Yang-Mills is a non-unitary toy model which retains integrability.
- One-loop dilatation operator $\mathfrak{D}$ is one of the simplest settings to understand integrability.
- In certain operator sectors $\mathfrak{D}$ in non-diagonalisable, which leads to logarithms in correlation functions determined by Jordan block structures.
- Bethe ansatz fails to describe the sizes and multiplicities of these Jordan blocks. What can be done?

Outline

1. Strongly Twisted $\mathcal{N} = 4$ SYM
2. Hypereclectic Spin Chain and Integrability
3. Spectral Problem and Data
4. Solution via q-Combinatorics

Strongly Twisted $\mathcal{N} = 4$ SYM

Start from γ-deformed $\mathcal{N} = 4$ SYM $\mathcal{L}_{\mathsf{SYM}}(A, \Psi, \Phi, g, N_c, \gamma_1, \gamma_2, \gamma_3)$ and take the limit [Gürdogan, Kazakov '15]

$$g = \frac{\sqrt{\lambda}}{4\pi} \to 0, \qquad q_j = e^{-i\gamma_j/2} \to \infty \quad \text{or} \quad 0,$$

where either $g q_j \equiv \xi_j^+$ or $g q_j^{-1} \equiv \xi_j^-$ are held fixed.

There are $2^3 = 8$ possible limits. For $(q_1, q_2, q_3) = (\infty, \infty, \infty)$ we have

$$\mathcal{L}_{\text{int}} = N_c \text{tr}\left((\xi_1^+)^2 \phi_2^\dagger \phi_3^\dagger \phi_2 \phi_3 + (\xi_2^+)^2 \phi_3^\dagger \phi_1^\dagger \phi_3 \phi_1 + (\xi_3^+)^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right) + N_c \text{tr}\left(i\sqrt{\xi_2^+ \xi_3^+} (\psi^3 \phi_1 \psi^2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic} \right).$$

Overall the 8 limits give 2 inequivalent models (6+2).

Electic Spin Chain

Consider local composite operators of length L built from ϕ_i

$$\mathcal{O}_{j_1,j_2,\dots,j_L}(x) = \mathrm{tr}(\phi_{j_1}\phi_{j_2}\dots\phi_{j_L}(x)).$$

The dilatation operator reads [Ahn, Staudacher '20]

$$\mathfrak{D} = \mathfrak{D}_0 + g^2 H_{\mathrm{ec}} + O(g^4).$$

For $(q_1, q_2, q_3) = (\infty, \infty, \infty)$ we have

$$H_{\mathrm{ec}} = \sum_{i=1}^{L} \mathcal{H}_{\mathrm{ec}}^{i,i+1}, \qquad H_{\mathrm{ec}} : (\mathbb{C}^3)^{\otimes L} \to (\mathbb{C}^3)^{\otimes L},$$

where

$$\mathcal{H}_{\mathrm{ec}}|11\rangle = 0, \qquad \mathcal{H}_{\mathrm{ec}}|22\rangle = 0, \qquad \mathcal{H}_{\mathrm{ec}}|33\rangle = 0,$$

$$\mathcal{H}_{\mathrm{ec}}|12\rangle = 0, \qquad \mathcal{H}_{\mathrm{ec}}|23\rangle = 0, \qquad \mathcal{H}_{\mathrm{ec}}|31\rangle = 0,$$

$$\mathcal{H}_{\mathrm{ec}}|21\rangle = \xi_3^+|12\rangle, \qquad \mathcal{H}_{\mathrm{ec}}|32\rangle = \xi_1^+|23\rangle, \qquad \mathcal{H}_{\mathrm{ec}}|13\rangle = \xi_2^+|31\rangle.$$

Hypereclectic Spin Chain

For $\xi_1^+ = \xi_2^+ = 0, \xi_3^+ = 1$ we get the **hypereclectic** spin chain:

$$H_{\text{hec}} = \sum_{i=1}^{L} \mathcal{H}_{\text{hec}}^{i,i+1},$$

where

$$\begin{aligned}
&\mathcal{H}_{\text{hec}} |11\rangle = 0, && \mathcal{H}_{\text{hec}} |22\rangle = 0, && \mathcal{H}_{\text{hec}} |33\rangle = 0, \\
&\mathcal{H}_{\text{hec}} |12\rangle = 0, && \mathcal{H}_{\text{hec}} |23\rangle = 0, && \mathcal{H}_{\text{hec}} |31\rangle = 0, \\
&\mathcal{H}_{\text{hec}} |21\rangle = |12\rangle, && \mathcal{H}_{\text{hec}} |32\rangle = 0, && \mathcal{H}_{\text{hec}} |13\rangle = 0.
\end{aligned}$$

Interestingly, this simple model appears to contain most of the information of the eclectic model.

Integrability

H_{hec} can be realised as the logarithmic derivative of a transfer matrix

$$H_{\text{hec}} = \frac{d}{du} \log(t(u))\Big|_{u=0}.$$

$t(u)$ is constructed from an R−matrix

$$R(u) = \mathcal{P} + u\mathcal{H}_{\text{hec}}$$

which satisfies YBE, and thus

$$[t(u), t(u')] = 0.$$

However H_{hec} is nilpotent, and therefore non-diagonalisable.

Spectral Problem

For diagonalisable Hamiltonians $H : (\mathbb{C}^3)^{\otimes L} \to (\mathbb{C}^3)^{\otimes L}$ we know there are 3^L eigenstates $|\psi_j\rangle$ such that

$$H\,|\psi_j\rangle = E_j\,|\psi_j\rangle\,, \qquad j = 1, 2, \ldots, 3^L.$$

For the hypereclectic model this must be replaced by

$$(H_{\text{hec}} - E_j)^{m_j}\,|\psi_j^{m_j}\rangle = 0, \quad j = 1, \ldots, N, \quad m_j = 1, \ldots, l_j.$$

There are N Jordan blocks labelled by j, of length l_j.

Jordan blocks leads to logarithms in correlation functions. For example

$$\mathfrak{D}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix} = \begin{pmatrix}\Delta & 1\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\end{pmatrix} \to \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \sim \frac{1}{|x|^{2\Delta}}\begin{pmatrix}\log x^2 & 1\\ 1 & 0\end{pmatrix}.$$

Spectral Problem

H_{hec} nilpotent $\rightarrow$ all generalised eigenvalues E_j are 0.

H_{hec} is block diagonal with respect to sectors of fixed numbers $L - M$ of fields ϕ_1, $M - K$ fields ϕ_2, and K fields ϕ_3.

Goal: given L, M, K, find sizes and multiplicities of the Jordan blocks.

(The limit of the) Bethe ansatz fails to describe this.

Let's see how it works combinatorially for $K = 1$ in the cyclic sector.

$L = 3, M = 2, K = 1$

Work in 'cyclic sector', where all states are invariant under the shift operator $t(0)$. For $L = 3, M = 2, K = 1$ there are 2 states

$$|123\rangle + |312\rangle + |231\rangle\,, \qquad |213\rangle + |321\rangle + |132\rangle\,,$$

which we write simply as

$$|123\rangle_c\,, \qquad |213\rangle_c\,.$$

We clearly identify single a Jordan block of size 2

$$|213\rangle_c \xrightarrow{H} |123\rangle_c \xrightarrow{H} 0,$$

$$H^{\text{cyc}}_{3,2,1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

General L, $M = 2$, $K = 1$

Things are also trivial for higher L, $M = 2, K = 1$. There is a single Jordan block of size $L - 1$

$$|211\ldots13\rangle \to |121\ldots13\rangle \to |112\ldots13\rangle \to \cdots \to |111\ldots23\rangle \to 0,$$

Things become much more intricate for higher M.

Data for $M = 5$

L	Sizes of Jordan Blocks
8	$1\ 5\ 7\ 9\ 13$
9	$1\ 5^2\ 9^2\ 11\ 13\ 17$
10	$1\ 5^2\ 7\ 9^2\ 11\ 13^2\ 15\ 17\ 21$
11	$1^2\ 5^2\ 7\ 9^3\ 11\ 13^3\ 15\ 17^2\ 19\ 21\ 25$
12	$1\ 5^3\ 7\ 9^3\ 11^2\ 13^3\ 15^2\ 17^3\ 19\ 21^2\ 23\ 25\ 29$
13	$1^2\ 5^3\ 7\ 9^4\ 11^2\ 13^4\ 15^2\ 17^4\ 19^2\ 21^3\ 23\ 25^2\ 27\ 29\ 33$
14	$1^2\ 5^3\ 7^2\ 9^4\ 11^2\ 13^5\ 15^3\ 17^4\ 19^3\ 21^4\ 23^2\ 25^3\ 27\ 29^2\ 31\ 33\ 37$
15	$1^2\ 5^4\ 7\ 9^5\ 11^3\ 13^5\ 15^3\ 17^6\ 19^3\ 21^5\ 23^3\ 25^4\ 27^2\ 29^3\ 31\ 33^2\ 35\ 37\ 41$
16	$1^2\ 5^4\ 7^2\ 9^5\ 11^3\ 13^6\ 15^4\ 17^6\ 19^4\ 21^6\ 23^4\ 25^5\ 27^3\ 29^4\ 31^2\ 33^3\ 35\ 37^2\ 39\ 41\ 45$
17	$1^3\ 5^4\ 7^2\ 9^6\ 11^3\ 13^7\ 15^4\ 17^7\ 19^5\ 21^7\ 23^4\ 25^7\ 27^4\ 29^5\ 31^3\ 33^4\ 35^2\ 37^3\ 39\ 41^2\ 43\ 45\ 49$
18	$1^2\ 5^5\ 7^2\ 9^6\ 11^4\ 13^7\ 15^5\ 17^8\ 19^5\ 21^8\ 23^6\ 25^7\ 27^5\ 29^7\ 31^4\ 33^5\ 35^3\ 37^4\ 39^2\ 41^3\ 43\ 45^2\ 47\ 49\ 53$

Table 3: Structures of Jordan blocks for the sector of $M = 5, K = 1, k = 0$ (cyclic states). Exponents denote multiplicities.

$L = 7, M = 3, K = 1$ (15 states)

As before, there is a natural 'top state' for a Jordan block

$$
\begin{aligned}
&|2211113\rangle && H^0 \\
&\to |2121113\rangle && H^1 \\
&\to |2112113\rangle + |1221113\rangle && H^2 \\
&\to |2111213\rangle + 2\,|1212113\rangle && H^3 \\
&\to |2111123\rangle + 3\,|1211213\rangle + 2\,|1122113\rangle && H^4 \\
&\to 4\,|1211123\rangle + 5\,|1121213\rangle && H^5 \\
&\to 5\,|1112213\rangle + 9\,|1121123\rangle && H^6 \\
&\to 14\,|1112123\rangle && H^7 \\
&\to 14\,|1111223\rangle && H^8 \\
&\to 0 && H^9
\end{aligned}
$$

This gives a Jordan block of size 9, but state space is still not exhausted.

$L = 7, M = 3, K = 1$

$$
\begin{array}{ll}
|2211113\rangle & H^0 \\
\to |2121113\rangle & H^1 \\
\to |2112113\rangle + |1221113\rangle & H^2 \\
\to |2111213\rangle + 2\,|1212113\rangle & H^3 \\
\to |2111123\rangle + 3\,|1211213\rangle + 2\,|1122113\rangle & H^4 \\
\to 4\,|1211123\rangle + 5\,|1121213\rangle & H^5 \\
\to 5\,|1112213\rangle + 9\,|1121123\rangle & H^6 \\
\to 14\,|1112123\rangle & H^7 \\
\to 14\,|1111223\rangle & H^8 \\
\to 0 & H^9
\end{array}
$$

New ansatz for top state: $\alpha\,|2112113\rangle + \beta\,|1221113\rangle$.

$L = 7, M = 3, K = 1$

$$
\begin{aligned}
&\alpha\,|2112113\rangle + \beta\,|1221113\rangle \\
&\to \beta\,|2111213\rangle + (\alpha+\beta)\,|1212113\rangle \\
&\to \beta\,|2111123\rangle + (\alpha+2\beta)\,|1211213\rangle + (\alpha+\beta)\,|1122113\rangle \\
&\to (\alpha+3\beta)\,|1211123\rangle + (2\alpha+3\beta)\,|1121213\rangle \\
&\to (2\alpha+3\beta)\,|1112213\rangle + (3\alpha+6\beta)\,|1121123\rangle \\
&\to (5\alpha+9\beta)\,|1112123\rangle = 0
\end{aligned}
$$

if $\alpha = -9, \beta = 5$. This determines a Jordan block of length 5.

$L = 7, M = 3, K = 1$

$$\alpha\,|2112113\rangle + \beta\,|1221113\rangle$$
$$\to \beta\,|2111213\rangle + (\alpha+\beta)\,|1212113\rangle$$
$$\to \beta\,|2111123\rangle + (\alpha+2\beta)\,|1211213\rangle + (\alpha+\beta)\,|1122113\rangle$$
$$\to (\alpha+3\beta)\,|1211123\rangle + (2\alpha+3\beta)\,|1121213\rangle$$
$$\to (2\alpha+3\beta)\,|1112213\rangle + (3\alpha+6\beta)\,|1121123\rangle$$
$$\to (5\alpha+9\beta)\,|1112123\rangle = 0$$

New ansatz for top state: $\alpha'\,|2111123\rangle + \beta'\,|1211213\rangle + \gamma'\,|1122113\rangle$. This is an eigenstate for $\alpha' = -\beta' = \gamma' = 1$, giving a Jordan block of length 1.

Jordan block structure in cyclic sector for $L = 7, M = 3, K = 1$ is $(9, 5, 1)$.

L=7, M=3, K=1

$$
\begin{aligned}
&|\phi\rangle \equiv |2211113\rangle && H^0\\
&\to |2121113\rangle && H^1\\
&\to |2112113\rangle + |1221113\rangle && H^2\\
&\to |2111213\rangle + 2\,|1212113\rangle && H^3\\
&\to |2111123\rangle + 3\,|1211213\rangle + 2\,|1122113\rangle && H^4\\
&\to 4\,|1211123\rangle + 5\,|1121213\rangle && H^5\\
&\to 5\,|1112213\rangle + 9\,|1121123\rangle && H^6\\
&\to 14\,|1112123\rangle && H^7\\
&\to 14\,|1111223\rangle && H^8\\
&\to 0 && H^9
\end{aligned}
$$

Full Jordan block structure can be deduced by computing $\dim(H^k\,|\phi\rangle)$.

q-Combinatorics

Encode this structure in a partition function

$$Z_{7,3}(q) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8.$$

Problem is solved if we can calculate $Z_{L,M}(q)$. It turns out that it is a q-Binomial coefficient

$$Z_{L,M}(q) = \binom{L-1}{M-1}_q = \prod_{k=1}^{M-1} \frac{1 - q^{L-k}}{1 - q^k},$$

which is always a polynomial.

The q^k coefficient of these polynomials have combinatorial interpretation as counting number partitions of the integer k subject to certain restrictions.

More Examples

For $M = 2$ the partition functions are very simple

$$Z_{7,2}(q) = 1 + q + q^2 + q^3 + q^4 + q^5.$$

This indicates a single block of size 6.

It neatly encodes the involved structures in the previous table:

$$\begin{aligned} Z_{9,5}(q) =& 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 \\ &+ 7q^9 + 7q^{10} + 5q^{11} + 5q^{12} + 3q^{13} + 2q^{14} + q^{15} + q^{16} \end{aligned}$$

leads to a Jordan block spectrum $(17, 13, 11, 9^2, 5^2, 1)$.

General Situation (To Appear)

Can rewrite

$$Z(q) = \text{tr } q^{\hat{S}},$$

for an appropriate state-counting operator $\hat{S}$. Generalises naturally to higher K.

Eclectic: Universality hypothesis. Spectrum of hypereclectic matches that of the eclectic provided the filling conditions

$$L - M \geq M - K \geq K$$

are satisfied.

Conclusions and Outlook

- We devised a partition function which encodes Jordan block spectrum of the (hyper)eclectic spin chain.
- This in turn determines the logarithmic structure of certain correlation functions in strongly twisted $\mathcal{N} = 4$ SYM.
- Can we relate $Z(q)$ to objects in integrability?
- Prove rigorously universality hypothesis.
- Connect to general LCFT results.
- Higher loops?
- Other non-diagonalisable models: different strong twisting limits, different theories (chiral ABJM).