The Gaudin matrix and AdS/CFT

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Based on:

- C.K., D. Müller & K. Zarembo, ArXiv:2011.12192[hep-th], JHEP 03 (2021) 100
- C.K., D. Müller & K. Zarembo, ArXiv:2106.08116[hep-th], JHEP 09 (2021) 004

Correlation Functions and Wave Functions in Solvable Models

IPhT, Saclay & ENS, Paris

September 12th, 2021

Motivation: Spin chain perspective

- Encodes the norm of a Bethe eigenstate via its determinant
- Encodes the overlap of a Bethe eigenstate with an integrable boundary state via its super determinant
- Encodes the representation theory of the underlying (super) Lie algebra including certain duality relations

Motivation: AdS/CFT beyond the spectral problem

- Operators need normalization
- Correlation functions can be formulated as overlaps
- Duality relations might constrain correlation functions

Plan of the talk

I. The Gaudin matrix

II. One-point functions in a BCFT (in AdS/CFT)

III. Fermionic duality relations (simple examples)

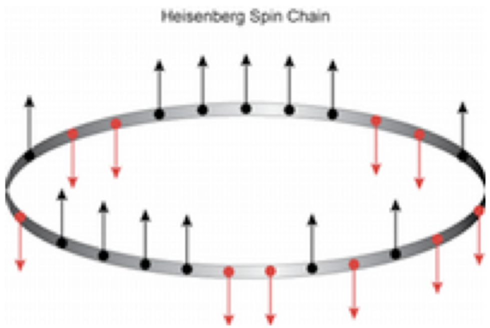
IV. Bosonic duality relations (simple examples)

V. Some technicalities (towards a proof, singular roots)

VI. Outlook

The Gaudin matrix and the norm

$$H = \sum_{n=1}^{L} (1 - P_{n,n+1})$$



$|\{u_i\}_{i=1}^{K}\rangle \equiv |\mathbf{u}\rangle$: Eigenstates with K excitations where

$$1 = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{j \neq k}^{K} \frac{u_k - u_j + \frac{i}{2}}{u_k - u_j - \frac{i}{2}} = e^{i\chi_k} \qquad k = 1, \ldots, K$$

Gaudin matrix $\boxed{G_{kj} = \frac{\partial \chi_k}{\partial u_j}}$ $K \times K$ matrix

$$\langle \mathbf{u} | \mathbf{u} \rangle \propto \det G$$

The Gaudin matrix and overlaps

Integrable boundary states $|B\rangle$:

$$\langle B|\mathbf{u}\rangle \quad \text{computable in closed form}$$

Matrix product states de Leeuw, CK, Zarembo '15

$$|B\rangle = |\text{MPS}\rangle = \sum_{\{s_i\}} \text{Tr}(t_{s_1} \ldots t_{s_L})|s_1 \ldots s_L\rangle$$

Valence Bond States

$$|\text{VBS}\rangle = |K\rangle^{\otimes \frac{L}{2}}, \qquad K = \sum_{s_1,s_2} K_{s_1,s_2}|s_1 s_2\rangle$$

Integrability understood in a scattering picture

$$\mathbf{Q}_{2n+1}|B\rangle = 0$$

Ghoshal & Zamolodchikov '94

Piroli, Pozsgay Vernier '17

Conserved parity-odd charges of spin chain

Integrable overlaps and the Gaudin determinant

$$\hat{\mathbf{Q}}_{\mathbf{2n+1}}|B\rangle = 0 \implies$$

$\langle B|\mathbf{u}\rangle \neq 0$ iff roots are paired $\{u_i, -u_i\}_{i=1}^{K_u}$

Gaudin matrix has block structure

$$\det G = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ B+A & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ 0 & A-B \end{vmatrix} = \det(A+B)\cdot\det(A-B)$$

$$= \det G_+ \cdot \det G_-$$

Quantity entering overlap formulas

$$\mathrm{SDet}\, G = \frac{\det G_+}{\det G_-} \equiv \mathbb{D}$$

Baxter polynomials

$$|\mathrm{VBS}\rangle = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)^{\otimes L/2}, \qquad \frac{\langle \mathrm{VBS}|\mathbf{u}\rangle^2}{\langle \mathbf{u}|\mathbf{u}\rangle} = \frac{Q(0)}{Q(\frac{i}{2})}\,\mathrm{SDet}G$$

Pozsgay ‘18

Integrable Super Spin Chains (of type $SU(M|N)$)

Cartan matrix M_{ab}, Dynkin labels q_a, $a,b = 1,\ldots,M+N-1$

Bethe equations

$$(-1)^{F_a+1} = \left(\frac{u_{a,j} - \frac{iq_a}{2}}{u_{a,j} + \frac{iq_a}{2}}\right)^L \prod_{b,k} \frac{u_{a,j} - u_{b,k} + \frac{iM_{ab}}{2}}{u_{a,j} - u_{b,k} - \frac{iM_{ab}}{2}} \equiv \mathrm{e}^{\,i\chi_{a,j}}$$

$u_{a,j}$: $a = 1,\ldots$ # of nodes in Dynkin diagram
$j = 1,\ldots,K_a$ (# of roots of type a)

$$G_{aj,bk} = \frac{\partial \chi_{a,j}}{\partial u_{b,k}}$$

$$\frac{\langle \mathrm{VBS}|\mathbf{u}\rangle^2}{\langle \mathbf{u}|\mathbf{u}\rangle} = \prod_a \frac{\prod_{j=1}^{n_a} Q_a(\frac{is_{a,j}}{2})}{\prod_{k=1}^{m_a} Q_a(\frac{ir_{a,k}}{2})}\ \mathrm{SDet}G$$

Gombor & Bajnok ’20
Komatsu & Want ‘20
C.K., Müller, Zarembo ‘20

AdS/CFT: N=M=4

QQ-system

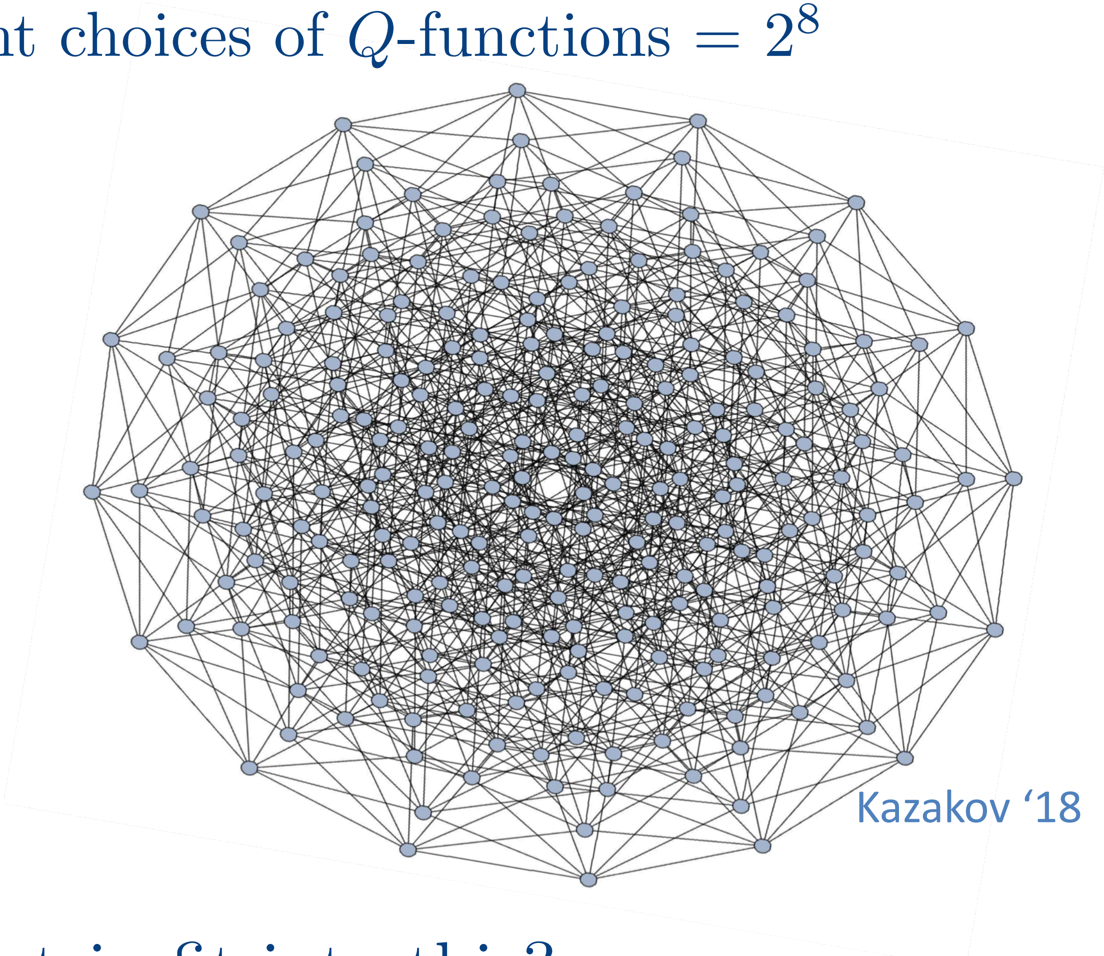
Q-functions and QSC optimal language for the spectral problem

Many equivalent ways of writing the Bethe equations

For $\mathcal{N} = 4$ SYM, # different choices of Q-functions $= 2^8$

Connected via dualities

- Fermionic (Change of Dynkin diagram)
- Bosonic



Kazakov '18

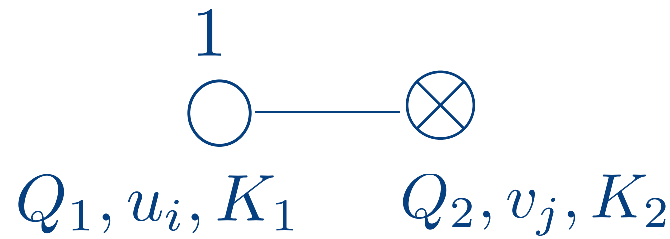
How does the Gaudin matrix fit into this?

Example: $SU(2|1)$ super spin chain

Encodes conformal single trace operators built from fields
X (bosonic), Ψ_1, Ψ_2 (fermionic) in $\mathcal{N} = 4$ SYM

Cartan matrix   Dynkin label

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$H = \sum_{n=1}^{L} (1 - \Pi_{n,n+1})$$

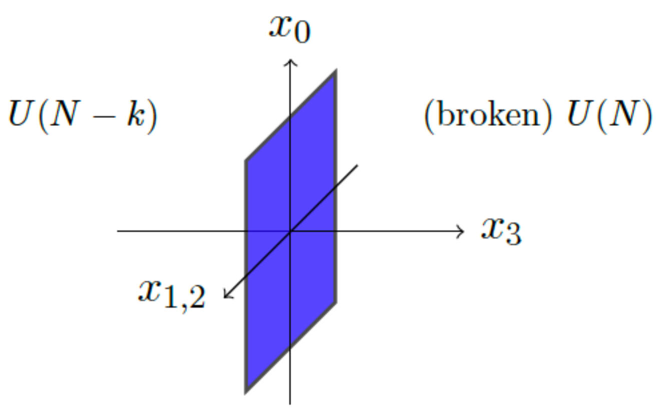
graded permutation

Baxter polynomials

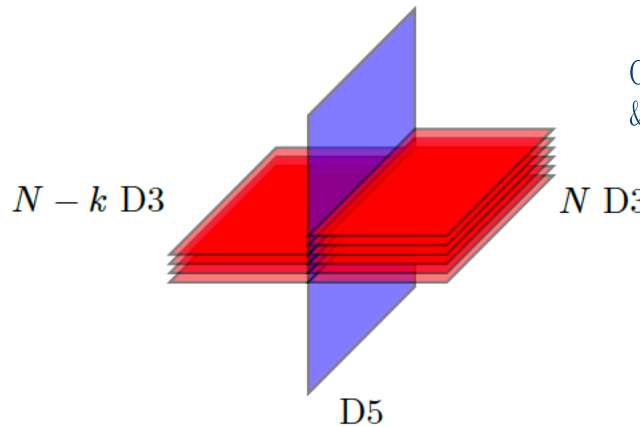
$$Q_1(u) = \prod_{i=1}^{K_1} (u - u_i), \qquad Q_2(u) = \prod_{j=1}^{K_2} (v - v_j)$$ (plus two trivial ones)

Vacuum: $|\Psi_1 \Psi_1 \ldots \rangle$,   Excitations at level 1 and 2: Ψ_2, X

AdS/dCFT and overlaps in $SU(2|1)$ spin chain



Gauge Theory



String Theory

Constable, Myers & Tafjord '99

The $k>1$ case: Fuzzy funnel solution, defect described by $|\mathrm{MPS}\rangle$ De Leeuw C.K, & Zarembo '15

The $k=1$ case: Defect described by $|\mathrm{VBS}\rangle$ C.K, Müller & Zarembo '20

For $x_3 > 0$:

$$A_\mu, \Phi_i, \Psi_\alpha = \begin{array}{c} \quad 1 \qquad N-1 \\ \left[\begin{array}{c|ccc} x & y & y & y \\ \hline y & z & z & z \\ y & z & z & z \\ y & z & z & z \end{array}\right] \end{array}$$

Boundary conditions (supersymmetric)

	$\Phi_{4,5,6}$	$\Phi_{1,2,3}$
x, y	Dirichlet	Neumann
z	no BCs	no BCs

One-point functions

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{C}{|x_3|^\Delta}$$

Propagators for complex scalars: $X = \phi_1 + i\phi_4,$ etc.

$$D_\kappa(x,y) = \frac{1}{4\pi^2}\left(\frac{1}{|x-y|^2} + \frac{\kappa}{|\bar{x}-y|^2}\right), \qquad \kappa = \begin{cases} 1 & \text{Neumann} \\ -1 & \text{Dirichlet} \\ 0 & \text{no BCs.} \end{cases}$$

$$\bar{x} = (x_0, x_1, x_2, -x_3)$$

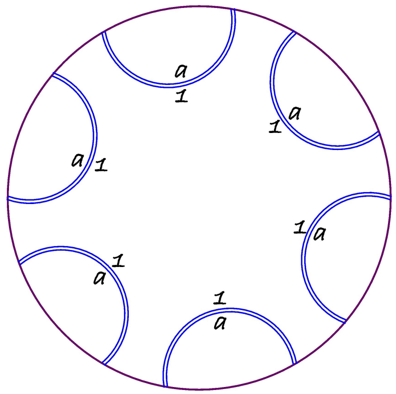
$$\left\langle X^{1a}(x) X^{b1}(y) \right\rangle = \frac{g_{\rm YM}^2 \delta^{ab}}{2}\Big(D_1(x,y) - D_{-1}(x,y)\Big) = \frac{g_{\rm YM}^2 \delta^{ab}}{4\pi^2 |\bar{x}-y|^2}\,,$$

Propagators for the fermions

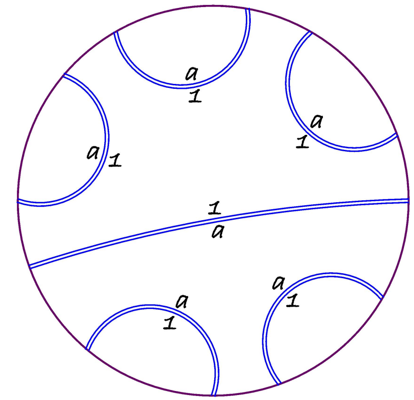
$$\langle \Psi_\alpha^{1a}(x) \Psi_\beta^{b1}(y) \rangle = \frac{g_{\rm YM}^2}{8\pi^2}\,\epsilon_{\alpha\beta}\,\delta^{ab} \cdot \frac{\bar{x}_3 - y_3}{|\bar{x}-y|^4}.$$

One-point functions and VBS

Feynman diagrams



Leading for large-N



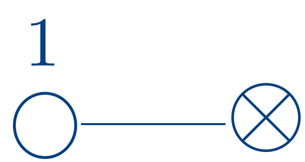
Sub-leading for large-N

C.K., Müller, Zarembo ‘20

Object to calculate $C_{k=1} = \dfrac{\langle \mathrm{VBS} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle^{1/2}}$

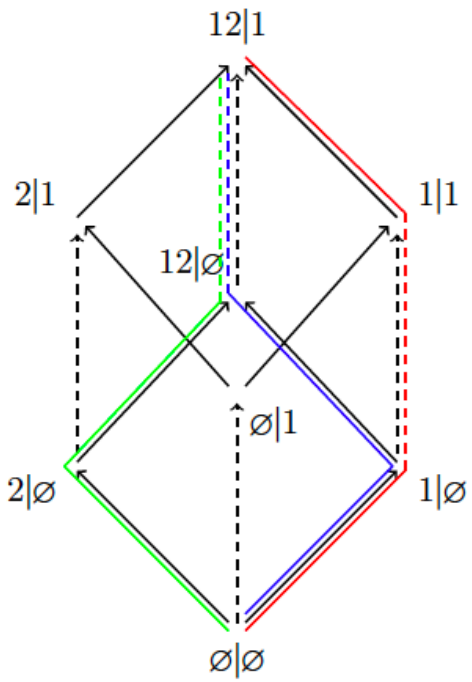
$$\langle \mathrm{VBS} | = \left(\langle XX | + \langle \Psi_1 \Psi_2 | - \langle \Psi_2 \Psi_1 | \right)^{\otimes L/2},$$

$$C_{k=1} = \frac{Q_1(0) Q_2(0)}{Q_1\left(\frac{i}{2}\right)} \mathrm{SDet} G$$



A Web of Dualities: Ex: $SU(1|2)$

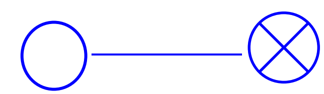
Tsuboi '98



2^3 Q-functions, 2 fixed

$$Q_{\emptyset|\emptyset} = u^L, \quad Q_{12|1} = 1$$

$6 = 3 \times 2$ versions of the BE's ($\sim$ paths)

Standard choice: Blue path 

Fermionic Duality 

(Change of variables
in the Bethe equations)

$$Q_{12|\emptyset} Q_{1|1} = Q^-_{1|\emptyset} - Q^+_{1|\emptyset}$$

Bosonic duality 

$$Q^+_{1|\emptyset} Q^-_{2|\emptyset} - Q^-_{1|\emptyset} Q^+_{2|\emptyset} = Q_{\emptyset|\emptyset} Q_{12|\emptyset}$$

Fermionic Duality: Ex: $SU(2|1)$

Beisert, Kazakov, ,
Sakai, Zarembo ‘05

$Q_1, u_i, K_1 \quad Q_2, v_j, K_2$

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix},\ q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$Q_1, u_i, K_1 \quad \widetilde{Q}_2, \tilde{v}_j, \widetilde{K}_2$

$$\widetilde{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \tilde{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Change of variables (from v_j to $\tilde{v}_j$)

K_2 roots v_j

$$\downarrow$$

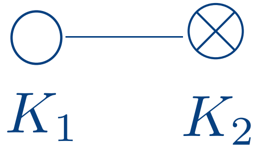
$$Q_1^-(v) - Q_1^+(v) = Q_2(v) \cdot \widetilde{Q}_2(v)$$

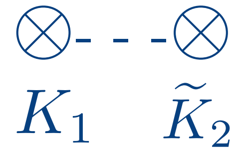
$\widetilde{K}_2 = K_1 - K_2 - 1$ roots $\tilde{v}_j$

$$1 = \frac{Q_1^-(v_k)}{Q_1^+(v_k)} \longrightarrow \frac{Q_1^+(\tilde{v}_k)}{Q_1^-(\tilde{v}_k)} = 1$$

$$-1 = \frac{Q_1^{++}(u_k)}{Q_1^{--}(u_k)} \cdot \frac{Q_2^-(u_k)}{Q_2^+(u_k)} \left(\frac{Q_\theta^-(u_k)}{Q_\theta^+(u_k)}\right)^L \longrightarrow \frac{\widetilde{Q}_2^+(u_k)}{\widetilde{Q}_2^-(u_k)} \left(\frac{Q_\theta^-(u_k)}{Q_\theta^+(u_k)}\right)^L = 1$$

Transformation formula: Ex: $SU(2|1)$





$K_1,\ K_2$ even $\implies \widetilde{K_2} = K_1 - K_2 - 1$ odd, i.e. $\tilde{v}$'s contain a single zero

Det $\widetilde{G}$ still factorizes

$Q_1^+(u) - Q_1^-(u) = i K_1\, u\, Q_2(u)\, \widetilde{Q}_2(u)$, with reduced Baxter polynomials

$$\boxed{\widetilde{\mathbb{D}} = K_1 \frac{\widetilde{Q}_2(0) Q_2(0)}{Q_1(\frac{i}{2})} \mathbb{D}}$$

Found numerically C.K., Müller, Zarembo '20

Analytical proof in progress

Notice:

- Holds semi-on-shell (the $\{u_i, -u_i\}$'s can be chosen at random)
- Covariance of overlap formula which involves $Q_2(0)\mathbb{D}$
- Factor K_1 signals that a hws is mapped to a descendent

de Leeuw, Ipsen, C.K., Vardinghus, Wilhelm 17

Fermionic dualities in general

- Allow one to move between any two Dynkin diagrams of a super Lie algebra (of type $SU(N|M)$)

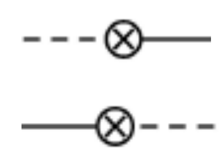
- Involve a fermionic node and its neighbours only

- Changes the nature of neighbouring nodes $\otimes \longleftrightarrow \bigcirc$
and the connections ——— $\longleftrightarrow$ - - -

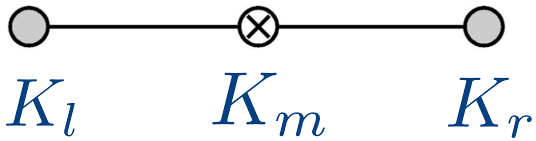
- Dualized node non-momentum carrying $\implies$ Dynkin labels unchanged
(Dynkin label equal to zero)

- Dualized node momentum carrying $\implies$ Dynkin labels change

$$\begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} V \pm 1 \\ -V \\ V \mp 1 \end{bmatrix} \quad \text{for}$$



Dualizing a non-momentum-carrying node



$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix}, \qquad q = \begin{bmatrix} V_l \\ 0 \\ V_r \end{bmatrix}, \qquad \begin{array}{l} \eta_1 \in \{-1, +1\} \\ \eta_2 \in \{0, -2\eta_1\} \\ \eta_3 \in \{0, 2\eta_1\} \end{array},$$

$$K_l, K_r, K_m \text{ all even} \implies \widetilde{K}_m = K_l + K_r - K_m - 1 \text{ odd}$$

$$Q_l^- Q_r^+ - Q_l^+ Q_r^- = i\eta_1 (K_r - K_l) u\, Q_m \widetilde{Q}_m\,,$$

C.K., Müller, Zarembo '20

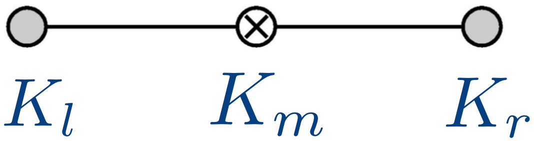
$$\widetilde{\mathbb{D}} = \mathrm{J}\,\mathbb{D} = (-\eta_1)^{K_l} \eta_1^{K_r} (\eta_1 K_r - \eta_1 K_l) \frac{Q_m(0) \widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)} \mathbb{D}$$

Found numerically
Analytical proof
in progress

K_l, K_r even, K_m odd

$$\widetilde{\mathbb{D}} = (-\mathrm{J})^{-1}\,\mathbb{D}\,,$$

Dualizing a momentum-carrying node



$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix}, \qquad \begin{aligned} &\eta_1 \in \{-1, +1\} \\ &\eta_2 \in \{0, -2\eta_1\} \\ &\eta_3 \in \{0, 2\eta_1\} \end{aligned}$$

$$K_l, K_r, K_m, L \text{ all even} \implies \widetilde{K}_m = L + K_l + K_r - K_m - 1 \text{ odd}$$

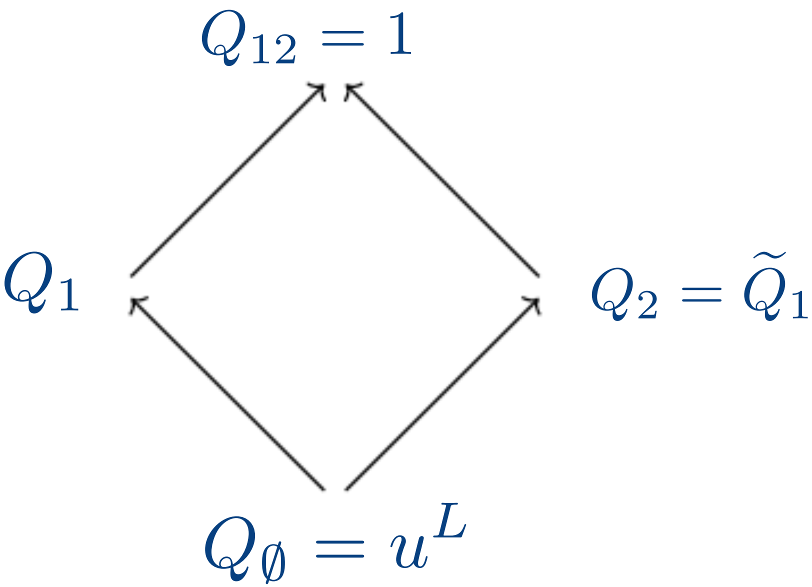
$$\left(u + V\tfrac{i}{2}\right)^L Q_l^- Q_r^+ - \left(u - V\tfrac{i}{2}\right)^L Q_l^+ Q_r^- = i(VL - \eta_1 K_l + \eta_1 K_r)\, u\, Q_m \widetilde{Q}_m,$$

$$\widetilde{\mathbb{D}} = \left(\frac{2i}{V}\right)^L (VL - \eta_1 K_l + \eta_1 K_r) \frac{Q_m(0)\widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)} \mathbb{D},$$

Found numerically
Analytical proof
in progress

NB: K_r odd or K_l odd requires regularization

Bosonic Dualities: A warm-up example: $SU(2)$



Bosonic duality eqn.

$$Q_1^+ \widetilde{Q}_1^- - Q_1^- \widetilde{Q}_1^+ = u^L$$

$$\widetilde{K} = L - K + 1$$

(States beyond the equator)

Dual roots at $0, \pm\frac{i}{2}$ call for regularization of $\det G$

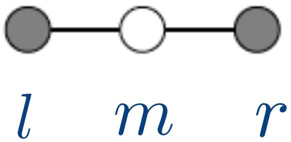
After regularization: Roots at $0, \pm\frac{i}{2}$ left out in $\widetilde{Q}$

$$\widetilde{\mathbb{D}} = \mathbb{A}_{L/2-K} \, \frac{Q(0)\widetilde{Q}(i/2)}{Q(i/2)\widetilde{Q}(0)} \, \mathbb{D}, \qquad \mathbb{A}_n = \frac{(2^n n!)^4}{2\,(2n)!\,(2n+1)!}$$

Overlaps with VBS Duality invariant

Bosonic dualities in general

- Involve a bosonic node and its neighbours only



- Do not change the Dynkin diagram or the Dynkin labels

- Momentum carrying bosonic node

$$\widetilde{\mathbb{D}} = \mathbb{A}_{(L+K_r+K_l)/2-K_m} \frac{Q_m(0)\widetilde{Q}_m(i/2)}{Q_m\left(\frac{i}{2}\right)\widetilde{Q}_m(0)} \mathbb{D}$$

- Non-momentum carrying bosonic node

$$\widetilde{\mathbb{D}} = \mathbb{A}_{(K_r+K_l)/2-K_m} \frac{Q_m(0)\widetilde{Q}_m(i/2)}{Q_m\left(\frac{i}{2}\right)\widetilde{Q}_m(0)} \mathbb{D}$$

- Overlaps in the scalar $SO(6)$ sector invariant (up to pre-factor)

A note on singular roots

C.K., Müller, Zarembo '21

$\langle \text{VBS}|\mathbf{u}\rangle \neq 0$: momentum carrying roots: $\{u_i, -u_i\}_{i=1}^{K_u}$

auxiliary roots: $\{v_i, -v_i\}_{i=1}^{K_v}$, possibly $\cup\{0\}$

Duality transformations introduce dual roots at $0, \pm\frac{i}{2}$

Need to know the Gaudin matrix, i.e. $G_\pm$ for these cases

Root at zero: Naturally included in G_+

Roots at $\pm\frac{i}{2}$: Lead to divergencies in G_+, G_-
Can be regulated by impurities and
removing row and column of singular root from G_-

Special challenge for bosonic dualities: $\det \widetilde{G}_+ = 0$

Regulated by removing the entries of the zero root

Summary

- We have determined the transformation properties of the Gaudin super determinants under all fermionic and bosonic dualities encoded in the QQ-system.

- We can translate overlap formulas between any two Dynkin diagrams (Kostya's talk).

Open Problems

- Analytical proof of the duality transformation formulas
  Easy to state --- difficult to prove

- Understand the pre-factors in the transformation formulas

- Understanding the rationale for the regularization
  in the bosonic case

- Express the overlaps entirely in terms of Q-functions
  and treat the overlaps by means of the Quantum Spectral Curve

Thank you