SEPARATION OF VARIABLES FOR THE FISHNET CFT

THE FUNCTIONAL METHOD

Based on hep-th/2103.15800 and works with N. Gromov and F. Levkovich-Maslyuk, +D.Grabner, A. Sever

ENS-IPhT workshop
“Correlators and wave functions in solvable models”

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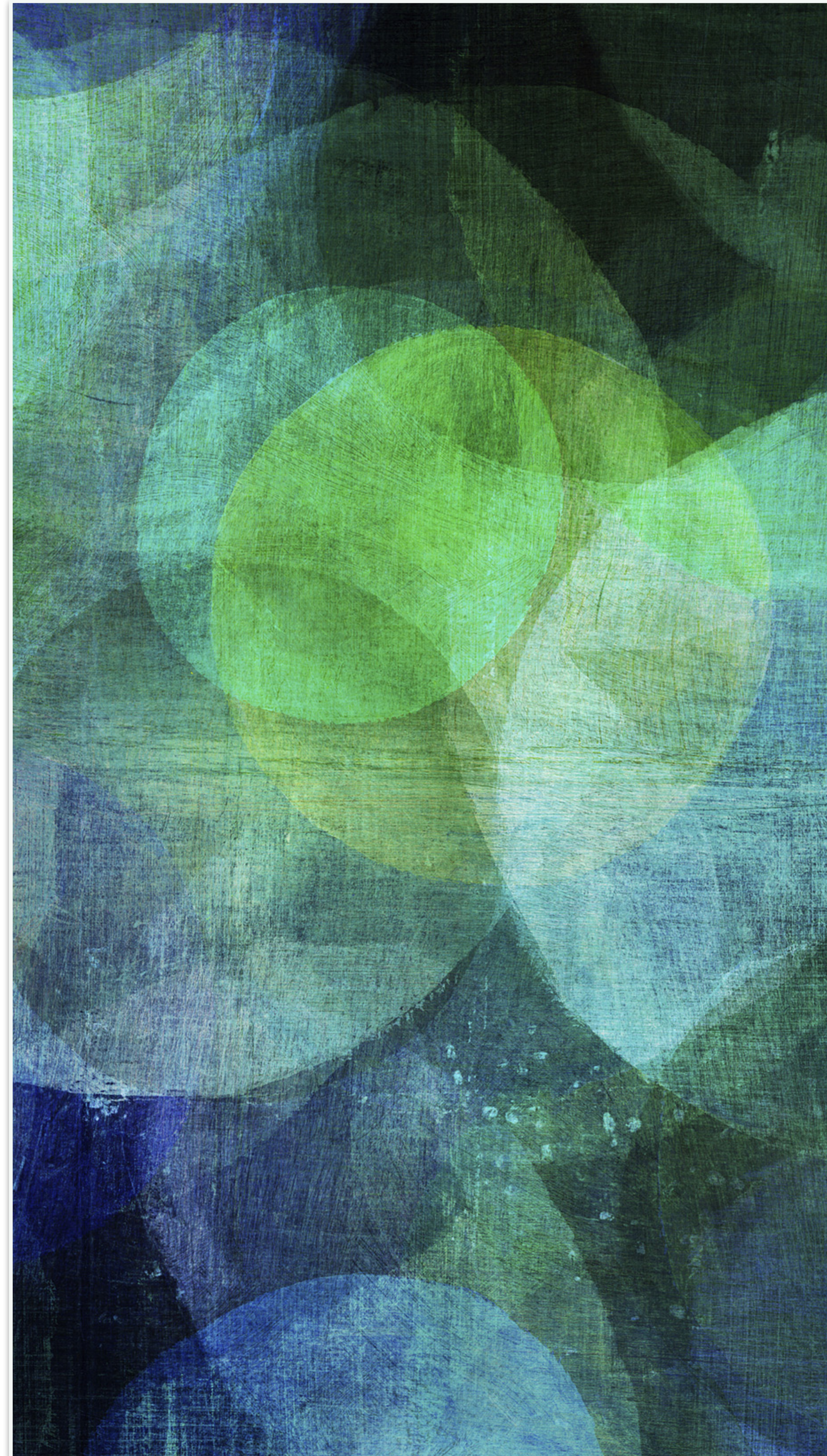
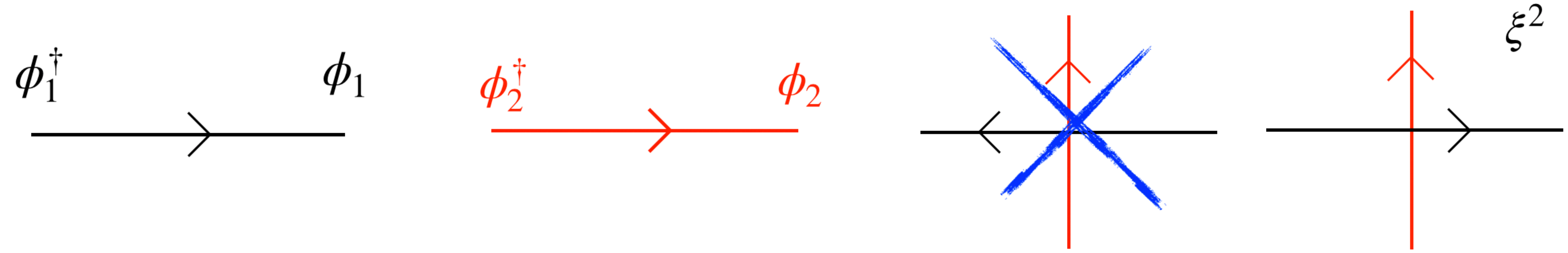


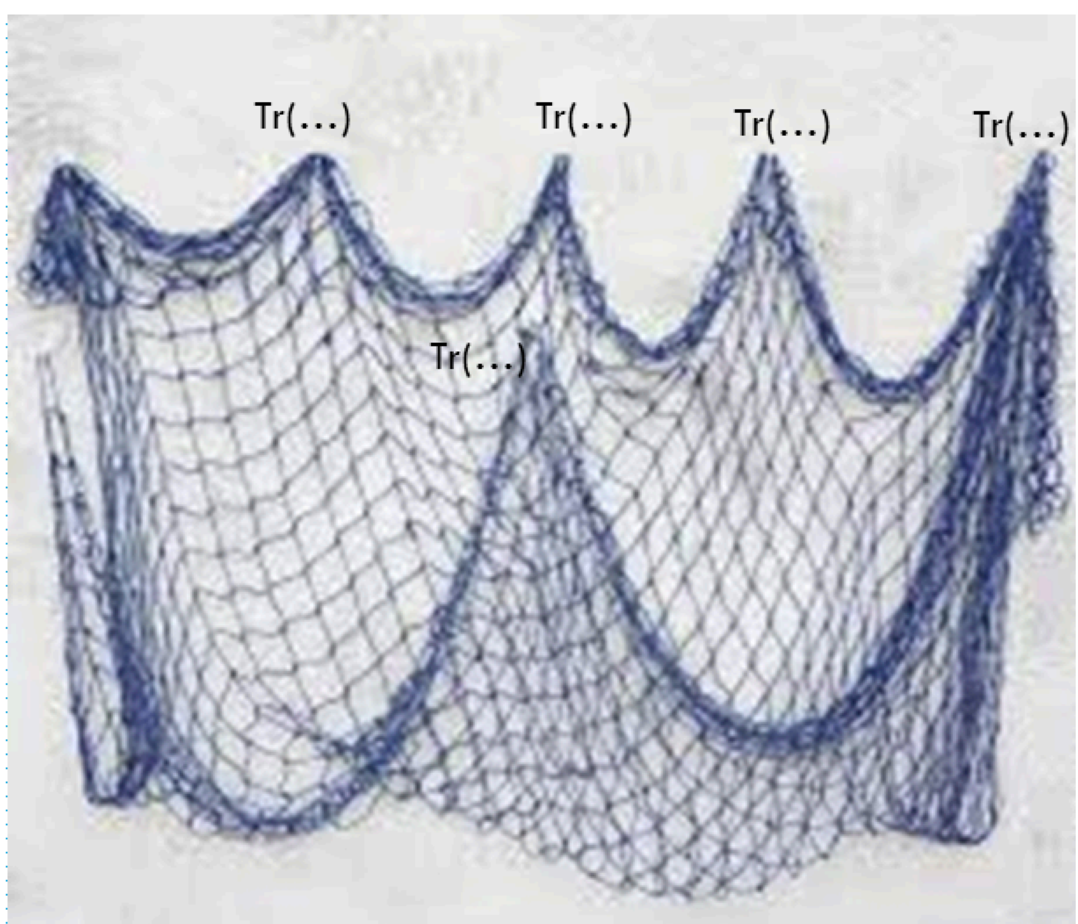
Fishnet theory (limit of marginal deform. of $\mathcal{N}=4$ SYM) [Gurdogan Kazakov '15]

$$\mathcal{L}_{bi-scalar}[\phi_1,\phi_2]=\frac{N_c}{2}tr\left(\partial^\mu\phi_1^\dagger\partial_\mu\phi_1+\partial^\mu\phi_2^\dagger\partial_\mu\phi_2+2\xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right)$$

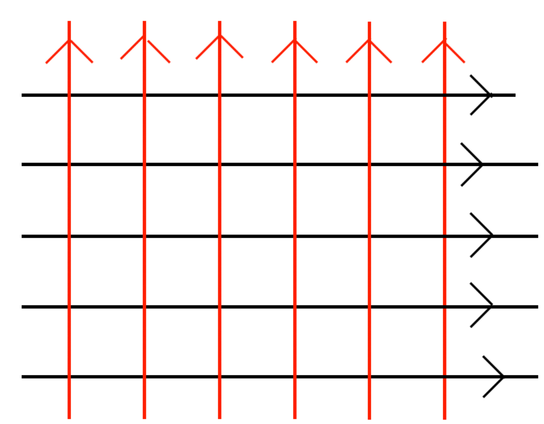
't Hooft coupling

Feynman rules:





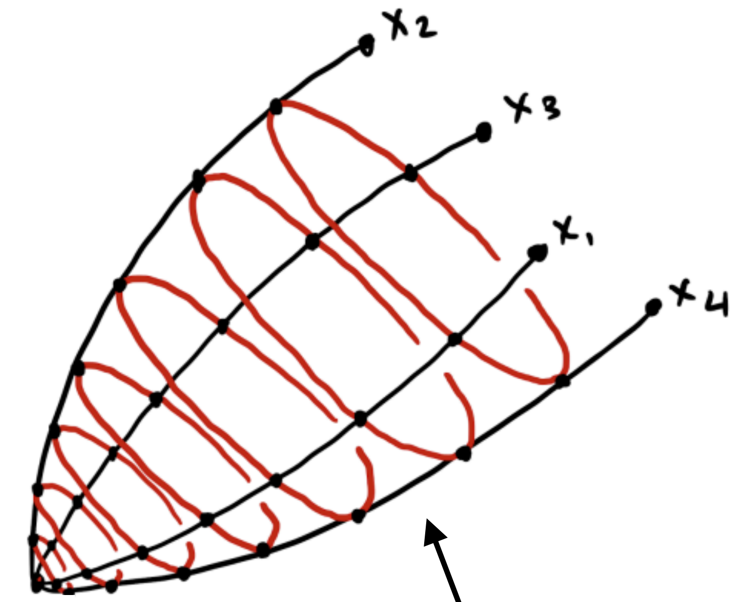
Only diagrams with "fishnet" structure in the bulk



Interesting observables: correlators of single traces

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\rangle$$

Key object: CFT wave function *[Gromov,Sever'19]*



$\mathcal{O}(x_0)$

Renormalised sum of all wheels

Operator/(*Spin chain*) state correspondence

$$\Psi(x_0\,|\,x_1, x_2, \ldots, x_J) \equiv \langle \mathcal{O}(x_0)\, \textbf{Tr}\left(\phi_1^\dagger(x_1)\ldots\phi_1^\dagger(x_J)\right)\rangle$$

Each site: representation of the 4D conformal group

$$\mathbb{P}_\mu \equiv -i\partial_\mu, \qquad \mathbb{K}_\mu \equiv i\left(x^2\partial_\mu - 2x_\mu(x\cdot\partial) - 2hx_\mu\right)$$
$$\mathbb{S}_{\mu\nu} \equiv ix_\mu\partial_\nu - ix_\nu\partial_\mu, \qquad \mathbb{D} \equiv -ix^\mu\partial_\mu - ih\,.$$

$h = 1$ or 2 at each site

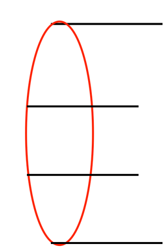
The wave function contains all information on the operator,

e.g. the scaling dimension $\sum_{\alpha=1}^{J} \mathbb{D}_\alpha \circ \Psi(0\,|\,x_1, \ldots, x_J) = \Delta\, \Psi(0\,|\,x_1, \ldots, x_J)$

… and is eigenstate of an **integrable Hamiltonian**

[Gromov,Sever'19]
+ [Grabner,Gromov,Kazakov,,Korchemsky,Negro…]

$$\hat{\mathbb{H}} \circ \Psi = \xi^{2J}\, \Psi$$

$$\hat{\mathbb{H}} = \Big(\ $$  $$\ \Big)^{-1}$$

Spin chain integrability

Commuting transfer matrices

conf. generators

$$\hat{\mathbb{L}}^{\mathbf{4}}_k(u) = u\,\mathbb{I}^{(\text{phys})} \otimes \mathbb{I}^{(\text{aux})} - \frac{i}{2}\hat{q}_k^{MN} \otimes \Sigma_{MN}$$

spectral parameter

4x**4** *Sigma matrices*

$$\hat{\mathbb{T}}^{\mathbf{r}}(u) = \mathrm{Tr}_{\mathbf{r}}\left[\hat{\mathbb{L}}^{\mathbf{r}}_J(u-\vartheta_J)\hat{\mathbb{L}}^{\mathbf{r}}_{J-1}(u-\vartheta_{J-1})\dots\hat{\mathbb{L}}^{\mathbf{r}}_1(u-\vartheta_1)\,G^{\mathbf{r}}\right]$$

twist

aux. space $\mathbf{r} \in \{\mathbf{1}, \mathbf{4}, \mathbf{6}, \bar{\mathbf{4}}, \bar{\mathbf{1}}\}$ *found in [Gromov,Sever '19]*

[AC,Grabner,Gromov,Sever '19]

inhomogeneities

$$\left[\hat{\mathbb{T}}^{\mathbf{r}}(u), \hat{\mathbb{T}}^{\mathbf{r}'}(v)\right] = 0 \qquad \hat{\mathbb{H}} = \hat{\mathbb{T}}^{\mathbf{6}}(0)$$

generate a complete set of 4xJ independent integrals of motion!

$$\mathbb{T}^{\bar{\mathbf{4}}}(u) \propto P_J^{\bar{\mathbf{4}}}(u)$$

$$\mathbb{T}^{\mathbf{6}}(u) \equiv P_{2J}^{\mathbf{6}}(u)$$

$$\mathbb{T}^{\mathbf{4}}(u) \equiv P_J^{\mathbf{4}}(u)$$

All the usual integrability structures are here,

but we cannot use the Bethe Ansatz.

Need to use another spell in the integrability book

TQ equations (quantum spectral curve for this model) *[Krichever-Lipan-Wiegmann-Zabrodin'96]*

$$0 = \left(\mathbb{T}^{\mathbf{1}}(u+i)D^4 - \mathbb{T}^{\mathbf{4}}(u+\tfrac{i}{2})\,D^2 + \mathbb{T}^{\mathbf{6}}(u) - \mathbb{T}^{\bar{\mathbf{4}}}(u-\tfrac{i}{2})\,D^{-2} + \mathbb{T}^{\bar{\mathbf{1}}}(u-i)\,D^{-4}\right)\circ Q(u)$$

$$D \circ f(u) = f(u+\tfrac{i}{2})$$

TQ equations for the fishnet model: *[Gromov-Sever]*

$$C_\theta(u) \equiv \prod_{\alpha=1}^{J}(u-\theta_\alpha)$$

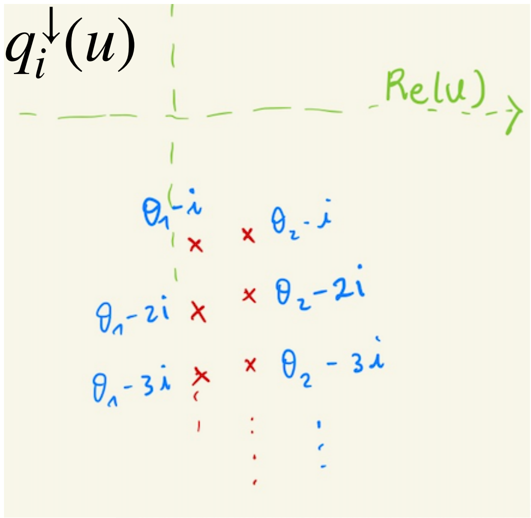
$$0 = \Big(C_\theta(u+i)\, D^4 - P^{\mathbf{4}}_J(u+\tfrac{i}{2})\, D^2 + P^{\mathbf{6}}_{2J}(u) - P^{\bar{\mathbf{4}}}_J(u-\tfrac{i}{2})\, D^{-2} + C_\theta(u-i)\, D^{-4} \Big) \circ q_a(u)$$

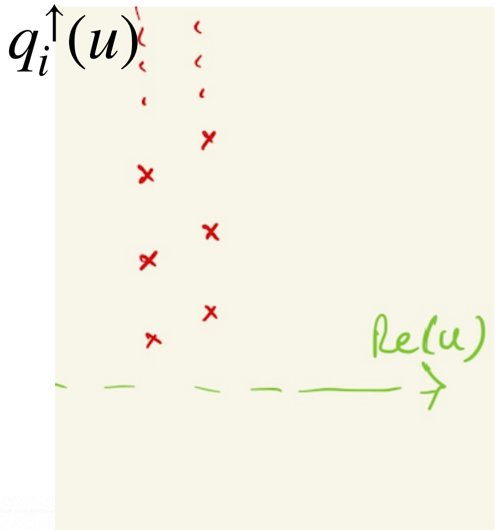
Contains 4J unfixed int. of motion (incl. ξ^2, Δ)

Two bases of solutions:

$$q_i^{\downarrow/\uparrow}(u) \simeq \lambda_i^{-iu}\, u^{\hat{M}_i} \left(1 + O(\tfrac{1}{u})\right)$$

$$\hat{M}_i = \left(\frac{\Delta - S_1 - S_2 - D_0}{2}, \frac{\Delta + S_1 + S_2 - D_0}{2}, \frac{-\Delta - S_1 + S_2 - D_0}{2}, \frac{-\Delta + S_1 - S_2 - D_0}{2}\right)$$



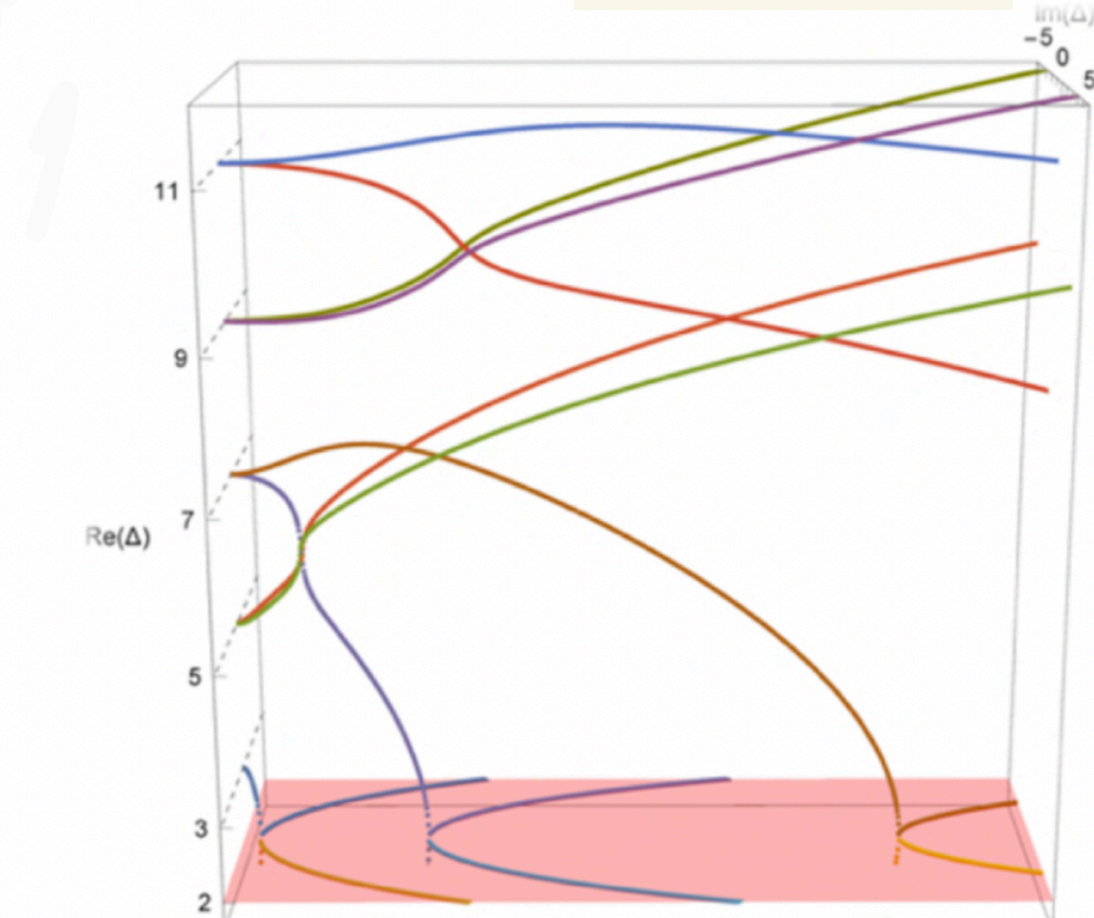


Quantisation condition: *[AC,Grabner,Gromov,Sever'20]*

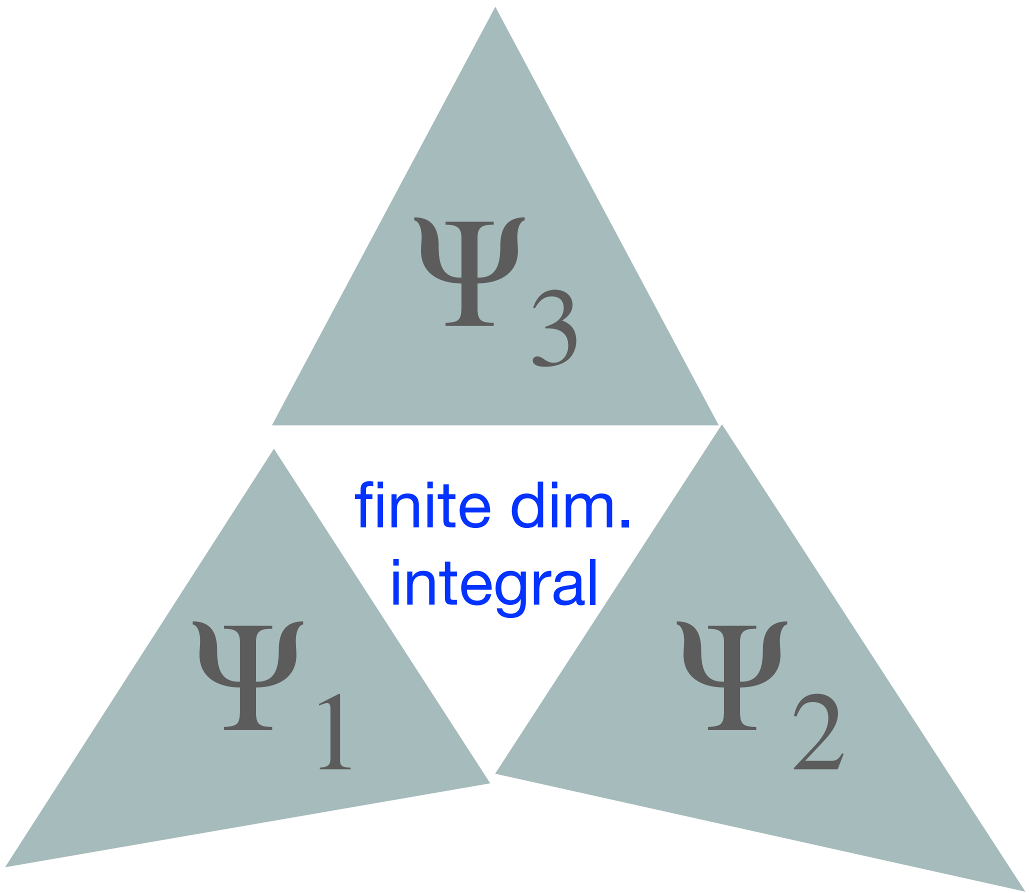
$$\Omega_1^{\ 2} = \Omega_2^{\ 1} = \Omega_3^{\ 4} = \Omega_4^{\ 3} = 0$$

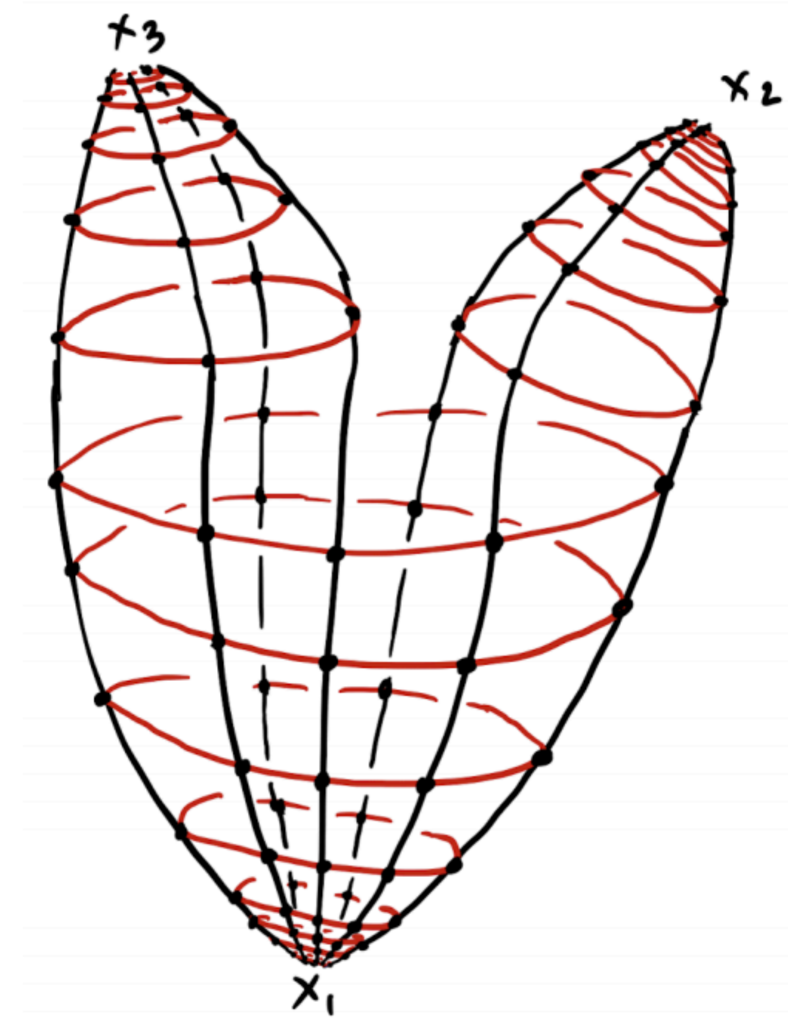
where $\Omega(u) = \Omega(u+i)$ s.t. $q_i^{\uparrow}(u) = \Omega_i^{\ j}(u) q_j^{\downarrow}(u)$

fixes the whole spectrum!

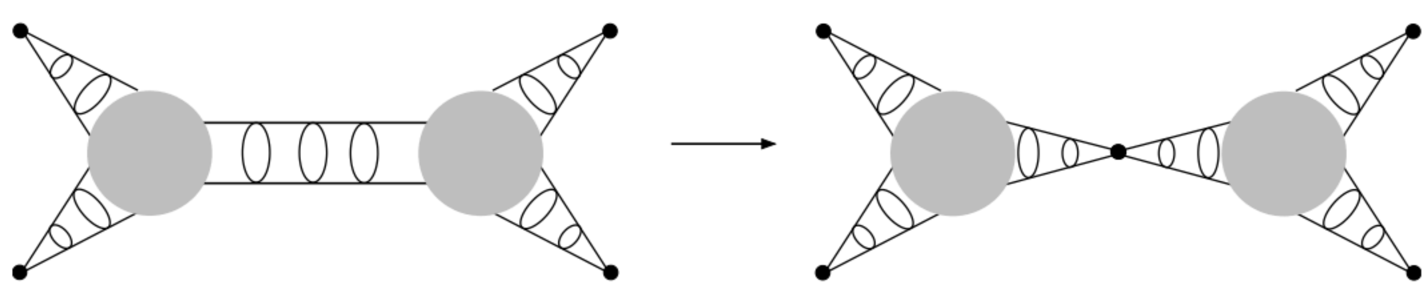


The wave functions are non-perturbative **building blocks for correlators**, joined by coupling-independent vertex





If one understands 3 pts, techniques of *[Gross Rosenhaus '17]* could become powerful



Can we compute the wave functions (and 3-point vertices) with Separation of Variables? *[Sklyanin '90s]+….*

$$\langle \overrightarrow{x} | \Psi \rangle = \prod_i^L Q_{(i)}(x_i)$$

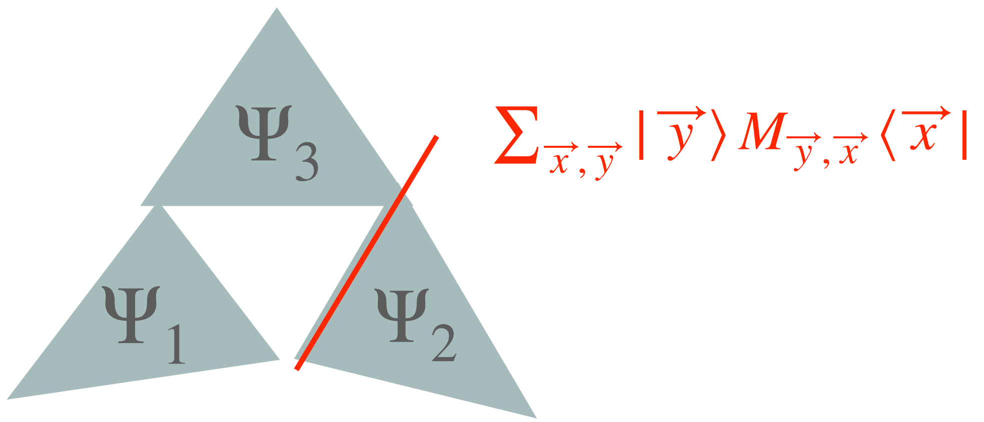
Then we could construct

$$\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle = \int \left(\prod Q^{\textbf{op. 1}} \right) \left(\prod Q^{\textbf{op. 2}} \ldots \right) \left(\prod Q^{\textbf{op. n}} \right) \, dM_n$$

From examples we can hope M_n would be simple

[AC,Gromov,Levkovich-Maslyuk'18][Giombi Komatsu '18]

An important and useful element of the construction is the completeness relation of SoV bases



Operatorial approach:

The goal is finding a good description of the SoV basis, and determine the form of the factorisation of eigenstates

$$\langle \vec{x} \, | \Psi \rangle = \prod_i^L Q_{(i)}(x_i)$$

(complicated, but with a clear paradigm)

e.g. **SoV basis** $\langle \vec{x} \, | =$ eigenstates of $\hat{\mathbb{B}} \simeq \hat{T}^K_{12}$ *[Sklyanin (SL(2)) '90]*

exciting developments for higher rank

[Sklyanin SL(3)]; [Smirnov][Gromov,Levkovich-Maslyuk,Sizov]SL(N)
[Maillet Niccoli'18][Ryan Volin'18] …
[Gromov,Levkovich-Maslyuk,Ryan,Volin'19]
[Gromov,Levkovich-Maslyuk,Ryan'20],….

open fishnet graphs *[Derkachov Olivucci '19,'21,'21]*

2D fishnets *[Derkachov,Korchemsky,Manashov'01]*
[Derkachov,Kazakov,Olivucci '19]

Functional approach:

The goal is deducing aspects of the SoV construction using only the TQ equations. Fixes the form of **SoV measure** and some "prototype" observables.

crucial to find measure beyond SL(2)

[AC,Gromov,Levkovich-Maslyuk'19]

[Gromov,Levkovich-Maslyuk,Ryan,Volin'19]

Today: functional approach and the fabric of SoV

Operatorial technology for the future

Integrable eigenstates should be orthogonal

$$\langle \Psi_A | \Psi_B \rangle \propto \mathcal{N}_A \, \delta_{AB}$$

Assume SoV $\langle \vec{x} | \Psi_B \rangle = \prod_i Q^B(x_i), \quad \langle \Psi_A | \vec{y} \rangle = \prod_i Q^A(y_i)$

$$M_{\vec{y},\vec{x}} = \langle \vec{x} | \vec{y} \rangle^{-1}$$

state-independent
SoV measure

$$1 = \sum_{\vec{x},\vec{y}} | \vec{y} \rangle M_{\vec{y},\vec{x}} \langle \vec{x} |$$

Non diagonal
[AC,Gromov,Levkovich-Maslyuk'19]
[Gromov,Levkovich-Maslyuk,Ryan,Volin'19]

$$\langle \Psi_A | \Psi_B \rangle = \sum_{\vec{x},\vec{y}} \prod_i Q^A(y_i) \, M_{\vec{y},\vec{x}} \prod_i Q^B(x_i) \propto \mathcal{N}_A \, \delta_{AB}$$

This gives infinitely many constraints!
Expected to fix the form of the SoV scalar product

SL(2) example

source term $Q_\theta = \prod_{i=1}^{L}(u-\theta_i)$

TQ: $$\hat{O} \circ Q \equiv Q_\theta^+ Q^{++} - TQ + Q_\theta^- Q^{--} = 0$$

$$T(u) = 2\cos\phi\, u^L + \sum_{n=0}^{L-1} I_n u^n .$$

T(u) contains integrals of motion

Ingredient: bilinear pairing of Q's

$$(gf)_j \equiv \int_{-\infty}^{+\infty} du\, g(u)\mu_j(u)f(u) \qquad \mu_j(u) = \frac{1}{1+e^{2\pi(u-\theta_j)}} \ , \quad j = 1,\dots,L$$

Key conjugation relation: $\langle F_1\, \hat{O} \circ F_2 \rangle = \langle F_2\, \hat{O} \circ F_1 \rangle$

$$\hat{O} \circ Q \equiv Q_\theta^+ Q^{++} - TQ + Q_\theta^- Q^{--} = 0$$

Take two states, A and B

$$0 = 0 - 0 = \langle Q_1^A \, \hat{O}^B \circ Q_1^B \rangle - \langle Q_1^B \, \hat{O}^A \circ Q_1^A \rangle = \langle Q_1^A \, (\hat{O}^B - \hat{O}^A) \circ Q_1^B \rangle$$

$$= \langle Q_1^A \, (T_A(u) - T_B(u)) \, Q_1^B \rangle$$

$$\Longrightarrow \quad \sum_{i=1}^{L} (I_{i-1}^A - I_{i-1}^B) \langle Q_1^A \, u^{i-1} Q_1^B \rangle_j = 0 \ , \quad j = 1, \ldots, L$$

Two different states have different IM eigenvalues

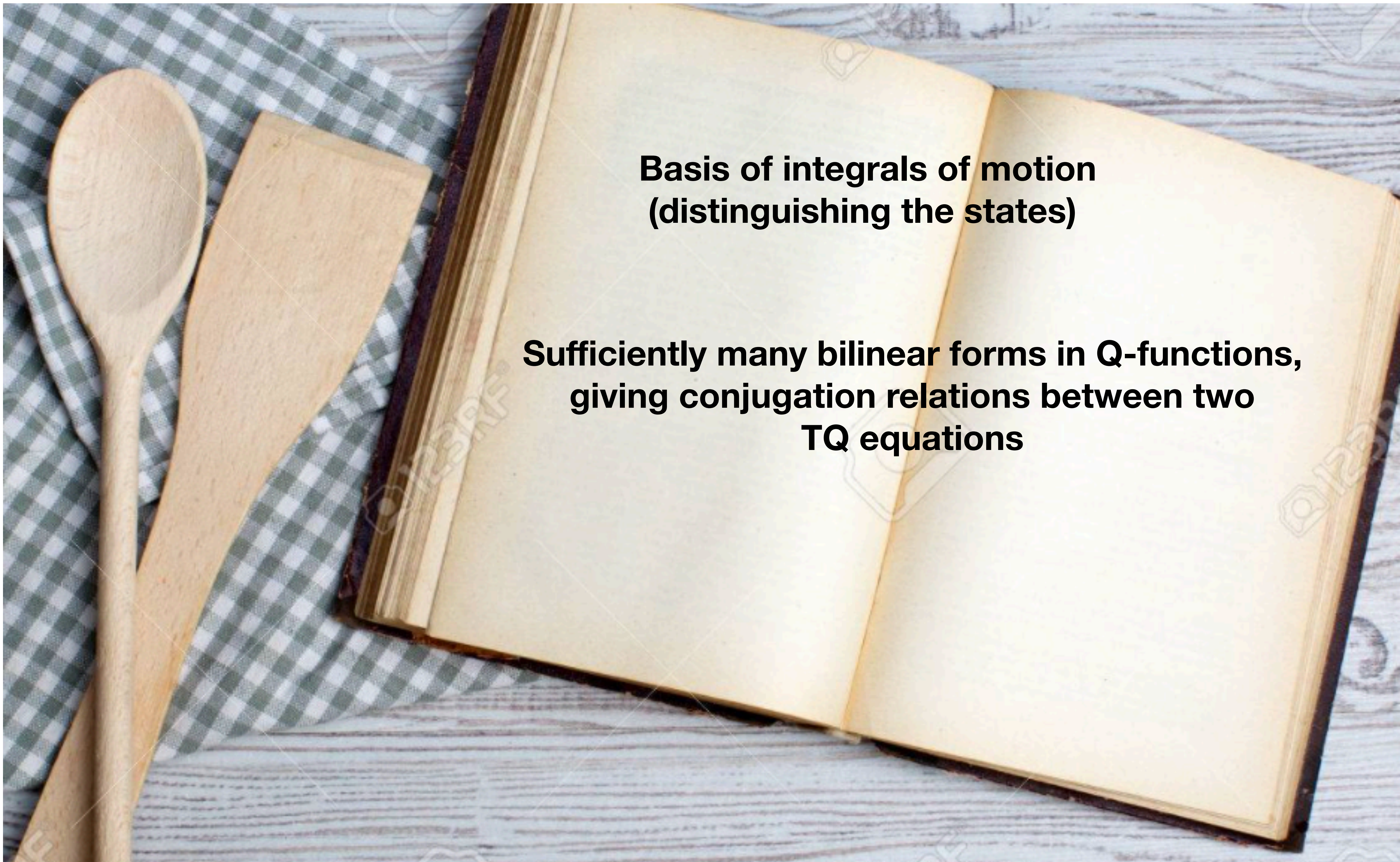
$$\det \left| \langle Q_1^A \, u^{i-1} Q_1^B \rangle_j \right|_{1 \leqslant i,j \leqslant L} \propto \delta_{AB}$$

$$\int \prod_{i=1}^{L} dx_i \, Q_1^A(x_i) M(\mathbf{x}) Q_1^B(x_i) \propto \delta_{AB} \qquad M(\mathbf{x}) = \frac{\prod_{j<k} (e^{2\pi x_j} - e^{2\pi x_k})(x_j - x_k)}{\prod_{j,k} (1 + e^{2\pi(x_j - \theta_k)})}$$

Matches with the SoV measure of *[Derkachov Korchemsky Manashov '01]*

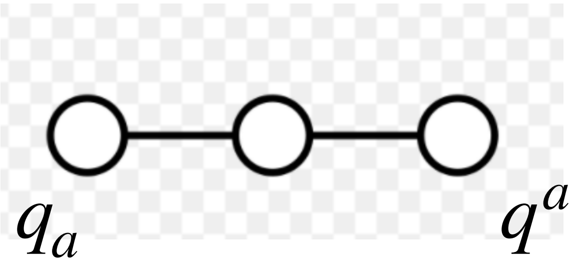
This procedure is understood in SL(N) *[AC,Gromov,Levkovich-Maslyuk'18]*

SoV scalar product cookbook



$$0=\underbrace{\left(C_\theta(u+i)\,D^4-P_J^{\mathbf{4}}(u+\tfrac{i}{2})\,D^2+P_{2J}^{\mathbf{6}}(u)-P_J^{\bar{\mathbf{4}}}(u-\tfrac{i}{2})\,D^{-2}+C_\theta(u-i)\,D^{-4}\right)}\circ q_a(u)$$

$\equiv \mathscr{B}$ finite diff. op.



$P_J^{\mathbf{4}}(u),\ P_J^{\bar{\mathbf{4}}}(u),\ P_{2J}^{\mathbf{6}}(u)$: contain int. of motion

"dual" Q-functions: $q^a(u)\equiv C_\theta(u)\,\epsilon^{abcd}q_b(u+i)q_c(u)q_d(u-i)$

$$0=\underbrace{\left(C_\theta(u+i)\,D^4-P_J^{\bar{\mathbf{4}}}(u+\tfrac{i}{2})\,D^2+P_{2J}^{\mathbf{6}}(u)-P_J^{\mathbf{4}}(u-\tfrac{i}{2})\,D^{-2}+C_\theta(u-i)\,D^{-4}\right)}\circ q^a(u)$$

$\equiv \mathscr{B}^{dual}$ $\quad(\mathbf{4}\leftrightarrow\bar{\mathbf{4}})$

$$(f(u),g(u))_\mu\equiv\int_{|} f(u)g(u)\,\mu(u)du \qquad \mu(u+i)=\mu(u)$$

$$\boxed{(f(u),\mathscr{B}\circ g(u))_\mu=(\mathscr{B}^{dual}\circ f(u),g(u))_\mu}$$

Orthogonality and Q-functions

$\mathscr{B} = \ldots + P_J^{\mathbf{4}}(u+\frac{i}{2})\, D^2 + \ldots$

depends on the state through
int. of motion = coefficients of polynomials

Different states A, B $\quad \mathscr{B}_A \circ q_A = 0\,, \qquad \mathscr{B}_B^{dual} \circ q_B^{dual} = 0$

$$\longrightarrow \quad (q_B^{dual}\left(\mathscr{B}_A - \mathscr{B}_B\right)\circ q_A)_\mu = 0$$

Basis of linearly indep. measures

$$\boxed{\begin{array}{c}\mu_\alpha(u) \propto \prod_{\beta\neq\alpha}(1-e^{2\pi(u-\theta_\beta)}),\\ \alpha = 1,\ldots,J\end{array}}$$

$16 \times J$ linear equations for the difference $\vec{I}_A - \vec{I}_B$

$$(\mathscr{M}^{AB})\cdot(\vec{I}_A - \vec{I}_B) = 0$$

$$\mathscr{M}^{AB} = \begin{pmatrix} \cdots & (q_B^{dual,1}(u+\frac{i}{2})^k D^2 \circ q_{A,1})_{\mu_1} & \cdots & (q_B^{dual,1}(u-\frac{i}{2})^{J-1} D^{-2} \circ q_{A,1})_{\mu_1} \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \cdots & (q_B^{dual,4}(u+\frac{i}{2})^k D^2 \circ q_{A,4})_{\mu_J} & \cdots & (q_B^{dual,4}(u-\frac{i}{2})^{J-1} D^{-2} \circ q_{A,4})_{\mu_J} \end{pmatrix}$$

$$\longrightarrow \quad \textbf{det}\left[\mathscr{M}^{AB}\right]_{4J\times 4J} = 0$$

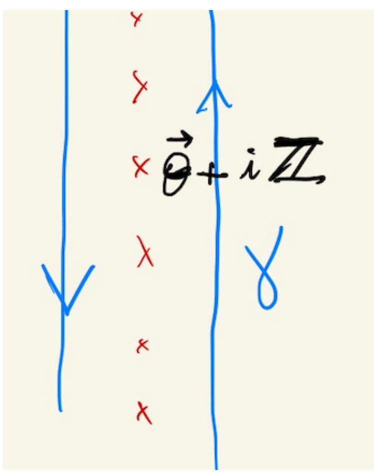
Can we actually compute the integrals?

$$\boxed{\begin{array}{c}\mu_\alpha(u) \propto \prod_{\beta\neq\alpha}(1-e^{2\pi(u-\theta_\beta)}),\\ \alpha=1,\ldots,J\end{array}}$$

Q-bilinear forms:

$$(q^a_{state\,A}(u), \hat{O}\circ q_{state\,B,\,b}(u))_\mu \equiv \int_{|} q^{a\uparrow}_{state\,A}(u)\,\hat{O}\circ q^{\downarrow}_{state\,B,\,b}(u)\,\mu(u)du$$

integration contour



Integral is often divergent. $q_i^{\downarrow/\uparrow}(u) \simeq \lambda_i^{-iu}\,u^{\hat{M}_i}\left(1+O(\frac{1}{u})\right)$

Take ζ-regularised sum of residues.

$$(q^B)^{c\downarrow}(u)\;u^k\;q_b^{A\downarrow}(u+im) \simeq (\lambda_b^A)^{-iu}(\lambda_c^B)^{+iu}u^{+\hat{M}_a^A-\hat{M}_c^B-D_0+k}\left(1+\mathcal{O}\left(\frac{1}{u}\right)\right)$$

$$\sum_{n=1}^{\infty}\frac{\lambda^n}{n^\alpha}=\mathrm{Li}_\alpha(\lambda)$$

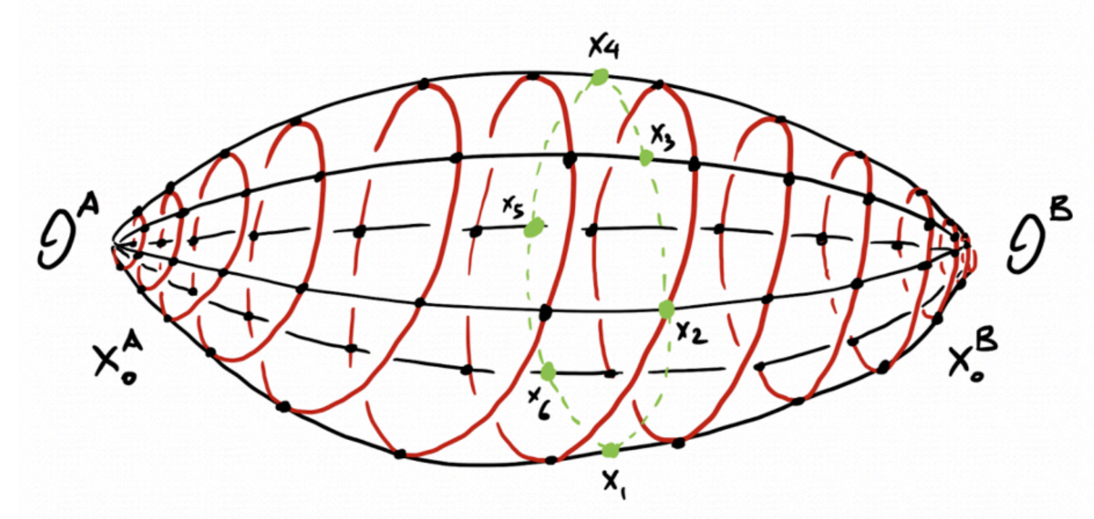
Finite for all different states $A\neq B$

Very fast numerical evaluation. Residues come for free from Ω.

Need only a finite number of parameters, and sampling Q's at $\overrightarrow{\theta}+i\mathbb{N}$

Special combination of indices is finite for A = B, due to $\Omega_1^{\ 2}=\Omega_2^{\ 1}=\Omega_3^{\ 4}=\Omega_4^{\ 3}=0$

$$\langle \Psi_{A}, \Psi_{B} \rangle \propto \mathbf{det}\left[\mathscr{M}^{AB}\right]_{4J\times 4J}$$



Does it match with the norm of the conf. invariant scalar product?

$$\langle \Psi_{A}, \Psi_{B} \rangle \propto \int \prod_{i=1}^{J} d^4 y_i \Psi_{A}(x_{\bar{0}} \,|\, y_1, \ldots, y_J) \prod_k \Box_{y_k}^{2-h_k} \Psi_{B}(x_0 \,|\, y_1, \ldots, y_J)$$

$$\Box^{\beta} f(x) \equiv \frac{(-4)^{\beta}\Gamma(2+\beta)}{\pi^2 \Gamma(-\beta)} \int d^4 y \frac{f(y)}{|x-y|^{4+2\beta}} \,.$$

$$\langle \Psi_A, \Psi_B \rangle \propto \int \prod_{i=1}^{J} d^4 y_i \Psi_A(x_{\bar{0}} \,|\, y_1, \ldots, y_J) \prod_k \Box_{y_k}^{2-h_k} \Psi_B(x_0 \,|\, y_1, \ldots, y_J)$$

$$\Psi_A(x_0 \,|\, \overrightarrow{x}) \sim \epsilon^{\Delta_A - h_{tot}} \times O(1) \qquad \Psi_B(x_{\bar{0}} \,|\, \overrightarrow{x}) \sim \epsilon^{-\Delta_B - h_{tot}} \times O(1)$$

$$\epsilon \sim |x_i - x_0|$$

Divergence for $\Delta_A = \Delta_B$: $\quad \langle \Psi_A, \Psi_B \rangle \propto \left(\log(\epsilon_{UV}) \times \langle \Psi_A, \Psi_B \rangle_{fin} + \ldots \right)$

$\langle \Psi_A, \Psi_B \rangle \propto$ **det** $\left[\mathscr{M}^{AB} \right]_{4J \times 4J}$

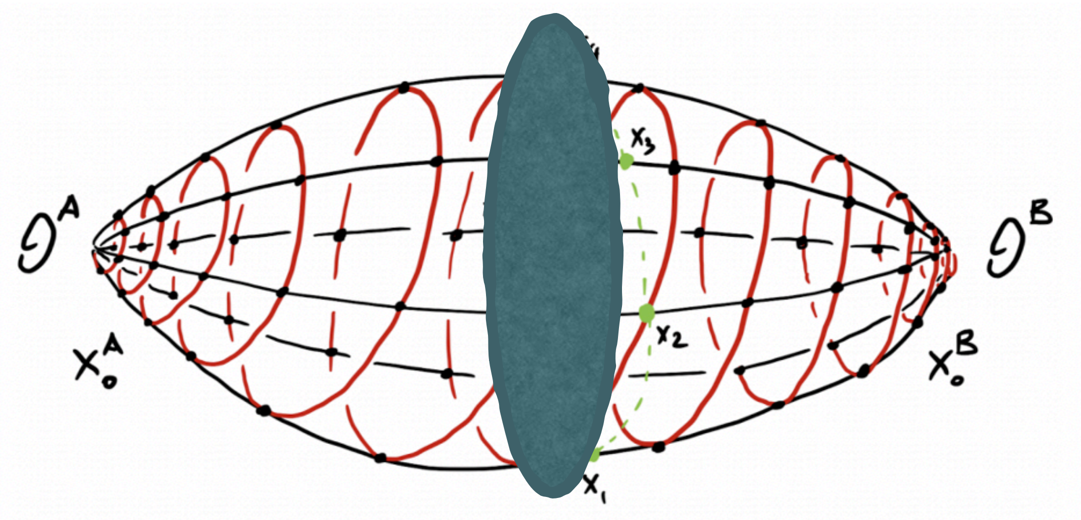
for $A = B$ we have a divergence (for all minors except one)
which matches the behaviour of the scalar product. Otherwise we get correctly zero!

Would be interesting to regularise (twists?) and match expressions for the norm

We can also compute some "prototype observables" *[AC,Gromov,Levkovich-Maslyuk '19]* *[Gromov,Levkovich-Maslyuk,Ryan'20]*

$$\langle \Psi | \partial_p \hat{I} | \Psi \rangle$$

parameter e.g. θ_α, ξ^2, h_α, twists,..
Roughly $\sim J^2$ operators



Diagonal form-factor

include special class of 3-point functions

$$\frac{\partial}{\partial \xi^2} \Delta_{\mathcal{O}} \simeq \langle \mathcal{O}^\dagger \, \mathcal{O} \, \textbf{Tr}(\phi_1 \phi_2 \phi_1^\dagger \phi_2^\dagger) \rangle$$

We obtain an informative expression as ratio of SoV-type determinants

Example - coupling variation

A **unique minor is finite**
thanks to the quantisation conditions

$$\mathscr{M}^{AB} = \begin{pmatrix} \cdots & (q_B^{dual,1}(u+\frac{i}{2})^k D^2 \circ q_{A,1})_{\mu_1} & \cdots & (q_B^{dual,1}(u-\frac{i}{2})^{J-1} D^{-2} \circ q_{A,1})_{\mu_1} \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \ddots & \vdots & \ddots & \vdots \\ \cdots & (q_B^{dual,4}(u+\frac{i}{2})^k D^2 \circ q_{A,4})_{\mu_J} & \cdots & (q_B^{dual,4}(u-\frac{i}{2})^{J-1} D^{-2} \circ q_{A,4})_{\mu_J} \end{pmatrix}$$

$$\left[\mathscr{M}^{\mathcal{O},\mathcal{O}}_{finite}\right]_{4J\times 4J} \cdot \partial_{\xi^2} \vec{I} = 0$$

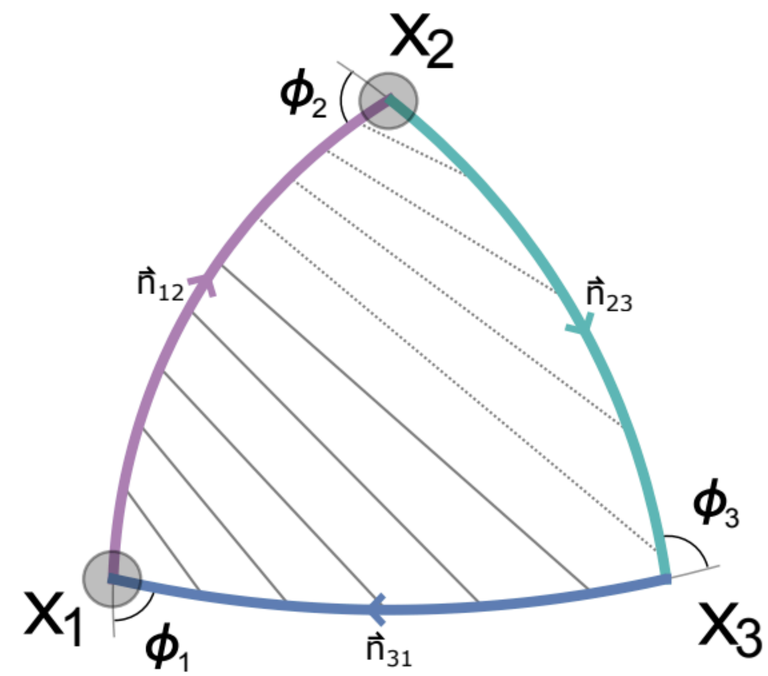
Unique solution,
(normalisation of null vector is fixed).

$$\boxed{\begin{aligned} &\frac{\partial}{\partial \xi^2}\Delta_{\mathcal{O}} \simeq \langle \mathcal{O}^\dagger\, \mathcal{O}\, \mathbf{Tr}(\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger)\rangle \\ &= \text{Ratio of SoV-type determinants} \end{aligned}}$$

Compare with

$$C_{123}^{\bullet\bullet\circ} = \frac{\langle q_1\, q_2\, e^{-\phi_3 u}\rangle}{\sqrt{\langle q_1^2\rangle\langle q_2^2\rangle}}$$

[AC Gromov Levkovich-Maslyuk'18]

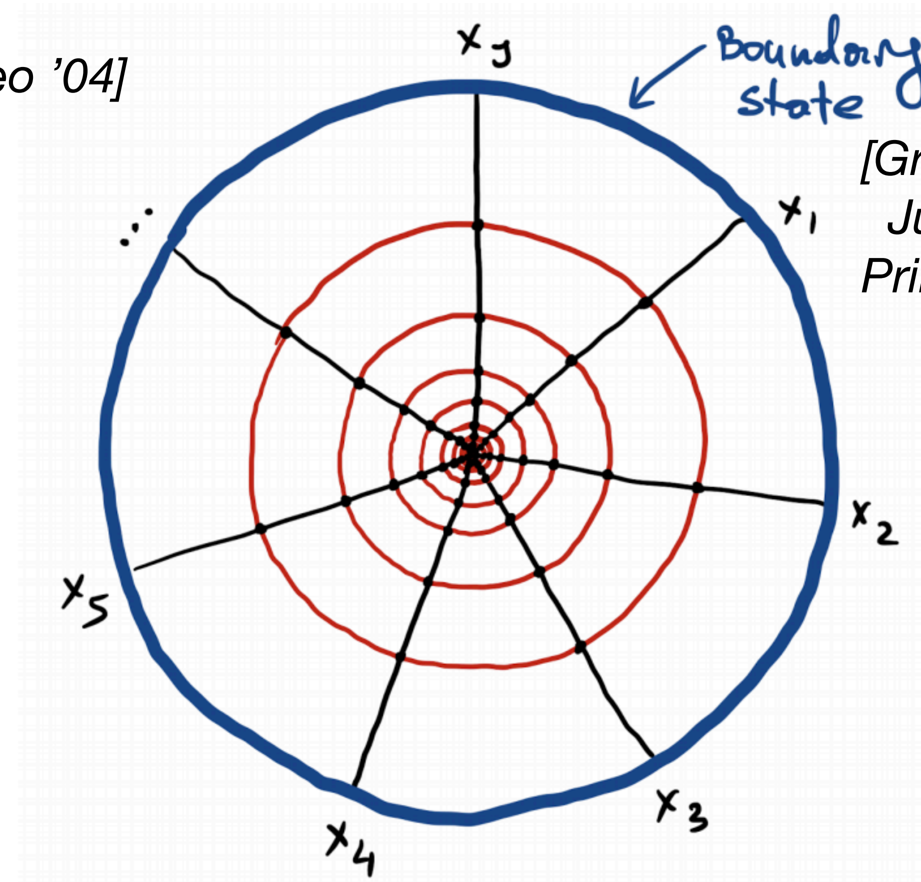


We now have ratios of determinants.
The ones in the denominator are likely a universal building block

It would be very nice to have examples of $C_{123}^{\bullet\bullet\bullet}$, where we expect new structures - operatorial SoV could be the way

Another observable: the **g-function**

[Affleck Ludwig'91]
[Dorey Fioravanti Rim Tateo '04]



[Gromov, Julius, Primi'21]

$$g_{B,\Psi}^{2} = \frac{\langle B \,|\, \Psi \rangle \langle \Psi \,|\, B \rangle}{\langle \Psi \,|\, \Psi \rangle}$$

eigenstate/operator

"integrable" boundary state = defect, e.g. Wilson line, 3D domain wall, …, or determinant operators

[Kristjansen,de Leeuw,Zarembo'15]
[Komatsu,Jiang,Vescovi'19]
[Bajnok Gombor '20] [Komatsu Wang '20]

Integrability condition

[Ghoshal Zamolodchikov '93]
[Piroli Pozsgay Vernier '17]

$$\hat{I}\_ \,|\, B \rangle = 0$$

$$\Pi \circ \hat{I}_{\pm} \circ \Pi = \pm \hat{I}_{\pm}$$

for the charges that are odd under chain-reflection

Integrability of boundary state $\hat{I}_{-}\,|B\rangle = 0$ $\Pi \circ \hat{I}_{\pm} \circ \Pi = \pm\,\hat{I}_{\pm}$

For us, chain-reflection will be:

$$\Pi \circ \Psi(x_1,\ldots,x_J) \to \Psi(\tilde{x}_J,\ldots,\tilde{x}_1) \qquad \tilde{x}_\alpha = (-x_{\alpha,1}, x_{\alpha,2}, -x_{\alpha,3}, x_{\alpha,4}).$$

Leaves invariant the Wilson loop state

For eigenstates with twists $\lambda_1 = 1/\lambda_2,\ \lambda_3 = 1/\lambda_4$ and $\theta_\alpha = -\theta_{J+1-\alpha}.$

Π acts as $u \to -u$

$$(\hat{\mathbb{L}}^{\mathbf{6}}(u))^{MN} = (\hat{\mathbb{L}}^{\mathbf{6}}(-u))^{NM} \quad , \quad (\hat{\mathbb{L}}^{\mathbf{4}}(u))^{a}{}_{b} = -(\hat{\mathbb{L}}^{\bar{\mathbf{4}}}(-u))_{b}{}^{a}$$

$$\Lambda^{\mathbf{4}} = \mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \to \mathrm{diag}\,(\lambda_2,\lambda_1,\lambda_4,\lambda_3) = \Lambda^{\bar{\mathbf{4}}}$$

$$\mathbb{T}^{\mathbf{6}}(u) \to \mathbb{T}^{\mathbf{6}}(-u) \quad , \quad \mathbb{T}^{\mathbf{4}}(u) \to (-1)^J\,\mathbb{T}^{\bar{\mathbf{4}}}(-u)$$

$$q(u) \to \tilde{q}(u) = q(-u)$$

TBA-based formula for the g-function

[Dorey,Fioravanti,Rim,Tateo'04]
[Pozsgay'10]
nested case: *[Kostov,Serban,Vu'18]*

$$g = \underbrace{\exp\left(\int_0^\infty \Theta(u)\log(1+Y(u))du\right)}_{\text{boundary-dependent, simple}} \times \underbrace{\sqrt{\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}}}_{\text{universal factor, hard}}$$

It should be a simple combination of Q's!

See Charlotte's, Kostya's talks
[Kristjansen,Muller,Zarembo '21]

finite-volume version of

Gaudin SDet

[Komatsu,Jiang,Vescovi'19]

Let's discuss the SoV expression of the universal piece

Following *[Caetano Komatsu '20]* , it should be largely fixed by functional arguments

This is consistent with structures found rigorously in XXX model *[Gombor Pozsgay'21]*

Starting point

$$\langle B | \vec{\hat{I}}_- = 0$$ → **selection rules** $\langle B | \Psi \rangle \propto g_{B,\Psi} = 0$ **if** $\Pi | \Psi \rangle \neq | \Psi \rangle$

Consider the scalar product argument for $|\Psi\rangle$ (state A) and $\Pi \circ |\Psi\rangle$ (state $\bar{A}$)

$$\vec{I}^{A}_{\pm} = \pm \vec{I}^{\bar{A}}_{\pm}$$

$$\langle p_a^{-\bullet_a A} (\mathcal{B}_A - \mathcal{B}_{\tilde{A}}) \circ q_a^{\bullet_b \tilde{A}} \rangle_\mu = 0, \qquad \bullet_a \equiv (\uparrow, \downarrow, \uparrow, \downarrow)_a \,, \quad -\bullet_a \equiv (\downarrow, \uparrow, \downarrow, \uparrow)_a \,,$$

→ $$\tilde{\mathcal{M}}^{A\tilde{A}} \cdot \left(\frac{2\vec{I}_{(-)}}{0\vec{I}_{(+)}} \right) = 0 \qquad \tilde{\mathcal{M}}^{A\tilde{A}} = \left(\begin{array}{c|c} M_- & M'_+ \\ \hline M'_- & M_+ \end{array} \right)$$ Built with Q-bilinears

→ ***If*** $A \neq \bar{A}$ ***, nonzero solution, so*** $|M_-| = 0$

Conjecture: $$\langle B | \Psi \rangle \langle \Psi | B \rangle \propto |M_-|^2$$ **satisfies all selection rules !**

$$\langle B|\Psi\rangle\langle\Psi|B\rangle \propto |M_-|^2$$

Now divide by the norm

Symmetric case: block structure

(due to $q_1(u) = q_2(-u), \quad q_3(u) = q_4(-u)$)

$$\tilde{\mathcal{M}}^{A\tilde{A}}\Big|_{A=\tilde{A}} = \begin{pmatrix} M_- & 0 \\ 0 & M_+ \end{pmatrix},$$

$$\langle\Psi|\Psi\rangle \sim \det(\tilde{\mathcal{M}}^{AA}) \propto |M_-^{(A)}|\,|M_+^{(A)}|$$

non-vanishing

This piece does vanish (related to the zero-mode) - regularisation needed

$$(g_{\text{universal}})^2 \propto \frac{|M_-|}{|M_+|_*}$$

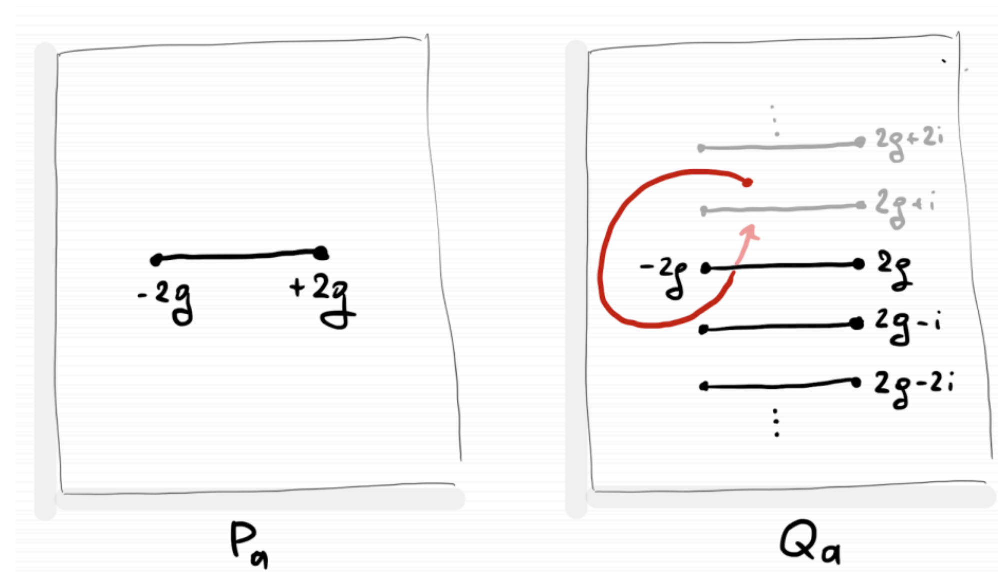
Ratio of SoV-type determinants

Fix details with TBA in fishnet theory?

[Basso Ferrando Kazakov Zhong '19]

Many analogies in $\mathcal{N} = 4$ SYM

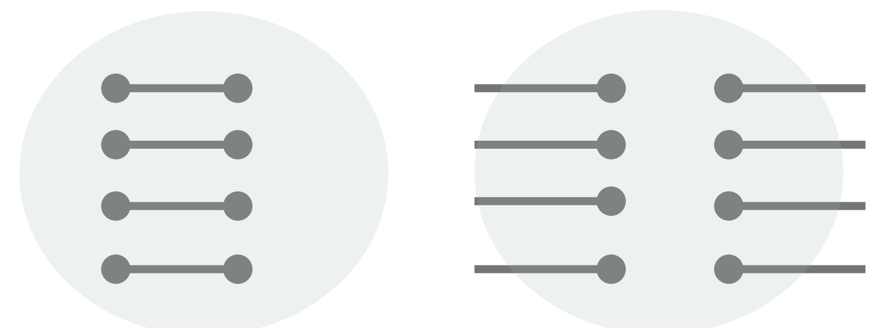
Poles become cuts.
Integrals regularised in the same way.



$$\mathcal{B} = \sum_{n=-2}^{2} \mathbb{A}_n D^{2n} \quad , \quad \mathcal{B} \circ \mathbf{Q}_i = 0$$

but now $\mathbb{A}_n \equiv$ determinants of $\mathbf{P}_a$'s

The coefficients are not polynomials, but still look special



We still have: $$\langle \mathbb{Q}^{B\,i} \, (\mathcal{B}^B - \mathcal{B}^A) \circ \mathbf{Q}_j^A \rangle_{\mu(u)} = 0$$

We expect infinitely many integrals of motion - infinite size determinants as in ShG

[Lukyanov '00]

$$\mathbb{A}_k(u) = \sum_{n=0}^{\infty} \frac{\mathcal{I}_{(k,n)}}{u^n}, \qquad |u| > R_*.$$

The main problem is finding a good basis of IM: linearly indep. and good for numerics

We have found the fabrics
of SoV in the fishnet theory

The operatorial setup would be very useful

Can we understand general ($\geq$3)-point correlators? Learn lessons for $\mathcal{N}$=4 SYM?

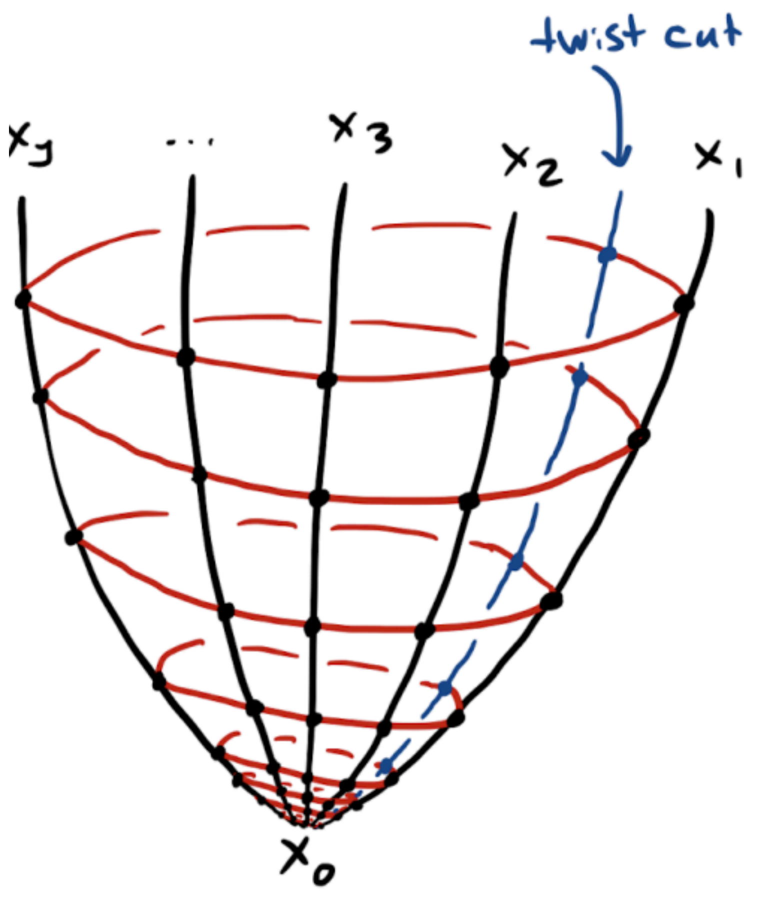
The functional approach for $\mathcal{N}$=4 SYM looks feasible, and would be useful for linking the QSC and g-functions.

Can we use the fishnet results to build a bridge with hexagons? *[Basso,Caetano,Fleury'19]*

Hopefully the fishnet model has many more surprises for us!

Thank you!

Simple construction of the twist in field theory



[AC,Grabner,Gromov,Sever '19]

**Its main role is splitting the conformal multiplets,
To allow integrability to parametrise all states**