New supergravity duals for marginal deformations

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Introduction

Overview

Some old questions in gauge/gravity dual

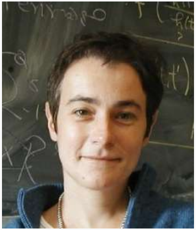
- what is the gravity dual of a generic marginal deformation?
- how can you count (short) operators from the geometry?

but with some new tools of "exceptional generalised geometry"

- natural flux extensions of standard geometries
- builds on long history of using G-structures and generalised complex geometry to analyse supersymmetric flux backgrounds

Anthony Ashmore

Ed Tasker



Michela Petrini

- `arXiv:2112.08375` – AA, ET, MP, DW
- `arXiv:2112.09167` – ET
- `arXiv:22xx.xxxx` – AA, ET, MP, DW

$\mathcal{N} = 1$ description of $\mathcal{N} = 4$

Recall superpotential and Kähler potential for chiral fields (Φ^1, Φ^2, Φ^3)

$$\mathcal{W} = \tfrac{1}{6}\epsilon_{ijk} \operatorname{tr} \Phi^i \Phi^j \Phi^k \qquad\qquad \mathcal{K} = \tfrac{1}{2}\delta_{i\bar{j}} \operatorname{tr} \Phi^i \bar{\Phi}^{\bar{j}}$$

$\frac{3}{2}R$-symmetry (and SU(3) flavour symmetry $\Phi^i \to M^i{}_j \Phi^j$)

$$\Phi^i \to \mathrm{e}^{\mathrm{i}\alpha}\Phi^i \qquad\qquad \mathcal{W} \to \mathrm{e}^{3\mathrm{i}\alpha}\mathcal{W}$$

F-term conditions mean Φ^i commute

$$\partial\mathcal{W}/\partial\Phi^1 = \tfrac{1}{2}[\Phi^2, \Phi^3] = 0 \quad \text{etc}$$

Chiral single-trace mesonic operators

$$\mathcal{O}_f = \sum_n f_{(i_1 \dots i_n)} \operatorname{tr} \Phi^{i_1} \dots \Phi^{i_n} \quad \leftrightarrow \quad f(z^1, z^2, z^3)$$

chiral ring $\leftrightarrow$ ring of holomorphic functions on $\mathbb{C}^3$

Hilbert series counts operators with R-charge $\frac{2}{3}k$

$$\tilde{H}(t) = \sum_k (\# \text{ of chiral ops.})\, t^{2k}$$

writing $\tilde{H}(t) = H(t^2)$ for unit charge gives

$$H(t) = 1 + 3t + 6t^2 + 10t^3 + \cdots = \frac{1}{(1-t)^3}$$

We also have the single-trace superconformal index

$$\mathcal{I}_{\text{s.t.}}(t,y) = \sum_{j_1, j_2, r=\frac{2}{3}k} (\#\text{ of single trace ops.})\,(-1)^F\, t^{3(2j_2+r)} y^{2j_1}$$

for spins j_1 and j_2 and only short multiplets with $E = 2j_2 + \frac{3}{2}r$ contribute

Large N limit, no y dependence if no adjoint chirals, and for $\mathcal{N} = 4$

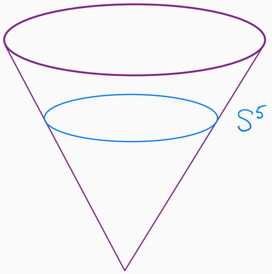
$$\mathcal{I}_{\text{s.t.}}(t) = 3t^2 + 3t^4 + 3t^6 + \ldots = \frac{3t^2}{1-t^2}$$

(here $\mathcal{I}_{\text{s.t.}}(t)$ is constant term in Laurent expansion in y of $\mathcal{I}_{\text{s.t.}}(t,y)$)

Gravity dual: S^5 with F_5 flux in IIB

Calabi–Yau cone $C(S^5) = \mathbb{C}^3$

- chiral operators dual to particular supergravity KK modes on S^5
- calculate using spherical harmonics (homogeneous space)



$\mathcal{N} = 1$ marginal deformations for $\mathcal{N} = 4$

Superpotential deformation (charge 3) [*Leigh, Strassler 95*]

$$\mathcal{W} = \tfrac{1}{6}\epsilon_{ijk}\,\mathrm{tr}\,\Phi^i\Phi^j\Phi^k + \tfrac{1}{6}f_{ijk}\,\mathrm{tr}\,\Phi^i\Phi^j\Phi^k$$

- ten complex marginal deformations f_{ijk}
- but beta-function constrains moment map for SU(3) symmetry

$$f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0 \qquad \text{adjoint of SU(3)}$$

- exactly marginal deform. as symplectic quotient [*Kol 02,03; Green et al 10*]

$$\widetilde{\mathcal{M}} = \{f_{ijk}\} /\!\!/ \mathrm{SU}(3) \qquad \text{“conformal manifold”}$$

Choose

$$\Delta\mathcal{W} = \tfrac{1}{2}\lambda_1 \,\mathrm{tr}\, \Phi^1\Phi^2\Phi^3 + \tfrac{1}{6}\lambda_2 \,\mathrm{tr}\left[(\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right]$$

F-terms now give non-commutative "Sklyanin" algebra

$$[\Phi^1, \Phi^2] + \lambda_1(\Phi^1\Phi^2 + \Phi^2\Phi^1) + \lambda_2(\Phi^3)^2 = 0 \qquad \text{etc}$$

Hilbert series for generic λ_i calculated by mathematicians [*van den Bergh 94*]

$$\begin{aligned} H(t) &= 1 + 3t + 3t^2 + 2t^3 + 3t^4 + 3t^5 + 2t^6 + 3t^7 + 3t^8 + \dots \\ &= \frac{(1+t)^3}{1-t^3} \end{aligned}$$

But index same as undeformed theory $\mathcal{I}_{\text{s.t.}}(t) = 3t^2/(1-t^2)$

Gravity dual?

Moduli space of AdS_5 solutions, deforming S^5 and more fluxes

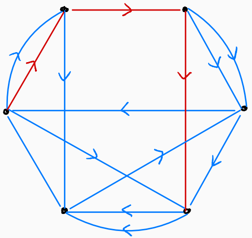
- $\lambda_2 = 0$: "β-deformation", $U(1)^3$ isometry, exact dual via solution-generating transformation [*Lunin, Maldacena 05*]
- $\lambda_1 \neq 0, \lambda_2 \neq 0$: perturbative tour de force to 2nd/3rd order [*Aharony, Kol, Yankielowicz 02*]

only $U(1)_R$ isometry, as hard as finding explicit Calabi–Yau metrics

KK expansion for duals of chiral on β-deformation formidable ...

> *But ... much of field theory quite simple, since only depends on holomorphic structure. Is there some supergravity analogue?*

More generally . . .



Field theory characterises "non-commutative alegbraic geometry"

- quiver Q for fields + superpotential $\mathcal{W}$
- $\mathrm{d}\mathcal{W} = 0$ defines "Calabi–Yau algebra", A for paths in Q [*Ginzburg 06*]
- short ops $\Leftrightarrow$ "cyclic homology" $\overline{HC}_n(A, k)$ [*Berenstein et al. 00*]

$$\mathcal{I}_{\text{s.t.}}(t) = \sum_{n=0,1,2} (-1)^n \dim \overline{HC}_n(A, k)$$ [*Eager et al. 12*]

Special case $(Q_{\mathrm{SE}}, \mathcal{W}_{\mathrm{SE}})$ dual to Sasaki–Einstein

- gives commutative algebra for Calabi–Yau cone $C(M)$
- short ops $\Leftrightarrow$ "transverse Dolbeault cohomology" $H^{(p,q)}_{\bar\partial}(k)$ on M [*Eager, Schmude, Tachikawa 12*]

$$\overline{HC}_n(A,k) = \bigoplus_{p-q=n} H^{(p,q)}_{\bar\partial}(k) \qquad k>0$$

Exactly marginal deformation $(Q_{\mathrm{SE}}, \mathcal{W}_{\mathrm{SE}} + \Delta\mathcal{W})$

- $\Delta\mathcal{W}$ characterised by holomorphic function f on $C(M)$ of charge 3

Problem to solve

What is the geometry in supergravity for generic field theory $(Q, \mathcal{W})$?

- can we find dual of superpotential?
- calculate chiral spectrum?

tool will be exceptional geometry ...

Introduction

Exceptional Sasaki–Einstein geometries

Dual of the superpotential, deformations and chiral operators

Exceptional Sasaki–Einstein geometries

IIB supersymmetric background: Killing spinor equations

We have $\mathcal{M}_{10} = \mathsf{AdS}_5 \times M$ with

- warped metric: $\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathsf{AdS}_5) + \mathrm{d}s^2(M)$
- bosonic fields on M: $g_{mn}, B^i_{mn}, C_{mnpq}, \phi, \chi, \Delta$

preserving supersymmetry, heuristically, for spinor ϵ on M, solve

$$\delta\psi = \nabla_m \epsilon + (\mathsf{flux})_m\, \epsilon = 0,$$
$$\delta\rho = (\mathsf{flux})\, \epsilon = 0.$$

where, for AdS_5 radius m^{-1},

$$\mathsf{flux} \in \{\mathrm{d}B^i, \mathrm{d}C, \mathrm{d}\phi, \mathrm{d}\chi, \mathrm{d}\Delta, m\}$$

Special case of Sasaki–Einstein

- $B^i = \Delta = 0$, ϕ and χ constant
- metric g and spinor ϵ equivalent to

$$\sigma \in \Gamma(T^*M), \quad \omega \in \Gamma(\Lambda^2 T^*M), \quad \Omega \in \Gamma(\Lambda^2 T^*_{\mathbb{C}}M)$$

where

$$\mathrm{d}\sigma = 2\omega, \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega, \qquad F = \mathrm{d}C = 4\,\mathrm{vol}_M,$$

and "Reeb" Killing vector $\xi = \sigma^\sharp = \partial/\partial\psi$

$$\mathcal{L}_\xi \Omega = 3\mathrm{i}\Omega \qquad \tfrac{3}{2}\text{R-symmetry}$$

Generically have all fluxes and no conventional geometry ...

$E_{6(6)} \times \mathbb{R}^+$ exceptional geometry

For conventional 5d metric

$$g \in \frac{GL(5,\mathbb{R})}{O(5)}$$

Can also package supergravity fields on M as generalised metric

$$G = \{g, \phi, \chi, B^i, C, \Delta\} \in \frac{E_{6(6)} \times \mathbb{R}^+}{USp(8)/\mathbb{Z}_2}$$

Rewrite supergravity using natural action of $E_{6(6)} \times \mathbb{R}^+ \supset GL(5,\mathbb{R})$

[*Hull 07; Pacheco, DW 08; Coimbra, Strickland-Constable, DW 11*][*Berman et al. 11*]

Generalised tensors

Generalised tangent space

$$E \simeq TM \oplus 2T^*M \oplus \Lambda^3 T^*M \oplus 2\Lambda^5 T^*M$$
$$V^M = (v^m, \lambda^i_m, \rho_{mnp}, \sigma^i_{m_1\dots m_5})$$

- transforms as $\mathbf{27}_1$ under $\mathrm{E}_{6(6)} \times \mathbb{R}^+$
- parametrises diffeomorphism and gauge symmetry
- $\mathrm{E}_{6(6)}$ cubic invariant in $\Gamma(\Lambda^5 T^*M)$

$$c(V,V,V) = \epsilon_{ij}(i_v\lambda^i)\sigma^j - \tfrac{1}{2} i_v\rho \wedge \rho - \tfrac{1}{2}\epsilon_{ij}\rho \wedge \lambda^i \wedge \lambda^j$$

The adjoint 78_0 includes potentials

$$\mathrm{ad}\,\tilde{F} \simeq 3\mathbb{R} \oplus (TM \otimes T^*M) \oplus 2\Lambda^2 T^*M \oplus 2\Lambda^2 TM \oplus \Lambda^4 T^*M \oplus \Lambda^4 TM$$

$$A^M{}_N = (\ldots, B^i_{mn}, \ldots, C_{mnpq})$$

"Twisting" of generalised tensors by gauge potentials acting in adjoint

$$V = \mathrm{e}^{B^i + C} \cdot \tilde{V} \qquad A = \mathrm{e}^{B^i + C} \cdot \tilde{A}$$

$\frac{1}{3}\mathbb{R}^+$-weight keeps track of tensor density, so 78_3

$$\mathrm{ad}\,\tilde{F} \otimes \det T^*M \simeq T^*M \oplus 2\Lambda^3 T^*M \oplus \ldots$$

Generalised Lie derivative

"Generalised diffeomorphism, GDiff" symmetries generated by

$$
\begin{aligned}
L_V &= \text{diffeo} + \text{gauge transf} \\
&= \mathcal{L}_v - (\mathrm{d}\lambda^i + \mathrm{d}\rho) \cdot
\end{aligned}
$$

where forms act via adjoint

- $L_V W$ defines "Leibniz algebroid" structure on E
- $L_V W \neq L_W V$
- by Leibniz can extend L_V action to any generalised tensor

Exceptional Sasaki–Einstein structure

For supersymmetric background

$$\{\text{bosonic fields}\} + \epsilon \quad \Longleftrightarrow \quad (K, X)$$

- V structure : $K \in \Gamma(E)$ such that $c(K, K, K) > 0$
- H structure : $X \in \Gamma(\operatorname{ad} \tilde{F}_{\mathbb{C}} \otimes \det T^*M)$ highest root in $\mathfrak{e}_6$ such that

$$0 < -\operatorname{tr} X\bar{X} \in \Gamma((\det T^*M)^2)$$

The H and V structures are compatible

$$X \cdot K = 0 \qquad\qquad c(K, K, K)^2 = -\tfrac{1}{2} \operatorname{tr} X\bar{X}$$

[*Ashmore, DW 15; Ashmore, Petrini, DW 16*]

For Sasaki–Einstein, writing $\tau = \chi + \mathrm{i}\mathrm{e}^{-2\phi}$ and $u^i = \tau_2^{-2}(\tau, 1)^i$

$$X = -\tfrac{1}{2}\mathrm{i}u^i\,\mathrm{e}^{C + \frac{1}{4}\mathrm{i}\Omega\wedge\bar{\Omega}} \cdot \sigma \wedge \Omega, \qquad \text{“Cauchy–Riemann structure”}$$

with $\operatorname{ad}\tilde{F} \otimes \det T^*M \simeq T^*M \oplus 2\Lambda^3 T^*M \oplus \dots$

$$K = \mathrm{e}^{C} \cdot (\xi - \sigma \wedge \omega)\,, \qquad \text{“contact structure”}$$

with $E \simeq TM \oplus 2T^*M \oplus \Lambda^3 T^*M \oplus 2\Lambda^5 T^*M$.

And $L_K = \mathcal{L}_\xi$ generates the R-symmetry

Differential conditions

- Exceptional complex structure : X defines decomposition

$$E_{\mathbb{C}} \simeq L_{+} \oplus L_{-} \oplus L_{0}$$

and susy implies L_{+} is involutive (generalises Cauchy–Riemann)

$$L_V W = -L_W V \in \Gamma(L_+) \qquad \forall V, W \in \Gamma(L_+)$$

- R-charges :

$$L_K X = 3\mathrm{i}X, \qquad\qquad L_K K = 0.$$

X is charge 3 and K is contact structure [*Gabella, Gauntlett, Sparks, DW 09*]

- Moment map : $\mu = K^*$

Moment map

Define space of structures $\mathcal{Z} = \{X : L_+ \text{ is involutive}\}$

$\mathcal{Z}$ is infinite-dimensional Kähler space

with infinitesimal action of GDiff parametrised by $V \in \Gamma(E) \simeq \mathfrak{gdiff}$

$$\delta X = L_V X$$

preserves Kähler structure on $\mathcal{Z}$ giving moment map

$$\mu(V) = -\frac{\mathrm{i}}{4} \int_M \frac{\operatorname{tr} X(L_V \bar{X})}{(\operatorname{tr} X\bar{X})^{1/2}}$$

so susy condition is $\mu(V) = K^*(V) := \int_M c(K, K, V)$, for all $V \in \mathfrak{gdiff}$

Dual of the superpotential, deformations and chiral operators

Nice formalism but is it useful?

Beautiful SCFT result [*Kol 02,03; Green, Komargodski, Seiberg, Tachikawa, Wecht 10*]

> *all marginal deformations are in the superpotential and are all exactly marginal unless there is a global symmetry*

Gravity dual follows directly from moment map structure since deformations of K and X directly related to supermultiplets [*Ashmore, Gabella, Graña, Petrini, DW 16*]

What about the missing deformed solutions or counting chiral operators?

Idea: analogue with Calabi–Yau

Given explicit Kähler SU(3) structure (Ω, ω) with $\frac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega} = \frac{1}{6}\omega^3$

$$\mathrm{d}\Omega = 3\mathrm{i}\alpha \wedge \Omega, \qquad \mathrm{d}\omega = 0$$

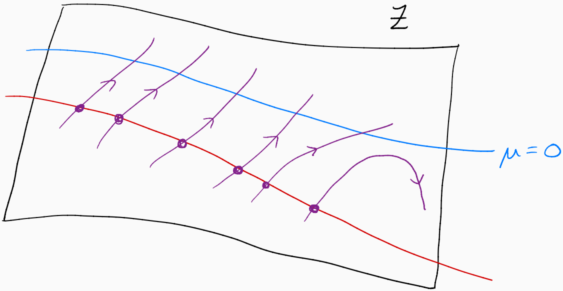
then vary Kähler form within cohomology class

$$\omega' = \omega + \mathrm{i}\partial\bar{\partial}h, \qquad \Omega' = \lambda\Omega$$

there exists unique Calabi–Yau solution (Ω_*, ω_*)

Symplectic quotient/GIT

Typical of supersymmetry conditions: first solve F-terms (holomorphic)



- $\mathcal{Z}$ is Kähler (infinite-dimensional) with group action G
- orbits for $G_{\mathbb{C}}$ intersect $\mu = 0$ (if "stable" – algebraic condition)

Kähler–Einstein, Sasaski–Einstein, Hermitian Yang-Mills, ...

[*Yau; Tian; Donaldson, ...*]

Exceptional Sasaki geometry

Relax moment map condition (analogue of Kähler on cone)

$$\mu_3 \neq K^*$$

and $-\frac{1}{2}\operatorname{tr} X\bar{X} \neq c(K,K,K)^2$

- Sasaki $\subset$ Exceptional Sasaki.
- complex $\mathrm{GDiff}_{\mathbb{C}}$ orbit generated by

$$\delta X = L_V X \qquad V \in \Gamma(E_{\mathbb{C}}) \simeq \mathfrak{gdiff}_{\mathbb{C}}$$

- intersects moment map condition on susy ESE background (X_*, K_*)

Physical interpretation

From supersymmetric multiplet structure

$$\boxed{\text{orbit } [X] = \{X' : X' = \mathrm{GDiff}_{\mathbb{C}} \cdot X\} \text{ encodes superpotential } \mathcal{W}}$$

- $\delta X = L_V X$ with $V \in \Gamma(E_{\mathbb{C}})$ part of long vector mulitplet deforming Kähler potential
- motion in orbit is renormalisation flow of Kähler potential; class $[X]$ does not change for domain wall flow – non-renormalization of $\mathcal{W}$

$$X' = -\tfrac{2}{3}\mathrm{i} L_K X, \qquad K^{*\prime} = \mu$$

Relation $L_K X = 3\mathrm{i}X$ fixes marginal condition $\Delta = 3$

Existence is implied by SCFT result of [*Kol 02,03; Green et al. 10*]

Solution for marginal deformation of Sasaki–Einstein

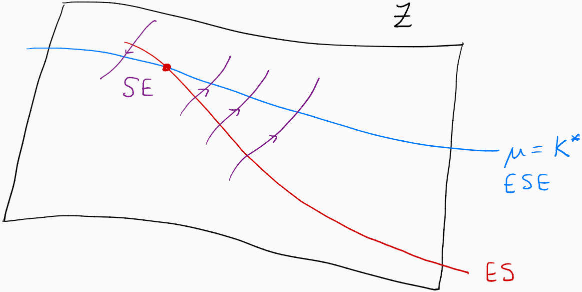
We find new family of Exceptional Sasaki solutions with $\mathcal{L}_\xi f = 3\mathrm{i}f$

$$\boxed{\begin{aligned} K &= \mathrm{e}^{C} \cdot (\xi - \sigma \wedge \omega) \\ X &= \mathrm{e}^{b^i(\tau,f)+C} \cdot \big(\mathrm{d}f + v^i(\tau,f)\,\sigma \wedge \Omega\big) \end{aligned}}$$

with $b^i \in \Gamma(\Lambda^2 T^*_{\mathbb{C}} M)$ linear and v^i quadratic in f

- very complicated deformed metric g, axion-dilaton and fluxes
- valid for deformation of any Sasaki–Einstein
- for S^5 matches to 2nd order [*Aharony et al 02*] and $\mathrm{GDiff}_{\mathbb{C}}$ action gives $\{f_{ijk}\} /\!\!/ \mathrm{SU}(3)$
- can also check if $f = z^1 z^2 z^3$ then is $\mathrm{GDiff}_{\mathbb{C}}$ of LM solution

ESE solution exists in open neighbourhood of Sasaki–Einstein point



- moment map $\tilde{\mu} = \mu - K^*$ for GDiff_K (preserves K)
- stable points form open set in $\mathcal{Z}$ (Kempf–Ness + no additional sym)
- global? family has same discrete symm. as $\Delta\mathcal{W}$ [*cf Baggio et al. 17*]
- Monge–Ampère-type equation?

Chiral spectrum

Come from deformations of superpotential

$$\text{space of chiral ops, } \mathcal{C} = \frac{\{\delta X : L_+ + \delta L_+ \text{ involutive}\}}{\{\delta X = \mathfrak{gdiff}_C \cdot X\}}$$

Counted by cohomology groups $H^p(\Lambda^* L_+, \mathrm{d}_L)$ of complex defined by L_+

$$\cdots \xrightarrow{\mathrm{d}_L} \Lambda^p L_+^* \xrightarrow{\mathrm{d}_L} \Lambda^{p+1} L_+^* \xrightarrow{\mathrm{d}_L} \cdots$$

$$\boxed{\text{independent of choice of } X \text{ in class } [X]}$$

- reflects dependence on holomorphic data
- means can calculate at ES point

Calculating the cohomology

If $\eta = \mathrm{d}f$ nowhere vanishing (not β-deformation, not $Y^{p,q}$) then

$$X = \mathrm{e}^{\tilde{b}^i(f,\tau)+\tilde{c}(f,\tau)}\,\mathrm{d}f$$

and cohomology reduces to calculating "η-cohomology" $H^n_{\mathrm{d}_\eta}(k)$

$$\cdots \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^p T^*_{\mathbb{C}} M \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^{p+1} T^*_{\mathbb{C}} M \xrightarrow{\mathrm{d}} \cdots$$

which can be calculated using $H^{(p,q)}_{\bar{\partial}}(k)$ of Sasaki–Einstein

$$\dim H^0_{\mathrm{d}_\eta}(k) = [k \equiv_3 0], \qquad \dim H^1_{\mathrm{d}_\eta}(k) = 0,$$
$$\dim H^2_{\mathrm{d}_\eta}(k) = \mathrm{ind}_{\bar{\partial}}(k) - [k \equiv_3 0]$$

where $\mathrm{ind}_{\bar{\partial}}(k) = \sum_{p,q}(-1)^{p+q} \dim H^{(p,q)}_{\bar{\partial}}(k)$ and [true] $= 1$, [false] $= 0$

[*Tasker 21*]

New results

Chirals counted by $H^2_{\mathrm{d}_\eta}(k)$ giving general Hilbert series

$$\tilde{H}(t) = 1 + \mathcal{I}_{\text{s.t.}}(t) - \sum_k [k \equiv_3 0, k > 0] t^{2k}$$

For regular Sasaki–Einstein

$$S^5: \qquad H(t) = \frac{(1+t)^3}{1-t^3}$$ matches [*van den Bergh*]

$$T^{1,1}: \qquad H(t) = \frac{1+4t+2t^2}{1-t^2}$$ prediction (checked to level 7)

$$\mathrm{dP}_6: \qquad H(t) = \frac{1+7t}{1-t}$$ prediction

- first spectrum for non-SE/non-homogeneous (cf [*Bobev et al 20*])
- mesonic moduli spaces are one-dimensional ($\mathrm{d}f = 0$)
- conjecture: $\overline{HC}_n(A,k) = H^{2-n}_{\mathrm{d}_\eta}(k)$ so no $\overline{HC}_1(A,K)$ short mults.

Summary and extensions

New framework based on holomorphic structure $[X]$

- exceptional geometry has allowed us to implicitly solve for a large class of new supergravity duals
- $[X]$ determines the spectrum of short-multiplets, index
- existence = stability = IR fixed point

Extensions

- other IIB cases: vanishing η, flows, relevant deformations, toric, ...
- exact same formalism for M-theory: $\mathcal{N} = 1$, class $\mathcal{S}$
- generic dual of a-maximisation, localisation for index
- analogous formalism for AdS_4 with $E_{7(7)}$