Complex spins in BPS/CFT,

or:

Do replicants dream of electric sheaf?

Some remarks on symmetries of quantum field theory

NIKITA NEKRASOV

29 April 2022

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In quantum field theory and statistical mechanics
one often uses the trick of analytic continuation from $\mathbb{Z}$ to $\mathbb{C}$

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In quantum field theory and statistical mechanics
one often uses the trick of analytic continuation from $\mathbb{Z}$ to $\mathbb{C}$

Particles as S-matrix poles in complex angular momentum l

T.Regge

Replica trick: $\langle \log Z \rangle \to \langle Z^n \rangle$

G.Parisi

Dimensional regularization: spacetime dimension D

G.'tHooft

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Is there any physical meaning to these complexifications?

What physical system realizes complex spin representations of $\mathfrak{sl}_N$?

Which physical system's partition function is equal to Z^n for complex n?

In string paradigm the number of species is the spacetime dimension

$D \sim c$, the central charge of the matter sector of the worldsheet theory

What is the physical realization of Virasoro representations with complex c?

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Is there any physical meaning to these complexifications?

We shall argue the answer is in extra dimensions and supersymmetry!

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Let us start with the simple representation theory

of $\mathfrak{sl}_2$ algebra

$$L_+ = x^2\partial_x - 2sx\,,\;\; L_0 = x\partial_x - s\,,\;\; L_- = \partial_x$$

Realized in $\psi(x)dx^{-s}$ tensors in one dimension.

For $2s \in \mathbb{Z}_+$ there is a finite dimensional $SL(2,\mathbb{C})$ group representation

$$\psi(x) = f_0 + f_1 x + \ldots + f_{2s}x^{2s}$$

$$\psi(x)dx^{-s} \mapsto f\left(\frac{ax+b}{cx+d}\right)(cx+d)^{2s}dx^{-s}$$

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For $2s \in \mathbb{Z}_+$ there is a finite dimensional $SL(2,\mathbb{C})$ group representation

$$\psi(x) = \psi_0 + \psi_1 x + \ldots + \psi_{2s} x^{2s}$$

The space of states of a quantum mechanics of a particle on a sphere S^2

$\textit{Geometric quantization, Kirillov}-\textit{Kostant}-\textit{Souriau}$

$$\int DpDq\, e^{\mathrm{i}\int p\dot{q}}$$

$$dp \wedge dq = \mathrm{i}s\frac{dx \wedge d\bar{x}}{(1+x\bar{x})^2}$$

The symmetry of quantum mechanics is $SU(2)$

The wavefunction $\psi(x)$ is a globally defined holomorphic section of $\mathcal{O}(2s)$

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Once $s \in \mathbb{C}$ the group action is lost

There are various options for the nature of the $\psi(x)$ functions

Verma modules $\mathcal{V}_s^+$: $\psi(x) =$ a polynomial in x

Verma modules $\mathcal{V}_s^-$: $\psi(x) = x^{2s}\cdot$ a polynomial in x^{-1}

Heisenberg-Weyl modules $\mathcal{HW}_s^a$: $\psi(x) = x^{s+a}\cdot$ a polynomial in x, x^{-1}

No hermitian invariant product

Only the Lie algebra $\mathfrak{sl}_2$ acts

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We encounter these representations when we think about invariants

$$\mathcal{J}^{s_1,s_2,s_3} = (x_1 - x_2)^{s_1+s_2-s_3}(x_2 - x_3)^{s_2+s_3-s_1}(x_1 - x_3)^{s_1+s_3-s_2}$$

Is invariant under $L_n^{(1)} + L_n^{(2)} + L_n^{(3)}$

Expand $\mathcal{J}^{s_1,s_2,s_3}$ in the region

$$|x_1| \ll |x_2| \ll |x_3|$$

to see $\mathcal{J}^{s_1,s_2,s_3} \in \mathcal{V}_{s_1}^{+} \otimes \mathcal{HW}_{s_2}^{s_1-s_3} \otimes \mathcal{V}_{s_3}^{-}$

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Moving ahead, the next stop is the Knizhnik-Zamolodchikov equation

$$\mathbf{\Psi} = \mathcal{J}^{s_0, s_1, \ldots, s_{n+1}} \in \left(\mathcal{V}^{+}_{s_0} \otimes \mathcal{HW}^{a_1}_{s_1} \otimes \mathcal{HW}^{a_2}_{s_2} \otimes \ldots \otimes \mathcal{V}^{-}_{s_{n+1}} \right)^{\mathfrak{sl}_2}$$

depending on additional parameters $z_0, z_1, \ldots, z_{n+1} \in \mathbb{CP}^1$ obeying a system of compatible(!) equations

$$\nabla_i \mathbf{\Psi} \equiv (k+2) \frac{\partial}{\partial z_i} \mathbf{\Psi} + \widehat{H}_i \mathbf{\Psi} = 0$$

with z-dependent Gaudin Hamiltonians

$$\widehat{H}_i = \sum_{j \neq i} \frac{1}{z_i - z_j} \left(x_{ij}^2 \frac{\partial^2}{\partial x_i \partial x_j} - 2 x_{ij} \left(s_i \frac{\partial}{\partial x_j} - s_j \frac{\partial}{\partial x_i} \right) - 2 s_i s_j \right)$$

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For $2s_i \in \mathbb{Z}_+$ one can restrict $\boldsymbol{\Psi}$ to be polynomials in x_i of degree $2s_i$

For $k \in \mathbb{Z}_+$ finite dimensional space of solutions

conformal blocks of $SU(2)_k$ Wess-Zumino-Novikov-Witten theory

$$(k+2)\frac{\partial}{\partial z_i}\boldsymbol{\Psi} + \widehat{H}_i\boldsymbol{\Psi} = 0$$

with z-dependent Gaudin Hamiltonians

$$\widehat{H}_i = \sum_{j\neq i} \frac{1}{z_i - z_j}\left(L_+^{(i)}L_-^{(j)} + L_+^{(j)}L_-^{(i)} - 2L_0^{(i)}L_0^{(j)}\right)$$

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Mathematicians and physicists have studied these equations for generic $k \in \mathbb{C}$

Feigin–Frenkel, Reshetikhin, Babujian–Flume, Varchenko–Schekhtman...

conformal blocks of level k $\widehat{\mathfrak{sl}_2}$ current algebra

What is the physics? For complex s_i's and k's?

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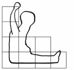
Another story: Generalization of Dyson-Macdonald identities

$$\eta(\mathfrak{q})^{-\dim(G)} = \sum_{\lambda} \tau_{\lambda} \mathfrak{q}^{|\lambda|}$$

to



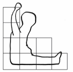
Picture of arms and legs by Ugo Bruzzo

$$\eta(\mathfrak{q})^{\frac{(m+\varepsilon_1)(m+\varepsilon_2)}{\varepsilon_1 \varepsilon_2}} =$$

$$= \sum_{\lambda} \prod_{\square \in \lambda} \frac{(m + \varepsilon_1(a_\square + 1) - \varepsilon_2 l_\square)(m - \varepsilon_1 a_\square + \varepsilon_2(l_\square + 1))}{(\varepsilon_1(a_\square + 1) - \varepsilon_2 l_\square)(-\varepsilon_1 a_\square + \varepsilon_2(l_\square + 1))} \mathfrak{q}^{|\lambda|}$$

Generalization of Dyson-Macdonald identities

to



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SUPERSYMMETRIES AND REPLICAS

is there a physical realization of the replica trick? could one refine it?

since the replica symmetry is often broken,

could one introduce some chemical potentials for different S(n) representations?

SUPERSYMMETRIES AND REPLICAS

$\lambda \in \mathbb{C}$

$$\frac{1}{\eta(q)^{\lambda}} = \sum_{\rho} q^{|\rho|} \times \mu_{\rho}(\lambda)$$

For $\lambda \in \mathbb{Z}$ $|\lambda|$ chiral bosons/fermions

flavor

$$\lambda = \frac{(m+\epsilon_1)(m+\epsilon_2)}{\epsilon_1 \epsilon_2}$$

spacetime rotations

Partition function of 4d gauge theory $\mathcal{N}=2$ / 6d tensor theory $(2,0)$

can be refined to

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For theories with O(n) or U(n) symmetry one can use Deligne category to define "representations" for complex n... (Binder-Rychkov'2016)

For Chern-Simons theory with a simple Lie gauge group one can use Vogel plane to define universal CS theory (Mkrtchyan-Veselov'2012)

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Partition function of 4d gauge theory $\mathcal{N}=2$ / can be refined to 6d tensor theory $(2,0)$

The refined replica of 2d chiral bosons/fermions = 6d (2,0) tensor multiplet

Q: Refined replica of 2d chiral WZW ADE theory = nonabelian 6d (2,0) SCFT theory?

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Q: Refined replica of 2d chiral WZW ADE theory = nonabelian 6d (2,0) SCFT theory?

The refined replica of 3d conformally coupled scalar = 11d linearized supergravity

(NN conjecture 2004, A.Okounkov proof 2015)

Q: what is the ``non-abelian'' 3d theory whose *replicant is* M-theory?

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Armed with the previous example
Let us study four dimensional super-Yang-Mills theory

$$\mathcal{S} \; = \; -\frac{1}{4g_{\mathrm{ym}}^2}\int_{M^4} \mathrm{tr}\left\{ F_A \wedge \star F_A + D_A\sigma \wedge \star D_A\bar{\sigma} + [\sigma, \bar{\sigma}]^2 \right\}$$

$$+\frac{\mathrm{i}\vartheta}{8\pi^2}\mathrm{tr} F_A \wedge F_A +$$

$$+ \; matter \;\; fields \;\; and \;\; fermions$$

its string/M theory realizations,

and connections to quantum theories in lower dimensions

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Super-Yang-Mills subject to Ω-deformation

$$\mathcal{S}_{\varepsilon_1,\varepsilon_2} = $$

$$-\frac{1}{4g_{\mathrm{ym}}^2}\int_{M^4} \mathrm{tr}\left\{ F_A \wedge \star F_A + (D_A\sigma + \iota_V F_A)\wedge\star(D_A\bar\sigma + \iota_{\bar V}F_A) \right\}$$

$$+\frac{\mathrm{i}\vartheta}{8\pi^2}\mathrm{tr} F_A \wedge F_A +$$

$$+\ \mathrm{tr}\left([\sigma,\bar\sigma] + \ldots\right)^2 + \ \textit{matter fields and fermions}$$

$$V = \varepsilon_1\partial_{\varphi_1} + \varepsilon_2\partial_{\varphi_2}\,,\ \ \bar V = \bar\varepsilon_1\partial_{\varphi_1} + \bar\varepsilon_2\partial_{\varphi_2}$$

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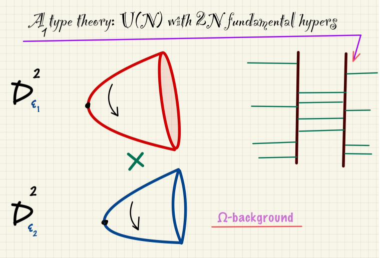
Super-Yang-Mills with fundamental matter
subject to Ω-deformation

$$ds^2 = ds^2_{\mathcal{D}^2_1} + ds^2_{\mathcal{D}^2_2}$$

$$ds^2_{\mathcal{D}^2_i} = f_i(r_i)\left(dr_i^2 + r_i^2 d\varphi_i^2\right)$$

$$i = 1, 2$$

$$V = \varepsilon_1 \partial_{\varphi_1} + \varepsilon_2 \partial_{\varphi_2}$$
$$\bar{V} = \bar{\varepsilon}_1 \partial_{\varphi_1} + \bar{\varepsilon}_2 \partial_{\varphi_2}$$



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First of all, we can compute exactly quite a few things

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We can compute its super-partition function

$$\mathcal{Z}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathfrak{q}\right)$$

$$= \int_{gauge\ \ fields\ +\ matter+\ superpartners} DAD\psi D\sigma D\bar{\sigma} D\chi D\eta\ e^{-\mathcal{S}_{\varepsilon_1,\varepsilon_2}}$$

where we fix the asymptotics $\sigma(x) \to \mathrm{diag}\left(a_1,\ldots,a_N\right)$ as $x \to \infty$

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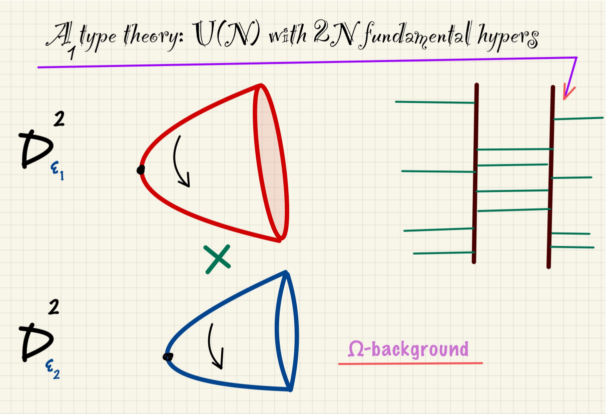
We can compute its super-partition function using localization and other clever tricks

$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right)$$



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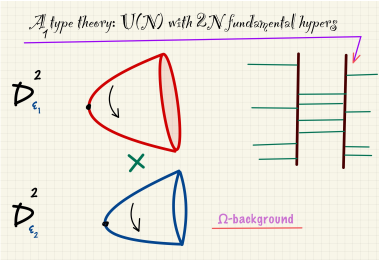
$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right)$$



$$\mathbf{a} = (a_1, \ldots, a_N)\ ,\ \mathbf{m} = \left(m_1^{\pm}, \ldots, m_N^{\pm}\right)\ ,\ \mathfrak{q} = e^{2\pi \mathrm{i}\tau}\ ,\ \tau = \frac{\vartheta}{2\pi} + \frac{4\pi \mathrm{i}}{g_{\mathrm{ym}}^2}$$

We can compute its super-partition function
using localization and other clever tricks

$$\mathcal{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) = \mathcal{Z}^{\mathrm{pert}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})\mathcal{Z}^{\mathrm{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})$$

$$\mathcal{Z}^{\text{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) = \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}} \prod_{\alpha=1}^{N} \mathfrak{q}^{|\lambda^{(\alpha)}|} \times$$

$$\times \prod_{\alpha,\beta=1}^{N} \frac{\prod_{(i,j)\in\lambda^{(\alpha)}} \left(a_\alpha - m_\beta^+ + c_{i,j}\right)\left(m_\beta^- - a_\alpha - c_{i,j}\right)}{\prod_{(i,j)\in\lambda^{(\alpha)}} \prod_{(i',j')\in\lambda^{(\beta)}} \left(a_\alpha - a_\beta + d_{i,j;i',j'}\right)}$$

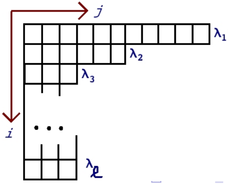
$\mathbf{a} = (a_1, \ldots, a_N)$ Coulomb moduli

$\mathbf{m} = (m_1^\pm, \ldots, m_N^\pm)$ Masses of fundamental hypers

$\mathfrak{q} = e^{2\pi i\tau}$ Instanton fugacity

$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_{\text{ym}}^2}$ Complexified gauge coupling

$$c_{i,j} = \varepsilon_1(i-1) + \varepsilon_2(j-1)$$



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$\mathbf{a} = (a_1, \ldots, a_N)$	Coulomb moduli
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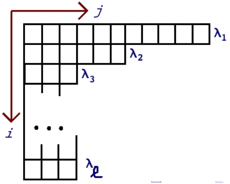

In the *classical* limit $\varepsilon_1, \varepsilon_2 \to 0$

$$\mathcal{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) = \exp \frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F}(\mathbf{a}, \mathbf{m}; \mathfrak{q})$$

with the special geometry of an algebraic integrable system emerging

genus zero $SL(N)$ Hitchin system $=$ classical Gaudin

Prepotential $\mathcal{F}(\mathbf{a}, \mathbf{m}; \mathfrak{q})$ of classical Gaudin:

$$\Phi(\xi) = \sum_I \frac{\Phi_I}{\xi - \xi_I} = \frac{\Phi_0}{\xi} + \frac{\Phi_{\mathfrak{q}}}{\xi - \mathfrak{q}} + \frac{\Phi_1}{\xi - 1}$$

Spectral curve $\mathcal{C}_{\mathbf{u}}$: $\mathrm{Det}\left(\Phi(\xi) - \eta \cdot \mathbf{1}_N\right) = 0$

$$a_i = \oint_{A_i} \eta d\xi \,, \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B^i} \eta d\xi$$

Prepotential $\mathcal{F}(\mathbf{a}, \mathbf{m}; \mathfrak{q})$ of classical Gaudin:

Spectral curve $\mathcal{C}_{\mathbf{u}}$: $\mathrm{Det}\left(\sum_I \frac{\Phi_I}{\xi - \xi_I} - \eta \cdot \mathbf{1}_N\right) = 0$

$$\Phi_0 + \Phi_{\mathfrak{q}} + \Phi_1 + \Phi_\infty = 0$$

$$\begin{aligned}
\Phi_0 &\sim \mathrm{diag}\left(m_1^+ - m^+, \ldots, m_N^+ - m^+\right), \\
\Phi_{\mathfrak{q}} &\sim \mathrm{diag}\left(m^+, \ldots, m^+, m^+(1-N)\right) \\
\Phi_1 &\sim \mathrm{diag}\left(m^-, \ldots, m^-, m^-(1-N)\right), \\
\Phi_\infty &\sim \mathrm{diag}\left(m_1^- - m^-, \ldots, m_N^- - m^-\right)
\end{aligned}$$

$$\begin{aligned}
Nm^+ &= m_1^+ + \ldots + m_N^+, \\
Nm^- &= m_1^- + \ldots + m_N^-
\end{aligned}$$

$$a_i = \oint_{A_i} \eta d\xi\,, \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B^i} \eta d\xi$$

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For finite $\varepsilon_2 \to 0$, $\varepsilon_1 = \hbar$ finite

$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right) \;=\; \exp \frac{1}{\varepsilon_2} \widetilde{\mathcal{W}}\left(\mathbf{a}, \mathbf{m}, \hbar; \mathfrak{q}\right)$$

With $\widetilde{\mathcal{W}}$ describing the monodromy data of a family of $PGL(N)$-opers

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Quantum version of isomonodromic deformation

*N. Reshetikhin'*91

Knizhnik-Zamolodchikov/quantum differential equation

Two dimensional version of instanton partition function *Givental'*94

$$\kappa \frac{\partial \mathbf{\Psi}}{\partial z_i} = \widehat{H}_i \cdot \mathbf{\Psi}$$

Two quasiclassical limits

- $\kappa \to 0$, $\mathbf{\Psi} = e^{\frac{\tilde{W}}{\kappa}} \cdot \chi$

$$\widehat{H}_i \chi = E_i \chi \,, \; E_i = \frac{\partial \tilde{W}}{\partial z_i}$$

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Quantum version of isomonodromic deformation

Knizhnik-Zamolodchikov/quantum differential equation

$$\kappa \frac{\partial \mathbf{\Psi}}{\partial t^i} = \widehat{H}_i \cdot \mathbf{\Psi}$$

Two quasiclassical limits

- $\kappa \to \infty$, $\mathbf{\Psi} = e^{\kappa S} \cdot \tilde{\chi}$

$$\frac{\partial S}{\partial z_i} = H_i \left(\frac{\partial S}{\partial \mathbf{x}}, \mathbf{x}; \mathbf{z} \right)$$

up to little symplectic subtleties of keeping something fixed

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BACK TO FOUR DIMENSIONS

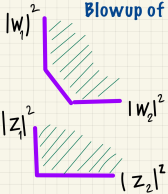
New tool: blowup equations

Idea: compare the theory on M^4 and $\widehat{M}^4$ its blowup

Taubes'93, Kronheimer+Mrowka'94, Fintushell+Stern'96

Losev+NN−Shatashvili'97

Nakajima−Yoshioka'03



Blowup of a point in $\mathbb{C}^2$

$$\widehat{\mathbb{C}^2} \longrightarrow \mathbb{C}^2$$

$$(w_1, w_2, u) \sim (w_1 t, w_2 t, u t^{-1})$$

$$\downarrow$$

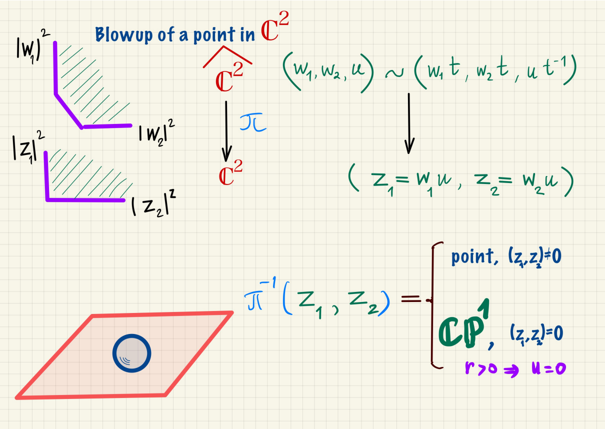
$$(z_1 = w_1 u, \; z_2 = w_2 u)$$

$$\frac{r^2}{4} + |z_1|^2 + |z_2|^2 = \left(r/2 + |u|^2\right)^2$$

$$|w_1|^2 + |w_2|^2 - |u|^2 = r > 0$$

BACK TO FOUR DIMENSIONS

New tool: blowup equations



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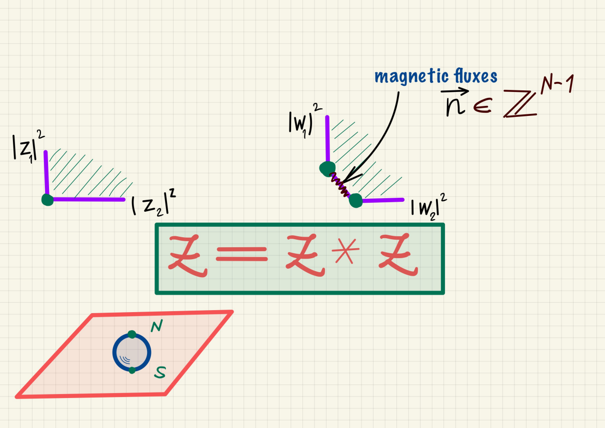
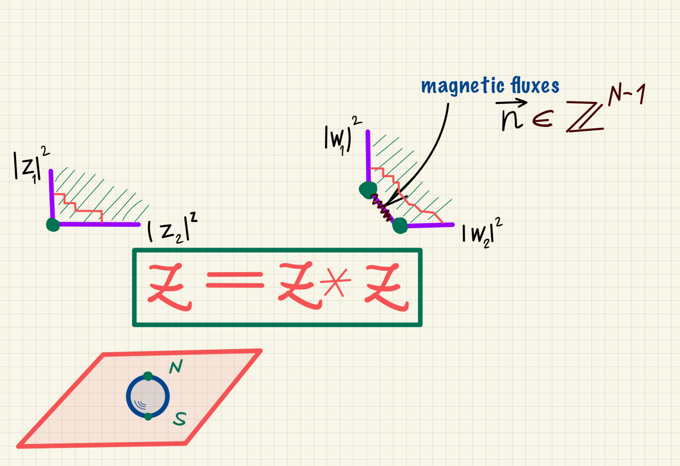


TOOL from FOUR DIMENSIONS: blowup equations

$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right) =$$

$$\sum_{\mathbf{n} \in \mathbb{Z}^{N-1}} \mathcal{Z}\left(\mathbf{a} + \varepsilon_1 \mathbf{n}, \mathbf{m}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathfrak{q}\right) \mathcal{Z}\left(\mathbf{a} + \varepsilon_2 \mathbf{n}, \mathbf{m}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}\right)$$



FOUR DIMENSIONAL TOYS

Surface defects

Kronheimer+Mrowka'93–95

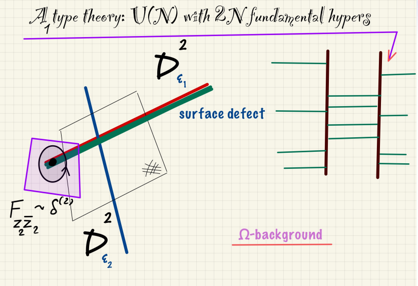
Losev+Moore+NN+Shatashvili'95

NN'95, NN'04

Braverman'04

Gukov+Witten'08

Kanno+Tachikawa'11



$$\boldsymbol{\Psi}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right) = \boldsymbol{\Psi}^{\mathrm{pert}}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)\boldsymbol{\Psi}^{\mathrm{inst}}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)$$

$$= \mathfrak{q}^{\frac{\mathbf{a}^2}{2\varepsilon_1\varepsilon_2}} \prod_{\omega} w_{\omega}^{\frac{a_\omega - a_{\omega+1}}{\varepsilon_1}} \times \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}} \prod_{\omega} w_{\omega}^{k_\omega(\lambda)} \mathfrak{q}^{k_{\mathrm{bulk}}(\lambda)} \times$$

$$\times \left(\prod_{\alpha,\beta=1}^{N} \frac{\prod_{(i,j)\in\lambda^{(\alpha)}} \left(a_\alpha - m^+_\beta + c_{i,j}\right)\left(m^-_\beta - a_\alpha - c_{i,j}\right)}{\prod_{(i,j)\in\lambda^{(\alpha)}} \prod_{(i',j')\in\lambda^{(\beta)}} \left(a_\alpha - a_\beta + d_{i,j;i',j'}\right)} \right)^{\mathbb{Z}_N}$$

BPS/CFT correspondence

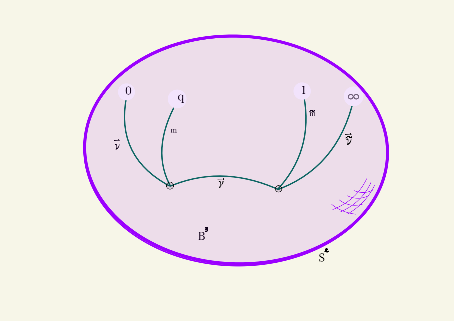
$NN'04$

Regular surface defect partition function

$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) \;=$$

Solves 4-point Knizhnik-Zamolodchikov equation

Theorem by NN+Tsymbalyuk′17−21



BPS/CFT correspondence

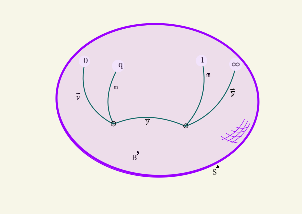
NN'04

Regular surface defect partition function

$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) \;=$$

Solves 4-point Knizhnik-Zamolodchikov equation
with $\boldsymbol{\Psi} \in \left(\mathcal{V}^{+} \otimes \mathcal{HW} \otimes \mathcal{HW} \otimes \mathcal{V}^{-}\right)^{\mathfrak{sl}_N}$

Theorem by NN+Tsymbalyuk'17–21



For $n=4$, $N=2$ it is PVI

Using BPZ equations observed earlier by Teschner'15

Litvinov+Lukyanov+NN+Zamolodchikov'16

Corollary:

$\varepsilon_1 \to 0$ *isomonodromic τ–function*

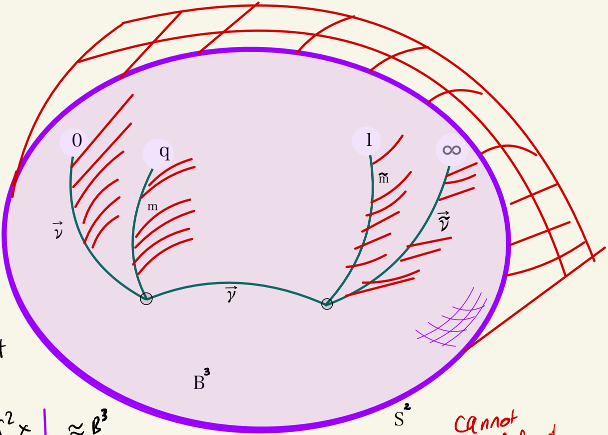
$$\boldsymbol{\Psi} \sim e^{\frac{\log \tau}{\varepsilon_1}}$$

Regular surface defect in $\mathcal{N}=2$ vs surface junction in $\mathcal{N}=4$

$$\mathbf{\Psi}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right) \in \left(\mathcal{V}^+\otimes\mathcal{HW}\otimes\mathcal{HW}\otimes\mathcal{V}^-\right)^{\mathfrak{sl}_N}$$

Solves 4-point Knizhnik-Zamolodchikov equation

Surface defects in $\mathcal{N}=4$ theory on $S^2\times$



up to Q-exact terms $\approx$

CS $\quad S^2\times | \approx B^3$

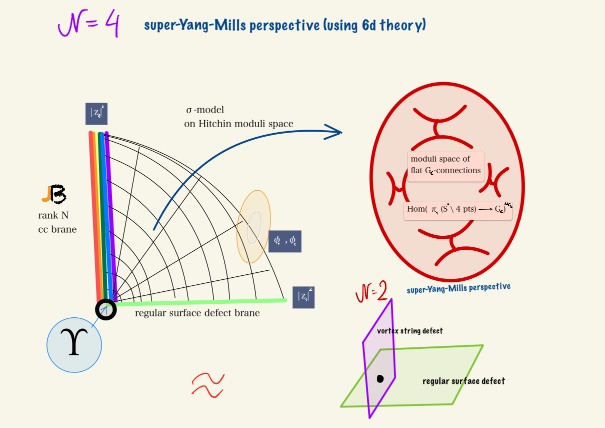
but electric charges in ∞-dim reps $\mathcal{V}^{\pm}, \mathcal{HW} \ldots$

cannot be defined as Wilson lines! Surface ops!

Intersecting regular and folded surface defect partition function

$$\widehat{\Psi}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)\in\mathbb{C}^N$$

Solves 5-point Knizhnik-Zamolodchikov equation



Mixed complex spins and finite dimensional reps

Parallel regular and folded surface defect partition function

$\tilde{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right)$

Solves 5-point Knizhnik-Zamolodchikov equation

In progress by Jeong+Lee+NN′21

$$0 = (k+2)\frac{d}{dz_0}\Psi + \sum_{i\neq 0} \frac{x_{i0}^2 \partial_{x_i}\partial_{x_0} - 2x_{i0}\left(s_i \partial_{x_0} - s_0 \partial_{x_i}\right) - 2s_0 s_i}{z_0 - z_i}\Psi$$

$$s_0 = \frac{k}{2} \qquad \parallel$$

twisted vacuum module field $w=1$

$$\Psi = \left\langle \mathbb{V}^{x_0}(z_0)\, \mathbb{V}^{s_1,x_1}(z_1)\, \mathbb{V}^{s_2,x_2}(z_2) \cdots \mathbb{V}^{s_n,x_n}(z_n) \right\rangle$$

$$(k+2)\nabla_0 \Psi + \partial_{x_0}\left(\tilde{\nabla}_0 \Psi\right) = 0$$

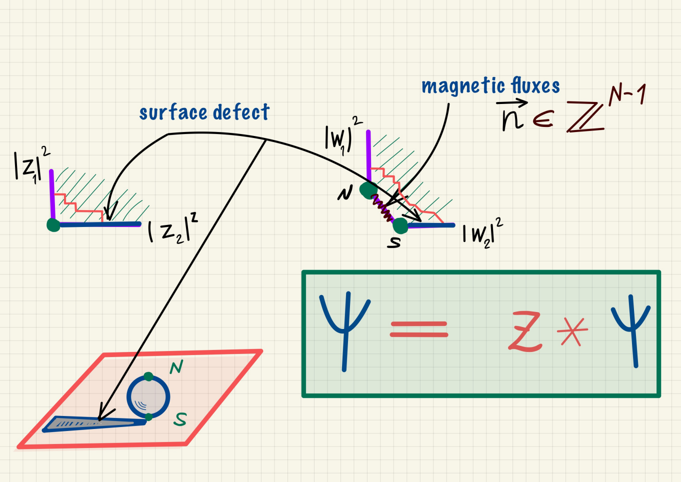
$$\tilde{\nabla}_0 = \sum_{i\neq 0} \frac{x_{i0}^2 \partial_{x_i} - 2x_{i0}s_i}{z_0 - z_i} \qquad\qquad \nabla_0 \Psi = \frac{d\Psi}{dz_0} + \sum_i \frac{x_{i0}\partial_{x_i} - s_i}{z_0 - z_i}\Psi$$

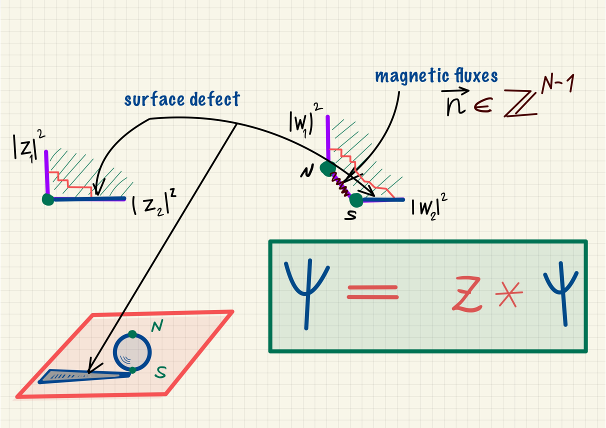
$$\left[\nabla_0, \tilde{\nabla}_0\right] = 0$$

$$\tilde{J}^0_n = J^0_n + \frac{k}{2} w\, \delta_{n,0} \qquad \tilde{J}^{\pm}_n = J^{\pm}_{n\pm w}$$

long strings in AdS_3

The power of four dimensions: Blown up Surface defects





$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) =$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^{N-1}} \mathcal{Z}\left(\mathbf{a} + \varepsilon_2 \mathbf{n}, \mathbf{m}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}\right) \boldsymbol{\Psi}\left(\mathbf{a} + \varepsilon_1 \mathbf{n}, \mathbf{m}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{w}, \mathfrak{q}\right)$$

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Limit $\varepsilon_1 \to 0$: higher rank analogue of GIL "Kyiv" formula

$N = 2, n = 4$ *case*: *Gamayun–Iorgov–Lysovyy*'12

$$\textit{Schematically}, \ \tau^{\mathrm{PVI}}_{\vec{\nu}}(a, b; \mathfrak{q}) = \sum_{n \in \mathbb{Z}} e^{nb} \mathcal{Z}_{\vec{\nu}}(a + n; \mathfrak{q})^{c=1}$$

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BPS/CFT correspondence

$\mathcal{Z}$-partition function $=$ W_N-conformal block

Alday+Gaiotto+Tachikawa'09, *Wyllard*'09

For $N = 2$: Virasoro with

$$c = 1 + 6Q^2\,, \;\; Q^2 = \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1 \varepsilon_2}$$

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BPS/CFT correspondence

The $Spin(4) \to SU(2)$ reduction $\varepsilon_1 + \varepsilon_2 = 0$

corresponds to $c = 1$ conformal blocks

For $\varepsilon_1 + \varepsilon_2 = 0$, $\mathcal{Z}$ is expressed in terms of free fermions $\psi, \tilde{\psi}$

NN+Okounkov'03

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BPS/CFT correspondence

The $\mathbb{R}^4 \to \mathbb{R}^2$ reduction $\varepsilon_1 \to 0$

corresponds to $c \to \infty$, i.e. classical conformal blocks

A. and Al. Zamolodchikov, late eighties

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Thus, using the four dimensional side
of the BPS/CFT correspondence

We managed to establish the $c = 1/c = \infty$ duality

$\approx$ fermion/boson dualuty in two dimensions

This is one of the few almost proven dualities,

which might someday help establish

The existence of a $4 + 2$-dimensional superconformal field theory

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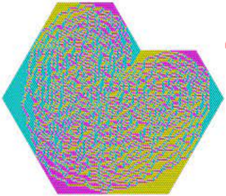
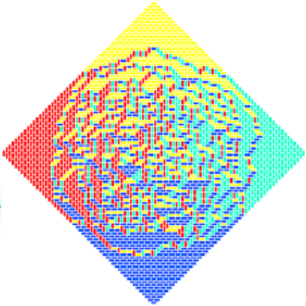
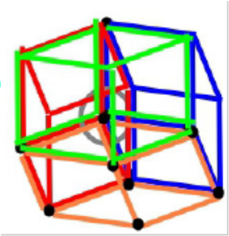
BEYOND Lie algebra symmetries..

From sums over partitions







To higher dimensional suprises to come....

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