Topological Phases within CFT Interfaces

Im Memory of Krzysztof Gawedzki

Volker Schomerus, Eurostrings 2022, Lyon, 28. April 2022

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Krzysztof Gawedzki

2. July 1947 – 21. January 2022

Krzysztof Gawedzki

2. July 1947 (Zarki, Poland) – 21. January 2022 (Lyon, France)

- **1971 PhD in Warsaw, Poland**
- **1975-76 Univ. Göttingen**
- **1978-79 Univ. Gdansk**
- **1979-80 Harvard Univ.**
- **1982-2000 CNRS Researcher**

  at IHES Bures-sur-Yvette
- **2001-2014 at ENS Lyon**

Krzysztof Gawedzki

Selected Contributions

Early contributions in

- **Constructive Quantum Field Theory**

Iconic series of papers on

- **WZW coset Conformal Field Theory**

- **Boundaries and defects in 2D CFT,**

  in particular for WZW cosets

- **Non-abelian bundle gerbes**

  & relation with topological insulators

- **Tolopogical Chern-Simons theory**
- **Turbulence**

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Commun. Math. Phys. 99, 197–252 (1985)

Communications in Mathematical Physics

Massless Lattice ϕ_4^4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model

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Nuclear Physics B320 (1989) 625–668
North-Holland, Amsterdam

COSET CONSTRUCTION FROM FUNCTIONAL INTEGRALS

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Received 26 September 1988

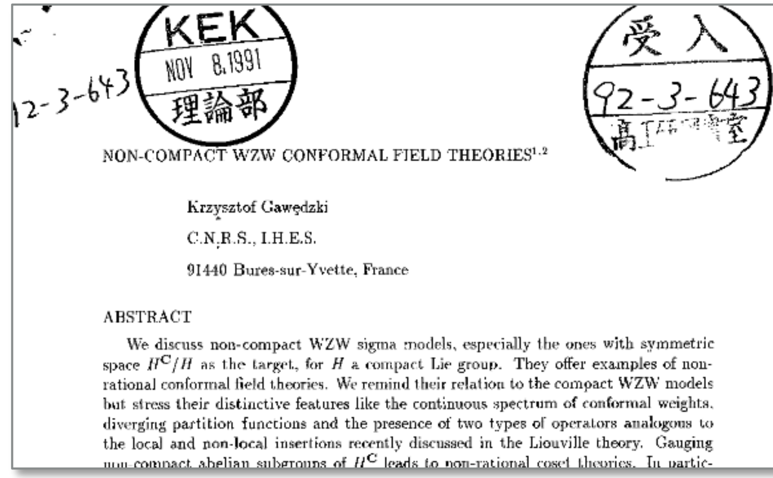
A detailed study of the gauged Wess–Zumino–Witten models is presented. These models are shown to be conformal field theories realizing the Goddard–Kent–Olive coset construction. Partition functions are computed for an arbitrary group G with a subgroup H gauged. Correlation functions are shown to be computable in terms of WZW ones. Explicit cases of the minimal models and parafermionic theories are worked out.

Krzysztof Gawedzki

Selected Contributions

Pioneering work on

irrational CFT w. current algebra symmetry


KEK NOV 8,1991 理論部

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NON-COMPACT WZW CONFORMAL FIELD THEORIES[1,2]

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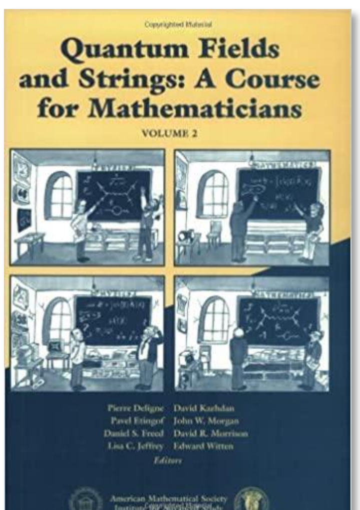
ABSTRACT

We discuss non-compact WZW sigma models, especially the ones with symmetric space $H^{\mathbb{C}}/H$ as the target, for H a compact Lie group. They offer examples of non-rational conformal field theories. We remind their relation to the compact WZW models but stress their distinctive features like the continuous spectrum of conformal weights, diverging partition functions and the presence of two types of operators analogous to the local and non-local insertions recently discussed in the Liouville theory. Gauging non-compact abelian subgroups of $H^{\mathbb{C}}$ leads to non-rational coset theories. In partic-

Strings in AdS_3 with pure NSNS flux



Unique teacher

&

Perfect editor-in-chief

of *Annales Henri Poincare*

On Nov 24 2021 the AIP & APS announced that

2022 Dannie Heineman Prize for Mathematical Phyics

is awarded to

Krzysztof Gawedzki and Antti Kupianen

for ``*fundamental contributions to quantum field theory, statistical mechanics, and fluid dynamics using geometric, probabilistic, and renormalization group ideas.*''

Topological Phases

within CFT Interfaces

Interfaces in Conformal Field Theory

Interfaces (and boundaries) in CFT have fascinating features:

- **Significant freedom even for fixed bulk**
- **Host a lot of information about the bulk**

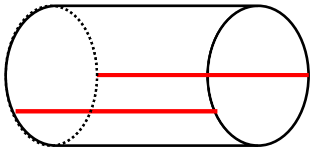
Illustration in 2D:

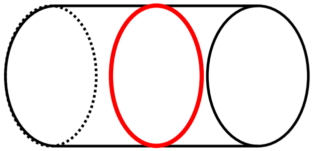
- **Even in rational 2D CFT moduli space of boundary conditions often continuous (e.g. in WZW models)**

D-brane moduli

- **Through modular bootstrap one can recover bulk from boundary**

open string spectrum





closed string couplings

Half-BPS interface of 4D N=4 SYM

A host for supergroup Chern-Simons theory

Half-BPS boundaries in N = 4 in 4D SU(N) SYM theory are consistent with

Kapustin-Witten topological twist $\rightarrow$ SU(N) Chern-Simons theory [Witten]

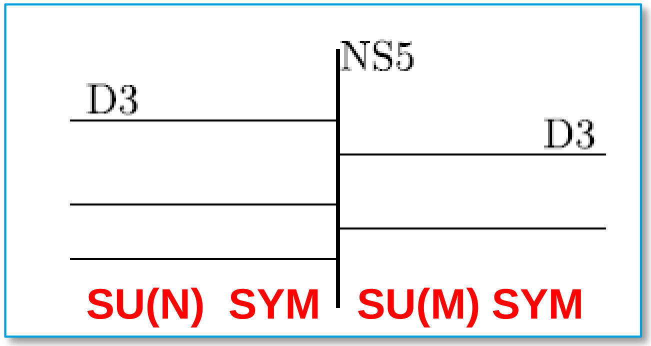
→ S,T duality provide new insight into Chern-Simons theory

Half-BPS (Janus) interface between

4D N=4 SYM with G = SU(N), SU(M)

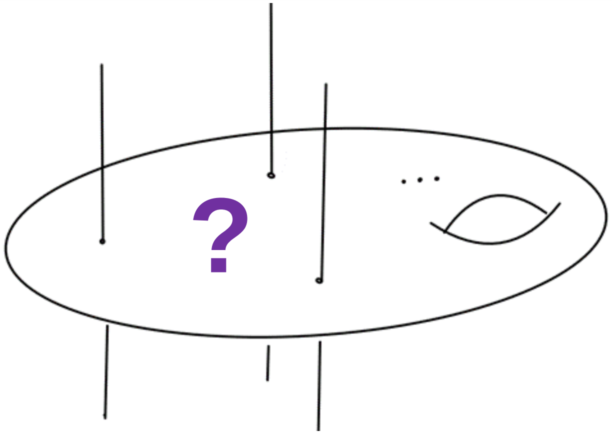
hosts SU(N|M) Chern-Simons theory.

[Gaiotto, Witten] [Kapustin,Witten]



Supergroup CS theory

is poorly understood.



[Rozansky, Saleur] … [Mikhaylov]

Quantum Holonomy

The Weyl Algebra of Quantum CS Theory

Chern–Simons Holonomies

Classical description

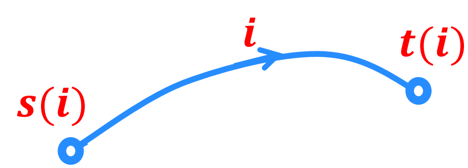
Chern-Simons gauge field $A^a_\mu(x)$ equipped with the Atiyah-Bott PB:

$$\{ A^a_\mu(x), A^b_\nu(y)\} \sim \frac{1}{k}\, \delta^{ab} \epsilon_{\mu\nu}\, \delta(x-y) \qquad A \mapsto g^{-1} A\, g + g^{-1} dg$$

$$\{A^1_\mu(x), A^2_\nu(y)\} \sim \frac{1}{k}\, \kappa^{12}\, \epsilon_{\mu\nu}\, \delta(x-y)$$

Gauge invariant observables built from holonomies along paths i :

$$U(i) = P \exp\left(\int_{s(i)}^{t(i)} A \right) \qquad U \mapsto g^{-1}\big(s(i)\big)\, U(i)\, g\big(t(i)\big)$$



Poisson bracket takes the form

$$\{U^1, U^2\} = \frac{1}{k}\left(r^{12}_-\, U^1 U^2 + U^1\, U^2\, r^{12}_+ \right)$$

$$\left[r^{12}, r^{23}\right] + \left[r^{12}, r^{13}\right] + \left[r^{13}, r^{23}\right] = 0 \qquad r_\pm = \sum_{i\ simple} h^i \otimes h^i + 2 \sum_{\alpha\ pos} t^{\pm\alpha} \otimes t^{\mp\alpha}$$

Chern–Simons Holonomies

Quantization

Suppose we have a solutions $R_\pm$ of quantum Yang-Baxter equation s.t.

$$R_\pm = 1 + \frac{1}{k} r_\pm + \cdots$$

$\rightarrow$ quantization of holonomy along open curve gives rise to link algebra

<u>Definition</u>: [Alekseev, Grosse, VS] The link algebra is generated by matrix elements of U satisfying quadratic relations

$$R_-^{12} U^1 U^2 = U^2 U^1 R_+^{12}$$

- $R_\pm$ can be constructed for all simple Lie superalgebras g from the generators of the quantum deformed enveloping algebra $\mathcal{G} = U_q(g)$
- Link algebra admits two (graded) commuting actions of $\mathcal{G}$

quantization gauge trafos in s and t

Quantum Holonomies

Example: GL(1|1)

$q = e^{\frac{2\pi i}{p}}$

$\mathcal{G} = U_q(\, gl(1|1)\,)$

$k_\alpha \,,\, k_\beta \,,\, e_\pm$

bosonic

fermionic

$$
\begin{array}{ll}
[k_\alpha, k_\beta] = 0 & k_\alpha^p = k_\beta^p = 1 \\
[k_\alpha, e_\pm] = 0 & k_\beta e_\pm = q^{\pm 1} e_\pm k_\beta \\
\{e_\pm, e_\pm\} = 0 & \{e_+, e_-\} = \dfrac{k_\alpha - k_\alpha^{-1}}{q - q^{-1}}
\end{array}
$$

$$R = \frac{1}{p^2}\left(1 \otimes 1 - (q - q^{-1}) e_+ \otimes e_-\right) \sum_{n,m=0}^{p-1} \sum_{s,t=0}^{p-1} q^{nt+ms} k_\alpha^n k_\beta^m \otimes k_\alpha^{-s} k_\beta^{-t}$$

Link algebra $\mathcal{U}$ is

a q-deformation of

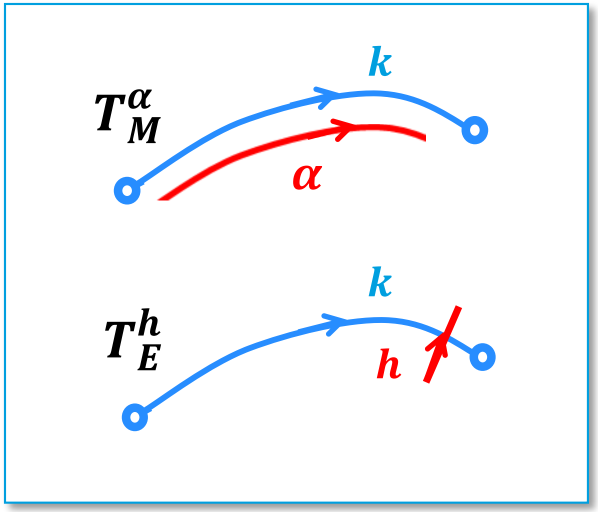
$\mathfrak{Fun}(\, GL(1/1)\,)$

$$
\begin{array}{ll}
\ell_\alpha \ell_\beta = \ell_\beta \ell_\alpha & \ell_\alpha^p = 0 = \ell_\beta^p \\
\ell_\alpha \xi_\pm = \xi_\pm \ell_\alpha & \ell_\beta \xi_\pm = q^{\mp 1} \xi_\pm \ell_\beta \\
\{\xi_\pm, \xi_\pm\} = 0 & \{\xi_+, \xi_-\} = q - q^{-1}
\end{array}
$$

Quantum Holonomies

Realization on Kitaev Spin DOF

Suppose that $\mathcal{G}$ = D($\mathcal{H}$) is Drinfel'd double of some Hopf algebra $\mathcal{H}$ $\quad \mathcal{G} \sim \mathcal{H} \otimes \mathcal{H}^*$

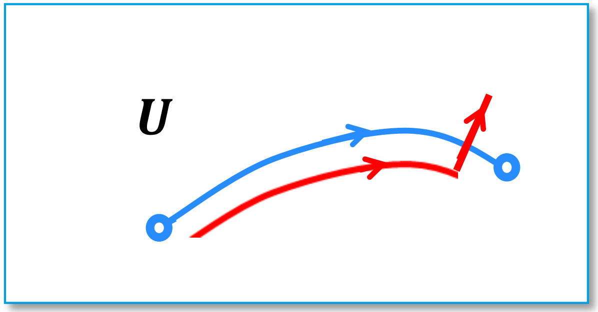


Kitaev's triangle operators T : $\mathcal{H} \rightarrow \mathcal{H}$

$T_E^h\, k = \; h\, k \qquad h \in \mathcal{H}$ **``electric"**

$T_M^\alpha\, k = \langle \alpha, k_2 \rangle\, k_1 \qquad \alpha \in \mathcal{H}^*$ **``magnetic"**

<u>Theorem</u>: [Meusburger] $U \sim T_E\, T_M : \mathcal{H} \rightarrow \mathcal{H}$ **behaves like quantum holonomy for algebra $\mathcal{G}$ = D($\mathcal{H}$) of gauge trafos given by DrinfeldD**



Super-Chern-Simons

Observables & States

Quantum Group Lattice Gauge Theory

The Combinatorial Quantization Approach

[Fock, Rosly]
[Alekseev, Grosse, VS]

Draw graph Γ on the surface $\Sigma_{n,g}$

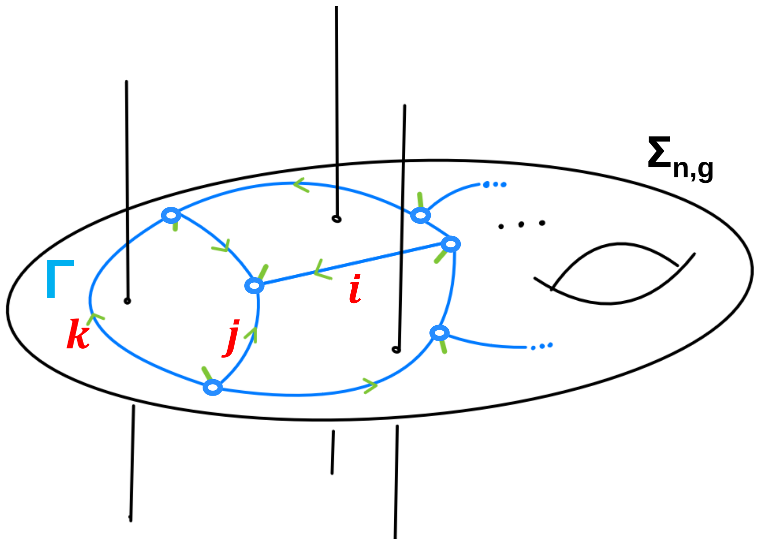
& glue link algebras accordingly:

choose cilia, orient links

$$U^1(i)U^2(j) = U^2(j)U^1(i)\, R^{12}$$

$$U^1(i)U^2(k) = U^2(k)U^1(i)_{\partial i \cap \partial k = \emptyset}$$

$$U(j)U(-j) = v \qquad \text{ribbon element}$$

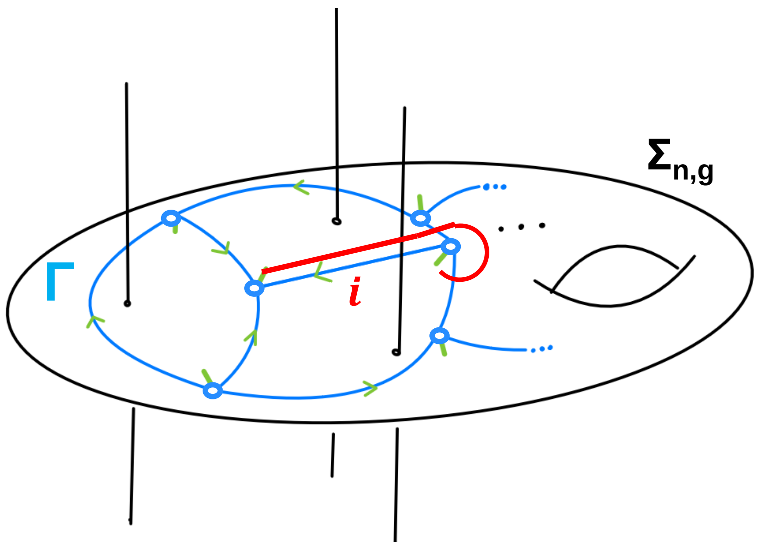


- Defines graph algebra $\mathcal{F}(\Gamma_{cil})$ → **Factorization algebras**
- Consistent with local deformed gauge symmetry at the vertices
- Chern-Simons observables: algebra $\mathcal{A}(\Sigma_{n,g})$ of gauge invariants

Kitaev Models

A special case of combinatorial CS theory

[Kitaev] [Levin,Wen]



If $\mathcal{G}$ = D($\mathcal{H}$) Drinfeld double there

exists a $\mathcal{H}$-spin model realization:

$$U(i):\ \mathcal{H}^{\otimes\,\mathrm{E}} \ \rightarrow\ \mathcal{H}^{\otimes\,\mathrm{E}}$$

E = # of edges

Kitaev's time evolution operator

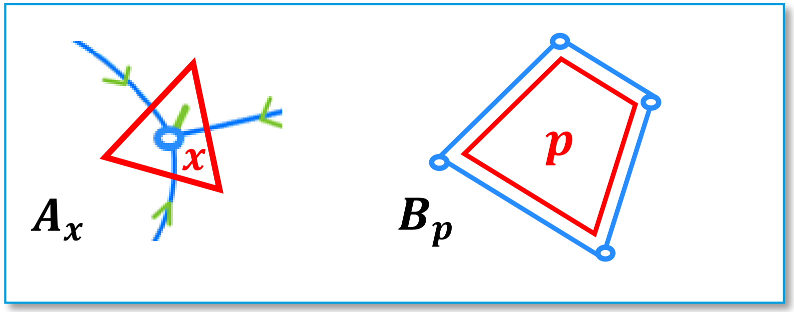
$$U_K \sim \prod_{x\in V} A_x^m \circ \prod_{p\in P} B_p^\eta$$

$\mathcal{H} = \mathbb{C}\,(\mathbb{Z}_2) \rightarrow$ Kitaev's toric code

constructed from

Vertex operators $A_x = \prod_{t(j)=x} T_{E,j}$

Face operators $B_p = \prod_{j\in\partial p}\ T_{M,j}$



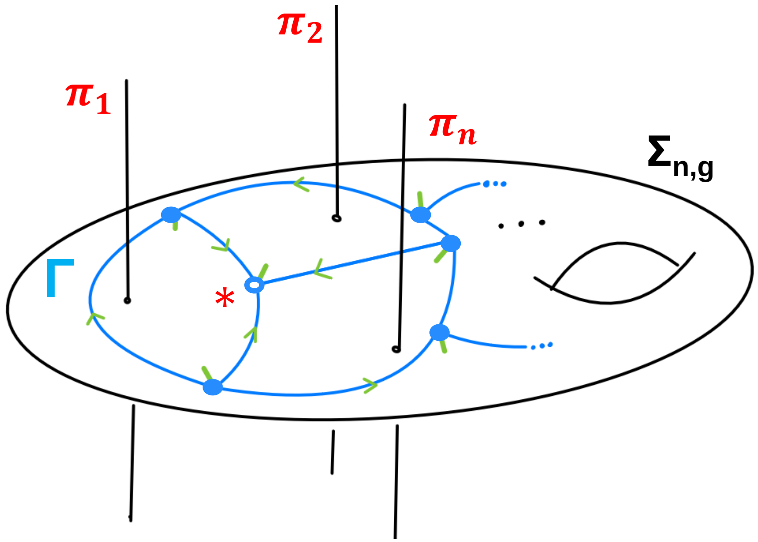
Chern-Simons Observables & Ground States

General construction

Pick irrep π_a of $\mathcal{G}$ for each vertical WL

and introduce the space

$$V^{\pi_a,g} := V^{\pi_1} \otimes V^{\pi_2} \ldots \otimes V^{\pi_n} \otimes \mathcal{G}^{\otimes g}$$



Theorem: [Aghaei, Gainutdinov, Pawelkiewicz, VS]

One can define an action Chern-Simons observables $\mathcal{A}(\Sigma_{n,g})$ on the space $V^{\pi_a,g}$ that centralizes the canonical action of $\mathcal{G} = \mathcal{G}_*$. ↔ like SU(N) and permutation group

- Provides complete control of representation theory of CS observables — as intricate as representation theory of $\mathcal{G}$
- Algebra $\mathcal{A}(\Sigma_{n,g})$ of observables contains Dehn twist generators of the mapping class group → reps of MCG/``quantum gates"

Chern-Simons Observables & Ground States

Example: GL(1|1) Chern-Simons Theory in a torus

Space $W^{g=1}$ of ground states given by $\mathcal{G}$ invariants within $V^{g=1} = \mathcal{G}$

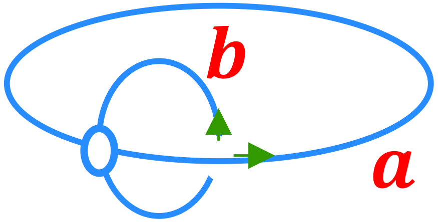
$$\dim W^{g=1} = p^2 + 1$$

$$\mathcal{G} \cong (p^2 - 1)\, P_0 \oplus 2\,\pi_0 \oplus \pi_1 \oplus \pi_{-1}$$

Carries (p^2+1) –dim. projective representation of modular group *SL(2,Z)*

$$S = \hat{v}(b)\, \hat{v}(a)\, \hat{v}(b) \qquad T = \hat{v}(a)^{-1}$$

generators of Dehn twists along a,b cycles



$$Sw_{e,n} = \frac{i}{p}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} q^{-2s(n-\frac{1}{2})-2e(t-\frac{1}{2})} w_{s,t} - \frac{i}{p}(q^e - q^{-e})\sum_{t=0}^{p-1} q^{-2et} w_t$$

$$Sw_n = \frac{i}{p(q-q^{-1})}\sum_{\substack{s,t=0\\ s\neq 0}}^{p-1} \frac{q^{-2ns}}{[s]_q} w_{s,t} - \frac{i}{2p(q-q^{-1})} w$$

$$Sw = 2i(q-q^{-1})\sum_{t=0}^{p-1} w_t \qquad W^{g=1} = \text{span}\,\{w_{e,n}, w_n, w\}_{e=1,n=0}^{p-1,p-1}$$

[Aghaei, Gainutdinov, Pawelkiewicz, VS]

↔ [Lyubashenko, Majid]

... [Mikhaylov] for gl(1|1)

Conclusions and Outlook

Efficient algebraic algorithm to determine the space of ground states along w. representations of MCG for supergroup Chern-Simons theory

$\mathcal{G}$ = **gl(1|1) → sl(1|2), psl(2|2) ?**

Applications include calculation of topological invaraints for knots and and 3-manifolds

From MCG reps using Heegaard representation & Dehn surgery

Extension of Kitaev/Levin-Wen string-net models to graded spin DOF $\mathcal{H}$

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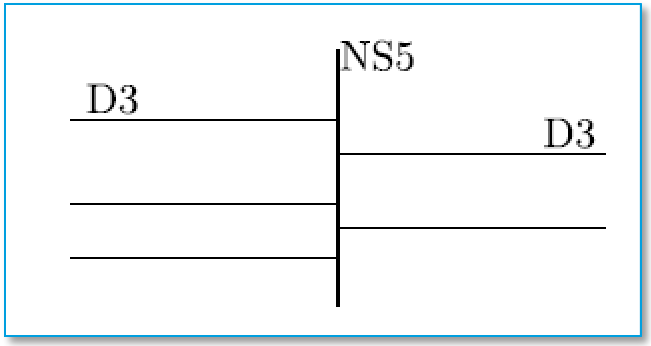

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