Spindles, gravitational blocks and geometric extremization

Dario Martelli

University of Torino

work with D. Cassani, F. Faedo, P. Ferrero, J. Gauntlett, M. Inglese, J. Perez Ipinã, J. Sparks, ...and more

Eurostrings 2022

Ecole Normale Supérieure de Lyon

25 - 29 April 2022

Outline

1. Spindles from supersymmetric black holes
2. Branes wrapped on spindles
3. "Spindly" gravitational blocks
4. Geometric extremization formalism
5. Summary

Motivations/supersymmetric black holes

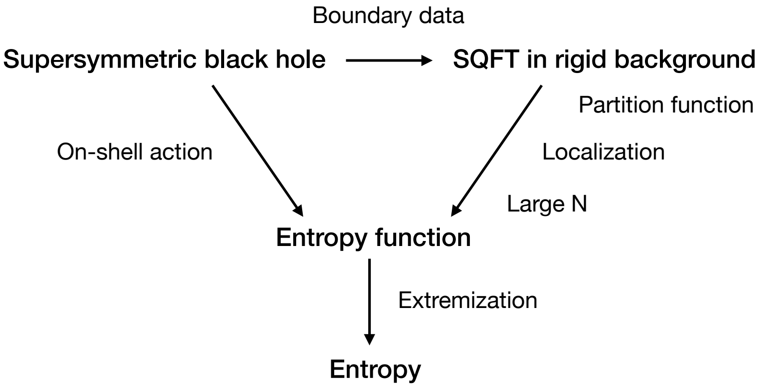
- Holography motivates the interest in asympototically locally AdS black holes
- Conformal boundary data $\longrightarrow$ precise identification of dual QFT + background where it lives (metric, R-symmetry gauge field $\boldsymbol{A_R}$, ...)
- When the black holes are supersymmetric, in principle we can employ localization to compute the appropriate partition function of the dual SQFT, that counts the states making up the black hole entropy

Motivations/supersymmetric black holes

- Holography motivates the interest in asympototically locally AdS black holes
- Conformal boundary data $\longrightarrow$ precise identification of dual QFT + background where it lives (metric, R-symmetry gauge field $\boldsymbol{A_R}$, ...)
- When the black holes are supersymmetric, in principle we can employ localization to compute the appropriate partition function of the dual SQFT, that counts the states making up the black hole entropy
- This is conceptually one of the most useful approaches for improving our understanding of quantum gravity, using the holographic duality!

The entropy function

The central object in the comparison between the black hole and its dual QFT setup is given by the notion of entropy function



Near the horizon

- General fact: any supersymmetric black hole (in AdS or flat space), near its horizon $\mathcal{H}$ approaches the space-time $\mathrm{AdS}_2 \times \mathcal{H}$

  E.g. in $D = 4$: $\mathcal{H} = \Sigma_g$ (Riemann surface with genus $g \geq 0$)

  in $D = 5$: $\mathcal{H} = S^3$

- At least for spherical horizons, black holes can rotate (angular momentum) $\rightarrow$ the near-horizon geometry is a "twisted product"

- Focussing on $D = 4$, there are 2 distinct ways to preserve susy

  Static $\mathcal{H} = \Sigma_g$: $\frac{1}{2\pi}\int_{\Sigma_g} dA_R = 2(1 - \mathrm{g})$, topological twist

  Rotating $\mathcal{H} = S^2$: $\frac{1}{2\pi}\int_{\Sigma_g} dA_R = 0$, no twist

- Although the topologically twisted black holes were studied first [Benini et al] [see FB's talk], I will jump to the rotating BHs

Kerr-Newman black holes: rotating and dyonically charged

- Context: $D = 4$ "STU" gauged ($U(1)^4$) supergravity
- Electric charges e_i and magnetic fluxes n_i defined as

$$e_i = \frac{1}{2\pi}\int_{S^2} *dA_i \qquad\qquad n_i = \frac{1}{2\pi}\int_{S^2} dA_i \ \in \mathbb{Z}$$

subject to the "no twist" condition, imposed by supersymmetry:

$$\sum_i n_i = \frac{1}{2\pi}\int_{S^2} dA_R = 0 \qquad \text{where} \qquad A_R \equiv \sum_i A_i$$

- The entropy (here written for the $U(1)^2$ sub-case and $G_4 = 1$) is

$$S(n_i, e_i) \ = \ \frac{\pi}{2}\left(-1 + \sqrt{1 + n_1 n_2 + e_1 e_2}\right)$$

The entropy function

- The entropy can be obtained by extremizing the entropy function

$$\mathcal{S}(\varphi_i, \epsilon; n_i, e_i, J) = \mathcal{E}(\varphi_i, \epsilon; n_i) - (\epsilon J + \varphi_1 e_1 + \varphi_2 e_2)$$

with

$$\mathcal{E}(\varphi_i, \epsilon; n_i) = \frac{1}{2i}\left(\frac{1}{4} n_1 n_2 \epsilon + 16\frac{\varphi_1 \varphi_2}{\epsilon}\right)$$

and where the fugacities $\varphi_1, \varphi_2, \epsilon$ are subject to the constraint

$$2\sum_i \varphi_i - \frac{\epsilon}{2} = \pi i$$

Warning: numerical factors in conventions vary wildly in the literature

Spindles from supersymmetric accelerating black holes

- The KN black holes (supersymmetric and otherwise) can be deformed:
  - The horizon S^2 is replaced by a spindle Σ: a two-sphere with two orbifold singularities $\mathbb{R}^2/\mathbb{Z}_{n_+}, \mathbb{R}^2/\mathbb{Z}_{n_-}$ at the poles
  - The total magnetic flux

$$\sum_i n_i = \frac{1}{2\pi}\int_\Sigma dA_R = \frac{1}{n_-} - \frac{1}{n_+} \neq 0$$

- $n_+ - n_-$ is interpreted as acceleration of the black hole
- Previously known only in minimal theory [Plebanski,Demianski] and later (partially) extended to $U(1)^4$ in [Ferrero,Inglese,DM,Sparks]
- Differently from KN, we can set $J = 0 \rightarrow$ static supersymmetric black holes with $AdS_2 \times \Sigma$ near-horizon geometry

Topological twist, twist and anti-twist

- In general, there exist only two types of supersymmetric twists on the spindle [Ferrero,Gauntlett,Sparks]

$$\sum_i n_i = \frac{1}{2\pi}\int_\Sigma dA_R = \frac{1}{n_-} - \frac{1}{n_+} \quad \textbf{(anti} - \textbf{twist)}$$

$$\sum_i n_i = \frac{1}{2\pi}\int_\Sigma dA_R = \frac{1}{n_-} + \frac{1}{n_+} \quad \textbf{(twist)}$$

- Note: for $n_+ = n_- = 1$ the anti-twist reduces to S^2 with no twist while the twist reduces to S^2 with the topological twist!

- Let me focus on the solution in the minimal theory (where only the anti-twist is realised) [Ferrero,Gauntlett,DM,Perez Ipiñ̃a,Sparks]

Entropy function [Cassani,Gauntlett,DM,Sparks]

- The total (only) magnetic flux and electric charge are

$$Q_e = \frac{1}{4\pi}\int_\Sigma *dA_R \qquad Q_m = \frac{1}{4\pi}\int_\Sigma dA_R = \frac{1}{n_-} - \frac{1}{n_+}$$

- The Bekenstein-Hawking entropy reads

$$S = \frac{\pi}{4}\left(-\chi + \sqrt{\chi^2 + 16\left(Q_e^2 + Q_m^2\right)}\right) \qquad \chi \equiv \frac{1}{n_-} + \frac{1}{n_+}$$

  and it can be obtained from the Legendre transform of the on shell action/entropy function

$$\mathcal{E}(\varphi, \epsilon; Q_m) = \frac{1}{2i}\left(Q_m^2\epsilon + \frac{\varphi^2}{\epsilon}\right) \qquad \varphi - \frac{\chi}{4}\epsilon = \pi i$$

- Note: this entropy function can be thought of a function encoding properties (the entropy) of the near-horizon $AdS_2 \times \Sigma$ solution

Embedding in $D = 11$ supergravity and M2 branes

- Both the black holes solutions and their near-horizon $\mathrm{AdS}_2 \times \Sigma$ geometries can be uplifted to $D = 11$ supergravity
  - Solutions of the minimal theory can be uplifted to $M_4 \times SE_7$ solutions where SE_7 is any regular Sasaki-Einstein seven-manifold
  - Solutions of the STU theory can be uplifted to $M_4 \times S^7$ solutions
- From now on we will focus on the near-horizon $\mathrm{AdS}_2 \times \Sigma$ solutions
- Can be interpreted as arising from wrapping N M2 branes on a spindle
- They are supersymmetric solutions of the type $\mathrm{AdS}_2 \times Y_9$ where Y_9 are fibered manifolds $SE_7 \to Y_9 \to \Sigma$

  Static solutions ($J = 0$) $\to$ Y_9 is a "GK geometry" [Kim,Park]

(More) branes wrapped on the spindle [DM+various]

- M2 branes can be replaced with D3, D4, M5 branes
- $\mathrm{AdS}_2 \times \mathbb{\Sigma}$ solutions of STU $D = 4$ gauged supergravity are replaced by $\mathrm{AdS}_p \times \mathbb{\Sigma}$ solutions of STU-like $D = p + 2$ gauged supergravities
- In each case, solutions can be uplifted to $\mathrm{AdS}_p \times Y_{D-p}$ solutions of Type IIB, massive Type IIA, D=11 supergravity, where Y_{D-p} are fibered manifolds/orbifolds $X_{D-p-2} \to Y_{D-p} \to \mathbb{\Sigma}$

  for $p = 3, D = 10$: $X_5 = \mathrm{SE}_5$ (D3 branes on $\mathbb{\Sigma}$)
  for $p = 4, D = 10$: $X_4 = \frac{1}{2} S^4$ (D4 branes on $\mathbb{\Sigma}$)
  for $p = 5, D = 11$: $X_4 = S^4$ (M5 branes on $\mathbb{\Sigma}$)
- The gravitational observables are c_{2d}, $\mathcal{F}_{3d}$, a_{4d}, respectively. The central charges can be reproduced by reducing anomaly polynomials of the parent theories on $\mathbb{\Sigma}$, but I will not discuss this today

D3 branes wrapped on the spindle

- In the $D = 5$ STU gauged ($U(1)^3$) supergravity the solutions are $\mathrm{AdS}_3 \times \Sigma$, with n_i fluxes for the dA_i through the spindle
- The fluxes associated to the gauge fields A_i gauging $U(1)^3$ satisfy

$$n_i = \frac{1}{2\pi}\int_{\Sigma} dA_i \qquad n_1 + n_2 + n_3 = \frac{n_+ + \sigma n_-}{n_+ n_-}$$

with $n_+ n_- n_i \in \mathbb{Z}$ and $\sigma = \pm 1$ corresponding to the twist/anti-twist

- The gravitational central charge of the dual 2d SCFT reads

$$c_{\mathrm{grav}} = \frac{3N^2 n_1 n_2 n_3}{\frac{\sigma}{n_+ n_-} - (n_1 n_2 + n_1 n_3 + n_2 n_3)}$$

D3 branes wrapped on the spindle

- This can be obtained extremizing the trial off-shell central charge

$$\mathcal{E}(\varphi_i, \epsilon; n_i) = -3N^2 \left(n_1\varphi_2\varphi_3 + \varphi_1 n_2 \varphi_3 + \varphi_1\varphi_2 n_3 + n_1 n_2 n_3 \epsilon^2 \right)$$

  with φ_i, ϵ satisfying the constraint

$$\varphi_1 + \varphi_2 + \varphi_3 - \frac{n_+ - \sigma n_-}{n_+ n_-}\epsilon = 2$$

- Uplift to Type IIB solutions of the type $\mathrm{AdS}_3 \times Y_7$ where Y_7 are fibered manifolds $S^5 \to Y_7 \to \Sigma$: Y_7 is a "GK geometry" [Kim]
- Solutions for D4 and M5 branes work similarly. Importantly, in all cases one can write down a function $\mathcal{E}$ such that its critical points reproduce the relevant gravitational observables [Faedo,DM]

Off-shell trial free energies

- Conjecture [Faedo,DM]: the off-shell (trial) free energies for all the "conformal" (M2,D3,D4,M5) branes wrapped on spindles can be written in terms of "gravitational blocks" as follows

$$\boxed{\mathcal{E}^{\pm}(\varphi_i, \epsilon; n_i, n_+, n_-, \sigma) = \frac{1}{\epsilon}\Big(\mathcal{F}_d(\varphi_i + n_i\epsilon) \pm \mathcal{F}_d(\varphi_i - n_i\epsilon)\Big)}$$

  where the variables φ_i, ϵ and the magnetic fluxes n_i obey

$$\boxed{\sum_{i=1} \varphi_i - \frac{n_+ - \sigma n_-}{n_+ n_-}\epsilon = 2 \qquad \sum_{i=1} n_i = \frac{n_+ + \sigma n_-}{n_+ n_-}}$$

- This reproduces the anomaly polynomial computations for D3 and M5 branes, while it predicts the large N limit of free energies on spindles for M2 and D4 branes

Spindly gravitational blocks

The building blocks $\mathcal{F}_d$ for (field theory) dimensions $d = 3, 4, 5, 6$

	$d=3$	$d=4$	$d=5$	$d=6$
$\mathcal{F}_d$	$b_3(\Delta_1\Delta_2\Delta_3\Delta_4)^{1/2}$	$b_4(\Delta_1\Delta_2\Delta_3)$	$b_5(\Delta_1\Delta_2)^{3/2}$	$b_6(\Delta_1\Delta_2)^2$
b_d	$-\frac{\sqrt{2}\pi}{3}N^{3/2}$ $-F_{S^3}$	$-\frac{3}{2}N^2$ $-6a_4$	$-\frac{2^{5/2}\pi}{15}\frac{N^{5/2}}{\sqrt{8-N_f}}$ $\frac{4}{27}F_{S^5}$	$-\frac{9}{256}N^3$ $-\frac{63}{4096}a_6$
$\pm$	$-\sigma$	$-$	$-\sigma$	$-$

They are proportional to: $[d=3]$ the S^3 off-shell free energy of the ABJM theory, $[d=4]$ the trial central charge of the $\mathcal{N}=4$ SYM theory, $[d=5]$ the S^5 off-shell free energy of the $d=5$, $\mathcal{N}=1$ SCFT, $[d=6]$ the trial central charge of the $d=6$, $(2,0)$ SCFT

Can we prove this conjecture? How should we go about it?

Spindly gravitational blocks

The building blocks $\mathcal{F}_d$ for (field theory) dimensions $d = 3, 4, 5, 6$

	$d=3$	$d=4$	$d=5$	$d=6$
$\mathcal{F}_d$	$b_3(\Delta_1\Delta_2\Delta_3\Delta_4)^{1/2}$	$b_4(\Delta_1\Delta_2\Delta_3)$	$b_5(\Delta_1\Delta_2)^{3/2}$	$b_6(\Delta_1\Delta_2)^2$
b_d	$-\frac{\sqrt{2}\pi}{3}N^{3/2}$ $-F_{S^3}$	$-\frac{3}{2}N^2$ $-6a_4$	$-\frac{2^{5/2}\pi}{15}\frac{N^{5/2}}{\sqrt{8-N_f}}$ $\frac{4}{27}F_{S^5}$	$-\frac{9}{256}N^3$ $-\frac{63}{4096}a_6$
$\pm$	$-\sigma$	$-$	$-\sigma$	$-$

They are proportional to: $[d=3]$ the S^3 off-shell free energy of the ABJM theory, $[d=4]$ the trial central charge of the $\mathcal{N}=4$ SYM theory, $[d=5]$ the S^5 off-shell free energy of the $d=5$, $\mathcal{N}=1$ SCFT, $[d=6]$ the trial central charge of the $d=6$, $(2,0)$ SCFT

Can we prove this conjecture? How should we go about it?

For M2 and D3 branes, we can make a lot of progress employing the geometric extremization formalism [Boido,Gauntlett,DM,Sparks] to appear

Supersymmetric $\mathrm{AdS}_3 \times Y_7$ solutions

- The class of solutions of Type IIB supergravity we consider comprises

$$\mathrm{d}s^2_{10} = L^2 \mathrm{e}^{-\frac{B}{2}} \left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_7 \right) \qquad F_5 = -L^4 \left(\mathrm{vol}_{\mathrm{AdS}_3} \wedge F + *_7 F \right)$$

  where B is a function and F is a closed 2-form on Y_7

- For a consistent solution, we must impose the appropriate F_5 flux quantization conditions through any five-cycle $\Sigma_a \in H_5(Y_7;\mathbb{Z})$

$$\frac{1}{(2\pi\ell_s)^4 g_s} \int_{\Sigma_a} F_5 = N_a \in \mathbb{Z}$$

- (0,2) supersymmetry implies that there exists a Killing vector ξ on Y_7 that is dual to the R-symmetry of the dual $(0,2)$ $2d$ SCFT. Moreover

$$ds^2_{Y_7} = e^B ds^2_T + \eta^2 \qquad \text{with} \qquad \eta \equiv \tfrac{1}{2}(dz + P)$$

GK geometries

- The coordinate z parameterises the Killing vector as $\xi = 2\partial_z$, ds^2_T is a Kähler metric, with Kähler form J and Ricci two-form ρ. Moreover

$$\rho = dP \qquad F = -2J + d(e^{-B}\eta) \qquad e^B = \frac{R}{8}$$

- To obtain a full AdS$_3 \times Y_7$ solution of Type IIB supergravity, we must impose also $dF_5 = 0$: this is referred to as "going on-shell"
- It turns out that this is equivalent to imposing the non-linear PDE

$$\Box R = \frac{1}{2}R^2 - R_{ij}R^{ij}$$

with R and R_{ij} the Ricci scalar and Ricci tensor of the metric ds^2_T

- Solutions to this equation must extremize a certain functional that we call "supersymmetric action" S_{SUSY} [Couzens,Gauntlett,DM,Sparks]

The extremal problem

- A definition of S_{SUSY} in any dimension $2n+1$ is the following

$$S_{SUSY} = \int_{Y_{2n+1}} \eta \wedge \rho \wedge e^{J}$$

- An important property of S_{SUSY} is that it depends only on the vector ξ and the transverse Kähler class $[J] \in H^2_B(\mathcal{F}_\xi)$ *i.e.*

$$S_{SUSY} = S_{SUSY}(\xi, [J])$$

- For a supersymmetric solution we require that S_{SUSY} is extremized with respect to ξ: $\nabla_\xi S_{SUSY} = 0$. Subject to the constraints
  - $\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge e^{J} = 0$ (integral of RHS of "box equation" must vanish)
  - $\int_{\Sigma_a} \eta \wedge \rho \wedge e^{J} = \nu N_a$ (flux quantization conditions)

Toric GK geometry and the master volume

- Toric: Y_{2n+1} admits $U(1)^{n+1}$ isometry, with $\xi = b_i \partial_{\phi_i}$, $b_i \in \mathbb{R}^{n+1}$
- Kähler class $[J] = -2\pi \sum_{a=1}^{d} \lambda_a c_a$, with $c_a \in H^2_B(\mathcal{F}_\xi)$, $\lambda_a \in \mathbb{R}^d$
- Everything needed in the extremal problem can be expressed in terms of a master volume $\mathcal{V} = \mathcal{V}(b_i; \lambda_a)$ and its λ_a-derivatives

$$\mathcal{V} = \int_{Y_{2n+1}} \eta \wedge e^J \qquad S_{SUSY} = -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a}$$

$$0 = \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} \qquad \nu N_a = \frac{1}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b}$$

- Given a toric diagram, there are explicit expressions for $\mathcal{V} = \mathcal{V}(b_i; \lambda_a) \rightarrow$ practically <u>useful</u> [Gauntlett,DM,Sparks]

Geometric duals of c-extremization and $\mathcal{I}$-extremization

- In Type IIB: at the extremum $\mathscr{Z} \equiv \frac{3L^8}{(2\pi)^6 g_s^2 \ell_s^8} S_{SUSY}$ reduces to the holographic central charge c of the dual, large N, SCFT$_2$
- In $D = 11$: at the extremum $\mathscr{S} \equiv \frac{8\pi^2 L^9}{(2\pi \ell_p)^9} S_{SUSY}$ reduces to the entropy, corresponding to the large N index $\mathcal{I}$ of the dual SCFT$_3$
- For solutions interpreted as wrapped D3/M2, the relevant Y_{2n+1} are

$$X_{2n-1} \;\to\; Y_{2n+1} \;\to \Sigma$$

  where X_{2n-1} are toric Sasakian fibres and Σ are Riemann surfaces
- For smooth Riemann surfaces the R-symmetry vector ξ "does not mix" with Σ, namely there is no component of ξ along Σ

Fibrations over the spindle

[Boido,Gauntlett,DM,Sparks] to appear

- In order to account for solutions interpreted as D3/M2 wrapped on a spindle $\mathbb{\Sigma}$, we consider a more general setup, where Y_{2n+1} are

$$X_{2n-1} \ \rightarrow \ Y_{2n+1} \ \rightarrow \mathbb{\Sigma}$$

- In this case, the R-symmetry vector can mix with Σ, since the spindle has a $U(1)$ symmetry

- Therefore both the fiber X_{2n-1} and the total space Y_{2n+1} are toric

- Idea: relate the toric data of X_{2n-1} and Y_{2n+1} and express S_{SUSY} in terms of data of X_{2n-1}

Spindly gravitational blocks from GK geometry

- One of our main results is the following general relation

$$S_{SUSY} \propto \mathcal{V}_{2n-1}(b_i^+;\lambda_a^+) - \mathcal{V}_{2n-1}(b_i^-;\lambda_a^-)$$

  where $\mathcal{V}_{2n-1}$ is the master volume of X_{2n-1} and $b_i^\pm$, $\lambda_a^\pm$ are "shifted" (by ϵ) versions of the b_i, λ_a (depending on the toric data)

- In Type IIB: $\mathcal{V}_5(b_i;n_a) \propto a_4(\Delta_a)$ where a_4 is the trial central charge of the dual SCFT$_4$ [Hosseini,Zafffaroni] (proved for any toric quiver)

- In $D=11$: $\mathcal{V}_7(b_i;n_a) \propto \mathcal{F}_{S^3}(\Delta_a)$ where $\mathcal{F}_{S^3}$ is the trial S^3 free energy of the dual SCFT$_3$ [Gauntlett,DM,Sparks] (many examples)

  1. Proves the conjectural gravitational block formulas for D3 and M2 branes wrapped on spindles with transverse spaces $\mathbb{C}^3$ and $\mathbb{C}^4$
  2. Predicts the large N factorizations in SCFTs compactified on spindles whose gravity duals are outside the gauged supergravities in $D=5,4$

Summary

- In the past 2-3 years there have been two sets of developments in holography, that were so far unrelated

  1. Geometric extremization formalism: extremal problems in the context of GK geometry led to new powerful tools for holography
  2. A number of new supergravity solutions with orbifold singularities and novel types of twists (spindles) with neat holographic interpretations

- Our recent work will synthetize these two developments, leading to a geometric understanding of the gravitational block formulas

- There are clearly still many exciting directions to explore!

Thank you!