Review:
Black Hole Microstates in AdS

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The AdS/CFT correspondence

provides a CONSISTENT and NON-PERTURBATIVE definition

of QUANTUM GRAVITY in ANTI-DE-SITTER SPACE

Quantum Gravity in asymptotically-AdS spacetime	AdS/CFT $\Longleftrightarrow$	Ordinary QFT living at conformal boundary

Semiclassical Regime for Gravity

⋆ In AdS:

Gravity is weakly coupled
(AdS much larger than Planck scale)

and close to Einstein gravity
(scale of higher-derivative corr.'s much higher than AdS scale)

⋆ In QFT:

large "central charge"
(large N)

QFT is strongly coupled

Need to take advantage of non-perturbative methods in QFT:

- conformal bootstrap
- integrability (for certain CFT's)
- supersymmetric localization
- numerics (lattice and/or Montecarlo)
- ...

Review of recent developments
in study of black holes in AdS: entropy and microstates

This topic has been extensively studied in asymptotically-flat spacetimes

⋆ String theory reproduced the Bekenstein-Hawking entropy [Strominger, Vafa 96]
of BPS black holes in asymptotically-flat spacetimes

Since AdS/CFT grants us a fully non-perturbative definition of Quantum Gravity, it is interesting to study black hole entropy in AdS

AdS_3 and AdS_2 are special.

Here AdS_d with $d \geq 4$

Black Hole Entropy

$$\boxed{S_{\text{Bekenstein-Hawking}} = \frac{\text{Horizon Area}}{4\,G_N\,\hbar/c^3}}$$

[Bekenstein 72, 73, 74; Hawking 74, 75]

Black hole $=$ Ensemble of states in quantum gravity $\stackrel{\text{AdS/CFT}}{=}$ Ensemble of states in boundary QFT

$$S_{\text{micro}} = \log N_{\text{micro}} = \frac{\text{Area}}{4\,G_N} + \textit{corrections}: \begin{cases} \text{perturbative} \\ \text{higher derivative} \\ \text{non-perturbative (classical sol's)} \\ \text{non-perturbative (branes)} \\ \ldots \end{cases}$$

⋆ Some caveats:

- Consider *large* black holes in AdS
- Boundary QFT captures *all* states in AdS
- SUSY, rather than thermal, entropy

Strategies

Count states in boundary QFT employing a grand canonical partition function

$$\mathcal{I}(y) = \sum_{\text{states}} y^Q = \sum_{\text{charges } Q} d(Q)\, y^Q$$

- Lorentzian: extract the degeneracy

$$d(Q) = \frac{1}{2\pi i} \oint \frac{dy}{y^{Q+1}}\, \mathcal{I}(y) = \oint d\Delta \;\; e^{\log \mathcal{I}(\Delta) - 2\pi i Q \Delta} \qquad\qquad y = e^{2\pi i \Delta}$$

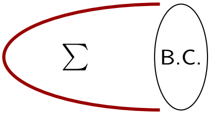
Assuming large degeneracies, saddle-point approximation $\rightarrow$ Legendre transform

$$\text{entropy } S \;=\; \log d(Q) \;\;\simeq\;\; \log \mathcal{I}(\Delta) - 2\pi i Q \Delta \,\Big|_{\Delta\,=\,\text{extremum}}$$

- Euclidean:

$$\mathcal{I} = Z_{M_{d-1}\times S^1} \quad \overset{\text{AdS/CFT}}{=}$$

Euclidean "gravitational path integral" with fixed boundary conditions



Partition function at strong coupling: very hard!

⋆ Employ a SUSY partition function, or index: $$\mathcal{I} = \sum_{\text{states}} (-1)^F \, y^Q$$

Ofter computable *exactly* with localization techniques

Index counts BPS states: applicable to BPS black holes

⋆ Does an index capture the *full* entropy? [FB, Hristov, Zaffaroni 16]

At least at leading order, yes! [Boruch, Heydeman, Iliesiu, Turiaci 22]

$\mathcal{I}$-extremization: near-horizon AdS_2 with $\mathfrak{su}(1,1|1)$

Similar to asyptotically-flat black holes [Sen 09]

Magnetically-charged BPS black holes in AdS_4

BPS Magnetically-Charged Black Holes in AdS$_4$

⋆ Spherically symmetric, static black holes in

4d $\mathcal{N}=2$ $U(1)^4$ gauged SUGRA (STU model)

uplift: 11d supergravity (M-theory) on AdS$_4 \times S^7$

[Cacciatori, Klemm 09]

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + g(r)\,ds^2_{S^2}$$

$$X^i = X^i(r) \qquad F^{a=1,2,3,4} = \mathfrak{n}_a\, d\mathrm{vol}_{S^2} \qquad \sum_{a=1}^{4} \mathfrak{n}_a = 2$$

Magnetic charge for an R-symmetry (+ possibly electric flavor charges $\mathfrak{q}_a$ & angular momentum)

BPS: 1 cplx supercharge $\mathcal{Q}$

Bekenstein-Hawking entropy:

$$S_{BH} \sim N^{\frac{3}{2}} \sqrt{\Lambda + \sqrt{\Lambda^2 - 4\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3\mathfrak{n}_4}} \qquad \Lambda = \sum_{a<b} \frac{\mathfrak{n}_a\mathfrak{n}_b}{2} - \sum_a \frac{\mathfrak{n}_a^2}{4}$$

⋆ Boundary theory: 3d $\mathcal{N}=8$ ABJM gauge theory $U(N)_1 \times U(N)_{-1}$

Boundary theory is topologically twisted $\qquad \ell^2_{\text{AdS}}/G_N \sim N^{\frac{3}{2}}$

Topologically Twisted Index

[FB, Zaffaroni 15]

Grand canonical partition function at strong coupling:

protected observable for 3d $\mathcal{N}=2$ SUSY gauge theories with R-symmetry

$$Z_{\mathsf{TTI}}[y_a,\mathfrak{n}_a] = \mathrm{Tr}_{\mathcal{H}}\, (-1)^F\; e^{-\beta H}\; e^{iJ^a A_a^{\mathsf{bkgd}}}$$

$\mathcal{H}$: Hilbert space of states on S^2 with R-symmetry background (top. twist)

$\mathcal{Q}^2 = H - m_a^{\mathsf{bkgd}} J^a$ only BPS states with $\mathcal{Q}^2=0$ contribute

Complex fugacities: $y_a = e^{i\Delta_a} = e^{i(A_a^{\mathsf{bkgd}} + i\beta m_a^{\mathsf{bkgd}})}$

⋆ Computable exactly with localization techniques. For ABJM:

$$Z = \frac{1}{(N!)^2} \sum_{\mathfrak{m},\widetilde{\mathfrak{m}}\in\mathbb{Z}^N} \int_{\mathcal{C}} \prod_{i=1}^N \frac{dx_i}{2\pi i x_i}\,\frac{d\tilde{x}_i}{2\pi i \tilde{x}_i}\, x_i^{\mathfrak{m}_i}\, \tilde{x}_i^{-\widetilde{\mathfrak{m}}_i} \times \prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\left(1-\frac{\tilde{x}_i}{\tilde{x}_j}\right)\times$$

$$\times \prod_{i,j=1}^N \prod_{a=1,2}\left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j}\,y_a}}{1-\frac{x_i}{\tilde{x}_j}\,y_a}\right)^{\mathfrak{m}_i-\widetilde{\mathfrak{m}}_j-\mathfrak{n}_a+1} \prod_{b=3,4}\left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i}\,y_b}}{1-\frac{\tilde{x}_j}{x_i}\,y_b}\right)^{\widetilde{\mathfrak{m}}_j-\mathfrak{m}_i-\mathfrak{n}_b+1}$$

TT Index at Large N

[FB, Hristov, Zaffaroni 15]

⋆ Compute contour integral as a multi-dimensional residue

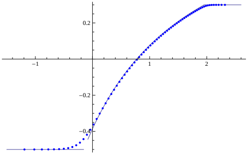
Distribution of poles at large N: "Bethe Ansatz Equations"

$$1 = x_i \prod_{j=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)} = \tilde{x}_j \prod_{i=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)}$$

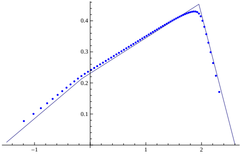
From a numerical analysis, ansatz: $\log x_j = \sqrt{N}\, t_j + i\, v_j$

⋆ Use a continuous distribution of poles:

$v_j(t) - \tilde{v}_j(t)$:



$\rho(t)$:



Partition Function and Entropy

[FB, Hristov, Zaffaroni 15]

Grand canonical partition function, at leading order in large N:

$$\boxed{\log Z_{\mathsf{TTI}}(\Delta_a, \mathfrak{n}_a) = -\frac{N^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}\sum_a \frac{\mathfrak{n}_a}{\Delta_a} \;+\; \dots}$$

Here $y_a = e^{i\Delta_a}$, $\quad 0 \leq \Delta_a \leq 2\pi$ and $\sum \Delta_a = 2\pi$.

Bekenstein-Hawking entropy from a (constrained) Legendre transform:

$$\log Z(\Delta_a, \mathfrak{n}_a) - i\Delta_a \mathfrak{q}_a \Big|_{\Delta_a = \mathsf{crit}} \;=\; S_{\mathsf{BH}}(\mathfrak{q}_a, \mathfrak{n}_a)$$

- ⋆ $\mathcal{I}$-extremization: dual to attractor mechanism in AdS_4

[Gauntlett, Martelli, Sparks 19][Hosseini, Zaffaroni 19]

[Kim, Kim 19][van Beest, Cizel, Schafer-Nameki, Sparks 20]

Many Examples in Various Dimensions

- Similar strategies reproduce the Bekenstein-Hawking entropy of various types of BPS black holes in different dimensions from various indices:

BPS Kerr-Newman rotating black holes	BPS black holes with R-symmetry magnetic charge
(possibly with electric and magnetic flavor charges)	(possibly rotating and with electric/magnetic flavor charges)
$\downarrow$	$\downarrow$
superconformal indices	topologically twisted indices

(Partial) List of Examples

Magnetically-charged BPS black holes:

- in AdS_4, with addition of electric charges [FB, Hristov, Zaffaroni 16]
  and/or angular momentum (only BHs) [Hristov, Katmadas, Toldo 18]
- with exotic horizon Σ_g [FB, Zaffaroni 16][Closset, Kim 16]
- in massive Type IIA on S^6 [Hosseini, Hristov, Passias 17][FB, Khachatryan, Milan 17]
- other QFTs and internal manifolds [Hosseini, Zaffaroni 16]
- from M5-branes, within 3d-3d correspondence [Dimofte, Gaiotto, Gukov 11]
  [Gang, Kim, Pando Zayas 19][Bobev, Crichigno 19]
- in AdS_5 with hyperbolic horizon [Bae, Gang, Lee 19]
- in AdS_6 with toric-Kahler or $\Sigma_{g_1} \times \Sigma_{g_2}$ horizon [Hosseini, Yaakov, Zaffaroni 18]
  [Crichigno, Jain, Willett 18][Suh 18] + Hristov, Passias, Fluder, Uhlemann]

Rotating Kerr-Newman BPS black holes:

- in AdS_4 (only Cardy limit) [Choi, Hwang, Kim 19][Nian, Pando Zayas 19]
- in AdS_5 (see later) [many...]
- in AdS_6 (only Cardy limit) [Choi, Kim 19]
- in AdS_7 (only Cardy limit) [Kantor, Papageorgakis, Richmond 19][Nahmgoong 19]

Other examples:

- black strings (holographic RG flows) [many...]
  [Hosseini, Nedelin, Zaffaroni 16][Hosseini, Hristov, Tachikawa, Zaffaroni 20]
- spindles [Ferrero, Gauntlett, Perez Ipiña, Martelli, Sparks 20][Hosseini, Hristov, Zaffaroni 21]
  [see D.Martelli's talk]

Perturbative and Higher-Derivative Corrections

★ In QFT, we can compute corrections to the TT index at large N.

Analytic computations turn out to be too difficult, [Liu, Pando Zayas, Rathee, Zhao 17]

$\rightarrow$ resort to numerical evaluations and fitting: [Bobev, Hong, Reys 22]

$$\log Z_0 = -\frac{N^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}\sum_a \frac{\mathfrak{n}_a}{\Delta_a}$$

$$\log Z = \log Z_0 \;+\; \underbrace{N^{\frac{1}{2}} f_1(\Delta_a, \mathfrak{n}_a)}_{\text{higher deriv.}} \;-\; \underbrace{\frac{1}{2}\log N}_{\text{1-loop}} \;+\; f_2(\Delta_a, \mathfrak{n}_a) \;+\; \mathcal{O}\big(N^{-\frac{1}{2}}\big)$$

In the specially symmetric case $\Delta_a = \frac{\pi}{2}$, and for generic k: $\Rightarrow \; S^7/\mathbb{Z}_k$

$$\log Z \;=\; -\frac{\pi\sqrt{2k}}{3}N^{\frac{3}{2}} \;+\; \frac{\pi}{\sqrt{2k}}\,\frac{k^2+32}{24}\,N^{\frac{1}{2}} \;-\; \frac{1}{2}\log N \;+\; \dots$$

Perturbative 1-Loop Correction

[Liu, Pando Zayas, Rathee, Zhao 17]

⋆ $\log$ corrections to entropy have been extensively studied for asymptotically-flat black holes [Sen 08]

Window into QG: 1-loop effect from matter fields in near-horizon region

⋆ For AdS_4 black holes: contribution from *whole space* [Jeon, Lal 17]

[Liu, Pando Zayas, Rathee, Zhao 17]

Arbitrary genus: $$Z_{\Sigma_g \times S^1}(\mathfrak{n}_a, \Delta_a) = \ \ldots\ - \frac{1-g}{2} \log N \ + \ \ldots$$

Computation in 11d SUGRA on $M_4 \times S^7$: (M_4 is reg. Euclidean black hole)

- in odd dimensions, only zero-modes contribute
- examine fields of 11d SUGRA, including ghosts
- M_4 is non-compact $\rightarrow$ space of L^2 harmonic forms $\mathcal{H}^p_{L^2}(M_4, \mathbb{R})$
  $\dim_{\mathsf{reg}} \mathcal{H}^{p=2}_{L^2} = 2(1-g)$

First Higher-Derivative Correction

[Bobev, Charles, Hristov, Reys 20/21]

⋆ Add 4-derivative corrections to 4d $\mathcal{N}=2$ minimal gauged SUGRA

Higher-derivative couplings: Weyl multiplet, T-log multiplet

- Any solution to 2-∂ action is also a solution of 4-∂ action, and preserves same amount of SUSY (special to AdS$_4$)

$$\log Z = -\pi\,\mathcal{F}\left(AN^{\frac{3}{2}} + BN^{\frac{1}{2}}\right) + \pi\left(\mathcal{F}-\chi\right)C\,N^{\frac{1}{2}}$$

  - $\mathcal{F},\chi$ depend on boundary geometry of asymtpotically-locally-AdS$_4$ solution
    Mag. AdS BH: $\mathcal{F}=(1-g) \qquad \chi = 2(1-g)$
  - A,B,C depend on theory: fix them with selected localization computations
    ABJM$_k$: $A=\frac{\sqrt{2k}}{3} \qquad B=-\frac{k^2+8}{24\sqrt{2k}} \qquad C=-\frac{1}{\sqrt{2k}}$

⋆ Grav. evaluation of TT index:

$$\log Z_{\Sigma_g\times S^1} = -(1-g)\frac{\pi\sqrt{2k}}{3}\left(N^{\frac{3}{2}} - \frac{32+k^2}{16k}\,N^{\frac{1}{2}}\right) + \ \dots$$

⋆ In order to address generic fugacities, need to add vector multiplets

Some interesting conjectures

[see V.Reys' talk]

By a very careful numerical analysis, [Bobev, Hong, Reys 22]

conjecture for TT index of ABJM$_k$, to all perturbative orders in $\frac{1}{N^{1/2}}$:

$$\log Z_{S^1\times\Sigma_{\mathfrak{g}\neq 1}} = -\frac{\sqrt{2k\Delta_1\Delta_2\Delta_3\Delta_4}}{3}\sum_{a=1}^{4}\frac{\mathfrak{n}_a}{\Delta_a}\left(\hat{N}_\Delta^{\frac{3}{2}} - \frac{\mathfrak{c}_a}{k}\,\hat{N}_\Delta^{\frac{1}{2}}\right) - \frac{1-\mathfrak{g}}{2}\log\hat{N}_\Delta$$

$$+ f_0(k,\Delta,\mathfrak{n}) \quad + \mathcal{O}\big(e^{-\sqrt{Nk}}\big) + \mathcal{O}\big(e^{-\sqrt{N/k}}\big)$$

where $$\hat{N}_\Delta = N - \frac{k}{24} + \frac{\pi}{12k}\sum_{a=1}^{4}\frac{1}{\Delta_a} \qquad \mathfrak{c}_a = \frac{\prod_{b(\neq a)}(\Delta_b+\Delta_a)}{8\Delta_1\Delta_2\Delta_3\Delta_4}\sum_{b(\neq a)}\Delta_b$$

Some conjectures on the prepotential [Hristov 21/22]

corrected to all higher-derivative orders,

compatible with this formula, have also appeared.

Kerr-Newman rotating BPS black holes in AdS_5

Kerr–Newman BPS black holes in AdS$_5$

Rotating & electrically-charged $\frac{1}{16}$-BPS black holes in AdS$_5$ [Gutowski, Reall 04] [Chong, Cvetic, Lu, Pope 05][Kunduri, Lucietti, Reall 06]

- Constructed in: 5d $\mathcal{N}=2$ $U(1)^3$ gauged SUGRA (STU model)
  or uplift to: 10d type IIB SUGRA on AdS$_5 \times S^5$

- Angular momentum — Here: J_1, J_2
  Electric charges — Charges for $U(1)^3 \subset SO(6)$: R_1, R_2, R_3

- SUSY (1 cplx supercharge $\mathcal{Q}$)

- Bekenstein-Hawking entropy (S^3 horizon):
$$S_{\text{BH}} = \frac{\text{Area}}{4G_N} = \pi\sqrt{R_1R_2 + R_1R_3 + R_2R_3 - 2N^2(J_1+J_2)}$$

- Angular momenta, charges and entropy scale $\sim N^2$

4d Superconformal Index

[Romelsberger 05; Kinney, Maldacena, Minwalla, Raju 05]

- Dual boundary theory: 4d $\mathcal{N} = 4$ $SU(N)$ SYM

⋆ Superconformal index:

Counts (with sign) BPS states on S^3 = protected operators on flat space

Index of $\mathcal{N} = 4$ SYM:

$$\mathcal{I}(p, q, y_1, y_2) = \mathrm{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{1}{2}R_3}\, q^{J_2+\frac{1}{2}R_3}\, y_1^{\frac{1}{2}(R_1-R_3)}\, y_2^{\frac{1}{2}(R_2-R_3)}$$

Write: $p = e^{2\pi i\tau}$ $\quad q = e^{2\pi i\sigma}$ $\quad y_a = e^{2\pi i\Delta_a}$

Introduce Δ_3 such that: $\Delta_1 + \Delta_2 + \Delta_3 - \tau - \sigma \in \mathbb{Z}$

⋆ Equals the Euclidean partition function on $S^3 \times S^1$ with background flat connections:

$$\mathcal{I} = Z_{S^3\times S^1}(\tau, \sigma, \Delta_1, \Delta_2)$$

⋆ *Exact* integral formula: [Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk 03] [Sundborg 99][Romelsberger 05][Kinney, Maldacena, Minwalla, Raju 05]

$$\mathcal{I} = \kappa_N \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i} \times \frac{\prod_{a=1}^{3} \prod_{\rho \in \mathfrak{R}_{\mathsf{adj}}} \widetilde{\Gamma}\big(\rho(u) + \Delta_a; \tau, \sigma\big)}{\prod_{\alpha \in \mathfrak{g}} \widetilde{\Gamma}\big(\alpha(u); \tau, \sigma\big)}$$

with

$$z = e^{2\pi i u} \qquad \kappa_N = \frac{(p;p)_\infty^{\mathrm{rk}(G)} (q;q)_\infty^{\mathrm{rk}(G)}}{|\mathsf{Weyl}_G|} \qquad \widetilde{\Gamma}(u;\tau,\sigma) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1} q^{n+1}/z}{1 - p^m q^n z}$$

⋆ Taking the large N limit turns out to be tricky...

One saddle point with $u_i \in \mathbb{R}$ found long ago [Kinney, Maldacena, Minwalla, Raju 05]
but it describes gas of gravitons in AdS_5, not black holes

- Many different approaches have been devised by now

Cardy Limit

[Choi, Kim, Kim, Nahmgoong 18][Honda 19][Arabi Ardehali 19]

⋆ Integrand simplifies in a (high temperature) Cardy limit:

- angular chemical potentials $\tau, \sigma \to 0$ (with $\tau/\sigma \in \mathbb{R}_+$ fixed)
- electric chemical potentials $\mathbb{Im}\,\Delta_a \to 0$ with $\mathbb{Re}\,\Delta_a$ fixed

$$\mathcal{I} = Z_{S^3 \times S^1} \underset{\tau,\sigma\to 0}{\simeq} \int d^{\mathrm{rk}(G)}u \;\; e^{\frac{i\pi}{6\tau\sigma}V_2(u) + \frac{i\pi(\tau+\sigma)}{2\tau\sigma}V_1(u)}$$

$V_{1,2}(u)$: piecewise polynomial functions, that depends on gauge/matter rep.

[Di Pietro, Komargodski 14][Arabi Ardehali 15][Di Pietro, Honda 16]

⋆ Next, take large N limit:

(here $[x] = x - \lceil x \rceil$)

$$\log \mathcal{I} = -i\pi N^2 \frac{[\Delta_1][\Delta_2][\Delta_3]}{\tau\sigma} + \ldots$$

⋆ (Constrained) Legendre transform reproduces Bekenstein-Hawking entropy:

$$S_{\mathsf{BH}} = \log \mathcal{I} - 2\pi i \left(\sum X_a \tfrac{R_a}{2} + 2\tau J \right) \Big|_{\substack{\text{constrained}\\ \text{extremum}}}$$

Cardy limit captures limit of: charges $\gg$ central charge

Cardy limit can be studied at generic (small) N

- ⋆ In 2d, Cardy limit follows from modular invariance
- ⋆ In 4d no modular invariance, yet central charges control the Cardy limit

  Obtained by reduction on S^1, careful treatment of
  3d EFT of massive and zero modes

- E.g., for the so-called "R-charge index", or "index on 2nd sheet":

$$\mathcal{I} = \mathrm{Tr}\ e^{-i\pi R}\ e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\ e^{2\pi i\tau\left(J_1+\frac{1}{2}R\right)}\ e^{2\pi i\sigma\left(J_2+\frac{1}{2}R\right)}$$

where R is superconformal R-symmetry, [Cassani, Komargodski 21]

[see also: Kim, Kim, Song 19; Cabo-Bizet, Cassani, Martelli, Murthy 19]

[Lezcano, Hong, Liu, Pando Zayas 20; Amariti, Fazzi, Segati 21]

[Arabi Ardehali, Murthy 21]

$$\log\mathcal{I} = \frac{i\pi}{24\tau\sigma}\Big[\big(\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R\big) + (\tau+\sigma)\big(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R\big)$$
$$+ (\tau+\sigma)^2 3\,\mathrm{Tr}\,R^3 - (\tau^2+\sigma^2)\,\mathrm{Tr}\,R\Big] + \log|G_{\text{1-form}}| + \mathcal{O}(\tau)$$

⋆ Relax any limit on fugacities, only large N

This analysis reveals interesting *non-perturbative* corrections

Various tools:

- Bethe Ansatz formula $\longleftarrow$
- Non-analytic extension
- Direct saddle-point approximation
- ...

Bethe Ansatz Formula for 4d Superconformal Index

Alternative formula: (set $\tau = \sigma$)

[Closset, Kim, Willett 17]
[FB, Milan 18]
[FB, Rizi 21]

$$\mathcal{I} = \sum_{u \,\in\, \mathfrak{M}_{\mathsf{BAE}}} \mathcal{Z}(u; \Delta, \tau, \tau)\; H(u; \Delta, \tau)^{-1}$$

1. $\mathfrak{M}_{\mathsf{BAE}}$ are solutions to "Bethe Ansatz Equations" for $\mathrm{rk}(G)$ complexified holonomies $[u_i]$ living on a complex torus T^2_τ of modular parameter τ:

   $\mathfrak{M}_{\mathsf{BAE}}$:
   $SU(N)$ $\mathcal{N} = 4$ SYM

   $$Q_i(u) = \prod_{a=1}^{3} \prod_{j=1}^{N} \frac{\theta(\Delta_a - u_{ij}; \tau)}{\theta(\Delta_a + u_{ij}; \tau)} = 1$$

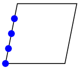
   $u_{ij} =$
   $u_i - u_j \neq 0$

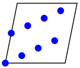
   Equations are defined on T^2_τ and are invariant under $SL(2, \mathbb{Z})$

2. $\mathcal{Z}$: same integrand as in integral formula $\quad H$: Jacobian $\quad H = \det_{ij} \frac{\partial Q_i}{\partial u_j}$

★ *Discrete* family of exact solutions [Hosseini, Nedelin, Zaffaroni 16][Hong, Liu 18]

Labelled by $\{m, r\}$ with $m \cdot n = N$ and $r \in \mathbb{Z}_n$

- BASIC SOLUTION $\{1, 0\}$: $u_j \sim \frac{\tau}{N}\, j$ 

- Other solutions – forming $SL(2,\mathbb{Z})$ orbits: *e.g.* 

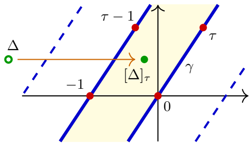
★ Contrib. of BASIC SOLUTION reproduces Bekenstein-Hawking entropy:

[FB, Milan 18]

$$\lim_{N\to\infty} \mathcal{I}(\tau, \Delta_1, \Delta_2)\Big|_{\substack{\text{BASIC}\\ \text{SOLUTION}}} \simeq \exp\left(-i\pi N^2 \frac{[\Delta_1]_\tau\, [\Delta_2]_\tau\, [\Delta_3]_\tau}{\tau^2}\right)$$

Large N limit is a *discontinuous* analytic function: Stokes phenomenon

$$[\Delta]_\tau \equiv \Delta + n \quad \text{s.t.} \in \text{STRIP}$$



Non-Perturbative Corrections from QFT

Expansion of the index at large N: $$\mathcal{I} = \sum_{\text{solutions} \in \mathfrak{M}_{\text{BAE}}} e^{\mathcal{O}(N^2) + \dots}$$

It looks like a semiclassical expansion

⋆ Large N contribution of $\{m, r\}$ solutions (with fixed m, r):

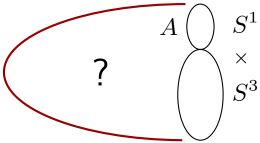
$$\log \mathcal{I}_{\{m,r\}} = \underbrace{-\frac{i\pi N^2}{m} \frac{[m\Delta_1]_{\check{\tau}}[m\Delta_2]_{\check{\tau}}[m\Delta_3]_{\check{\tau}}}{(m\tau + r)^2}}_{\text{on-shell action}} + \underbrace{\log N + \mathcal{O}(1)}_{\text{1-loop + higher-deriv. ?}}$$

$$+ \underbrace{\sum e^{\frac{2\pi i N}{m} \frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}} + \dots}}_{\text{Euclidean D3-branes}} + \quad \dots$$

where $\check{\tau} = m\tau + r$ [Lezcano, Hong, Liu, Pando Zayas 20][Aharony, FB, Mamroud, Milan 21]

"Classical" Non-Perturbative Corrections

Fill-in bulk geometry
for given boundary conditions
[Witten 98; Dijkgraaf, Maldacena, Moore, Verlinde 00]
[Maloney, Witten 07]



- $\exists$ infinite family of complex Euclidean solutions (including orbifolds) of 10d type IIB supergravity
  - ⋆ SUSY, but *not* extremal,

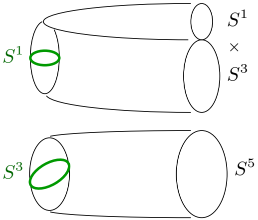
  with correct boundary conditions [Cabo-Bizet, Cassani, Martelli, Murthy 18] [Aharony, FB, Mamroud, Milan 21]

- (Renormalized) on-shell action F_{grav}
  Reproduces $\mathcal{O}(N^2)$ contribution to $\log \mathcal{I}_{\{m,r\}}$

"Stringy" Non-Perturbative Corrections

- A class of non-perturbative corrections from Euclidean SUSY D3-branes wrapped on 10d geometry at the horizon

  [Aharony, FB, Mamroud, Milan 21]



  On-shell action:

$$S_{\mathsf{D3}} = 2\pi N \, \frac{\Delta_a^{\mathrm{g}}}{\tau^{\mathrm{g}}} \qquad \text{or} \qquad S_{\mathsf{D3}} = 2\pi N \, \frac{\Delta_a^{\mathrm{g}}}{\sigma^{\mathrm{g}}}$$

- Effect of D3-brane corrections:

$$\mathcal{I} = Z_{S^3\times S^1} \;\simeq\; e^{-F_{\mathsf{grav}}} + \sum_k e^{-F_{\mathsf{grav}}}\, e^{ikS_{\mathsf{D3}}} \;\simeq\; \exp\Big\{ - \underbrace{F_{\mathsf{grav}}}_{\mathcal{O}(N^2)} + \sum_k \underbrace{e^{ikS_{\mathsf{D3}}}}_{\mathcal{O}(e^{-N})} \Big\}$$

- Criterium to retain a complex saddle:

$$\mathbb{Im}\, S_{\mathsf{D3}} > 0 \qquad \text{for all (SUSY) D3-brane embeddings}$$

Violation implies "D3-brane condensation" towards some other saddle point.

Expected to signal that complex saddle point does *not* contribute to the integral.

Perturbative and Higher-Derivative Corrections

Perturbative and higher-derivative corrections poorly understood in this example

⋆ $\log N$ term in BASIC SOLUTION [David, Lezcano, Nian, Pando Zayas 21]

Can be reproduced by Kerr/CFT applied to near-horizon

Computation in Lorentizian signature and microcanonical ensemble

Solid in the Cardy limit

No contribution from the bulk

⋆ $\mathcal{O}(1)$ terms or smaller

Expected 1-loop contribution from gas of gravitons in black hole background

[*cfr.* Kinney, Maldacena, Minwalla, Raju 05]

Higher derivative corrections? [Melo, Santos 20][Bobev, Hristov, Reys 21]

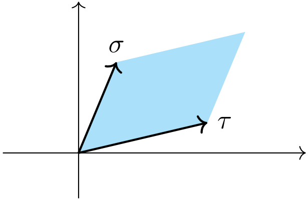
⋆ Why it is so difficult to find saddle points of the integral formula?

After some massaging, at large N [Choi, Jeong, Kim, Lee 21]

saddle points with

uniform distribution of eigenvalues on

a *parallelogram* in the u-plane



Some Open Questions

- AdS Black hole entropy beyond SUSY
- Near-horizon JT gravity from field theory [*cfr.* Boruch, Heydeman, Iliesiu, Turiaci 22]
- Classification of Euclidean gravitational saddles

  Multi-center black holes? [see C.Toldo's talk]
- Structure of higher-derivative corrections
- Relation to topological string?
- Resummation of contributions and phase transitions