Bounds on KK spin-two fields

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2109.11560 + work in progress
with De Luca, N. De Ponti, A. Mondino

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Introduction

- KK spectrum: one of the most important piece of data associated to a compactification

  - Full spectrum: relevant for holography

    [Kim, Romans, Van Nieuwenhuizen '85; Fabbri, Fré, Gualtieri, Termonia '99; Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

  - Smallest masses: scale separation, massive graviton models

    [Lüst, Palti, Vafa '19; Klaewer, Lüst, Palti '18...]

- Dimensional analysis:

$$m_{\rm KK} \sim \frac{1}{\text{diam}(M)} \sim \frac{1}{(\text{vol}(M))^{1/n}}$$

diameter: maximum distance between any two points in internal space

n: internal dimension

Hard to compute in general.

- gauge fixing; disentangling different spins; ...

⇨ problem is reduced to eigenvalues of internal diff. operators

Example: Freund–Rubin

Table 5
Mass operators from the Freund–Rubin ansatz

review: [Duff, Nilsson, Pope '86]

Spin	Mass operator
2^+	Δ_0
$(3/2)^{(1),(2)}$	$\not{D}_{1/2} + 7m/2$
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m(\Delta_1 + 4m^2)^{1/2}$
1^+	Δ_2
$(1/2)^{(4),(1)}$	$\not{D}_{1/2} - 9m/2$
$(1/2)^{(3),(2)}$	$3m/2 - \not{D}_{3/2}$
$0^{+(1),(3)}$	$\Delta_0 + 44m^2 \pm 12m(\Delta_0 + 9m^2)^{1/2}$
$0^{+(2)}$	$\Delta_L - 4m^2$
$0^{-(1),(2)}$	$Q^2 + 6mQ + 8m^2$

Laplace–Beltrami

Laplace–de Rham

Lichnerowicz

- now compute somehow eigenvalues of these internal operators

- Homogeneous spaces

[Kim, Romans, Van Nieuwenhuizen '85;
Fabbri, Fré, Gualtieri, Termonia '99;
Ceresole, Dall'Agata, D'Auria, Ferrara '99...]

- Exceptional/generalized geometry:

[Malek, Samtleben, '19;
Malek, Nicolai, Samtleben, '20...]

Spin-two fields: easiest operator

Table 5
Mass operators from the Freund–Rubin ansatz

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More generally for warped compactifications

spin-two operator =

weighted Laplacian

$$\Delta_f(\psi) \equiv -\frac{1}{\sqrt{\bar g}} e^{-f} \partial_m \left(\sqrt{\bar g} \bar g^{mn} e^f \partial_n \psi\right)$$

[Csaki, Erlich, Hollowood, Shirman'00; Bachas, Estes '11]

total dimension

$$f = (D-2)A$$

$$ds^2_D = e^{2A}(ds^2_d + ds^2_n)$$

warping

internal 'de-warped' metric

- Computed explicitly in several examples

[Klebanov, Pufu, Rocha '09; Richard, Terrisse, Tsimpis '14; Passias, AT '16; Pang, Rong, Varela '17...]

- If we are interested in 'scale separation' $m_{KK} \gg \sqrt{|\Lambda|}$, enough to focus on this spin-two tower

⇨ no scale separation for susy AdS7, AdS6

[Apruzzi, De Luca, Gnecchi, Lo Monaco, AT '19]
[Apruzzi, De Luca, Lo Monaco, Uhlemann '21]

But:

- In the past, theorems existed only about Laplace–Beltrami

smallest mass

for example: Ricci positive definite $\Rightarrow \frac{\pi^2}{4\text{diam}^2} \leqslant m_1^2 \leqslant \frac{2n(n+4)}{\text{diam}^2}$ [Li, Yau '80] [Cheng '75]

- Unclear how the equations of motion would put a bound on Ricci

attempts e.g. in [Gautason, Schillo, Van Riet, Williams '15]

This talk: these two problems solve each other

- Ricci+warping combine in EoM in 'right' mathematical way [De Luca, AT '21]
- Bakry–Émery geometry; optimal transport [Bakry, Émery '85] [Sturm '06; Lott, Villani '07; Ambrosio, Gigli, Savaré 14]

Plan

- The Ricci bound
  - The ‘synthetic’ view
    - Theorems on eigenvalues
      - Examples and applications

Ricci bound

Consider a higher-dimensional gravity $m_D^{D-2} \int \mathrm{d}^D x \sqrt{-g_D} R_D + \text{matter}$

and a compactification $\mathrm{d}s_D^2 = \mathrm{e}^{2A}(\mathrm{d}s_d^2 + \mathrm{d}s_n^2)$

max. symmetric

'de-warped' internal

EoM: $R_{MN} = \frac{1}{2} m_D^{2-D} \left(T_{MN} - \frac{1}{D-2} g_{MN} T \right) \equiv \hat{T}_{MN}$

$\Lambda - \frac{1}{d} \hat{T}_{(d)}$ external

internal:

$$R_{mn} + (D-2)(-\nabla_m \nabla_n A + \partial_m A \partial_n A) = ((D-2)|\mathrm{d}A|^2 + \nabla^2 A) g_{mn} + \hat{T}_{mn}$$

sign?

$$= \Lambda g_{mn} + (\hat{T}_{mn} - \tfrac{1}{d} g_{mn} \hat{T}_{(d)})$$

non-negative

["Reduced Energy Condition"]

- for all bulk fields in type II and $d = 11$ sugra
- for brane sources

"Bakry–Émery curvature":

[Bakry, Émery '85]

$$R^{N,f}_{mn} \equiv R_{mn} - \nabla_m \nabla_n f - \frac{1}{N-n} \partial_m f \partial_n f$$

it appears naturally in a 'warped' Raychaudhuri equation

$$\underbrace{R_{mn} + (D-2)(-\nabla_m \nabla_n A}_{= R^{N,f}_{mn}} + \underbrace{\partial_m A \partial_n A}_{\geqslant |\mathrm{d}A|^2 g_{mn}}) \geq \Lambda g_{mn}$$

$$f = (D-2)A$$

$$N = 2 - d < 0$$

actually still good!

$$\underbrace{R_{mn} - (D-2)\nabla_m \nabla_n A}_{= R^{\infty,f}_{mn}} \geqslant -K g_{mn}$$

$$\sigma \geqslant (D-2)|\mathrm{d}A|$$

'sup of the warping'

$$K \equiv |\Lambda| + \frac{\sigma^2}{D-2}$$

enough to derive eigenvalue bounds in the smooth case.

The 'synthetic' view

But: D-branes, O-planes ⇨ singularities

The field of optimal transport suggests a natural generalization:

$R^{N,f}_{mn} \geq -K g_{mn}$ → "RCD$(-K, N)$" space

RCD='Riemann-Curvature-Dimension' condition

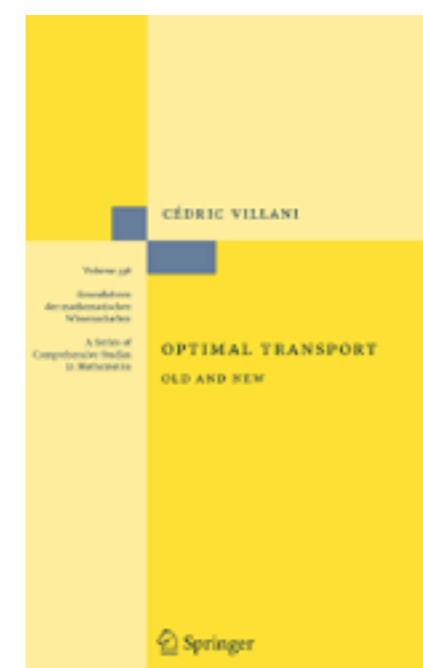
possibly singular!

[Sturm '06; Lott, Villani '07; Ambrosio, Gigli, Savaré 14]

$R^{\infty,f}_{mn} \geq -K g_{mn}$ → "RCD$(-K, \infty)$" space

⇩

- self-adjoint weighted Laplacian
- bounds on eigenvalues

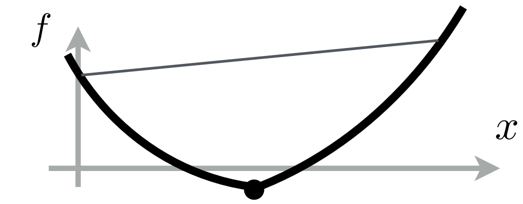


Rough analogy
for functions of one variable:

$$f'' \geq 0$$

generalize to
non-smooth functions:

convexity



$$f((1-t)x + ty) \leqslant (1-t)f(x) + tf(y)$$

$$R_{mn} - \nabla_m \partial_n f \geq 0$$

generalize to
non-smooth manifolds:

$\mathrm{RCD}(0, \infty)$:

[oversimplification!]

convexity of 'entropy' while moving particles geodesically

In more detail:

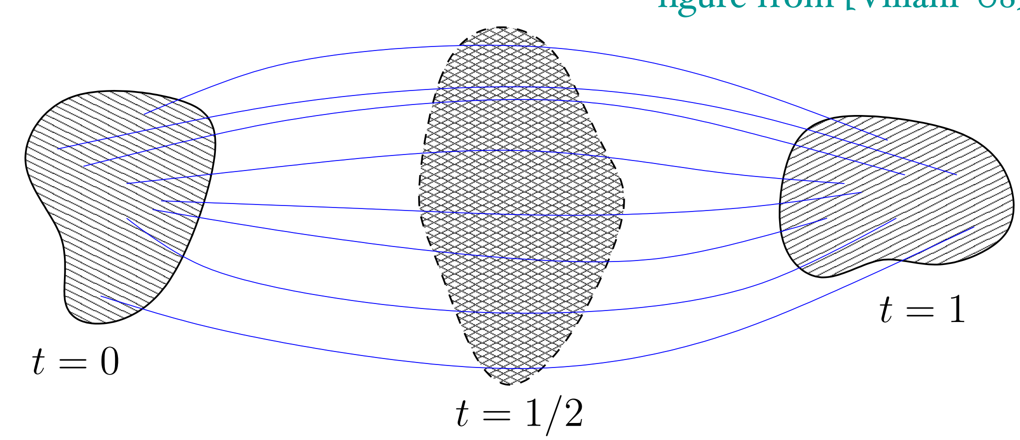
$\forall\, \rho_0, \rho_1$ non-neg. such that $\int \rho_i e^f \sqrt{g} d^n x = 1$ ["probability distributions"]

$\exists\, \rho_t$ with the same property that connects them 'geodesically'

[a geodesic with respect to natural distance of probability distributions:
"Kantorovich–Wasserstein distance"]

and 'entropy' $-\int \rho \log \rho$ is convex on this path

figure from [Villani '08]



Are string theory singularities RCD?

[De Luca, De Ponti, Mondino, AT '21]

$$ds^2 = e^{2A}(ds^2_d + ds^2_n)$$

$$ds^2_n \sim dx^2_{p+1-d} + H(dr^2 + r^2 ds^2_{\mathbb{S}^{8-p}})$$

[with usual caveats about supergravity singularities]

$\sigma \geqslant (D-2)|dA|$
'sup of the warping'

- Dp-branes, $p \leq 5$: [also M2,M5]

  $r = 0$ at infinite distance! ✓ $\sigma < \infty$.

- D6:

  math proof for exact solution; plausible in general.

  $\sigma = \infty$, but $R_{mn} - 8\nabla_m \partial_n A \geq 0$ anyway.

- D7, D8:

  math proof for exact solution; plausible in general. $\sigma < \infty$.

- Op-planes:

  $R^{\infty,f}_{mn} < 0$ for $p \geqslant 5$;
  $R^{2-d,f}_{mn} < 0$ for all p

  likely $\in \mathrm{RCD}(-K, 2-d)$

  [De Luca, De Ponti, Mondino, AT: *WIP*]

$$\mathrm{RCD}(K, N<0) \cup \mathrm{RCD}(K,\infty) \cup \mathrm{RCD}(K, N>0)$$

Eigenvalue bounds

[De Luca, AT '21]
using [Hassannezhad '12]

- a bound in terms of the Planck masses m_D, m_d [M_n smooth]

$D = d + n$

total dimension

$$m_k^2 \leqslant \alpha \max\left\{\sigma^2, \frac{1}{n-1}\left(|\Lambda| + \frac{\sigma^2}{D-2}\right)\right\} + \beta\left(k\,\frac{\sup(\mathrm{e}^{(D-2)A})}{\int \mathrm{d}^n y \sqrt{g_n}\,\mathrm{e}^{(D-2)A}}\right)^{2/n}$$

$[\alpha, \beta, \gamma \sim 10^4]$

$$\frac{\sup(\mathrm{e}^{(D-2)A})}{\int \mathrm{d}^n y \sqrt{g_n}\,\mathrm{e}^{(D-2)A}} = (m_D^{D-2} m_d^{2-d})^{2/n}$$

[Planck masses]

$\sigma \geqslant (D-2)|\mathrm{d}A|$
'sup of the warping'

doesn't exclude scale separation:

e.g. $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_p \to$ large second term

[Gautason, Schillo, Van Riet, Williams '15]
[Cribiori, Junghans, Van Hemelryck, Van Riet, Wrase '21]

- bounds in terms of the diameter d [M_n smooth]

[De Luca, AT '21] using [Setti '98, Charalambous, Lu, Rowlett '14]

$$m_k^2 \leqslant n\left(|\Lambda| + \frac{D-1}{D-2}\sigma^2\right) + \gamma\frac{k^2}{d^2} \qquad m_1^2 \geqslant \frac{\pi^2}{d^2}\exp\left(-c(n)\,d\sqrt{|\Lambda| + \frac{\sigma^2}{D-2}}\right)$$

empirical bound on d among SE's:
[Collins, Jafferis, Vafa, Xu, Yau '22]

likely to admit extension: RCD singularities, no σ

[De Luca, De Ponti, Mondino, AT: *WIP*]

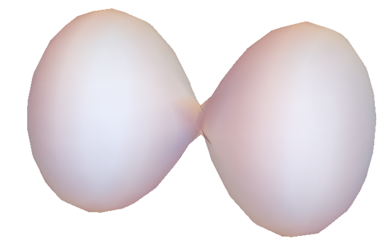
- bounds in terms of **Cheeger constant**

$$h_1(M_n) \equiv \inf_B \frac{\int_{\partial B} \sqrt{\bar{g}_{\partial B}}\, e^{(D-2)A}\, d^{n-1}x}{\int_B \sqrt{\bar{g}}\, e^{(D-2)A}\, d^n x}$$

[De Luca, De Ponti, Mondino, AT '21]

'min. of $\frac{\text{perimeter}}{\text{area}}$'

a space where h_1 is small has a small 'neck':



- smallest mass: $\frac{1}{4} h_1^2 \leqslant m_1^2 \leqslant \max\left\{\frac{21}{10} h_1 \sqrt{K},\ \frac{22}{5} h_1^2\right\}$

also for O-planes

RCD(K,∞) sing.
[recall: includes D-branes]

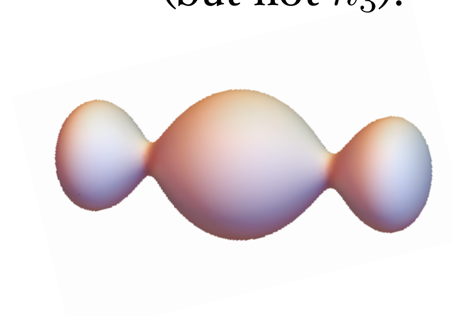
adapting [De Ponti, Mondino '19]

$$K \equiv |\Lambda| + \frac{\sigma^2}{D-2}$$

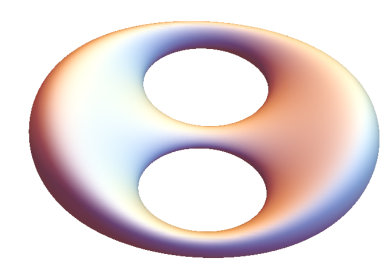
- higher masses: $\frac{h_k^2}{Ck^6} < m_k^2 < 600 k^2 \max\left\{K, 2\sqrt{K} h_k, 5 h_k^2\right\}$

$$h_k \equiv \inf_{B_0,\ldots,B_k} \max_{0 \leqslant i \leqslant k} \frac{\int_{\partial B_i} e^f\, d\overline{\text{vol}}_{n-1}}{\int_{B_i} e^f\, d\overline{\text{vol}}_n}$$

a space where h_2 is small (but not h_3):



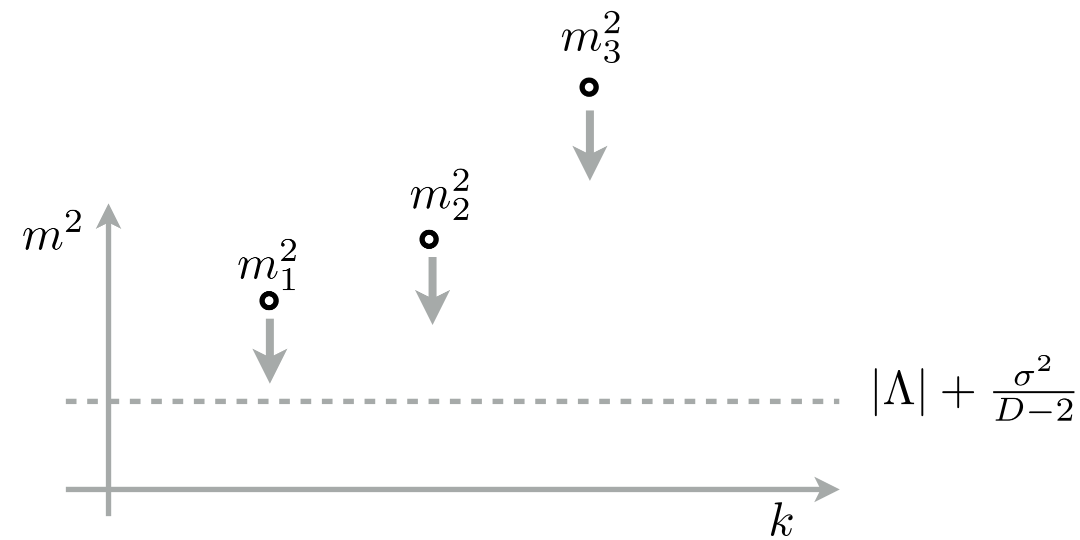
here h_1 small, h_2 large:



- work in progress: extend upper bounds to RCD$(-K, 2-d)$, get rid of σ [De Luca, De Ponti, Mondino, AT: *WIP*]

- **Application:** $$m_k^2 < 600k^2 \ \max\left\{m_1^2, |\Lambda| + \tfrac{\sigma^2}{D-2}\right\}$$

$$m_1^2 > |\Lambda| + \tfrac{\sigma^2}{D-2} \quad \Rightarrow \quad m_k^2 < 600k^2 m_1^2$$



If one lowers m_1^2,
it drags down all the higher m_k^2

so far in agreement with the Spin-2 conjecture

$\Rightarrow$ KK scale $\sim m_1$

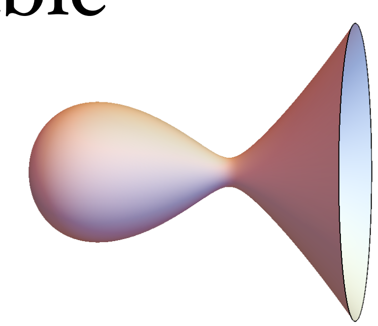
[Klaewer, Lüst, Palti '18]
[de Rham, Heisenberg, Tolley '18]
[Bachas '19]

it was formulated with Minkowski vacua in mind;
reasonable that it holds far above $|\Lambda|$

- **Non-compact analogues of these bounds also available**

for 'massive gravity' models

[Karch, Randall '01; Bachas, Lavdas '18...]



Examples

- A large class of $\mathcal{N} = 4$ IIB AdS_4 vacua with a small 'neck':

[Bachas, Lavdas '17, '18]

Compactifications with light spin-two fields.

Cheeger constant has
a holographic interpretation:

$$h_1 \propto \frac{\mathcal{F}_0(\mathrm{CFT}_4)}{\mathcal{F}_0(\mathrm{CFT}_3)}$$

- ‘Twisted compactifications’ on Riemann surfaces [Maldacena, Nuñez ’00]

$\mathrm{AdS}_5 \times M_6$ in $d = 11$ sugra dual to CFT_6 on Σ_g

$$\text{top. } S^4 \hookrightarrow M_6 \to \Sigma_g$$

- Small neck in Σ_g $\Rightarrow$ small neck in M_6 $\quad h_k(M_6) = h_k(\Sigma_g)$

- More explicit analysis: small masses = Laplacian eigenvalues on Σ_g (also follows from [Chen, Gutperle, Uhlemann ’19])

  [contributions from S^4 cannot be made small]

- Now recall pant decomposition [Fenchel–Nielsen coordinates on moduli space]

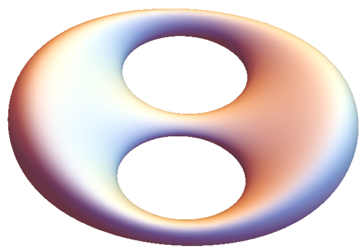
  as many necks as we want can be made arbitrarily small

  e.g. we can make h_1, ... h_{2g-2} small

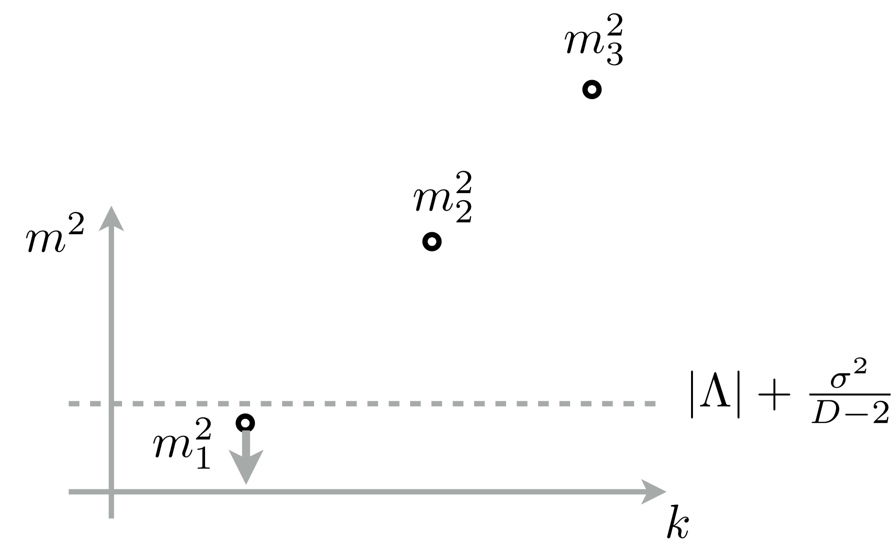
$$\frac{h_k^2}{Ck^6} < m_k^2 < 600k^2 \max\left\{K, 2\sqrt{K}h_k, 5h_k^2\right\} \quad \Rightarrow$$

tunable number of light spin-2 fields

here h_1 small, h_2 large:



example where m_1 very small,
but m_2 doesn't go down



'counterexample' to Spin-2 conjecture;
but we are now in regime beyond where it was expected to hold

[Klaewer, Lüst, Palti '18]

Massive-AdS-graviton conjecture?

$\Rightarrow$ mass spacing $\sim (m_1^2 m_d^{D-2})^{\frac{1}{D+2}}$ in AdS units

[Bachas '19]

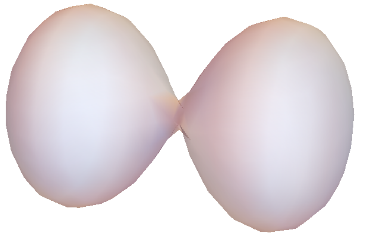
Conclusions

- Smooth case: mass bounds in terms of 'volume' and diameter

  [really $\int \mathrm{d}^n y \sqrt{g_n} \mathrm{e}^{(D-2)A}$]

- With brane singularities: bounds in terms of Cheeger constant

  e.g. $\frac{h_k^2}{Ck^6} < m_k^2 < 600k^2 \ \max\left\{K, 2\sqrt{K}h_k, 5h_k^2\right\}$



$K \equiv |\Lambda| + \frac{\sigma^2}{D-2}$

- Work in progress: eliminate K, include O-planes

- Any other 'simple' KK towers beyond spin-2?

- Is the appearance of optimal transport of deeper significance?

  e.g. RG can be formulated in this language [Mondino, Suhr '19]