Symmetries and M-theory

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One virtue of doing this is that we'll learn how to obtain the symmetry structures for theories that do not have known Lagrangians.

This is a very recent but very fast growing field, so I'll only be able to survey parts of it. Lakshya's talk will cover some largely complementary aspects.

Some terminology

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So for me, "symmetry" always means global symmetry, and "anomaly" always means 't Hooft anomaly of some global symmetries.

Anomalies can then be "perturbative" or "non-perturbative", depending on whether you can detect them via Feynman diagram computations or not.

Refining the anomaly polynomial

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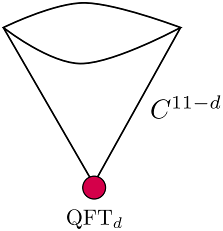
I am not including all possible anomalies and symmetries here, only internal ones. I don't know how to include things like Weyl anomalies in non-supersymmetric theories in this framework.

Anomaly inflow

In fact, anomaly theories arise very naturally when model building in M-theory:



Here the global anomalies of QFT_d are gauge symmetries from the point of view of M-theory, and we say that the full theory is anomaly free, even if QFT_d is anomalous, because there is *anomaly inflow*.

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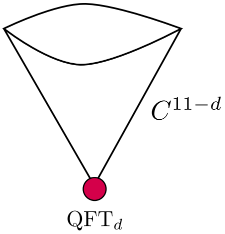

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If C^{11-d} is a cone (as is often the case in interesting examples) we can simplify the picture by integrating over the base of the cone, and we obtain anomaly inflow from a $d+1$ dimensional theory.

Symmetry inflow

I will sketch how integration on the base of the cone in M-theory does not quite produce an anomaly theory, but rather a *symmetry theory*, and object that includes information about all [*] possible theories QFT_d living on the singular point, and their anomalies.

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At the level of perturbative anomalies there is not much distinction, but the difference is crucial in resolving some puzzles when dealing with generalised symmetries, which are often discrete, and therefore only have non-perturbative anomalies.

Anomalies

What are anomalies?

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This is a problem whenever we are talking about gauge transformations: if a gauge transformation is anomalous then the theory is inconsistent. (For global symmetries anomalies are a good thing, they tell us information about the theory.)

Anomalies and the partition function

One concise way to state the problem is that it might not be possible to define the phase of the partition function in a well defined way, as a function of the background fields modulo gauge invariance:

$$Z[A^g] = e^{i\mathcal{A}(A,g)} Z[A]\,.$$

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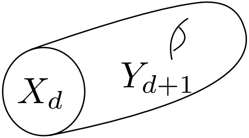
$$Z[A^g] = e^{i\mathcal{A}(A,g)} Z[A]\,.$$

[Dai, Freed '94] provide an alternative (anomaly inflow) viewpoint on this phase factor that is very fruitful.

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X_d is the boundary of some manifold Y_{d+1}, over which all the relevant structures on X_d extend.



We define the path integral of a fermion ψ on X_d as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_{Y_{d+1}})}$$

with

$$\eta(\mathcal{D}_{Y_{d+1}}) = \frac{\dim\ker\mathcal{D}_{Y_{d+1}} + \sum_{\lambda\neq 0}\operatorname{sign}(\lambda)}{2}\,.$$

[*] For the experts, this is the same η that appears in the APS index theorem.

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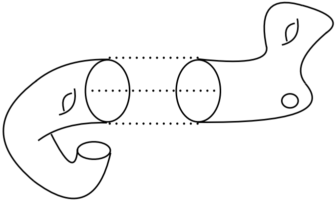
Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

$$e^{2\pi i\,\eta(\mathcal{D}_A)} = e^{-2\pi i\,\eta(\mathcal{D}_{\overline{A}})}$$

and it is "local", in the sense that η behaves nicely under gluing:

$$e^{2\pi i\eta(\mathcal{D}_A)}e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})}$$



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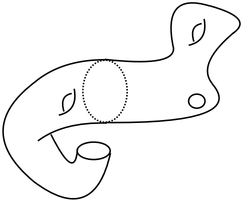

The Dai-Freed viewpoint on anomalies

Anomalies, in this language, come from situations in which the phase of the partition function depends on the choice of Y_{d+1}:

$$e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})} \neq e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})}$$

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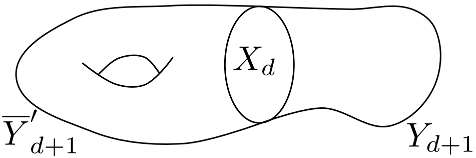
even if $\partial Y_{d+1} = \partial Y'_{d+1} = X_d$.

Gluing Y_{d+1} and $\overline{Y}'_{d+1}$ over X_d to form the closed manifold W_{d+1}, we find that the partition function is well defined as a function of the fields on X_d only if on every such W_{d+1}

$$e^{-2\pi i\,\eta(\mathcal{D}_{W_{d+1}})} = e^{-2\pi i\,\eta(\mathcal{D}_{Y_{d+1}})}/e^{-2\pi i\,\eta(\mathcal{D}_{Y'_{d+1}})} = 1$$



The Dai-Freed viewpoint on anomalies

The theory with partition function

$$Z^{\mathcal{A}}(Y_{d+1}, A) = e^{2\pi i\, \eta(\mathcal{D}_A)}$$

is an example of a topological field theory in $(d+1)$-dimensions, known in this context as the *anomaly theory*.

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We say that a theory in d-dimensions is anomaly-free if its associated anomaly theory (defined in $(d+1)$-dimensions) is trivial.

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So when talking about anomalies, it is very natural to consider topological theories in one dimension higher. Later on I will give examples of anomaly theories for 1-form symmetries.

Higher form symmetries

Classifying $\mathcal{N} = 4$ theories

Known $\mathcal{N} = 4$ theories in four dimensions are classified by a choice of gauge group G (with algebra $\mathfrak{g}$), and some discrete θ angles. [Aharony, Seiberg, Tachikawa '13]

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One can distinguish the different global forms by studying the partition function on four-manifolds $\mathcal{M}_4$ with $H^2(\mathcal{M}_4, C) \neq 0$, or by studying the properties and correlators of extended operators.

Classifying $\mathcal{N} = 4$ theories

When computing the partition function of a $\mathcal{N} = 4$ theory on some closed manifold $\mathcal{M}_4$ we do:

$$Z_{\mathcal{N}=4}[\mathcal{M}_4, \cdot] = \sum_{[F]} \int [DA][D\lambda][D\Phi] e^{-S_{\mathfrak{g}}[\tau, A, \lambda, \Phi]}$$

where $[F]$ denotes the homotopy class of the bundle over $\mathcal{M}_4$. Which classes $[F]$ should we include in the sum?

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There is a genuine choice to be made here.

Classifying $\mathcal{N} = 4$ theories

$SU(2)$ vs. $SO(3)$

I will briefly illustrate this in the case $\mathfrak{g} = \mathfrak{su}(2)$. There are two Lie groups with algebra $\mathfrak{su}(2)$: $SU(2)$ and $SO(3) = PSU(2) = SU(2)/\mathbb{Z}_2$.

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Every $SU(2)$ bundle can be interpreted as a $SO(3)$ bundle, but in sufficiently complicated manifolds there are $SO(3)$ bundles that cannot be understood as $SU(2)$ bundles.

Classifying $\mathcal{N} = 4$ theories

$SU(2)$ vs. $SO(3)$

The obstruction to understanding $SO(3)$ bundles as $SU(2)$ bundles is encoded by elements $w_2 \in H^2(\mathcal{M}_4; \mathbb{Z}_2)$, known as *Stiefel-Whitney classes*. If a $SO(3)$ bundle E has $w_2(E) \neq 0$ then it cannot be lifted to $SU(2)$.

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or only over those with $w_2 = 0$ (the "$SU(2)$" theory):

$$Z_{SU(2)}[\mathcal{M}_4, \cdot] = \sum_{[F] \in SU(2)} \int [DA][D\lambda][D\Phi] e^{-S_{\mathfrak{g}}[\tau, A, \lambda, \Phi]} .$$

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- The $SU(N)$ theory has a $\mathbb{Z}_N$ electric 1-form symmetry, counting Wilson lines in the fundamental. Introducing a background for this 1-form symmetry means turning on $w_2(E)$.
- In the $SU(N)/\mathbb{Z}_N$ theory we gauge this electric 1-form symmetry by summing over all backgrounds for the symmetry. A magnetic 1-form symmetry counting 't Hooft loops emerges.

Holography and global structure

What is the holographic interpretation of the possible variants for the $\mathfrak{su}(N)$ $\mathcal{N} = 4$ theory in 4d?

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Answered in [Witten '98]. The key insight is that we view the possible 4-dimensional theories as states in the Hilbert space of a 5-dimensional topological "bulk" theory, taking the radial direction as "time". [Friedan, Shenker '87], [Verlinde '88], [Moore, Seiberg '88], [Witten '89], . . . , [Witten '98], . . . , [Belov, Moore], . . .

Quantization of the bulk TQFT

(Following [Witten '98])

The reduction of IIB on S^5 gives an effective action

$$L_{\mathsf{CS}} = \frac{N}{2\pi i} \int_{X_5} B_2 \wedge dC_2 \,.$$

The equations of motion are

$$dB_2 = dC_2 = 0$$

and B_2, C_2 are canonically conjugate ($B_2 = C_2 = 0$ is disallowed!):

$$[B_{ij}(x), C_{kl}(y)] = -\frac{2\pi i}{N} \epsilon_{ijkl} \delta^4(x-y) \,.$$

Quantization of the bulk TQFT

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In order to specify the boundary conditions, in addition to specifying the vevs of local gauge invariant operators, we need to specify

$$\alpha(R) = \exp\left(i\int_R B_2\right) \quad ; \quad \beta(S) = \exp\left(i\int_S C_2\right)$$

for any $R, S \subset \mathcal{M}_4$ near the boundary, $X_5 \approx \mathbb{R} \times \mathcal{M}_4$.

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They do not commute:

$$\alpha(R)\beta(S) = \beta(S)\alpha(R)\exp\left(\frac{2\pi i}{N} R \cdot S\right) .$$

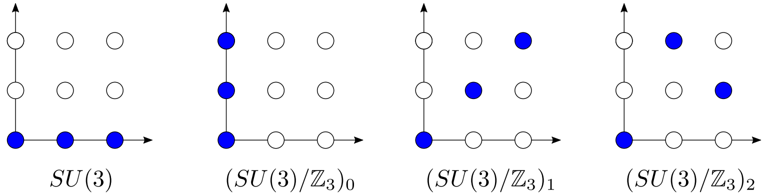
So a state cannot be a simultaneous eigenstate of both when $R \cdot S \neq 0 \bmod N$. In terms of boundary conditions, we cannot fix Dirichlet boundary conditions for both B_2 and C_2 simultaneously.

Quantization of the bulk TQFT

(Following [Witten '98])

The different global forms for $\mathfrak{su}(N)$ are then determined by the different boundary values of the B_2 and C_2 fields. In an appropriate duality frame:

- $\beta(R) = 1$ for all $R \mapsto SU(N)$.
- $\alpha(R) = 1$ for all $R \mapsto (SU(N)/\mathbb{Z}_N)_0$.
- $\alpha(R)\beta(R)^k = 1$ for all $R \mapsto (SU(N)/\mathbb{Z}_N)_k$.
- $\ldots$



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- Not every theory of interest admits a tractable large N limit. For instance the E_6 $(2,0)$ SCFT in $d=6$ is unlikely to be tractable in this way.

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- Not every theory of interest admits a tractable large N limit. For instance the E_6 $(2,0)$ SCFT in $d = 6$ is unlikely to be tractable in this way.
- Even theories that do are subtle. For example, the case of $\mathcal{N} = 4$ with algebra $\mathfrak{so}(N)$ has not yet been worked out.

Back to geometric engineering

Consider M-theory on $\mathbb{C}^2/\Gamma$. This gives rise to 7d SYM with gauge algebra $\mathfrak{g}_\Gamma$. The 1-form symmetry group of G_Γ (the simply connected form) is its centre:

$\Gamma \subset SU(2)$	$\mathfrak{g}_\Gamma$	G_Γ	$Z(G_\Gamma)$
$\mathbb{Z}_N$	$\mathfrak{su}(N)$	$SU(N)$	$\mathbb{Z}_N$
Binary dihedral $\mathrm{Dic}_{(2k-2)}$	$\mathfrak{so}(4k)$	$Spin(4k)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\mathrm{Dic}_{(2k-1)}$	$\mathfrak{so}(4k+2)$	$Spin(4k+2)$	$\mathbb{Z}_4$
Binary tetrahedral $2T$	$\mathfrak{e}_6$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	$\mathfrak{e}_7$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	$\mathfrak{e}_8$	E_8	1

Other global forms are possible, for instance $SU(N)/\mathbb{Z}_N$, which has a magnetic 4-form symmetry.

Where is the data for the global form?

The form of the singularity does not fully fix the global form of the gauge group, only the algebra. Either:

- There is a preferred global form of the gauge group (alternatively, a preferred set of higher form symmetries).

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- Or there is some extra data that we need to specify when constructing the string theory model.

In [IGE, Heidenreich, Regalado '19] we argued[1] that (as in holography) it is the second option that is realised: the choice of global form for the gauge group is encoded in a choice of boundary conditions at infinity for the supergravity fields, and all possible global forms can be obtained in this way. (Related work: [Del Zotto, Heckman, Park, Rudelius '15], [Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], . . .)

[1] The key observation is that fluxes do not commute in spaces with torsion [Freed, Moore, Segal '06].

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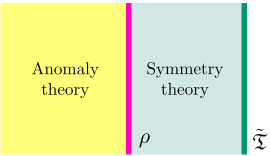
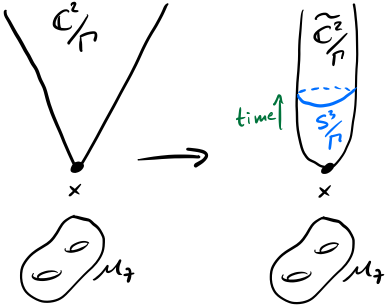
A picture (suggested by D. S. Freed) makes this precise



where $\tilde{\mathfrak{T}}$ encodes the (relative [Freed, Teleman '12]) theory of local dynamics, and ρ is a gapped interface encoding the choice of global form.

How symmetry theories appear in string theory

The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.



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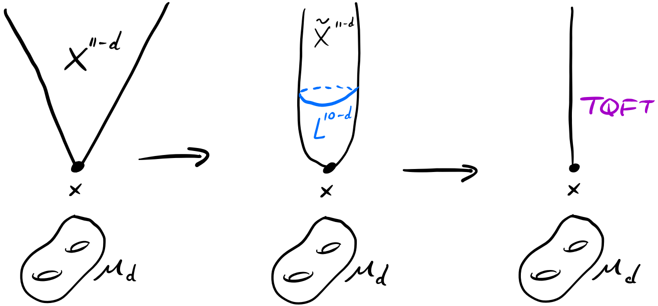

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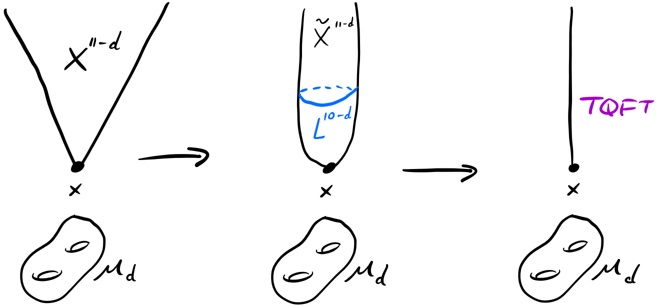

In this picture the boundary conditions at infinity that we need to specify in string theory correspond to ρ, so the object that arises naturally is the symmetry theory. ("Symmetry inflow" instead of "anomaly inflow".)

The anomaly theory

As an example, for 5d SCFTs the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the (5d) link.

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$$K_{11} = \gcd(p,q) \ ; \ \Omega_{111} = \frac{q\, p\,(p-1)\,(p-2)}{6\gcd(p,q)^3} \ ; \ \Upsilon_{111} = \frac{p\,(p-1)}{2\gcd(p,q)^2}$$

in agreement with [Gukov, Pei, Hsin '20].

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String theory provides a beautiful geometrisation of these developments. In some simple examples in 7d and 5d we could derive systematically the symmetry theory from doing dimensional reduction of the M-theory Chern-Simons sector on the link of the singularity. We did not need any Lagrangian information about the theory, only the geometry!

Future directions I

- Higher groups almost certainly fit well in this framework too, although the symmetry theory has not been obtained from reduction yet. But an analysis in terms of extended operators shows that all necessary structures are indeed present in the geometry of the link. [Apruzzi, Bhardwaj, Oh, Schäfer-Nameki '21] [Bhardwaj '21], [Del Zotto, Heckman, Meynet, Moscrop, Zhang '22], [Del Zotto, I.G.-E., Schäfer-Nameki '22], [Cvetič, Heckman, Hübner, Torres '22]

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- Higher categorical structures should also be visible from string theory, but this has not been done yet.
- We need to develop a classification of gapped interfaces for the kind of symmetries theories that arise in string theory.

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  - Relatedly, what is the imprint of the K-theoretical structure of type II fluxes on the geometrically engineered theories?
- Gravity is a big question mark. Gravitational anomalies fit well in the framework, but how do we modify the previous discussion to account for topology change? [Banerjee, Moore '22]