The string dual of free N=4 SYM

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AdS/CFT correspondence

The relation between the parameters of string theory on AdS and the dual CFT is

$$g_s \sim \frac{1}{N} \qquad\qquad \frac{R}{l_s} \sim g_{\mathrm{YM}}^2 N = \lambda$$

string coupling constant

AdS radius in string units

't Hooft parameter

AdS/CFT correspondence

In particular, weakly coupled (planar) gauge theory corresponds to the tensionless regime of string theory

$$g_s \sim \frac{1}{N}$$

small

$$\frac{R}{l_s} \sim g^2_{\mathrm{YM}} N = \lambda$$

small

$l_{\mathrm{s}} \to \infty$ `tensionless strings'

[Sundborg '01] [Witten '01]
[Sezgin,Sundell '01]

Tensionless limit

In tensionless limit AdS/CFT correspondence becomes potentially perturbative:

tensionless strings on AdS	$\longleftrightarrow$	weakly coupled/free SYM theory

- very stringy (far from sugra)
- higher spin symmetry
- maximally symmetric phase of string theory

Tensionless AdS/CFT

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tensionless strings on AdS ⟷ weakly coupled/free SYM theory

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Could it have a free worldsheet description?

Paradigm example: AdS3

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \ \longleftrightarrow \ \mathrm{Sym}_N(\mathbb{T}^4)$$

For pure NS-NS flux, string theory can be described by solvable WZW model.

Tensionless limit: level of WZW model is small.

The WZW model string with k=1 is exactly dual to symmetric orbifold theory.

[MRG, Gopakumar ’18]
[Eberhardt, MRG, Gopakumar ’18]
[Eberhardt, MRG, Gopakumar ’19]
[Dei, MRG, Gopakumar, Knighton ’20]

Hybrid formalism

[Berkovits, Vafa, Witten '99]

AdS3 theory at k=1 best described in hybrid formalism: for pure NS-NS flux, hybrid string consists of WZW model based on

$$\mathfrak{psu}(1,1|2)_k$$

together with the (topologically twisted) sigma model for T4. For generic k, this description agrees with the NS-R description a la Maldacena-Ooguri.

[Troost '11], [MRG, Gerigk '11]
[Gerigk '12]

Free field realisation

The level k=1 theory has a free field realisation

$$\mathfrak{u}(1,1|2)_1 \;\cong\; \begin{cases} 4 \textbf{ symplectic bosons } \xi^{\pm}\,,\ \eta^{\pm} \\ 4 \textbf{ real fermions } \psi^{\pm}\,,\ \chi^{\pm} \end{cases}$$

with

$$\boxed{\{\psi^{\alpha}_r\,,\chi^{\beta}_s\} = \epsilon^{\alpha\beta}\,\delta_{r,-s}} \qquad \boxed{[\xi^{\alpha}_r\,,\eta^{\beta}_s] = \epsilon^{\alpha\beta}\,\delta_{r,-s}}$$

Generators of $\mathfrak{u}(1,1|2)_1$ are bilinears in these free fields.

In order to reduce this to $\mathfrak{psu}(1,1|2)_1$ one has to gauge by the `diagonal' u(1) field

$$Z = \tfrac{1}{2}\big(\eta^-\xi^+ - \eta^+\xi^- + \chi^-\psi^+ - \chi^+\psi^-\big)\ .$$

Free field realisation

The only highest weight representations are:

- NS sector: all fields half-integer moded
  [vaccum rep of psu(1,1|2)]
- R sector: all fields integer moded
  [short rep of psu(1,1|2)]

Here positive modes annihilate the ground state.

The full worldsheet spectrum consists then of R-sector representation, together with its spectrally flowed images (both for left- and right-movers). These representations are then not highest weight.

Physical spectrum

Since $\mathfrak{psu}(1,1|2)_1$ has many (!) null-vectors, it has effectively only 2 bosonic + fermionic oscillator degrees of freedom.

[Gotz, Quella, Schomerus ’06]
[Ridout ’10]

Thus after imposing the physical state conditions, only the degrees of freedom of $\mathbb{T}^4$ survive, and we get exactly the (single-particle) spectrum of

$$\mathrm{Sym}_N(\mathbb{T}^4)$$

in the large N limit, where w-cycle twisted sector comes from w spectrally flowed sector.

[Eberhardt, MRG, Gopakumar ’18]

BMN operators

[MRG, Gopakumar, in progress]

In fact, can identify the BMN operators explicitly on the worldsheet. For example, for the AdS3 factor we propose that in w-cycle twisted/w spectrally flowed sector

$$a_n^{1\,\dagger} \longleftrightarrow J^+_{-n}\,(J^+_w)^{\frac{n}{w}} \longleftrightarrow \mathcal{L}_{-1-\frac{n}{w}}$$

BMN operator [BMN '02] — worldsheet — sym. orbifold [Gomis, Motl, Strominger '02] [Gava, Narain '02]

and similarly for the right-movers

$$a_n^{\bar{1}\,\dagger} \longleftrightarrow \bar{J}^+_n\,(\bar{J}^+_w)^{-\frac{n}{w}} \longleftrightarrow \bar{\mathcal{L}}_{-1+\frac{n}{w}}$$

Fractional powers as in [Malikov, Feigin, Fuchs '86]

BMN operators

$$a_n^{1\,\dagger} \longleftrightarrow J^+_{-n}\,(J^+_w)^{\frac{n}{w}} \longleftrightarrow \mathcal{L}_{-1-\frac{n}{w}}$$

$$a_n^{\bar{1}\,\dagger} \longleftrightarrow \bar{J}^+_{n}\,(\bar{J}^+_w)^{-\frac{n}{w}} \longleftrightarrow \bar{\mathcal{L}}_{-1+\frac{n}{w}}$$

The BMN momentum conservation condition then translates into

$$\sum_n nN_n = 0 \longleftrightarrow \underbrace{(J^+_w)^a(\bar{J}^+_w)^b}_{a-b\in\mathbb{Z}} \longleftrightarrow \text{orb. invariance}$$

appears in worldsheet spectrum

[MRG, Gopakumar, in progress]

Correlators

The worldsheet correlators localise to those configurations that admit holomorphic covering map

$$\Big\langle \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i;z_i)\Big\rangle = \sum_{\Gamma} c_\Gamma\, \delta\big(z_i \text{ compatible with cov. map } \Gamma\big)$$

$$\Gamma(z) = x_i + a_i(z-z_i)^{w_i} + \cdots \qquad (z\sim z_i)$$

After integral over worldsheet moduli recover

$$\int_{\mathcal{M}} d\mu(z_i)\,\Big\langle \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i;z_i)\Big\rangle = \sum_{\Gamma} \tilde{c}_\Gamma \cong \Big\langle \prod_{i=1}^{n} \mathcal{O}_{h_i}^{w_i}(x_i)\Big\rangle$$

symmetric orbifold correlators

[Lunin, Mathur '00]
[Pakman, Rastelli, Razamat '09]

[Eberhardt, MRG, Gopakumar '19]
[Dei, MRG, Gopakumar, Knighton '20]

Correlation functions

In the above free field realisation this localisation can be deduced from the identity

$$\left\langle \left(\xi^-(z) + \underset{\substack{\uparrow \\ \text{covering map}}}{\Gamma(z)}\, \xi^+(z)\right) \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i; z_i) \right\rangle_{\text{phys}} = 0 \ .$$

[Dei, MRG, Gopakumar, Knighton '20]

Very reminiscent of incidence relation in twistor space…

Generalisation to AdS5

How much of the previous analysis can be generalised to AdS5?

Free fields behave like twistor fields; suggests that analogous AdS5 description should be `twistorial'.

Spectral flow w — w'th twisted sector of sym. orb.

↕ (BMN)

single-trace of length w

Ansatz for worldsheet

These ideas suggest that the dual to free N=4 SYM in 4d could be described by a worldsheet theory consisting of what can be interpreted as components of ambitwistor fields [MRG, Gopakumar '21]

$$
\begin{aligned}
Y_I &= (\mu^\dagger_\alpha, \lambda^\dagger_{\dot\alpha}, \psi^\dagger_a) \qquad && \alpha, \dot\alpha \in \{1,2\} \\
Z^I &= (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a) \qquad && a \in \{1,2,3,4\}
\end{aligned}
$$

see also [Berkovits '04]

with defining relations

$$
\boxed{
\begin{gathered}
[\lambda^\alpha_r, (\mu^\dagger_\beta)_s] = \delta^\alpha_\beta\, \delta_{r,-s} \ , \qquad [\mu^{\dot\alpha}_r, (\lambda^\dagger_{\dot\beta})_s] = \delta^{\dot\alpha}_{\dot\beta}\, \delta_{r,-s} \ , \\
\{\psi^a_r, (\psi^\dagger_b)_s\} = \delta^a_b\, \delta_{r,-s} \ .
\end{gathered}
}
$$

Free fields on worldsheet

The bilinears

$$\mathcal{J}^I_J =: Y_J Z^I : \qquad \begin{array}{rcl} Y_I & = & (\mu^\dagger_\alpha, \lambda^\dagger_{\dot\alpha}, \psi^\dagger_a) \\ Z^I & = & (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a) \end{array}$$

generate $\mathfrak{u}(2,2|4)_1$, and in order to obtain $\mathfrak{psu}(2,2|4)_1$ we need to gauge by the overall u(1) field

$$\mathcal{C} = \frac{1}{2} Y_I Z^I = \frac{1}{2} \left(\mu^\dagger_\gamma \lambda^\gamma + \lambda^\dagger_{\dot\gamma} \mu^{\dot\gamma} + \psi^\dagger_c \psi^c \right) .$$

[MRG, Gopakumar ’21]

This is the current algebra version of oscillator construction of $\mathfrak{psu}(2,2|4)$ which enters into spin chain discussion. see e.g. [Beisert thesis], [Alday, David, Gava, Narain ’06]

Spectral flow

As in the case for AdS_3, all non-trivial aspects come from spectral flow, and in the sector with w units of spectral flow the non-zero modes acting on $|0\rangle_w$ are the wedge modes

$$\boxed{\mu_r^{\dot{\alpha}}\ ,\ (\mu_{\alpha}^{\dagger})_r\ ,\ (\psi_{1,2}^{\dagger})_r\ ,\ \psi_r^{3,4}\ ,\qquad (-\tfrac{w-1}{2} \leq r \leq \tfrac{w-1}{2})}$$

as well as the `out-of-the-wedge' modes

$$Z_r^I \text{ and } (Y_J)_r \qquad \text{with } r \leq -\tfrac{w+1}{2}$$

Wedge modes

[MRG, Gopakumar ’21]

Postulate: physical state conditions remove all out-of-the-wedge modes.

Retain only:

generalised zero modes = (one copy of) wedge modes.

cf. [Dolan, Goddard ’07], [Nair ’08]

On the resulting (wedge) Fock space, we furthermore need to impose the residual gauge conditions

$$\mathcal{C}_n \phi = 0 \ \ (n \geq 0) \qquad (L_0 + pw)\phi = 0 \ \ (p \in \mathbb{Z}) \ .$$

Key ingredients

Resulting spectrum reproduces exactly that of free N=4 SYM in 4d in planar limit.

More specifically, wedge modes can be thought of as momentum modes of w position space generators

$$\hat{Z}^I{}_j = \frac{1}{\sqrt{w}} \sum_{r=-(w-1)/2}^{(w-1)/2} Z^I_r \, e^{-2\pi i \frac{rj}{w}} \qquad (j = 1, \ldots, w) \ ,$$

and similarly for $(\hat{Y}_I)_j$. These position modes then satisfy

$$[\hat{Z}^I_{j_1}, (\hat{Y}^\dagger_J)_{j_2}]_\pm = \delta^I_J \, \delta_{j_1,j_2} \ .$$

[MRG, Gopakumar '21]

Key ingredients

[MRG, Gopakumar '21]

The residual gauge conditions imply

$$\mathcal{C}_n \phi = 0 \quad (n \geq 0) \qquad\qquad (L_0 + pw)\phi = 0 \quad (p \in \mathbb{Z}) \ .$$

*

at each site j: $\hat{\mathcal{C}}_j = 0$

cyclic invariance

singleton rep

Get w-fold tensor product of singleton rep of $\mathfrak{psu}(2,2|4)$, subject to cyclicity condition: spectrum of free N=4 SYM.

*: Strictly speaking, this follows from $(\mathcal{C}_n + \mathcal{C}_{n-w})\phi = 0$, but this difference does not seem to matter for the calculation of the spectrum.

Conclusions and Outlook

The free field realisation of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ worldsheet theory dual to the symmetric orbifold suggests a natural generalisation to $\mathrm{AdS}_5 \times \mathrm{S}^5$.

With some assumptions about the structure of the physical state conditions, we have managed to reproduce the exact single-trace spectrum of free SYM in 4d from our worldsheet model.

This **opens the door for a proof of the AdS/CFT correspondence** for this most relevant case.

Future directions

- Understand physical state condition from first principles. [MRG, Gopakumar, Naderi, Sriprachyakul, in progress]
- Study structure of correlation functions for AdS_5 . [MRG, Gopakumar, Knighton, Maity, in progress]
- Analyse perturbation away from free case.
- D-branes and non-perturbative effects [MRG, Knighton, Vosmera '21]
- Worldsheet description of N=2 orbifolds [MRG, Galvagno, in progress]
- Study $\mathfrak{hs}(2,2|4)$ higher spin & Yangian symmetry from worldsheet perspective cf [Beisert, Bianchi, Morales, Samtleben '04]
- Relation of twistor correlators to hexagon approach cf [Basso, Komatsu, Vieira '15]
- …

Thank you!

Key ingredients

Get w-fold tensor product of singleton rep of $\mathfrak{psu}(2,2|4)$, subject to cyclicity condition: spectrum of free N=4 SYM.

w-spectrally flowed sector $\longleftrightarrow$ $\mathrm{Tr}\big(\underbrace{S_1 \cdots S_w}_{w\ \text{letters}}\big)$

$$S_l = \{\partial^s \phi^i, \partial^s \Psi^a_\alpha, \partial^s \Psi^{\dot\alpha}_a, \partial^s \mathcal{F}_{\alpha\beta}, \partial^s \mathcal{F}^{\dot\alpha\dot\beta}\}$$

String bit picture!

$$\hat{Y} = (\hat{\mu}^\dagger_\alpha, \hat{\lambda}^\dagger_{\dot\alpha}, \hat{\psi}^\dagger_a)\ , \quad \hat{Z} = (\hat{\lambda}^\alpha, \hat{\mu}^{\dot\alpha}, \hat{\psi}^a)$$

twistor-valued string bits

[MRG, Gopakumar '21]