



# The string dual of free $N=4$ SYM

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Based mainly on work with **Rajesh Gopakumar**



# AdS/CFT correspondence

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The [relation](#) between the parameters of string theory on AdS and the dual CFT is

$$g_s \sim \frac{1}{N}$$



string coupling  
constant

$$\frac{R}{l_s} \sim g_{\text{YM}}^2 N = \lambda$$



AdS radius in  
string units



't Hooft  
parameter



# AdS/CFT correspondence

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In particular, **weakly coupled (planar) gauge theory** corresponds to the **tensionless regime** of string theory

$$g_s \sim \frac{1}{N}$$

↑  
small

$$\frac{R}{l_s} \sim g_{\text{YM}}^2 N = \lambda$$

← small →

$l_s \rightarrow \infty$  'tensionless strings'

[Sundborg '01] [Witten '01]  
[Sezgin, Sundell '01]



# Tensionless limit

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In tensionless limit **AdS/CFT** correspondence becomes potentially perturbative:

tensionless strings  
on AdS



weakly coupled/free  
SYM theory

- ▶ very stringy (far from sugra)
- ▶ higher spin symmetry
- ▶ maximally symmetric phase of string theory



# Tensionless AdS/CFT

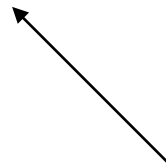
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**Could it have a free  
worldsheet description?**



# Paradigm example: AdS3

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$$\text{AdS}_3 \times S^3 \times \mathbb{T}^4 \longleftrightarrow \text{Sym}_N(\mathbb{T}^4)$$

For **pure NS-NS flux**, string theory can be described by solvable **WZW model**.

**Tensionless limit**: level of WZW model is small.

**The WZW model string with  $k=1$  is exactly dual to symmetric orbifold theory.**

[MRG, Gopakumar '18]

[Eberhardt, MRG, Gopakumar '18]

[Eberhardt, MRG, Gopakumar '19]

[Dei, MRG, Gopakumar, Knighton '20]



# Hybrid formalism

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[Berkovits, Vafa, Witten '99]

AdS3 theory at  $k=1$  best described in **hybrid formalism**:  
for pure NS-NS flux, hybrid string consists of WZW  
model based on

$$\mathfrak{psu}(1, 1|2)_k$$

together with the (topologically twisted) sigma model  
for T4. For generic  $k$ , **this description agrees with the  
NS-R description** a la Maldacena-Ooguri.

[Troost '11], [MRG, Gerigk '11]  
[Gerigk '12]



# Free field realisation

The level  $k=1$  theory has a **free field realisation**

$$\mathfrak{u}(1, 1|2)_1 \cong \begin{cases} 4 \text{ symplectic bosons } \xi^\pm, \eta^\pm \\ 4 \text{ real fermions } \psi^\pm, \chi^\pm \end{cases}$$

with

$$\{\psi_r^\alpha, \chi_s^\beta\} = \epsilon^{\alpha\beta} \delta_{r,-s}$$

$$[\xi_r^\alpha, \eta_s^\beta] = \epsilon^{\alpha\beta} \delta_{r,-s}$$

Generators of  $\mathfrak{u}(1, 1|2)_1$  are bilinears in these free fields.

In order to reduce this to  $\mathfrak{psu}(1, 1|2)_1$  one has to **gauge by the 'diagonal'  $\mathfrak{u}(1)$  field**

$$Z = \frac{1}{2} (\eta^- \xi^+ - \eta^+ \xi^- + \chi^- \psi^+ - \chi^+ \psi^-) .$$





# Free field realisation

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The only highest weight representations are:

▶ NS sector: all fields half-integer moded

[vacuum rep of  $\mathfrak{psu}(1,1|2)$ ]

▶ R sector: all fields integer moded

[short rep of  $\mathfrak{psu}(1,1|2)$ ]

Here positive modes annihilate the ground state.

The full worldsheet spectrum consists then of R-sector representation, together with its **spectrally flowed images** (both for left- and right-movers). These representations are then not highest weight.



# Physical spectrum

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Since  $\mathfrak{psu}(1, 1|2)_1$  has many (!) null-vectors, it has effectively only 2 bosonic + fermionic oscillator degrees of freedom.

[Gotz, Quella, Schomerus '06]  
[Ridout '10]

Thus after imposing the physical state conditions, only the degrees of freedom of  $\mathbb{T}^4$  survive, and we **get exactly the** (single-particle) **spectrum** of

$$\mathrm{Sym}_N(\mathbb{T}^4)$$

in the large  $N$  limit, where **w-cycle twisted sector** comes from **w spectrally flowed sector**.

[Eberhardt, MRG, Gopakumar '18]



# BMN operators

[MRG, Gopakumar, in progress]

In fact, can identify the **BMN operators** explicitly on the **worldsheet**. For example, **for the AdS3 factor** we propose that in  $w$ -cycle twisted/ $w$  spectrally flowed sector

$$\begin{array}{ccccc} a_n^{1\dagger} & \longleftrightarrow & J_{-n}^+ (J_w^+)^{\frac{n}{w}} & \longleftrightarrow & \mathcal{L}_{-1-\frac{n}{w}} \\ \uparrow & & \uparrow & & \uparrow \\ \text{BMN operator} & & \text{worldsheet} & & \text{sym. orbifold} \\ \text{[BMN '02]} & & & & \text{[Gomis, Motl, Strominger '02]} \\ & & & & \text{[Gava, Narain '02]} \end{array}$$

and similarly for the right-movers

$$a_n^{\bar{1}\dagger} \longleftrightarrow \bar{J}_n^+ (\bar{J}_w^+)^{-\frac{n}{w}} \longleftrightarrow \bar{\mathcal{L}}_{-1+\frac{n}{w}}$$

Fractional powers as in [Malikov, Feigin, Fuchs '86]



# BMN operators

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$$\begin{aligned} a_n^{1\dagger} &\longleftrightarrow J_{-n}^+ (J_w^+)^{\frac{n}{w}} \longleftrightarrow \mathcal{L}_{-1-\frac{n}{w}} \\ a_n^{\bar{1}\dagger} &\longleftrightarrow \bar{J}_n^+ (\bar{J}_w^+)^{-\frac{n}{w}} \longleftrightarrow \bar{\mathcal{L}}_{-1+\frac{n}{w}} \end{aligned}$$

The **BMN momentum conservation condition** then translates into

$$\sum_n n N_n = 0 \longleftrightarrow \underbrace{(J_w^+)^a (\bar{J}_w^+)^b}_{a-b \in \mathbb{Z}} \longleftrightarrow \text{orb. invariance}$$

↑  
appears in  
worldsheet  
spectrum

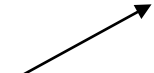
[MRG, Gopakumar, in progress]




# Correlators

The **worldsheet correlators localise** to those configurations that admit **holomorphic covering map**

$$\left\langle \prod_{i=1}^n V_{h_i}^{w_i}(x_i; z_i) \right\rangle = \sum_{\Gamma} c_{\Gamma} \delta(z_i \text{ compatible with cov. map } \Gamma)$$

$$\Gamma(z) = x_i + a_i(z - z_i)^{w_i} + \dots \quad (z \sim z_i)$$


After integral over worldsheet moduli recover

$$\int_{\mathcal{M}} d\mu(z_i) \left\langle \prod_{i=1}^n V_{h_i}^{w_i}(x_i; z_i) \right\rangle = \sum_{\Gamma} \tilde{c}_{\Gamma} \cong \left\langle \prod_{i=1}^n \mathcal{O}_{h_i}^{w_i}(x_i) \right\rangle$$


symmetric orbifold correlators

[Eberhardt, MRG, Gopakumar '19]

[Dei, MRG, Gopakumar, Knighton '20]

[Lunin, Mathur '00]

[Pakman, Rastelli, Razamat '09]



# Correlation functions

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In the above **free field realisation** this localisation can be deduced from the identity

$$\left\langle \left( \xi^-(z) + \underbrace{\Gamma(z)}_{\substack{\uparrow \\ \text{covering map}}} \xi^+(z) \right) \prod_{i=1}^n V_{h_i}^{w_i}(x_i; z_i) \right\rangle_{\text{phys}} = 0 .$$

[Dei, MRG, Gopakumar, Knighton '20]

Very reminiscent of **incidence relation in twistor space**...



# Generalisation to AdS5

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How much of the previous analysis can be generalised to AdS5?

Free fields behave like twistor fields; suggests that analogous AdS5 description should be 'twistorial'.

**Spectral flow  $w$**  — w'th twisted sector of sym. orb.

↕ (BMN)

**single-trace of length  $w$**



# Ansatz for worldsheet

These ideas suggest that the dual to free N=4 SYM in 4d could be described **by a worldsheet theory** consisting of what can be interpreted as **components of ambitwistor fields**

[MRG, Gopakumar '21]

$$\begin{aligned} Y_I &= (\mu_\alpha^\dagger, \lambda_{\dot{\alpha}}^\dagger, \psi_a^\dagger) & \alpha, \dot{\alpha} \in \{1, 2\} \\ Z^I &= (\lambda^\alpha, \mu^{\dot{\alpha}}, \psi^a) & a \in \{1, 2, 3, 4\} \end{aligned}$$

see also [Berkovits '04]

with defining relations

$$\begin{aligned} [\lambda_r^\alpha, (\mu_\beta^\dagger)_s] &= \delta_\beta^\alpha \delta_{r,-s} \ , & [\mu_r^{\dot{\alpha}}, (\lambda_{\dot{\beta}}^\dagger)_s] &= \delta_{\dot{\beta}}^{\dot{\alpha}} \delta_{r,-s} \ , \\ \{\psi_r^a, (\psi_b^\dagger)_s\} &= \delta_b^a \delta_{r,-s} \ . \end{aligned}$$





# Free fields on worldsheet

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The bilinears

$$\mathcal{J}_J^I =: Y_J Z^I : \quad \begin{array}{lcl} Y_I & = & (\mu_\alpha^\dagger, \lambda_{\dot{\alpha}}^\dagger, \psi_a^\dagger) \\ Z^I & = & (\lambda^\alpha, \mu^{\dot{\alpha}}, \psi^a) \end{array}$$

generate  $\mathfrak{u}(2, 2|4)_1$ , and in order to obtain  $\mathfrak{psu}(2, 2|4)_1$  we need to gauge by the overall  $\mathfrak{u}(1)$  field

$$\mathcal{C} = \frac{1}{2} Y_I Z^I = \frac{1}{2} (\mu_\gamma^\dagger \lambda^\gamma + \lambda_{\dot{\gamma}}^\dagger \mu^{\dot{\gamma}} + \psi_c^\dagger \psi^c) .$$

[MRG, Gopakumar '21]

This is the current algebra version of **oscillator construction** of  $\mathfrak{psu}(2, 2|4)$  which enters into **spin chain** discussion.

see e.g. [Beisert thesis], [Alday, David, Gava, Narain '06]



# Spectral flow

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As in the case for  $\text{AdS}_3$ , all non-trivial aspects come from **spectral flow**, and in the sector with  $w$  units of spectral flow the **non-zero modes** acting on  $|0\rangle_w$  are the **wedge modes**

$$\mu_r^{\dot{\alpha}}, (\mu_{\alpha}^{\dagger})_r, (\psi_{1,2}^{\dagger})_r, \psi_r^{3,4}, \quad \left(-\frac{w-1}{2} \leq r \leq \frac{w-1}{2}\right)$$

as well as the 'out-of-the-wedge' modes

$$Z_r^I \text{ and } (Y_J)_r \quad \text{with } r \leq -\frac{w+1}{2}$$



# Wedge modes

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[MRG, Gopakumar '21]

Postulate: physical state conditions remove all out-of-the-wedge modes.

Retain only:

generalised zero modes = (one copy of) wedge modes.

cf. [Dolan, Goddard '07], [Nair '08]

On the resulting (wedge) Fock space, we furthermore need to impose the residual gauge conditions

$$\mathcal{C}_n \phi = 0 \quad (n \geq 0) \qquad (L_0 + pw)\phi = 0 \quad (p \in \mathbb{Z}) \ .$$



# Key ingredients

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**Resulting spectrum reproduces exactly that of free N=4 SYM in 4d in planar limit.**

More specifically, wedge modes can be thought of as momentum modes of **w position space generators**

$$\hat{Z}^I_j = \frac{1}{\sqrt{w}} \sum_{r=-(w-1)/2}^{(w-1)/2} Z^I_r e^{-2\pi i \frac{rj}{w}} \quad (j = 1, \dots, w) ,$$

and similarly for  $(\hat{Y}_I)_j$  . These position modes then satisfy

$$[\hat{Z}^I_{j_1}, (\hat{Y}^\dagger_J)_{j_2}]_{\pm} = \delta^I_J \delta_{j_1, j_2} .$$



# Key ingredients

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[MRG, Gopakumar '21]

The residual gauge conditions imply

$$\mathcal{C}_n \phi = 0 \quad (n \geq 0) \qquad (L_0 + pw) \phi = 0 \quad (p \in \mathbb{Z}) .$$



at each site  $j$ :  $\hat{\mathcal{C}}_j = 0$



cyclic invariance

**singleton rep**

Get  $w$ -fold tensor product of singleton rep of  $\mathfrak{psu}(2, 2|4)$ ,  
subject to cyclicity condition: **spectrum of free N=4 SYM**.

\*: Strictly speaking, this follows from  $(\mathcal{C}_n + \mathcal{C}_{n-w})\phi = 0$ , but this difference does not seem to matter for the calculation of the spectrum.



# Conclusions and Outlook

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The **free field realisation** of the  $\text{AdS}_3 \times S^3$  worldsheet theory dual to the symmetric orbifold suggests a **natural generalisation to  $\text{AdS}_5 \times S^5$** .

With some assumptions about the structure of the physical state conditions, we have managed to reproduce the exact single-trace spectrum **of free SYM in 4d from our worldsheet model**.

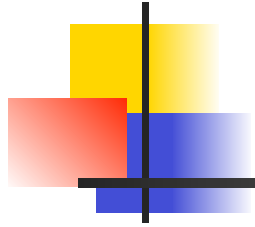
This **opens the door for a proof of the AdS/CFT correspondence** for this most relevant case.



# Future directions

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- Understand physical state condition from first principles.  
[MRG, Gopakumar, Naderi, Sriprachyakul, in progress]
- Study structure of correlation functions for  $\text{AdS}_5$  .  
[MRG, Gopakumar, Knighton, Maity, in progress]
- Analyse perturbation away from free case.
- D-branes and non-perturbative effects [MRG, Knighton, Vosmera '21]
- Worldsheet description of  $N=2$  orbifolds [MRG, Galvagno, in progress]
- Study  $\mathfrak{hs}(2, 2|4)$  higher spin & Yangian symmetry  
from worldsheet perspective cf [Beisert, Bianchi, Morales, Samtleben '04]
- Relation of twistor correlators to hexagon approach
- ... cf [Basso, Komatsu, Vieira '15]



Thank you!





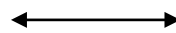


# Key ingredients

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Get  $w$ -fold tensor product of singleton rep of  $\mathfrak{psu}(2, 2|4)$ ,  
subject to cyclicity condition: **spectrum of free N=4 SYM**.

$w$ -spectrally  
flowed sector



$$\text{Tr} \left( \underbrace{S_1 \cdots S_w}_{w \text{ letters}} \right)$$

$$S_l = \{\partial^s \phi^i, \partial^s \Psi_\alpha^a, \partial^s \Psi_a^{\dot{\alpha}}, \partial^s \mathcal{F}_{\alpha\beta}, \partial^s \mathcal{F}^{\dot{\alpha}\dot{\beta}}\}$$

**String bit picture!**

$$\hat{Y} = (\hat{\mu}_\alpha^\dagger, \hat{\lambda}_{\dot{\alpha}}^\dagger, \hat{\psi}_a^\dagger), \quad \hat{Z} = (\hat{\lambda}^\alpha, \hat{\mu}^{\dot{\alpha}}, \hat{\psi}^a)$$

twistor-valued string bits

[MRG, Gopakumar '21]