Open-Closed Correspondence of K-theory and Cobordism

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Introduction

Quantum Gravity and string/M-theory

- Perturbative worldsheet description & spacetime picture
- quantum-mechanically consistent (no anomalies)
- dualities: open/closed strings, weak/strong coupling ...

However:

- Mathematical formulation?
- Signatures of QG in low energy EFTs?

The two questions can be addressed together!

Mathematical formulation

- In quantum gravity topology should fluctuate

- Cobordism: language to classify compact manifolds (**closed string backgrounds**) without fixing topology

- K-theory: proper language for D-branes (**open strings**) [Witten '98]

Q: Is there a **"open-closed"** correspondence between them?

Quantum Gravity signatures in EFTs
or
the Swampland program

[Vafa '05]

Not any EFT is consistent with quantum gravity

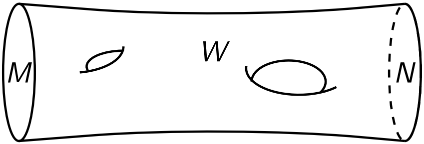
- Consistent EFTs obey swampland conjectures, which are meant to be QG "principles"
- Examples: no global symmetries [Misner, Wheeler '57; Banks, Dixon '88; Kallosh, Linde, Linde, Susskind '95; Harlow, Ooguri '18], gravity weakest force, ...
- Recent proposal: cobordism conjecture [McNamara, Vafa '19], generalising no global symmetries

Cobordism conjecture

Cobordism: definition

Consider n-dim compact manifolds M and N.
A **bordism** is a $(n+1)$-dim compact manifold W such that

$$\partial W = M \sqcup N$$



Being bordant is an equivalence relation, $[M] \sim [N]$.

Set of equivalence classes is an abelian group, **cobordism group**

$$\Omega_n = \{\text{compact } n\text{-dim manifolds}\}/\sim$$

Spin and Spin^c structures

Manifolds can be endowed with *structure*. This is inherited by the bordism group Ω_n^ξ. We will consider mainly

- **Spin structure**: $w_2(TM) = 0$

n	0	1	2	3	4	5	6	7	8	9	10
Ω_n^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$

- **Spin^c structure**: $W_3(TM) = 0$

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{Spin^c}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	0	$4\mathbb{Z}$

The cobordism conjecture

[McNamara, Vafa '19]

- String theory on compact n-dim manifold leads to $(d-n)$-dim EFT consistent with QG.

- Reversing the logic, for **any** $(d-n)$-dim EFT, ask:

  Swampland Cobordism Conjecture

  $\exists$ Quantum Gravity structure QG such that

  $$\Omega_n^{QG} = 0$$

  i.e. the group contains just the trivial element.

- **Motivation**: Non-zero bordism group leads to $(d-n-1)$-form global symmetry in the $(d-n)$-dim EFT. However, there are no global symmetries in QG. Therefore, we must have $\Omega_n^{QG} = 0$.

How to proceed

[McNamara, Vafa '19]

We do not know QG. Try with educated guess $\widetilde{QG}$.

Then, if $\Omega_n^{\widetilde{QG}} \neq 0$ we have a global symmetry.

- **Breaking**: $\exists$ defect with correct charge such that

$$\Omega_n^{\widetilde{QG}} \to \Omega_n^{\widetilde{QG}+\text{defects}} = 0 \qquad \text{killed}$$

$$d * J_{d-n} = \sum_{def\ i} \delta^{n+1}(\Delta_{d-n-1,\,i}) \neq 0$$

- **Gauging**: Compactification on $[M] = 0 \in \Omega_n^{\widetilde{QG}}$.
  Then, introduce gauge fields

$$0 = \Omega_n^{\widetilde{QG}+\text{gauging}} \to \Omega_n^{\widetilde{QG}} \qquad \text{co-killed}$$

An organising principle

Not clear in general how to get to QG. It seems that following the Whitehead tower helps. [NC, Andriot, Carqueville '22]

Whitehead tower: it organises topological structures according to the degree of "connectedness", while collecting information on the obstruction in lifting them.

Climbing the tower, cobordism groups generally become smaller

$$\begin{tikzcd}
 \\
\text{Fivebrane} \arrow[u] \\
\text{String} \arrow[u, "\frac{1}{6}p_2 = 0"'] \\
\text{Spin} \arrow[u, "\frac{1}{2}p_1 = 0"'] \\
\text{SO} \arrow[u, "w_2 = 0"'] \\
\text{O} \arrow[u, "w_1 = 0"']
\end{tikzcd}$$

K-theory and open strings

Intuitive picture

[Witten '98]

- Consider n $D9$-$\overline{D9}$ branes with U(n) bundles E and F

$$(E, F) = E - F$$

- Creation/annihilation of m pairs with *same* bundle H leaves (E,F) invariant

$$(E \oplus H, F \oplus H) \sim (E, F)$$

- The set of equivalence classes is the (reduced) K-theory group

$$K(X) = \{\text{vector bundles over X}\}/\sim$$

D-branes and K theory

D-branes are classified by K-theory. [Witten '98]

Dp-branes on $\mathbb{R}^{10}$ with $p = 9 - n$ are classified by

- **Type I**: real (reduced) K-theory $\widetilde{KO}(S^n)$

n	0	1	2	3	4	5	6	7	8	9	10
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D-brane	$D9$	$\widehat{D8}$	$\widehat{D7}$	-	$D5$	-	-	-	$D1$	$\widehat{D0}$	$\widehat{D(-1)}$

- **Type II**: complex (reduced) K-theory $\widetilde{K}(S^n)$

n	0	1	2	3	4	5	6	7	8	9
$\widetilde{K}(S^n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
D-brane	$D9$	-	$D7/D8$	-	$D5/D6$	-	$D3/D4$	-	$D1/D2$	-

Cobordism vs K-theory

n	Ω_n^{Spin}	$\widetilde{KO}(S^n)$	$\Omega_n^{\mathrm{Spin}^c}$	$\widetilde{K}(S^n)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
3	0	0	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$
5	0	0	0	0
6	0	0	$2\mathbb{Z}$	$\mathbb{Z}$
7	0	0	0	0
8	$2\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}$	$\mathbb{Z}$

Atiyah-Bott-Shapiro orientation

The existence of a relation between cobordism and K-theory dates back to the **ABS-orientation** [Atiyah, Bott, Shapiro '64]

$$\begin{aligned} \alpha \ &: \quad \Omega_n^{\mathrm{Spin}} \ \to \ \widetilde{KO}(S^n) \\ \alpha^c &: \quad \Omega_n^{\mathrm{Spin}^c} \ \to \ \widetilde{K}(S^n) \end{aligned}$$

Explicitly, they are given by the refined A-roof and Todd genus (see also [Hitchin '74])

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8k \\ \frac{1}{2}\hat{A}(M) & n = 8k+4 \\ \dim H \mod 2 & n = 8k+1 \\ \dim H^+ \mod 2 & n = 8k+2 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha_n^c([M]) = \mathrm{Td}(M)$$

This is the starting point to prove theorem by [Hopkins, Hovey '92], see also [Conner, Floyd '66; Landweber '76]

Physical consequences

- Cobordism and K-theory charges talk to each others.
  They must undergo the same fate in QG (see also [Uranga '00; Blumenhagen, Brinkmann, Makridou '19])

- Defects can carry K-theory charge

- The **combination** of cobordism and K-theory charge should be either gauged or broken. Schematically

$$\text{cobordism} + \text{K-theory} = 0$$

$$\text{closed strings} + \text{open strings} = 0$$

Application: tadpoles from bottom-up

[Blumenhagen, NC '21]

Gauging cobordism

- [McNamara, Vafa '19] mainly focused on breaking the cobordism symmetry. Gauging is less explored.

- In [NC, Blumenhagen '21] it is shown that gauging cobordism can lead to string theory tadpoles.

- **Tadpole:** integrated Bianchi identity of F_n. It means that the total charge on a compact manifold should vanish

$$0 = \int_M dF_{n-1} = \int_M J_n$$

Tadpoles from bottom-up

The map α is a natural candidate for the (integrated) current

$$0 = \int_M dF_{n-1} = \alpha_n(M) + \dots$$

however there can be additional contributions.

1. If α is surjective, we can also add the kernel

$$0 = \int_M dF_{n-1} = \alpha_n(M) + \ker \alpha_n(M) + \dots$$

2. There might be defects: branes classified by $\tilde{K}(S^n)$.

Thus we get a combination of cobordism and K-theory

$$0 = \int_{[M]} dF_{n-1} = a_1\alpha_n([M]) + a_2 \ker \alpha_n([M]) + \sum_i \int_{[M]} Q_i \delta^n(\Delta_{10-n,i})$$

An example: gauging $\Omega_6^{\mathrm{Spin}^c}$

- We have $\Omega_6^{\mathrm{Spin}^c} = \mathbb{Z} \oplus \mathbb{Z}$ generated by

$$\alpha_6 = td_6 = \frac{1}{24} c_1 c_2 , \qquad \ker \alpha_6 = \frac{1}{2} c_1^3$$

- This is a 5-form global symmetry, which can be gauged by C_4
- $\tilde{K}(S^6)$ classifies D3 branes

Putting everything together we get

$$\int_B \sum_i Q_i \, \delta^{(6)}(\Delta_{4,i}) = \int_B \left(\tfrac{a_1}{24} \, c_2(B) \, c_1(B) + \tfrac{a_2}{2} \, c_1^3(B) \right) \equiv \frac{\chi(Y)}{24}$$

This matches with the known D3-brane tadpole cancellation in F-theory for $a_1 = 12$ and $a_2 = 30$. [Sethi, Vafa, Witten '96]
Notice that c_3 cannot appear since it is not cobordism invariant.

Conclusion

- The absence of global symmetries seems to be a fact of QG. It holds true also when enlarging notion of symmetry, such as to include cobordism
- Cobordism and K-theory are closed-open string versions of global symmetries
- Their combination must be either broken or gauged
- This statement has predictive power.
  [Montero, Vafa '20; Dierigl, Heckmann '20; Hamada, Vafa, '21]

Outlook

- Clarify origin of tadpoles from bottom-up
- Cobordism groups with more structure (gauge fields, compact manifolds, ...)
  [Blumenhagen, NC, Kneißl, Makridou, to appear]
- Look at differential K-theory groups
- Is cobordism conjecture combined with K-theory enough to reconstruct tadpoles of string theory (SLP)?

Thank you!