BPS Invariants from Resurgence of Topological Strings

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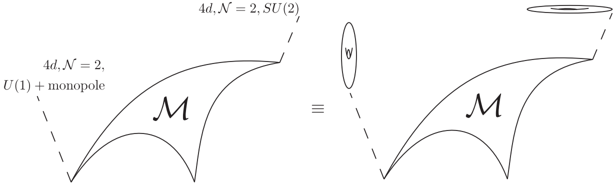
based on 2011.12759, 2102.07776 MA
2109.06878 +A. Saha + J. Teschner +I.Tulli
2203.08249 + L. Hollands and I.Tulli

Wonderful things happen at the intersection of mathematics and physics

- Packaging the data of physical theories in geometric terms has been of enormous benefit for mathematics and physics
- Parameter spaces become geometric moduli spaces $\mathcal{M}$, the change is captured by variation and wall-crossing problems

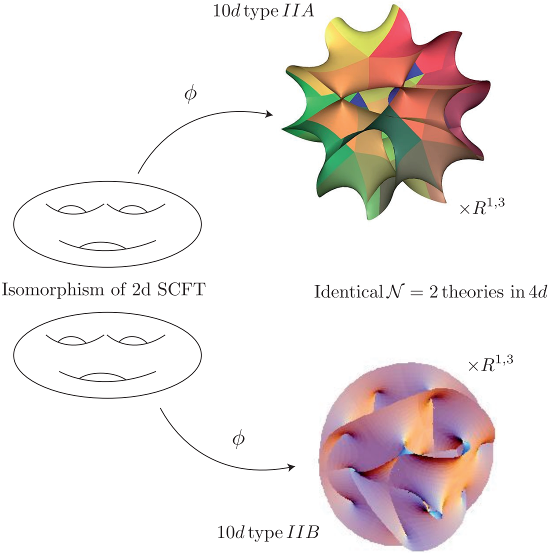


Seiberg and Witten '94

Physics provides organizing principles and connections within pure mathematics

- Mathematical invariants are organized in terms of physical partition functions.
- Gromov-Witten invariants (perturbative formulation), Donaldson-Thomas invariants (non-perturbative data)
- Generating functions of geometric invariants are naturally associated to quantization problems
- Mathematical relation between perturbative and non-perturbative invariants provides paths to go beyond perturbation theory

Mirror symmetry relates different Calabi-Yau geometries



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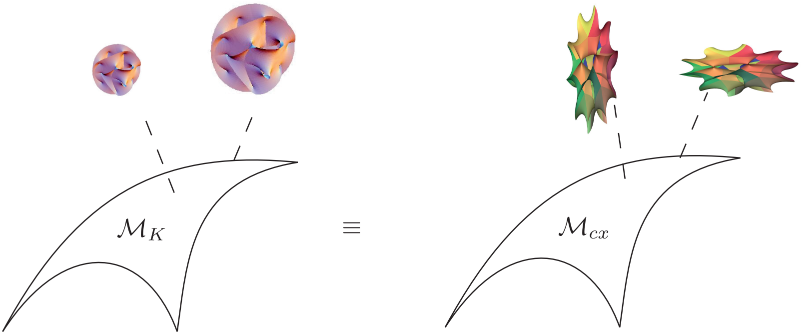

Topological string theory gives an asymptotic series

$$Z_{top}(\lambda, t) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^g(t)\right) .$$

- λ is a formal variable, the topological string coupling
- The series is asymptotic in λ
- $t = (t^1, \ldots, t^n)$ are local coordinates on $\mathcal{M}$,
- Mirror symmetry becomes

$$Z^A_{top}(X; \lambda, t) = Z^B_{top}(Y; \lambda, t(z))$$

The free energies encode Gromov-Witten invariants

In the A-model, the topological string free energies become the generating functions of higher genus Gromov-Witten invariants of X, a CY threefold:

$$F(\lambda, t) = \sum_{g\geq 0} \lambda^{2g-2} F^g(t) = \sum_{g\geq 0} \lambda^{2g-2} \sum_{\beta\in\Gamma} [GW]^g_\beta \, q^\beta \, ,$$

where $q^\beta := \exp(2\pi i t^\beta)$.

The Gopakumar-Vafa resummation provides integer invariants

- The GW potential can be furthermore written as:

$$F = F_{\beta=0} + \tilde{F}\,,$$

- $$F_{GV}(\lambda, t) = \sum_{\beta>0} \sum_{g\geq 0} n_\beta^g \sum_{k\geq 1} \frac{1}{k} \left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} q^{k\beta}$$

  is the Gopakumar-Vafa resummation, the (conjecturally) integer $n_\beta^g \in \mathbb{Z}$ count electrically charged M_2 branes in an M-theory setup.
- $$F_{GV}(\lambda, t) \sim \tilde{F}(\lambda, t)$$
- F_{GV} is an expression in $e^{i\lambda}$

Kontsevich formulates a mathematical conjecture

Kontsevich '94

Homological mirror symmetry conjecture

Identifies the categories of special objects on both sides

$$D^b(Coh(X)) \simeq DFuk(Y)$$

- What happened to $\mathcal{M}$?

BPS states are special objects

- We are interested in *minimal* (BPS) objects saturating a lower bound on the mass

$$\mathrm{mass}_\gamma(t) \geq |Z_\gamma(t)|\,, \quad \gamma \in \Gamma, t \in \mathcal{M}$$

- Physical stability condition Douglas, Fiol & Römelsberger
- Mathematical stability condition Bridgeland
- Stable sheaves $\equiv$ special Lagrangians (Donaldson-Thomas invariants)

What are the relations between Z_{BPS} and Z_{top}?

$$Z_{BPS}(t) = Z_{DT}(t)\,.$$

A relation has been expected

- MNOP: $Z_{GW} = Z_{DT}$ Maulik, Nekrasov, Okounkov, Pandharipande
- OSV conjecture a relation $Z_{top} \sim Z_{BH}(Z_{BPS})$ Ooguri, Strominger & Vafa

Can one be obtained from the other?

Topological strings from BPS wall crossing

BPS wall crossing as studied by Gaiotto, Moore and Neitzke leads to a mathematical Riemann-Hilbert problem put forward by Bridgeland which encodes Z_{top} as an asymptotic expansion

$$Z_{BPS} \Rightarrow Z_{top}$$

What about

$$Z_{top} \Rightarrow Z_{BPS}?$$

Yes! MA, Arpan Saha, Ivan Tulli, Jörg Teschner

Does Z_{top} have a life beyond the asymptotic expansion?

Topological strings are formulated perturbatively

$$Z_{top}(\lambda, t) = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2}\mathcal{F}^g(t)\right) .$$

- Matrix models and Chern Simons theory Pasquetti & Schiappa, Marino et al.
- Topological strings $\sim$ spectral theory Grassi, Hatsuda, Marino, Moriyama, Okuyama
- From superconformal theories Lockhart & Vafa
- Exact duality with Chern-Simons Krefl, Mkrtchyan
- Tau functions from quantum curves Coman, Longhi, Pomoni, Teschner

Is there an intrinsic characterization of the λ dependence?

A differential equation in λ?

- MA, Yau, Zhou, Airy equation in a universal limit
- In this talk: difference equations in λ based on arXiv:2011.12759,arXiv:2102.07776 and arXiv:2101.11672 with Arpan Saha

Resurgence and Borel resummation

- The methods of resurgence and Borel resummation can be applied to topological strings Pasquetti,Schiappa; Couso-Santamaria, Edelstein, Schiappa, Vonk; Hatsuda, Okuyama
- In this talk: analytic expressions for Borel sum, Stokes jumps and relation to BPS invariants based on arXiv:2109.06878 with Arpan Saha, Jörg Teschner and Iván Tulli

The conifold singularity can be resolved and deformed

The conifold singularity refers to a singular point in a threefold that locally looks like

$$(x_1\, x_2 - x_3\, x_4 = 0) \subset \mathbb{C}^4\,,$$

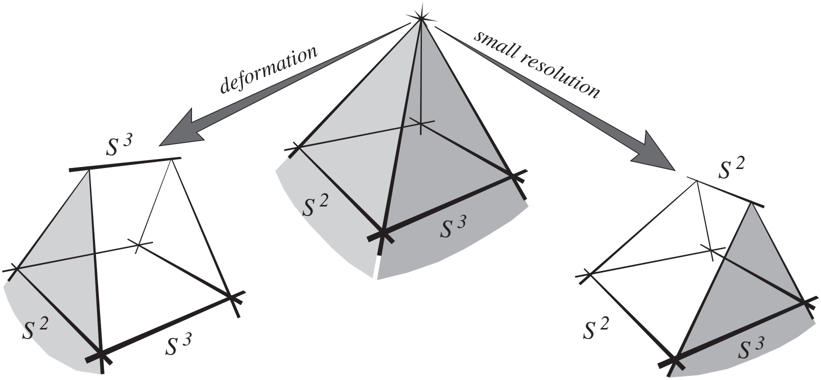

Illustration by Tristan Hübsch, from Wikipedia

The resolved conifold relates to Chern-Simons theory

The CY threefold given by the total space of the rank two bundle over the projective line:

$$X_t := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \quad (\mathbb{S}^2)\,,$$

is known as the resolved conifold.

$$t = \int_C B + i\omega\,.$$

From a large N duality with Chern-Simons theory, Gopakumar and Vafa obtained:

$$\tilde{F}(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \tilde{F}^g(t) = \frac{1}{\lambda^2}\mathrm{Li}_3(q) + \sum_{g=1}^{\infty} \lambda^{2g-2} \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} \mathrm{Li}_{3-2g}(q)\,,$$

where $q := \exp(2\pi i\, t)$ and the polylogarithm ist defined by:

$$\mathrm{Li}_s(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^s}\,, \quad s \in \mathbb{C}\,.$$

The resolved conifold has a single non-vanishing GV invariant

$$
\begin{aligned}
F_{GV}(\lambda, t) &= \sum_{\beta>0} \sum_{g\geq 0} n_\beta^g \sum_{k\geq 1} \frac{1}{k} \left(2 \sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} q^{k\beta} \\
&= \sum_{k=1}^{\infty} \frac{q^k}{k(2\sin k\lambda/2)^2}\,,
\end{aligned}
$$

is the Gopakumar-Vafa resummation, this is an expansion in $e^{i\lambda}$, singular

The free energies for the resolved conifold obey a difference equation

Theorem of MA 2011.12759 adapting techniques of Iwaki, Koike, Takei

$$\tilde{F}(\lambda, t+\check{\lambda}) - 2\tilde{F}(\lambda, t) + \tilde{F}(\lambda, t-\check{\lambda}) = \left(\frac{1}{2\pi}\frac{\partial}{\partial t}\right)^2 \tilde{F}^0(t)\,, \quad \check{\lambda} = \frac{\lambda}{2\pi}\,.$$

with

$$\tilde{F}_0(t) = \mathrm{Li}_3(q)\,.$$

Relates to integrable hierarchies MA, Saha 2101.11672

An analytic solution is obtained from the triple sine function

- Define

$$F_{np}(\lambda, t) := -\int_{\mathcal{C}} \frac{e^{(t+\check{\lambda})s}}{(e^s - 1)(e^{\check{\lambda}s} - 1)^2} \frac{ds}{s} \,,$$

  which is valid for $\mathrm{Re}\check{\lambda} > 0$ and $-\mathrm{Re}\check{\lambda} < \mathrm{Re}t < \mathrm{Re}(\check{\lambda} + 1)$.

- This is a solution of the difference equation, analytic in λ with the desired asymptotic expansion

The solution also provides a resummation

- $$F_{np}(\lambda, t) = \sum_{k=1}^{\infty} \frac{q^k}{k(2\sin k\lambda/2)^2} - \frac{\partial}{\partial \check{\lambda}} \left(\frac{\check{\lambda}}{4\pi} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(t-1/2)/\check{\lambda}}}{k^2 \sin(\pi k/\check{\lambda})} \right)$$

- We recognize the first part as:

$$F_{GV}(\lambda, t) = \sum_{k=1}^{\infty} \frac{q^k}{k(2\sin k\lambda/2)^2},$$

the second part is non-perturbative!

- This matches the computation of Hatsuda and Okuyama using a modified Borel resummation

Resurgence gives a systematic way to handle asymptotic series

Suppose

$$\phi(z) = \sum_{n\geq 0} a_n\, z^n\,,$$

is an asymptotic series around 0 with $a_n \sim n!$
Its Borel transform is then given by the series:

$$\mathcal{B}[\phi](\xi) = \sum_{n\geq 1} \frac{a_n}{(n-1)!}\xi^{n-1}\,.$$

The Laplace transform gives the Borel sum:

$$s[\phi](z) = \int_0^{e^{i\theta}\infty} e^{-\xi/z}\mathcal{B}[\phi](\xi)\,.$$

Singularities of the Borel transform know the BPS spectrum

In work with Arpan Saha, Jörg Teschner and Ivan Tulli we computed the Borel transform of the free energy for the conifold:

$$G(\xi, t) = -\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i)^2} \left(\frac{1}{m^3} \left(\frac{e^{2\pi i t + \xi/m}}{1 - e^{2\pi i t + \xi/m}} - \frac{e^{2\pi i t - \xi/m}}{1 - e^{2\pi i t - \xi/m}} \right) + \frac{\xi}{2m^4} \left(\frac{e^{2\pi i t + \xi/m}}{(1 - e^{2\pi i t + \xi/m})^2} + \frac{e^{2\pi i t - \xi/m}}{(1 - e^{2\pi i t - \xi/m})^2} \right) \right).$$

Its singularities at $\xi = 2\pi i m(k \pm t)$, $k, m \in \mathbb{Z}$ detect the BPS spectrum!

There are infinitely many Stokes rays

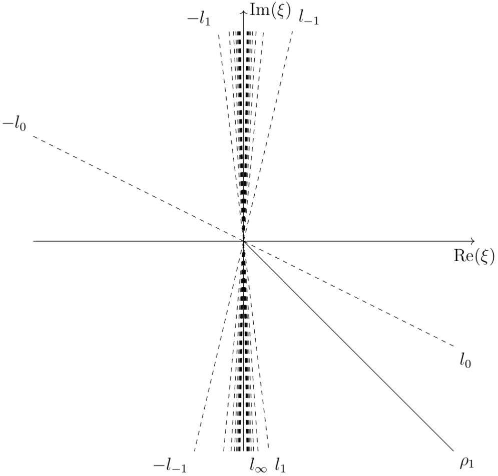

FIGURE 1. Illustration of the Stokes rays $l_k = \mathbb{R}_{<0} \cdot 2\pi\mathrm{i}(t+k)$ in the Borel plane, plotted for $t = \frac{1}{\pi}\left(1+\frac{\mathrm{i}}{2}\right)$ and $k = -10, \ldots, 10$, as well as a possible integration ray ρ_1.

There are infinitely many Borel resummations

- The jumps are given by:

$$F_{\pm\rho_{k+1}}(\lambda, t) - F_{\pm\rho_k}(\lambda, t) = \frac{1}{2\pi i}\partial_{\check{\lambda}}\left(\check{\lambda}\,\mathrm{Li}_2\left(e^{\pm 2\pi i(t+k)/\check{\lambda}}\right)\right) .$$

- These interpolate between:

$$F_{\rho_0} = F_{np}(\lambda, t) = \sum_{k=1}^{\infty} \frac{q^k}{k(2\sin k\lambda/2)^2} - \frac{\partial}{\partial\check{\lambda}}\left(\frac{\check{\lambda}}{4\pi}\sum_{k=1}^{\infty}\frac{e^{2\pi i k(t-1/2)/\check{\lambda}}}{k^2\sin(\pi k/\check{\lambda})}\right)$$

and

$$F_{\rho_\infty} = F_{GV}(\lambda, t) = \sum_{k=1}^{\infty} \frac{q^k}{k(2\sin k\lambda/2)^2} ,$$

The Stokes jumps encode the BPS spectrum

- Each Stokes ray is labeled by the charge of a BPS state γ
- The jumps encode the DT invariant associated to γ

$$\Delta_{l_\gamma} F(\lambda, t) = \frac{\Omega_{DT}(\gamma)}{2\pi i} \partial_{\check{\lambda}} \Big(\check{\lambda} \, \mathrm{Li}_2 \big(e^{\pm 2\pi i (t+k)/\check{\lambda}} \big) \Big) \, .$$

- Jumps correspond to transition functions of a holomorphic line bundle, Z to a section thereof

  This realizes a proposal of Coman, Longhi and Teschner and corresponds to the conformal limit (Gaiotto) of a construction of Neitzke and Alexandrov, Persson, Pioline

Topological string at strong coupling is revealed

- Having defined $F_{np}(\lambda, t)$, an expansion in $\lambda_D = \frac{4\pi^2}{\lambda}$ and $t_D = \frac{2\pi t}{\lambda}$ can be obtained
- This expansion is asymptotic in λ_D and leads to its own resurgent structure
- This problem is related to non-commutative DT invariants Szendröi and framed BPS states Jafferis and Moore

A dual difference equation captures the weak coupling jumps

- In recent joint work with L. Hollands and I. Tulli
- F_{np} satisfies a further difference equation:

$$F_{np}(\lambda, t+1) - F_{np}(\lambda, t) = \frac{1}{2\pi i}\partial_{\check{\lambda}}\Big(\check{\lambda}\,\mathrm{Li}_2\big(e^{2\pi i t/\check{\lambda}}\big)\Big)$$

- The inhomogeneous piece encodes the Stokes jumps!

The difference equation corresponds to a closed string quantum curve

- Introduce

$$\psi^c(\lambda, t) := \frac{Z(\lambda, t)}{Z(\lambda, t - \check{\lambda})},$$

- The difference equation for $\tilde{F}$ becomes:

$$\left(1 - \exp(2\pi i t) - \exp(\check{\lambda}\partial_t)\right) \psi^c(\lambda, t) = 0$$

- This ψ is related to the Nekrasov-Shatashvili limit:

$$\psi^c(\lambda, t) = \exp\left(-\frac{1}{2\pi}\partial_t F^{NS}(\lambda, t - \check{\lambda}/2)\right)$$

- Precise relation to exact WKB of the mirror curve, exponential networks, spectral problems.
- Related work Grassi, Hao, Neitzke

Mathematics allows us to tackle difficult physical questions

Summary

- Non-perturbative topological strings can be accessed by differential/difference equations
- Borel resummation reveals non-perturbative structure
- The mathematical structure of non-perturbative topological string theory gives DT invariants starting from GW
- Topological string S-duality wants to be understood!

Mathematics allows us to tackle difficult physical questions

Summary

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- Borel resummation reveals non-perturbative structure
- The mathematical structure of non-perturbative topological string theory gives DT invariants starting from GW
- Topological string S-duality wants to be understood!

Thank you for your attention!