

Gravitational binary dynamics from quantum scattering amplitudes

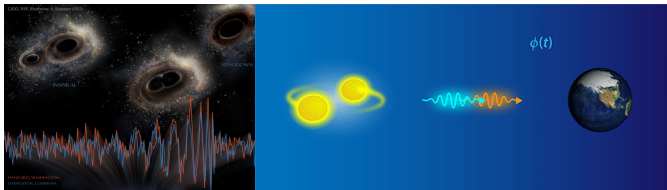
Pierre Vanhove



EuroString 2022, Lyon, France

based on work done in collaboration with
N.E.J. Bjerrum-Bohr, Poul Damgaard, Ludovic Planté, Stavros Mougiakakos

Einstein theory of gravity is the main paradigm for understanding the structure and dynamic of our observable Universe



Black holes, compact stars and gravitational waves are amongst the most spectacular predictions of general relativity and natural probes of the fundamental principles of Einstein's theory and its extension, e.g.

- ▶ The activation of scalar fields
- ▶ Gravitational leakage into large extra dimensions
- ▶ Variability of Newton's constant
- ▶ Propagation of gravitational waves
- ▶ gravitational Lorentz violation
- ▶ strong equivalence principle
- ▶ Higher-derivative corrections, ...

Classical gravity from quantum amplitudes

THE GENERATION OF GRAVITATIONAL WAVES.

IV. BREMSSTRAHLUNG*†‡

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AND

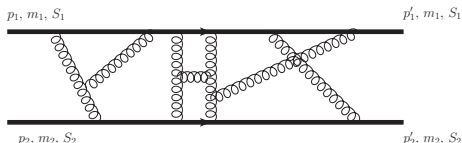
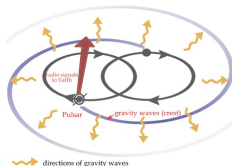
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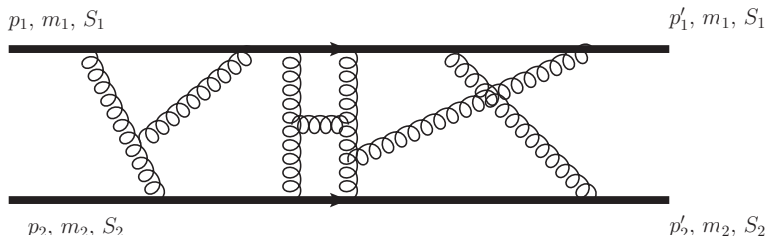
g) The Feynman-Diagram Approach

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta



Analytic expressions for the classical gravitational two-body interactions are obtained using techniques from quantum scattering amplitudes

Classical physics from quantum loops



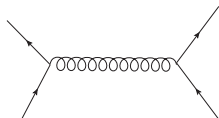
In the limit $\hbar, q^2 \rightarrow 0$ with $\underline{q} = \frac{q}{\hbar}$ fixed at each loop order of the quantum amplitude has the Laurent expansion¹ $\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$ and $q^2 = (p_1 - p_1')^2$

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2} + 2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2} + 2 - L}} + O(\hbar^0)$$

- ▶ The classical amplitude is the contribution of order $1/\hbar$
- ▶ A classical contribution of order $1/\hbar$ from all loop orders
- ▶ **classical gravity physics contributions** are determined by the **unitarity of the quantum scattering amplitudes**

¹ [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove; Kosower, Maybee, O'Connell]

One graviton exchange : tree-level amplitude



$$\mathfrak{M}_1 = -16\pi G_N \hbar \frac{[2(p_1 \cdot p_2)^2 - m_1^2 m_2^2 - |\hbar \underline{\vec{q}}|^2 (p_1 \cdot p_2)]}{|\hbar \underline{\vec{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude

$$\mathfrak{M}_1 = \frac{\mathcal{M}_1^{(-1)}(p_1 \cdot p_2)}{\hbar |\underline{q}|^2} + \hbar 4\pi G_N p_1 \cdot p_2$$

The higher order in q^2 are quantum with powers of $\hbar$

The *classical* potential is obtained by taking the 3d Fourier transform

$$E_i = \sqrt{p_i^2 + m_i^2}$$

$$\mathcal{V}_1(p_1 \cdot p_2, r) = \int \frac{d^3 \underline{\vec{q}}}{(2\pi)^3} \frac{\mathcal{M}_1^{(-1)}(\underline{\vec{q}}) e^{i \underline{\vec{q}} \cdot \vec{r}}}{4E_1 E_2} = \frac{G_N}{E_1 E_2} \frac{m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2}{r}$$

Exponentiation of the $\hat{S}$ -matrix

Using an exponential representation of the $\hat{S}$ matrix²

$$\hat{S} = \mathbb{I} + \frac{i}{\hbar} \hat{T} = \exp \left(\frac{i\hat{N}}{\hbar} \right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L \geq 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{l \geq 0} G_N^L \hat{T}_L^{\text{rad}}, \quad \hat{N} = G_N \sum_{L \geq 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{l \geq 0} G_N^L \hat{N}_L^{\text{rad}}$$

The classical radial action $\hat{N}^{\text{classical}}$ does not have any $\hbar$. The higher power of $1/\hbar$ more singular than the classical **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

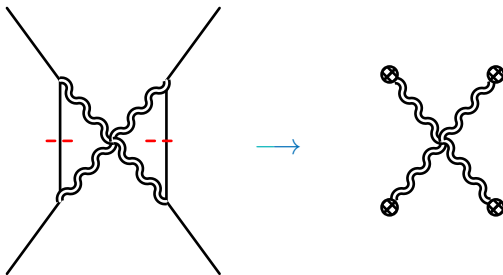
$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2} + 2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2} + 2 - L}} + \mathcal{O}(\hbar^0)$$

²[Damgaard, Planté, Vanhove]

Velocity cuts and classical radial action

At L -loop the classical part arises from imposing L -delta functions on the massive propagators

$$\mathcal{M}_L(\gamma, |\underline{q}|^2) \Big|_{\text{classical}} = \frac{1}{\hbar} \frac{\mathcal{M}_L^{(L-2)}(\gamma, D)}{|\underline{q}|^{2-(D-3)L}} \implies N_L^{\text{classical}}(\gamma, D)$$



In practice, we need only evaluate matrix elements in the soft $\underline{q}^2$ -expansion, this means that we expand genuine unitarity cuts around the velocity cuts introduced recently³

³[Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Classical black hole metric from quantum amplitudes

Quantum Tree Graphs and the Schwarzschild Solution

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(Received 7 July 1972)

I. INTRODUCTION

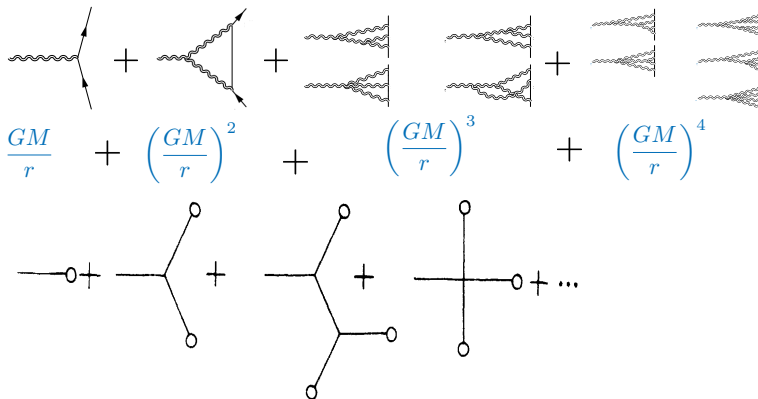
In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \rightarrow 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,¹ i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \rightarrow 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

In 1973 Duff asked the question about the classical limit of quantum gravity.^a He showed how to reproduce the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order

The double expansion in G_N and $\hbar$ give a new perspective on the classical limit of gravitational scattering amplitudes

^aM. J. Duff, "Quantum Tree Graphs and the Schwarzschild Solution," Phys. Rev. D **7** (1973), 2317-2326

Black hole metric from amplitudes



The image displays two rows of Feynman diagrams representing gravitational interactions. The top row shows the expansion of the metric tensor $g_{\mu\nu}$ in terms of the Newtonian potential $\frac{GM}{r}$. It consists of five terms, each with a diagram and a coefficient:

- Diagram 1: A single graviton exchange between two external lines (one wavy, one straight).
- Diagram 2: A graviton exchange with a self-energy loop on one of the external lines.
- Diagram 3: A graviton exchange with a graviton loop on the internal propagator.
- Diagram 4: A graviton exchange with a graviton loop on one of the external lines.
- Diagram 5: A graviton exchange with a graviton loop on the internal propagator and a graviton loop on one of the external lines.

 The coefficients are $\frac{GM}{r}$, $\left(\frac{GM}{r}\right)^2$, $\left(\frac{GM}{r}\right)^3$, and $\left(\frac{GM}{r}\right)^4$ respectively.

The bottom row shows tree-level graviton scattering diagrams. It includes:

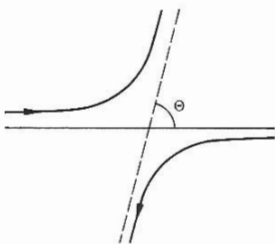
- A diagram with a single external line and a graviton loop.
- A diagram with two external lines and a graviton loop.
- A diagram with three external lines and a graviton loop.
- A diagram with four external lines and a graviton loop.
- A diagram with five external lines and a graviton loop.
- A diagram with six external lines and a graviton loop.
- A diagram with seven external lines and a graviton loop.
- A diagram with eight external lines and a graviton loop.
- A diagram with nine external lines and a graviton loop.
- A diagram with ten external lines and a graviton loop.
- A diagram with eleven external lines and a graviton loop.
- A diagram with twelve external lines and a graviton loop.
- A diagram with thirteen external lines and a graviton loop.
- A diagram with fourteen external lines and a graviton loop.
- A diagram with fifteen external lines and a graviton loop.
- A diagram with sixteen external lines and a graviton loop.
- A diagram with seventeen external lines and a graviton loop.
- A diagram with eighteen external lines and a graviton loop.
- A diagram with nineteen external lines and a graviton loop.
- A diagram with twenty external lines and a graviton loop.
- A diagram with twenty-one external lines and a graviton loop.
- A diagram with twenty-two external lines and a graviton loop.
- A diagram with twenty-three external lines and a graviton loop.
- A diagram with twenty-four external lines and a graviton loop.
- A diagram with twenty-five external lines and a graviton loop.
- A diagram with twenty-six external lines and a graviton loop.
- A diagram with twenty-seven external lines and a graviton loop.
- A diagram with twenty-eight external lines and a graviton loop.
- A diagram with twenty-nine external lines and a graviton loop.
- A diagram with thirty external lines and a graviton loop.
- A diagram with thirty-one external lines and a graviton loop.
- A diagram with thirty-two external lines and a graviton loop.
- A diagram with thirty-three external lines and a graviton loop.
- A diagram with thirty-four external lines and a graviton loop.
- A diagram with thirty-five external lines and a graviton loop.
- A diagram with thirty-six external lines and a graviton loop.
- A diagram with thirty-seven external lines and a graviton loop.
- A diagram with thirty-eight external lines and a graviton loop.
- A diagram with thirty-nine external lines and a graviton loop.
- A diagram with forty external lines and a graviton loop.
- A diagram with forty-one external lines and a graviton loop.
- A diagram with forty-two external lines and a graviton loop.
- A diagram with forty-three external lines and a graviton loop.
- A diagram with forty-four external lines and a graviton loop.
- A diagram with forty-five external lines and a graviton loop.
- A diagram with forty-six external lines and a graviton loop.
- A diagram with forty-seven external lines and a graviton loop.
- A diagram with forty-eight external lines and a graviton loop.
- A diagram with forty-nine external lines and a graviton loop.
- A diagram with fifty external lines and a graviton loop.
- A diagram with fifty-one external lines and a graviton loop.
- A diagram with fifty-two external lines and a graviton loop.
- A diagram with fifty-three external lines and a graviton loop.
- A diagram with fifty-four external lines and a graviton loop.
- A diagram with fifty-five external lines and a graviton loop.
- A diagram with fifty-six external lines and a graviton loop.
- A diagram with fifty-seven external lines and a graviton loop.
- A diagram with fifty-eight external lines and a graviton loop.
- A diagram with fifty-nine external lines and a graviton loop.
- A diagram with sixty external lines and a graviton loop.
- A diagram with sixty-one external lines and a graviton loop.
- A diagram with sixty-two external lines and a graviton loop.
- A diagram with sixty-three external lines and a graviton loop.
- A diagram with sixty-four external lines and a graviton loop.
- A diagram with sixty-five external lines and a graviton loop.
- A diagram with sixty-six external lines and a graviton loop.
- A diagram with sixty-seven external lines and a graviton loop.
- A diagram with sixty-eight external lines and a graviton loop.
- A diagram with sixty-nine external lines and a graviton loop.
- A diagram with seventy external lines and a graviton loop.
- A diagram with seventy-one external lines and a graviton loop.
- A diagram with seventy-two external lines and a graviton loop.
- A diagram with seventy-three external lines and a graviton loop.
- A diagram with seventy-four external lines and a graviton loop.
- A diagram with seventy-five external lines and a graviton loop.
- A diagram with seventy-six external lines and a graviton loop.
- A diagram with seventy-seven external lines and a graviton loop.
- A diagram with seventy-eight external lines and a graviton loop.
- A diagram with seventy-nine external lines and a graviton loop.
- A diagram with eighty external lines and a graviton loop.
- A diagram with eighty-one external lines and a graviton loop.
- A diagram with eighty-two external lines and a graviton loop.
- A diagram with eighty-three external lines and a graviton loop.
- A diagram with eighty-four external lines and a graviton loop.
- A diagram with eighty-five external lines and a graviton loop.
- A diagram with eighty-six external lines and a graviton loop.
- A diagram with eighty-seven external lines and a graviton loop.
- A diagram with eighty-eight external lines and a graviton loop.
- A diagram with eighty-nine external lines and a graviton loop.
- A diagram with ninety external lines and a graviton loop.
- A diagram with ninety-one external lines and a graviton loop.
- A diagram with ninety-two external lines and a graviton loop.
- A diagram with ninety-three external lines and a graviton loop.
- A diagram with ninety-four external lines and a graviton loop.
- A diagram with ninety-five external lines and a graviton loop.
- A diagram with ninety-six external lines and a graviton loop.
- A diagram with ninety-seven external lines and a graviton loop.
- A diagram with ninety-eight external lines and a graviton loop.
- A diagram with ninety-nine external lines and a graviton loop.
- A diagram with one hundred external lines and a graviton loop.

- ▶ The **tree skeleton graphs** are the one computed by Duff
- ▶ **Reproduces the Schwarzschild-Tangherlini metric in $d \geq 4$ dimensions**⁴

⁴ [Mougiakakos, Vanhove]

The scattering angle

One important observable that allows to analytically continue from the scattering regime to the bound state regime is the scattering angle



The scattering angle is obtained from the (classical) radiation action

$$N_{4\text{PM}}(\gamma, |b|) = \int_{\mathbb{R}^2} e^{i\underline{q} \cdot \underline{b}} \frac{N_{4\text{PM}}(\gamma, \underline{q}^2)}{4m_1 m_2 \sqrt{\gamma^2 - 1}} \frac{d^2 \underline{q}}{(2\pi)^2}$$

as $\chi_{4\text{PM}}(\gamma) = -\partial N_{4\text{PM}}(\gamma, J)/\partial J$ with the angular momentum $J = m_1 m_2 \sqrt{\gamma^2 - 1} b/E_{\text{C.M.}}$.

$$\begin{aligned} \left. \frac{\chi}{2} \right|_{1\text{PM}+2\text{PM}} &= \frac{(2\gamma^2 - 1)}{\gamma^2 - 1} \left(\frac{G_N m_1 m_2}{J} \right) \\ &\quad + \frac{3\pi(m_1 + m_2)(5\gamma^2 - 1)}{8(m_1^2 + m_2^2 + 2m_1 m_2 \gamma)} \left(\frac{G_N m_1 m_2}{J} \right)^2 \end{aligned}$$

Angle for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

$$\begin{aligned} \frac{\chi}{2} \Big|_{3\text{PM}} = & \left(\frac{G_N m_1 m_2}{J} \right)^3 \sqrt{\gamma^2 - 1} \left(\frac{(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3(\gamma^2 - 1)^2} \right. \\ & - \frac{4m_1 m_2}{3\mathcal{E}_{\text{C.M.}}^2} \gamma (14\gamma^2 + 25) + \frac{4m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4) \operatorname{arccosh}(\gamma)}{\mathcal{E}_{\text{C.M.}}^2 \sqrt{\gamma^2 - 1}} \\ & \left. + \frac{2m_1 m_2 (2\gamma^2 - 1)^2}{\mathcal{E}_{\text{C.M.}}^2 \sqrt{\gamma^2 - 1}} \left(-\frac{11}{3} + \frac{d}{d\gamma} \left(\frac{(2\gamma^2 - 1) \operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right) \right) \end{aligned}$$

At 3PM (two-loop) new phenomena arise

- ▶ The **conservative part** deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass⁵

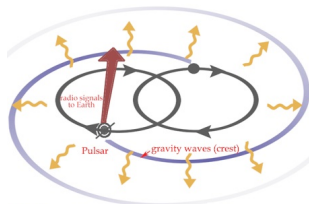
$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

- ▶ And the important **Radiation-reaction terms**⁶

⁵[Damour; Bern et al.; di Vecchia et al.; Bjerrum-Bohr et al.]

⁶[Damour; di Vecchia et al.; Bjerrum-Bohr et al.]

Radiation reaction



directions of gravity waves

The problem of **radiation reaction** has been one of the fundamental theoretical issues in general relativity. This is a needed contribution to match the binary-pulsar observations.

At 3PM a consistent derivation of radiation-reaction was missing. The amplitude approach clarified that

- ▶ The radiation-reaction from the soft region of the amplitude (not in the potential region of⁷)
- ▶ The radiation-reaction is needed for restoring a smooth continuity between the non-relativistic, relativistic and ultra-relativistic regimes⁸

The evaluation of the complete classical scattering amplitude gives a clear-cut unified and unambiguous resolution of these issues⁹

⁷ [Bern et al.]

⁸ [Damour, Veneziano et al.]

⁹ [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

It is satisfying to be able to embed such classical solutions in the new understanding of the relation between general relativity and the quantum theory of gravity

- 1 The 2-body gravitational scattering amplitude leads to the classical observables : potential, scattering angle, and radiation
- 2 The amplitude approach is much simpler than the traditional approach from solving Einstein's equation, and analytic relativistic expressions. The velocity cut method is very efficient in the probe regime
- 3 This is a very useful framework for studying subtle effects like radiation-reactions and memory effects where subtle non-linear effects arise from 5PN order [Blanchet; Damour; ...]
- 4 The approach applies to any EFT of gravity where one can compute amplitudes. Therefore this is a powerful approach to derive new constraints for modified gravity scenarios.