Gravitational binary dynamics from quantum scattering amplitudes

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based on work done in collaboration with
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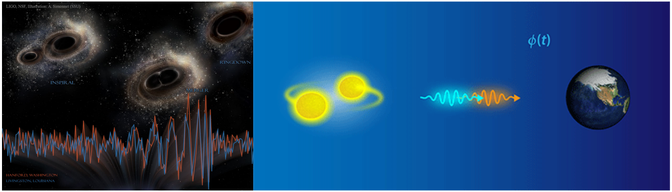
Einstein theory of gravity is the main paradigm for understanding the structure and dynamic of our observable Universe



Black holes, compact stars and gravitational waves are amongst the most spectacular predictions of general relativity and natural probes of the fundamental principles of Einstein's theory and its extension, e.g.

- The activation of scalar fields
- Gravitational leakage into large extra dimensions
- Variability of Newton's constant
- Propagation of gravitational waves
- gravitational Lorentz violation
- strong equivalence principle
- Higher-derivative corrections, …

Classical gravity from quantum amplitudes

THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

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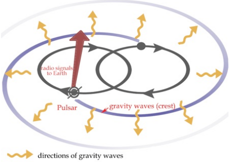
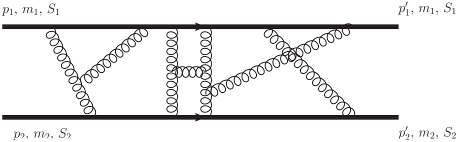
g) *The Feynman-Diagram Approach*

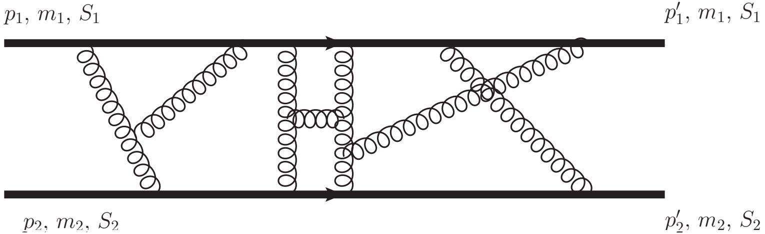
Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta





Analytic expressions for the classical gravitational two-body interactions are obtained using techniques from quantum scattering amplitudes

Classical physics from quantum loops



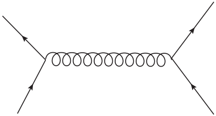
In the limit $\hbar, q^2 \to 0$ with $\underline{q} = \frac{q}{\hbar}$ fixed at each loop order of the quantum amplitude has the Laurent expansion[1] $\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$ and $q^2 = (p_1 - p'_1)^2$

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

- The classical amplitude is the contribution of order $1/\hbar$
- A classical contribution of order $1/\hbar$ from all loop orders
- **classical gravity physics contributions** are determined by the **unitarity** of the quantum scattering amplitudes

1 [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove; Kosower, Maybee, O'Connell]

One graviton exchange : tree-level amplitude



$$\mathfrak{M}_1 = -16\pi G_N \hbar \frac{[2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar \underline{\vec{q}}|^2 (p_1\cdot p_2)]}{|\hbar \underline{\vec{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude

$$\mathfrak{M}_1 = \frac{\mathcal{M}_1^{(-1)}(p_1\cdot p_2)}{\hbar |\underline{q}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The higher order in q^2 are quantum with powers of $\hbar$

The *classical* potential is obtained by taking the 3d Fourier transform

$E_i = \sqrt{p_i^2 + m_i^2}$

$$\mathcal{V}_1(p_1\cdot p_2, r) = \int \frac{d^3\underline{\vec{q}}}{(2\pi)^3} \frac{\mathcal{M}_1^{(-1)}(\vec{q})\, e^{i\underline{\vec{q}}\cdot\vec{r}}}{4E_1E_2} = \frac{G_N}{E_1E_2}\frac{m_1^2m_2^2 - 2(p_1\cdot p_2)^2}{r}$$

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix[2]

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geqslant 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{T}_L^{\rm rad}, \qquad \hat{N} = G_N \sum_{L\geqslant 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{N}_L^{\rm rad}$$

The classical radial action $\hat{N}^{\rm classical}$ does not have any $\hbar$. The higher power of $1/\hbar$ more singular than the classical **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

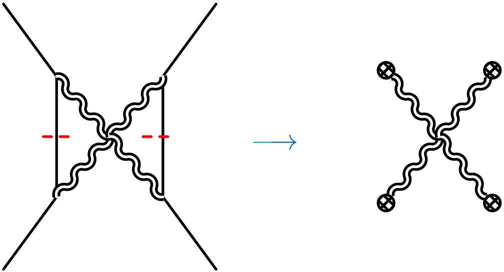
$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

[2] [Damgaard, Planté, Vanhove]

Velocity cuts and classical radial action

At L-loop the classical part arises from imposing L-delta functions on the massive propagators

$$\mathcal{M}_L(\gamma,|\underline{q}|^2)\Big|_{\text{classical}} = \frac{1}{\hbar}\frac{\mathcal{M}_L^{(L-2)}(\gamma,D)}{|\underline{q}|^{2-(D-3)L}} \Longrightarrow N_L^{\text{classical}}(\gamma,D)$$



In practice, we need only evaluate matrix elements in the soft $\underline{q}^2$-expansion, this means that we expand genuine unitarity cuts around the velocity cuts introduced recently[3]

[3] [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Classical black hole metric from quantum amplitudes

Quantum Tree Graphs and the Schwarzschild Solution

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I. INTRODUCTION

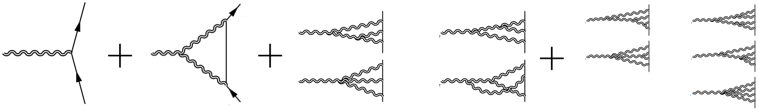
In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \to 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,[1] i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \to 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

In 1973 Duff asked the question about the classical limit of quantum gravity.[a] He showed how to reproduce the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order

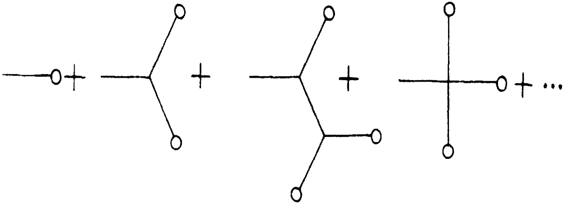
The double expansion in G_N and $\hbar$ give a new perspective on the classical limit of gravitational scattering amplitudes

[a] M. J. Duff, "Quantum Tree Graphs and the Schwarzschild Solution," Phys. Rev. D **7** (1973), 2317-2326

Black hole metric from amplitudes



$$\frac{GM}{r} + \left(\frac{GM}{r}\right)^2 + \left(\frac{GM}{r}\right)^3 + \left(\frac{GM}{r}\right)^4$$

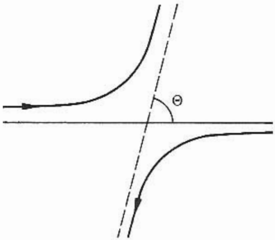


- The tree skeleton graphs are the one computed by Duff
- Reproduces the Schwarzschild-Tangherlini metric in $d \geqslant 4$ dimensions[4]

[4] [Mougiakakos, Vanhove]

The scattering angle

One important observable that allows to analytically continue from the scattering regime to the bound state regime is the scattering angle



The scattering angle is obtained from the (classical) radia action

$$N_{4PM}(\gamma, |b|) = \int_{\mathbb{R}^2} e^{i\underline{q}\cdot b} \frac{N_{4PM}(\gamma, \underline{q}^2)}{4m_1m_2\sqrt{\gamma^2-1}} \frac{d^2\underline{q}}{(2\pi)^2}$$

as $\chi_{4PM}(\gamma) = -\partial N_{4PM}(\gamma, J)/\partial J$ with the angular momentum $J = m_1m_2\sqrt{\gamma^2-1}b/E_{C.M.}$

$$\frac{\chi}{2}\Big|_{1PM+2PM} = \frac{(2\gamma^2-1)}{\gamma^2-1}\left(\frac{G_Nm_1m_2}{J}\right) + \frac{3\pi(m_1+m_2)(5\gamma^2-1)}{8(m_1^2+m_2^2+2m_1m_2\gamma)}\left(\frac{G_Nm_1m_2}{J}\right)^2$$

Angle for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The 3PM scattering angle [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

$$\frac{\chi}{2}\Big|_{3PM} = \left(\frac{G_N m_1 m_2}{J}\right)^3 \sqrt{\gamma^2-1}\left(\frac{(64\gamma^6-120\gamma^4+60\gamma^2-5)}{3\left(\gamma^2-1\right)^2}\right.$$
$$-\frac{4m_1m_2}{3\mathcal{E}_{\text{C.M.}}^2}\gamma\left(14\gamma^2+25\right)+\frac{4m_1m_2(3+12\gamma^2-4\gamma^4)\,\text{arccosh}(\gamma)}{\mathcal{E}_{\text{C.M.}}^2\sqrt{\gamma^2-1}}$$
$$\left.+\frac{2m_1m_2(2\gamma^2-1)^2}{\mathcal{E}_{\text{C.M.}}^2\sqrt{\gamma^2-1}}\left(-\frac{11}{3}+\frac{d}{d\gamma}\left(\frac{(2\gamma^2-1)\,\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}\right)\right)\right)$$

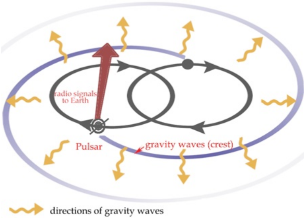
At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass[5] $\nu = \frac{m_1m_2}{(m_1+m_2)^2}$
- And the important Radiation-reaction terms[6]

[5] [Damour; Bern et al.; di Vecchia et al.;Bjerrum-Bohr et al.]

[6] [Damour; di Vecchia et al.;Bjerrum-Bohr et al.]

Radiation reaction



The problem of **radiation reaction** has been one of the fundamental theoretical issues in general relativity. This is a needed contribution to match the binary-pulsar observations.

At 3PM a consistent derivation of radiation-reaction was missing. The amplitude approach clarified that

- The radiation-reaction from the soft region of the amplitude (not in the potential region of[7])
- The radiation-reaction is needed for restoring a smooth continuity between the non-relativitic, relativistic and ultra-relativistic regimes[8]

The evaluation of the complete classical scattering amplitude gives a clear-cut unified and unambiguous resolution of these issues[9]

7 [Bern et al.]

8 [Damour, Veneziano et al.]

9 [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Outlook

It is satisfying to be able to embed such classical solutions in the new understanding of the relation between general relativity and the quantum theory of gravity

1. The 2-body gravitational scattering amplitude leads to the classical observables : potential, scattering angle, and radiation
2. The amplitude approach is much simpler that the traditional approach from solving Einstein's equation, and analytic relativistic expressions. The velocity cut method is very efficient in the probe regime
3. This is a very useful framework for studying subtle effects like radiation-reactions and memory effects where subtle non-linear effects arise from 5PN order [Blanchet; Damour; ...]
4. The approach applies to any EFT of gravity where one can compute amplitudes. Therefore this is a power approach to derive new constraints for modified gravity scenarios.