Invisibility cloaks and transformation optics

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In collaboration with... Frédéric Zolla (Université de Provence), Sébastien Guenneau (CNRS *et qui soutient son HDR en ce moment même!*), Christophe Geuzaine (Université de Liège).

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Outline

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- Various attempts to set up infinite elements...

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- Analytical map of the interior of a circle (disk) on the exterior of a circle (infinite domain), connecting the boundary to the boundary of the region of interest.
- Limited to 2D Laplace operator (harmonic problems).

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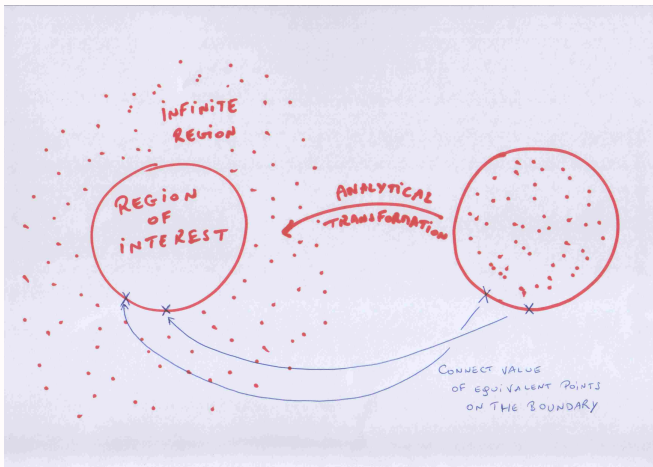
Solve a 2D harmonic problem on the disk.

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- Adapted to 3D problems and to various shapes, natural description of the geometry...
- Requires a modification of the operator.

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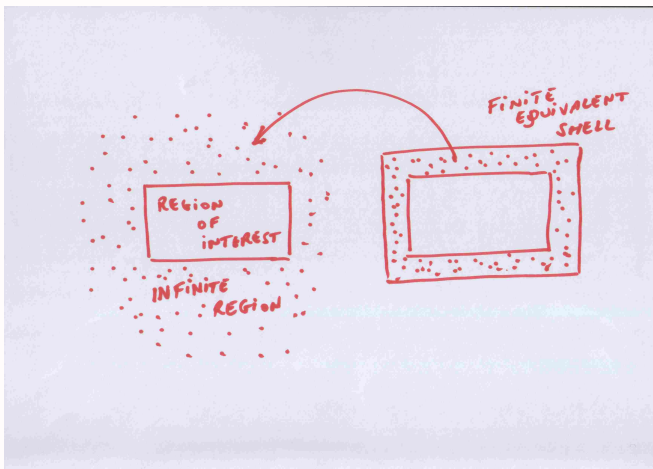
How to modify simply the code to introduce the new coefficients ?

Differential geometry of p-forms

- **p-forms** are rank p totally covariant tensors.
- **Exterior derivative** d = differential or gradient of functions, curl and divergence of vector fields...
- $dd = 0$ for **curl grad** $= 0$, div **curl** $= 0$ ...
- **Exterior product** $\wedge$ = multilinear antisymmetric map.
- **Geometric integration** $\int_{\Sigma} \alpha$ = line integrals of 1-forms, surface (flux) integrals of 2-forms, volume integrals of 3-forms...
- Stokes theorem $\int_{\Sigma} d\alpha = \int_{\partial\Sigma} \alpha$
- All are **metric free** operations!

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- **Maxwell's equations**

$$\begin{cases} \mathbf{curl}\, H = J + \partial_t D \\ \mathbf{curl}\, E = -\partial_t B \\ \operatorname{div} D = \rho \\ \operatorname{div} B = 0 \end{cases}$$

- **Poynting identity**

$$\operatorname{div}(E \times H) = J \cdot E + E \cdot \partial_t D + H \cdot \partial_t B$$

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Electrodynamics

- **Maxwell's equations**

$$\begin{cases} dH = J + \partial_t D \\ dE = -\partial_t B \\ dD = \rho \\ dB = 0 \end{cases}$$

- **Poynting identity**

$$d(E \wedge H) = J \wedge E + E \wedge \partial_t D + H \wedge \partial_t B$$

- These relations are metric free !

- In a general coordinate system $\{u, v, w\}$:
- The electric field is the 1-form $\mathbf{E} = E_u du + E_v dv + E_w dw$.
- The magnetic flux density is the 2-form
  $\mathbf{B} = B_u dv \wedge dw + B_v dw \wedge du + B_w du \wedge dv$.
- And $dE = -\partial_t B$ means
  $(\partial_u E_v - \partial_v E_u + \partial_t B_w) du \wedge dv +$
  $(\partial_v E_w - \partial_w E_v + \partial_t B_u) dv \wedge dw +$
  $(\partial_w E_u - \partial_u E_w + \partial_t B_v) dw \wedge du = 0$

- **Distance, angle...**
- Hodge star operator $*$ maps p-forms on $(3 - p)$-forms.
- Example of Euclidean metric in Cartesian coordinates :

$$\left\{ \begin{array}{l} *dx = dy \wedge dz \\ *dy = dz \wedge dx \\ *dz = dx \wedge dy \end{array} \right.$$

- This simplicity hides metric aspects in Cartesian coordinates but the relations are more complicated with a general coordinate system...

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- **... and electromagnetic constitutive laws !**
- For example in free space :

$$D = \varepsilon_0 * E$$

$$B = \mu_0 * H$$

- The Hodge star operator is necessary to transform fields (1-forms) into flux densities (2-forms) !

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- Considering a map from the coordinate system $\{u, v, w\}$ to the coordinate system $\{x, y, z\}$ given by the functions $x(u, v, w)$, $y(u, v, w)$, and $z(u, v, w)$,
- All the information is in the transformation of the differentials and is therefore given by the *chain rule* :

$$\begin{cases} dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw \\ dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw \\ dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw \end{cases}$$

All the information is in the Jacobian matrix **J** :

$$\left(\begin{array}{c} dx \\ dy \\ dz \end{array} \right) = \mathbf{J} \left(\begin{array}{c} du \\ dv \\ dw \end{array} \right)$$

with **J** defined as

$$\mathbf{J}(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \left(\begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right)$$

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Pullback

Direct map :
New coordinates $\longrightarrow$ Cartesian coordinates
u, v, w $x(u, v, w), y(u, v, w), z(u, v, w)$
=Modelling space   ="Physical space"

Pullback :
Forms in $\longleftarrow$ Forms in
new coordinates   Cartesian coordinates
$E_u\,du + E_v\,dv + E_w\,dw$ $E_x\,dx + E_y\,dy + E_z\,dz$
etc.

- Map the transformed domain Ω' (modelling space) on the original domain Ω (physical space usually in Cartesian coordinates) and pullback the covariant objects (physical equations) to the transformed domain (new model equations).

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- The **material properties**, involving all the **metric** aspects, are written with the Hodge operator and contain all the coordinate dependent information.

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Consequences

- Map the transformed domain Ω' (modelling space) on the original domain Ω (physical space usually in Cartesian coordinates) and pullback the covariant objects (physical equations) to the transformed domain (new model equations).

- **Maxwell's equations**, that express topological and differential metric free relations, may be written in a completely covariant **coordinate free** form.

- The **material properties**, involving all the **metric** aspects, are written with the Hodge operator and contain all the coordinate dependent information.

Any change of coordinates can be translated into equivalent materials given by :

$$\underline{\underline{\varepsilon}}' = \varepsilon \mathbf{T}^{-1} \, , \quad \text{and} \quad \underline{\underline{\mu}}' = \mu \mathbf{T}^{-1} \, .$$

- with the matrix $\mathbf{T} = \mathbf{J}^T \mathbf{J} / det(\mathbf{J})$.

Equivalent materials mean that

you can work in the new coordinate system just as if you were still in Cartesian coordinates but for the ϵ and μ that have been turned to new equivalent material properties

$$\underline{\underline{\varepsilon}}' = \varepsilon \mathbf{T}^{-1} \text{ and } \underline{\underline{\mu}}' = \mu \mathbf{T}^{-1}$$

with the matrix $\mathbf{T} = \mathbf{J}^{T}\mathbf{J}/det(\underline{\underline{\mathbf{J}}})$!

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- In the more general case where the initial $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\mu}}$ are tensors corresponding to anisotropic properties, the equivalent properties become

$$\underline{\underline{\varepsilon}}' = \mathbf{J}^{-1}\underline{\underline{\varepsilon}}\mathbf{J}^{-T}det(\mathbf{J}) , \quad \text{and} \quad \underline{\underline{\mu}}' = \mathbf{J}^{-1}\underline{\underline{\mu}}\mathbf{J}^{-T}det(\mathbf{J}) .$$

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where $\mathbf{J}^{-T}$ denotes the transpose of the inverse of $\mathbf{J}$.

- The two successive changes of coordinates are given by the Jacobian matrices $\mathbf{J}_{xX}$ and $\mathbf{J}_{Xu}$ so that

$$\mathbf{J}_{xu} = \mathbf{J}_{xX}\mathbf{J}_{Xu} \,.$$

This rule naturally applies for an arbitrary number of coordinate systems.

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$\mathbf{T}^{-1}$ for an open domain (exterior of a disk) mapped on a circular annulus.

Consider the radial transformation
$r = f(r') = (R_1 - R_2)r'/(r' - R_2)$
so that $r' = R_1 \Rightarrow r = R_1$ and $r' = R_2 \Rightarrow r \to \infty$.
Define $c_{11}(r') = \frac{df(r')}{dr'}$ and

$$\mathbf{T}^{-1} = \mathbf{R}(\theta')\mathbf{diag}(\frac{f(r')}{c_{11}(r')r'}, \frac{c_{11}(r')r'}{f(r')}, \frac{c_{11}(r')f(r')}{r'})\mathbf{R}(\theta')^T$$
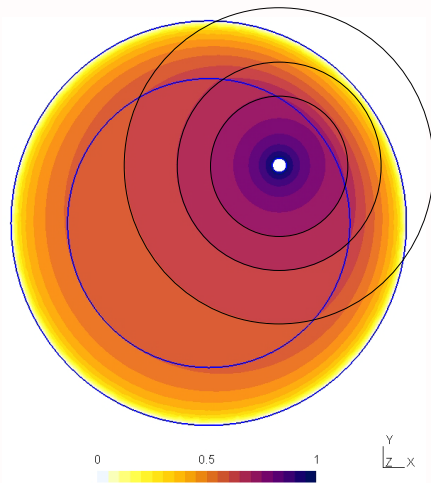
where r' and θ' are the well known functions
$r'(x', y') = \sqrt{x'^2 + y'^2}$ and $\theta'(x', y') = 2\arctan(\frac{y'}{x' + \sqrt{x'^2 + y'^2}})$ and
$\mathbf{R}(\theta')$ is a rotation matrix.

x', y' are the Cartesian coordinates in the annulus configuration.

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Electrostatic potential : circular cylinder $V = 1$, $V(r \to \infty) = 0$
$\Rightarrow$ circular equipotential lines !

0   0.5   1

- Previous technique in not workable for wave propagation problems because the wavelength is going to zero.

Open propagation problems : Perfectly Matched Layers (PML)

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- *J.-P. Bérenger, A Perfectly Matched Layer for the Absorption of Electromagnetic Waves, Journal of Computational Physics, 1994, 114, pp. 185-200.*

Open propagation problems : Perfectly Matched Layers (PML)

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- *J.-P. Bérenger, A Perfectly Matched Layer for the Absorption of Electromagnetic Waves, Journal of Computational Physics, 1994, 114, pp. 185-200.*
- Nowadays, may be introduced as a complex mapping

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- Consider polar coordinates with ρ the radial coordinate.

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- Perform the complex change of coordinates $\tilde{\rho} = \int_0^\rho s_\rho(\rho')d\rho'$ with s_ρ a complex valued function (**complex radial stretch**).

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A Complex Change of Coordinates

- Consider polar coordinates with ρ the radial coordinate.
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- Roughly turns $e^{-i\gamma\rho}$ into $e^{-i\gamma\tilde{\rho}}$ exponentially decreasing because $\tilde{\rho}$ is complex (with the good sign of the imaginary part of $\tilde{\rho}$) and it damps outgoing waves and other leaky modes...

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- Roughly turns $e^{-i\gamma\rho}$ into $e^{-i\gamma\tilde{\rho}}$ exponentially decreasing because $\tilde{\rho}$ is complex (with the good sign of the imaginary part of $\tilde{\rho}$) and it damps outgoing waves and other leaky modes...
- The global coordinate transformation is taken to be **identity ($s_\rho = 1$) inside a region of interest ($\rho < R^*$)** and **the complex change of coordinates (s_ρ complex valued) outside the region of interest ($\rho > R^*$)** so that the transformed problem provides directly the required fields in the region of interest (This will be more clear on an example !)

The inverse matrix $\mathbf{T}_{PML}^{-1}$ corresponding to the complex stretch is given by :

$$\begin{pmatrix} \frac{\rho s_\rho \sin(\theta)^2}{\tilde{\rho}} + \frac{\tilde{\rho} \cos(\theta)^2}{\rho s_\rho} & \sin(\theta)\cos(\theta)\left(\frac{\tilde{\rho}}{\rho s_\rho} - \frac{\rho s_\rho}{\tilde{\rho}}\right) & 0 \\ \sin(\theta)\cos(\theta)\left(\frac{\tilde{\rho}}{\rho s_\rho} - \frac{\rho s_\rho}{\tilde{\rho}}\right) & \frac{\rho s_\rho \cos(\theta)^2}{\tilde{\rho}} + \frac{\tilde{\rho} \sin(\theta)^2}{\rho s_\rho} & 0 \\ 0 & 0 & \frac{\tilde{\rho} s_\rho}{\rho} \end{pmatrix}.$$

All the quantities involved in the previous expression can be given as explicit functions of x and y "pseudo Cartesian modelling coordinates" : $\theta = 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)$,

$\rho = \sqrt{x^2 + y^2}$, $s_\rho(\rho) = s_\rho(\sqrt{x^2+y^2})$, and $\tilde{\rho} = \int_0^{\sqrt{x^2+y^2}} s_\rho(\rho')d\rho'$.

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A strong test for PML : leaky modes in Microstructured Optical Fibres

Cross section of a six air hole in silica MOF structure

The hole structure is with $\Lambda = 6.75\,\mu m$, $r_s = 2.5\,\mu m$, the surrounding annulus used to set up the PML has $R^* = 30\mu m$, $R^{trunc} = 40\mu m$

$\lambda_0 = 1.55 \, \mu m$ is considered for which the index of silica is about $\sqrt{\varepsilon_{\mathrm{Si}}} = n_{\mathrm{Si}} = 1.444024$.

The corresponding complex effective index $n_{\mathrm{eff}} = \beta/k_0$ is

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$1.4387741 + 4.3257457 \, 10^{-8} i$

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finite element method :

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Comparison with the Multipole Method

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multipole methods :

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finite element method :

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COMSOL Multiphysics®, about $16,800$ second order triangular elements, 150 seconds on a Pentium M 1.86 GHz, 1Go RAM laptop computer.

What happens if a Microstructured Optical Fibre is twisted ?

- From translational invariance...
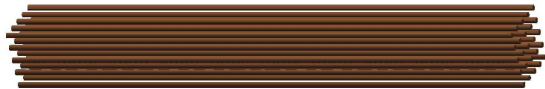
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What happens if a Microstructured Optical Fibre is twisted ?

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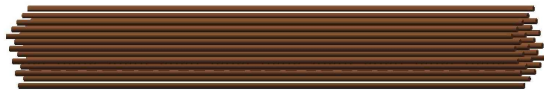
- ... to twisted structures :

- We are going to look for a simple and efficient model i.e. still 2D and rigorous !

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- Helicoidal coordinates :

$$
\begin{cases}
x_1 = \xi_1 \cos(\alpha\xi_3) + \xi_2 \sin(\alpha\xi_3)\ , \\
x_2 = -\xi_1 \sin(\alpha\xi_3) + \xi_2 \cos(\alpha\xi_3)\ , \\
x_3 = \xi_3\ ,
\end{cases}
\tag{1}
$$

where α is a parameter which characterizes the torsion of the structure.

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where α is a parameter which characterizes the torsion of the structure.

A twisted problem

is a problem whose cross section is independent from ξ_3.

- This coordinate system is characterized by the Jacobian matrix of the transformation :

$$\mathbf{J}(\xi_1, \xi_2, \xi_3) = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}$$

$$= \begin{pmatrix} \cos(\alpha\xi_3) & \sin(\alpha\xi_3) & \alpha\xi_2 \cos(\alpha\xi_3) - \alpha\xi_1 \sin(\alpha\xi_3) \\ -\sin(\alpha\xi_3) & \cos(\alpha\xi_3) & -\alpha\xi_1 \cos(\alpha\xi_3) - \alpha\xi_2 \sin(\alpha\xi_3) \\ 0 & 0 & 1 \end{pmatrix} ,$$

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- which does depend on the three variables ξ_1, ξ_2 and ξ_3.

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$\mathbf{T}_h$ matrix for equivalent material in helicoidal coordinates

- The transformation matrix $\mathbf{T}_h$ for helicoidal coordinates is :

$$\mathbf{T}_h(\xi_1, \xi_2) = \frac{\mathbf{J}^T \mathbf{J}}{det(\mathbf{J})} = \begin{pmatrix} 1 & 0 & \alpha\xi_2 \\ 0 & 1 & -\alpha\xi_1 \\ \alpha\xi_2 & -\alpha\xi_1 & 1 + \alpha^2(\xi_1^2 + \xi_2^2) \end{pmatrix} ,$$

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- which only depends on the first two variables ξ_1 and ξ_2...

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- which only depends on the first two variables ξ_1 and ξ_2...
- ... allowing a **two-dimensional formulation** of the twisted waveguide problem !

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Compose the transformation into helicoidal coordinates with a complex stretch to obtain the **twisted PML** inverse matrix $\mathbf{T}_{hPML}^{-1}$ given by :

$$\begin{pmatrix} \frac{\tilde{\rho}\cos^2(\varphi)}{\rho s_\rho} + \frac{\rho(1+\alpha^2\tilde{\rho}^2)s_\rho\sin^2(\varphi)}{\tilde{\rho}} & \frac{\sin(2\varphi)(\tilde{\rho}^2 - \rho^2(1+\alpha^2\tilde{\rho}^2)s_\rho^2)}{2\rho\tilde{\rho}s_\rho} & -\alpha\tilde{\rho}s_\rho\sin(\varphi) \\ \frac{\sin(2\varphi)(\tilde{\rho}^2 - \rho^2(1+\alpha^2\tilde{\rho}^2)s_\rho^2)}{2\rho\tilde{\rho}s_\rho} & \frac{\tilde{\rho}\sin^2(\varphi)}{\rho s_\rho} + \frac{\rho(1+\alpha^2\tilde{\rho}^2)s_\rho\cos^2(\varphi)}{\tilde{\rho}} & \alpha\cos(\varphi)\tilde{\rho}s_\rho \\ -\alpha\tilde{\rho}s_\rho\sin(\varphi) & \alpha\cos(\varphi)\tilde{\rho}s_\rho & \frac{\tilde{\rho}s_\rho}{\rho} \end{pmatrix}$$

All the quantities involved in the previous expression can be given as explicit functions of the two "helicoidal pseudo Cartesian modelling coordinates " ξ_1, ξ_2 :

$\varphi = 2\arctan\left(\frac{\xi_2}{\xi_1 + \sqrt{\xi_1^2 + \xi_2^2}}\right)$, $\rho = \sqrt{\xi_1^2 + \xi_2^2}$, $s_\rho(\rho) = s_\rho(\sqrt{\xi_1^2 + \xi_2^2})$,

and $\tilde{\rho} = \int_0^{\sqrt{\xi_1^2 + \xi_2^2}} s_\rho(\rho')d\rho'$.

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Evolution of the Real Part of Effective Index
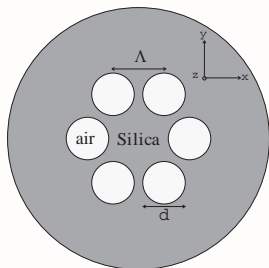
Parameters

- $\Lambda = 6.75\mu m$
- $d = 5\mu m$
- $n_{Si.} = 1.444024$
- $\lambda = 1.55\mu m$
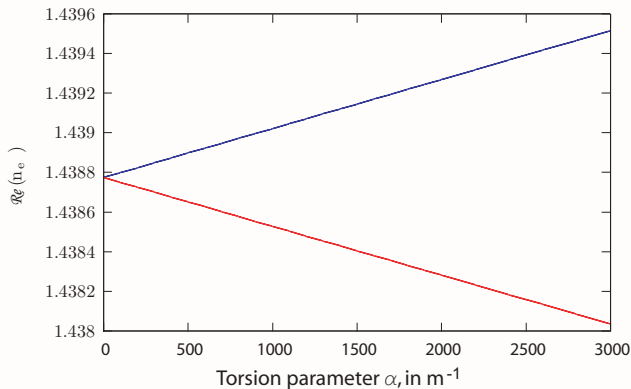
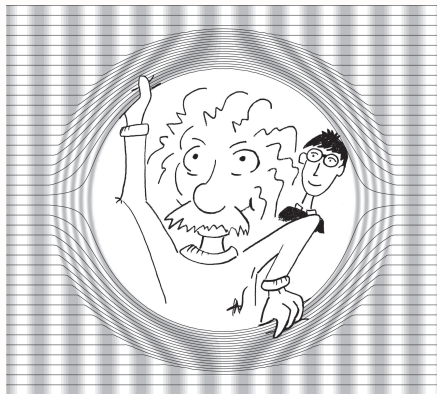# Evolution of the Imaginary Part of Effective Index
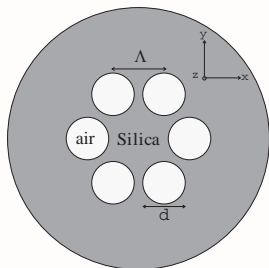
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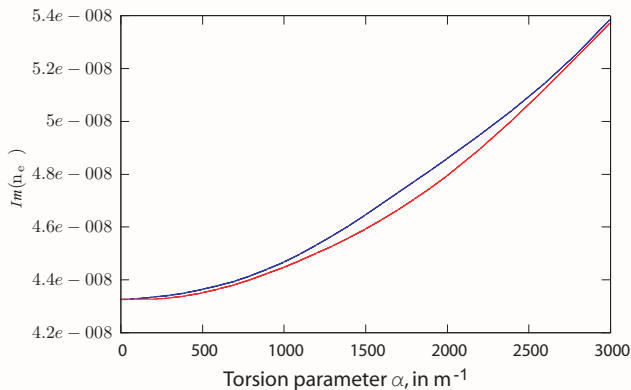
From invisibility cloak...


... to polyjuice potion !

Map an annulus on a circle and let the **EQUIVALENT MATERIAL** to become a **NEW PHYSICAL MATERIAL** ! Consider a geometric transformation which maps the field within the disk $r \leq R_2$ onto the annulus $R_1 \leq r \leq R_2$:

$$\begin{cases} r = f(r') = (r' - R_1)R_2/(R_2 - R_1) \text{ for } R_1 \leq r \leq R_2, \\ \theta = \theta', \ z = z'. \end{cases}$$

where r', θ' and z' are "radially contracted cylindrical coordinates". Moreover, this transformation maps the field for $r \geq R_2$ onto itself by the identity transformation.

r=f(r')

r'

Initial (Cartesian coordinates)   New
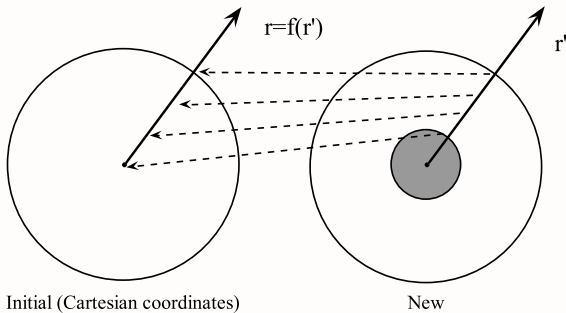
The material properties of the invisibility cloak are given by :

$$\mathbf{T}^{-1} = \mathbf{R}(\theta')\mathbf{diag}(\frac{r' - R_1}{r'}, \frac{r'}{r' - R_1}, c_{11}^2 \frac{r' - R_1}{r'})\mathbf{R}(\theta')^T$$

where $c_{11} = R_2/(R_2 - R_1)$, r' and θ' are the well known functions $r'(x', y') = \sqrt{x'^2 + y'^2}$ and $\theta'(x', y') = 2\arctan(\frac{y'}{x' + \sqrt{x'^2 + y'^2}})$, and $\mathbf{R}(\theta')$ is a rotation matrix.

x', y' are the Cartesian coordinates in the annulus configuration.

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- ... and we have got the recipe for the cylindrical circular cloak !
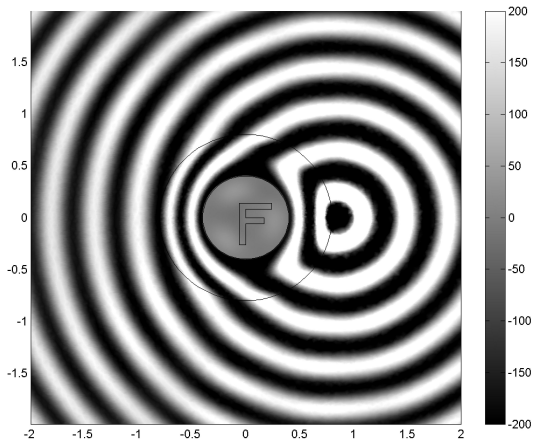
Mirage effect

- The light source can also be inside the cloak !
  In this case, the object in the cavity is still invisible and the light seems
  to be emitted from a shifted position (mirage effect).
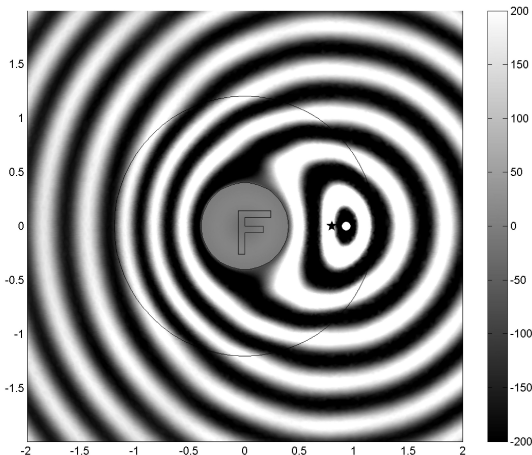
Cloak of arbitrary shape

Make the radii depend on θ : $R_1(\theta)$, $R_2(\theta)$.
The geometric transformation which maps the field within the full domain $r \leq R_2(\theta)$ onto the hollow domain $R_1(\theta) \leq r \leq R_2(\theta)$:

$$\begin{cases} r'(r,\theta) = R_1(\theta) + r(R_2(\theta) - R_1(\theta))/R_2(\theta) \ , \ 0 \leq r \leq R_2(\theta) \\ \theta' = \theta \ , \ 0 < \theta \leq 2\pi \\ z' = z \ , \ z \in \mathbb{R} \ , \end{cases}$$

and the transformation maps the field for $r \geq R_2(\theta)$ onto itself by the identity transformation. This leads to

$$\mathbf{J}_{rr'} = \frac{\partial(r(r',\theta'),\theta,z)}{\partial(r',\theta',z')} = \begin{pmatrix} c_{11}(r',\theta') & c_{12}(r',\theta') & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Cloak of arbitrary shape

$$\mathbf{J}_{rr'}(r', \theta') = \frac{\partial(r(r',\theta'),\theta,z)}{\partial(r',\theta',z')} = \begin{pmatrix} c_{11}(r',\theta') & c_{12}(r',\theta') & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $c_{11}(r',\theta') = R_2(\theta')/(R_2(\theta') - R_1(\theta'))$ for $0 \le r' \le R_2(\theta')$
and $c_{11} = 1$ for $r' > R_2(\theta')$

and $c_{12}(r',\theta') = \frac{(r'-R_2(\theta'))R_2(\theta')\frac{dR_1(\theta')}{d\theta'} - (r'-R_1(\theta'))R_1(\theta')\frac{dR_2(\theta')}{d\theta'}}{(R_2(\theta')-R_1(\theta'))^2}$
for $0 \le r' \le R_2(\theta')$
and $c_{12} = 0$ for $r' > R_2(\theta')$.

Finally, the properties of the cloak are given by :

$$\mathbf{T}^{-1} = \mathbf{R}(\theta') \begin{pmatrix} \frac{c_{12}^2 + f_r^2}{c_{12} f_r r'} & -\frac{c_{12}}{f_r} & 0 \\ -\frac{c_{12}}{f_r} & \frac{c_{11} r'}{f_r} & 0 \\ 0 & 0 & \frac{c_{11} f_r}{r'} \end{pmatrix} \mathbf{R}(\theta')^T$$

with

$$f_r = \frac{(r' - R_1) R_2}{(R_2 - R_1)}.$$

The central matrix depends on θ' !

- Parametric representation of the ellipse

$$R(\theta) = \frac{ab}{\sqrt{a^2 \cos(\theta)^2 + b^2 \sin(\theta)^2}}$$
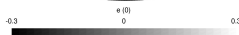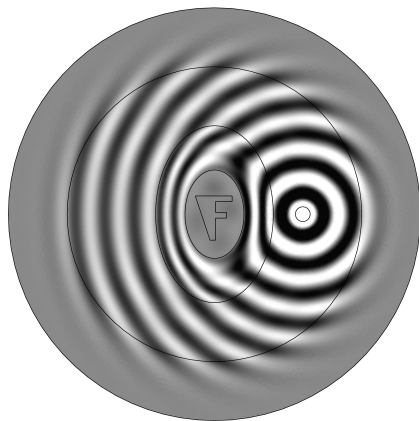
Cloak of arbitrary shape

Use Fourier series $R(\theta) = a_0 + \sum_{i=1}^{n}(a_i \cos(i\theta) + b_i \sin(i\theta))$ to obtain general shapes :

R_1 is with $a_0 = 1$, $b_1 = 0.1$, $a_2 = -0.15$, $b_3 = 0.2$, $a_4 = 0.1$,
R_2 is with $a_0 = 2$, $a_2 = -0.1$, $a_3 = -0.15$, $b_3 = 0.3$, $a_4 = 0.2$,
all the other coefficients $= 0$.

e (0)

-0.3   0   0.3
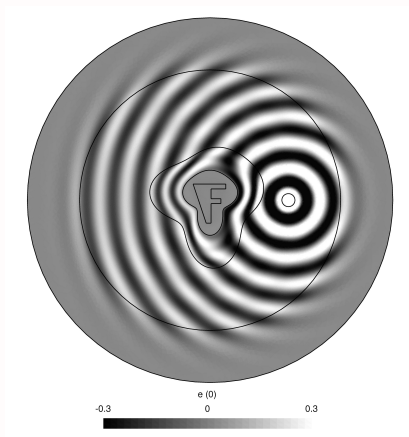
- Zoom on the cloak and the source...
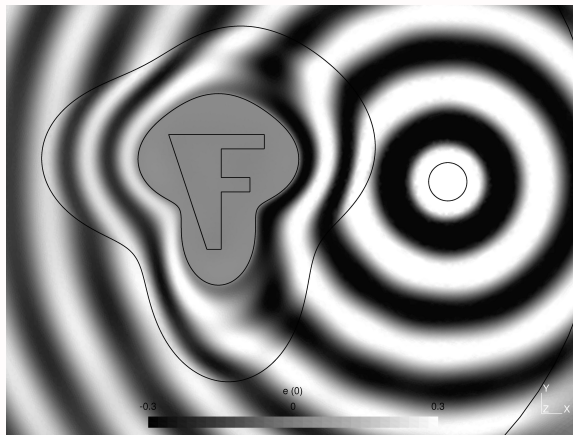
The geometric transformation is performed pointwise on the disk :

$$\underline{\underline{\varepsilon}}'(\mathbf{x}') = \mathbf{J}^{-1}(\mathbf{x}')\underline{\underline{\varepsilon}}(\mathbf{x}(\mathbf{x}'))\mathbf{J}^{-T}(\mathbf{x}')\det(\mathbf{J}(\mathbf{x}')),$$
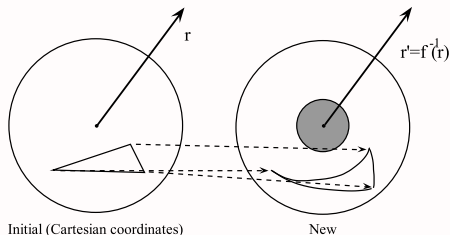$$\underline{\underline{\mu}}'(\mathbf{x}') = \mathbf{J}^{-1}(\mathbf{x}')\underline{\underline{\mu}}(\mathbf{x}(\mathbf{x}'))\mathbf{J}^{-T}(\mathbf{x}')\det(\mathbf{J}(\mathbf{x}')).$$

These formulae are valid for any initial content of the central disk !

Scattering of cylindrical waves (real part of E_z) by a conducting object of triangular cross section :
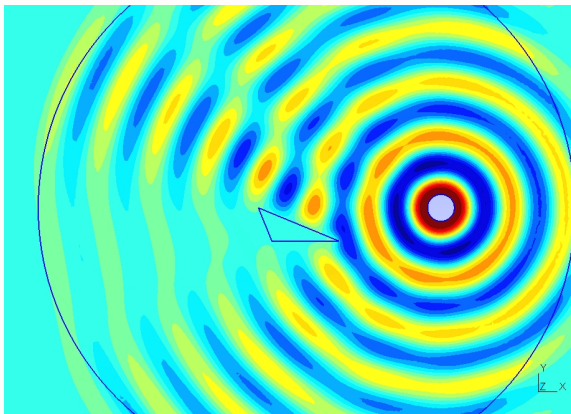
Transforming a non homogeneous region :


Initial (Cartesian coordinates)   New

- The cross sections of the boundaries of homogeneous regions (jumps of material properties) are curves (here, the triangle) and are therefore **contravariant**.
- They are **pushed forward** along the $r' = f^{-1}(r)$ inverse map (that is fortunately simple) :

$$\mathbf{x}'(t) = f^{-1}(\mathbf{x}(t)) = (\frac{R_2 - R_1}{R_2} + \frac{R_1}{\|\mathbf{x}(t)\|})\mathbf{x}(t),$$
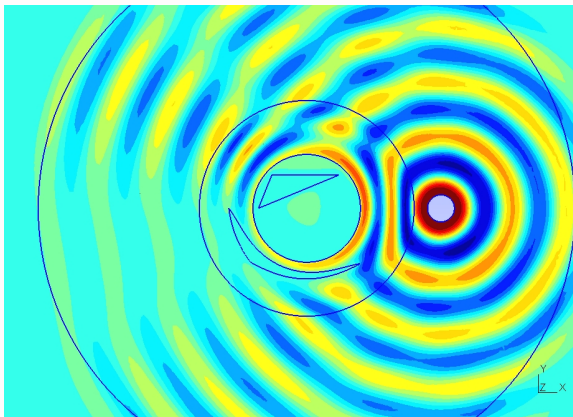
- The anamorphosis of the triangle is described by three splines interpolating each 40 points.

Masking effect

- Geometry description and meshing are performed with GetDP and Gmsh.
- The inner boundary of the cloak must be very finely meshed because of the singularity of the material properties.
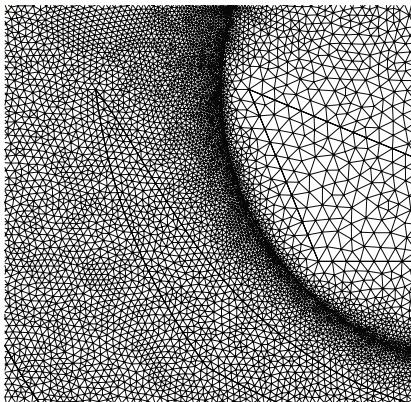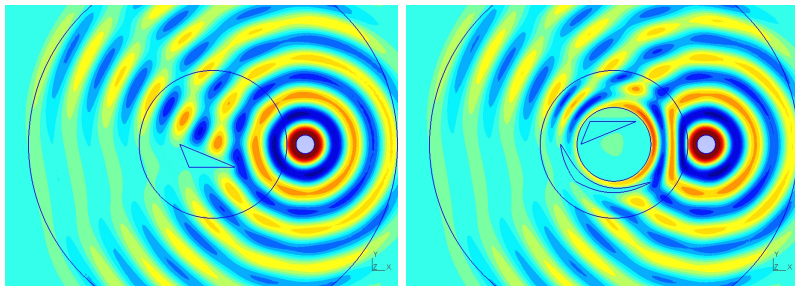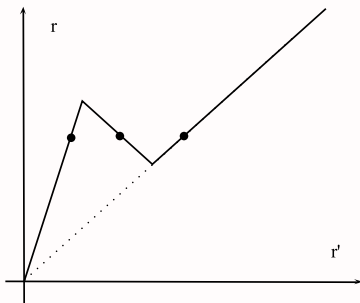
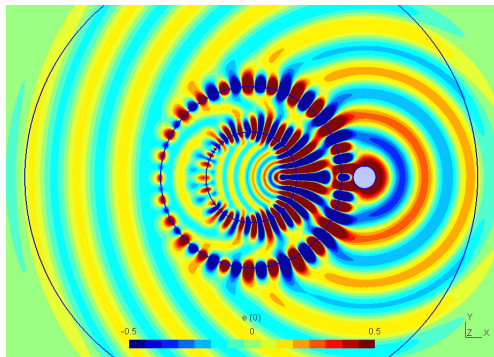Outside the cloak, the scattered field is left unchanged with respect to the initial situation !
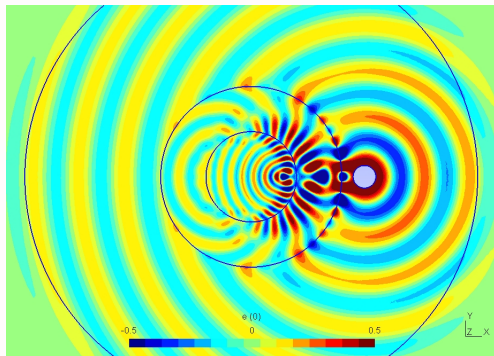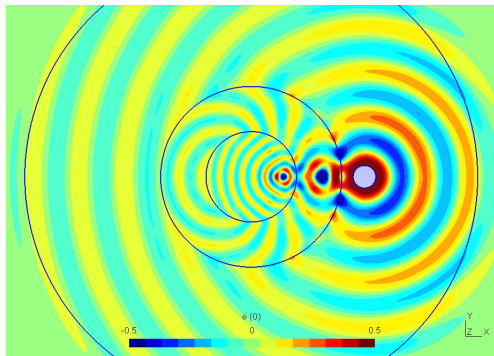
Compare the field outside the disks :

The negative slope corresponds to Negative Refraction Index materials.

Without losses : anomalous resonances are dazzling !

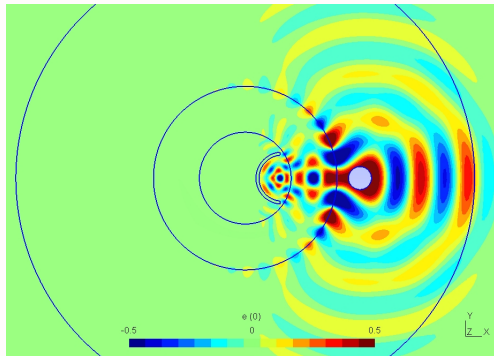With $1/1000$ losses in the superlens : the two images of the sources appear !

With $1/100$ losses in the superlens : the disturbance is still reasonable and the three copies of the source appear clearly !

A small scatterer close to the interior image of the source acts as a large one on the source (1/100 losses in the superlens) !

- Periodic structures

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  - 1D : gratings, Bragg mirrors...

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- Plasmonic - metallic nanostructures for optics ($Re(\varepsilon) < 0$, $Im(\varepsilon)$ very small, size $< \lambda$),
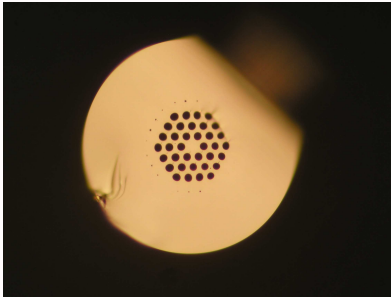 Split Ring Resonators for microwaves

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  Split Ring Resonators for microwaves
- **Wire media**.

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Microstructured optical fibre made of chalcogenide glass manufactured at the Université de Rennes (ext. diam. = 139 μm).
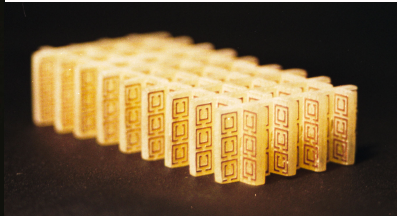
Split Ring Resonator device with printed copper "double C" for microwave experiments.

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Purpose : obtain new artificial properties :

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- Invisibility Cloaking

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Metamaterials in electromagnetism

Purpose : obtain new artificial properties :

- Negative Refraction Index for Perfect Lenses
- Photonic Band Gap and Epsilon Near Zero for guiding light in defects
- Invisibility Cloaking

Homogenization :

Find equivalent material properties (ε, μ ...) in order to represent the metamaterial as an homogeneous medium !

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  - Modelling of open problems including propagation problems and leaky modes, helicoidal geometries...
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- They also provide another argument for the use of differential geometry to manipulate Maxwell's equations !

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Conclusion

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  - Modelling of open problems including propagation problems and leaky modes, helicoidal geometries...
  - Design of new device/EM exotic properties : invisibility cloaks, masking devices, perfect lenses with negative refractive index materials...

- They also provide another argument for the use of differential geometry to manipulate Maxwell's equations !

- **Thank you for your attention !**
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  http ://www.fresnel.fr/perso/nicolet/
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