S-matrix bootstrap in 3+1 dimensions

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based on 2103.11484 with Martin Kruczenski

2-to-2 S-matrix in 3+1d

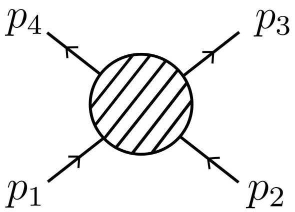
maximal analyticity

$$F(s,t,u)$$

cuts $s > 4 \quad t > 4 \quad u > 4$

crossing

$$F(s,t,u) = F(s,u,t) = F(u,t,s)$$



$$\pi(p_1) + \pi(p_2) \to \pi(p_3) + \pi(p_4)$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

$$s + t + u = 4$$

$$m^2 = 1$$

amplitude $F(s,t,u)$

physical region

$s > 4 \quad 4 - s < t < 0$

$$h_\ell(s) = \frac{\pi}{4}\sqrt{\frac{s-4}{s}}\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta) F(s^+, t, u)$$

physical partial waves

$$|h_\ell(s)|^2 \leq 2\,\mathrm{Im} h_\ell(s)$$

unitarity

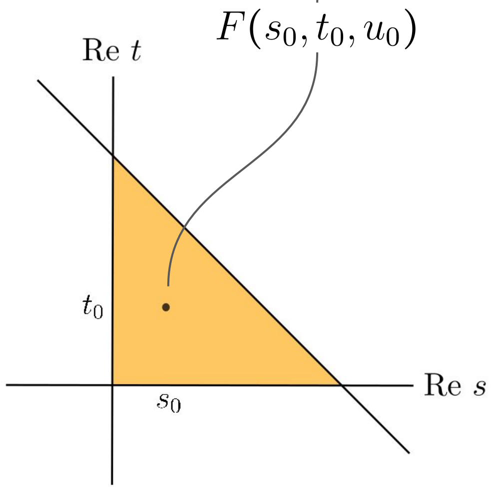
Modern S-matrix bootstrap

maximize linear functional → **space of S-matrices**

numerics

[Paulos, Penedones, Toledo, van Rees, Vieira, 2016 & 2017]

primal problem



1+1d dual problem

[Cordova, He, Kruczenski, Vieira, 2019]

approaching the space from outside

↓

strict outer boundary

alternative, efficient approach to map out the space S-matrices

3+1d needs the dual problem

an example: maximize $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ for single flavor pion

[Paulos, Penedones, Toledo, van Rees, Vieira, 2017]

maximum attained by amplitude with a threshold pole

included in setup	not included in setup
↓	↓
good convergence	bad convergence

we would like to explore the space without assuming such structures, use the bootstrap to discover them

dual problem $\boxed{\checkmark}$

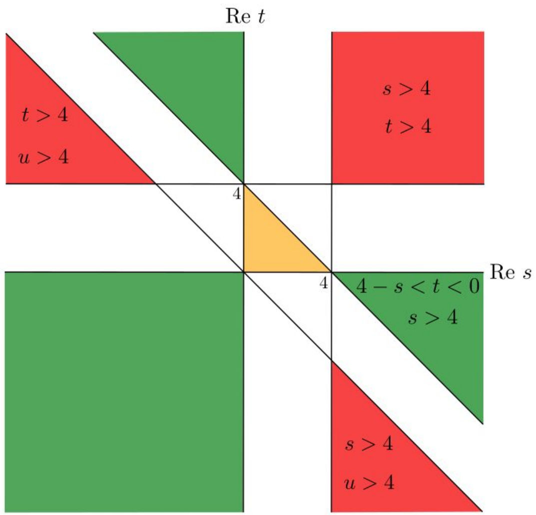
3+1d primal problem setup

maximal analyticity + crossing

parameters

$$F(s,t,u) = \boxed{\frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \left[\frac{1}{(x-s)(y-t)} + \frac{1}{(x-s)(y-u)} + \frac{1}{(x-u)(y-t)} + (x \leftrightarrow y)\right]\rho(x,y)}$$

+subtractions



unitarity

$$h_\ell(s) = \frac{\pi}{4}\sqrt{\frac{s-4}{s}}\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta) F(s^+,t,u)$$

$$|h_\ell(s)|^2 \le 2\,\mathrm{Im} h_\ell(s)$$

maximize

$\{\rho(x,y)\}$

$$F(s_0,t_0,u_0) \qquad (s_0,t_0,u_0) \in$$

Ill-posed problem & regularization

finding ρ given amplitude F constrained by unitarity in the physical region

data

$$F = \mathcal{K}\rho$$

Fredholm equation of first kind

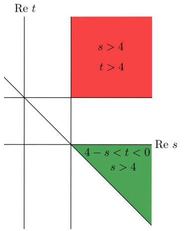
continuous kernel: ill-posed problem

a small error in the data can change the solution drastically

double dispersion: $$\mathcal{K}(s,t;x,y) \sim \frac{1}{(x-s)(y-t)}$$

$y > 4$

$4 - s < t < 0$



regularization: $$||\rho|| \leq M_{\mathrm{reg}}$$

suppress high oscillatory noise
with no physical interests

Generalized dispersion relation

$\Delta_{12}G(x,y)$

analytic function of two variables $$G(w_1,w_2) = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \frac{g(x,y)}{(x-w_1)(y-w_2)}$$

product of two such analytic functions: $$G(w_1,w_2) = H(w_1,w_2)K(w_1,w_2)$$

amplitude **dual amplitude**

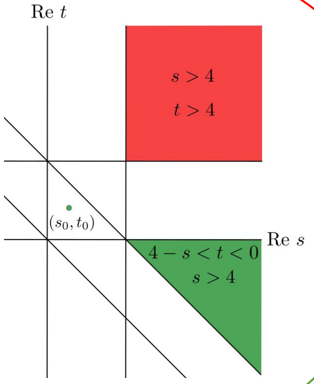
generalized dispersion relation:

$$\boxed{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, [\Delta_{12}H(x,y)K(x^-,y^+) - H(x^+,y^-)\Delta_{12}K(x,y)] = 0}$$

by choosing K, relate amplitude to physical region

primal → dual

$$\boxed{H(s,t) = \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \frac{\rho(x,y)}{(x-s)(y-t)}}$$



define dual partial waves:

$$k_\ell(s) = \frac{(2\ell+1)}{\pi^3}\sqrt{s(s-4)}\int_{-1}^{1} d\cos\theta P_\ell(\cos\theta) k(s,t)$$

regularization

$$M_{\text{reg}}||\text{Re}K||_*$$

$\leq$

$$H(s_0,t_0) = \frac{i}{\pi^2}\int_4^\infty dx \int_{4-x}^0 dy H(x^+,y)\bar{k}(x,y) - \frac{1}{\pi^2}\int_4^\infty dx \int_4^\infty dy \rho(x,y) K(x^-,y^+)$$

$\leq$

$$\sum_\ell \int_4^\infty ds \Big(|k_\ell(s)| - \text{Re}k_\ell(s)\Big)$$

unitarity

$$\boxed{K(s,t) = -\frac{1}{(s-s_0)(t-t_0)} + \frac{i}{\pi^2}\int_4^\infty dx \int_{4-x}^0 dy \frac{\bar{k}(x,y)}{(s-x)(t-y)}}$$

$$F(s_0,t_0,u_0) \leq \underbrace{\sum_{\ell\text{ even}} \int_4^\infty ds \Big(|k_\ell(s)| - \text{Re}k_\ell(s)\Big) + M_{\text{reg}}||\text{Re}K||_*}_{}$$

minimize

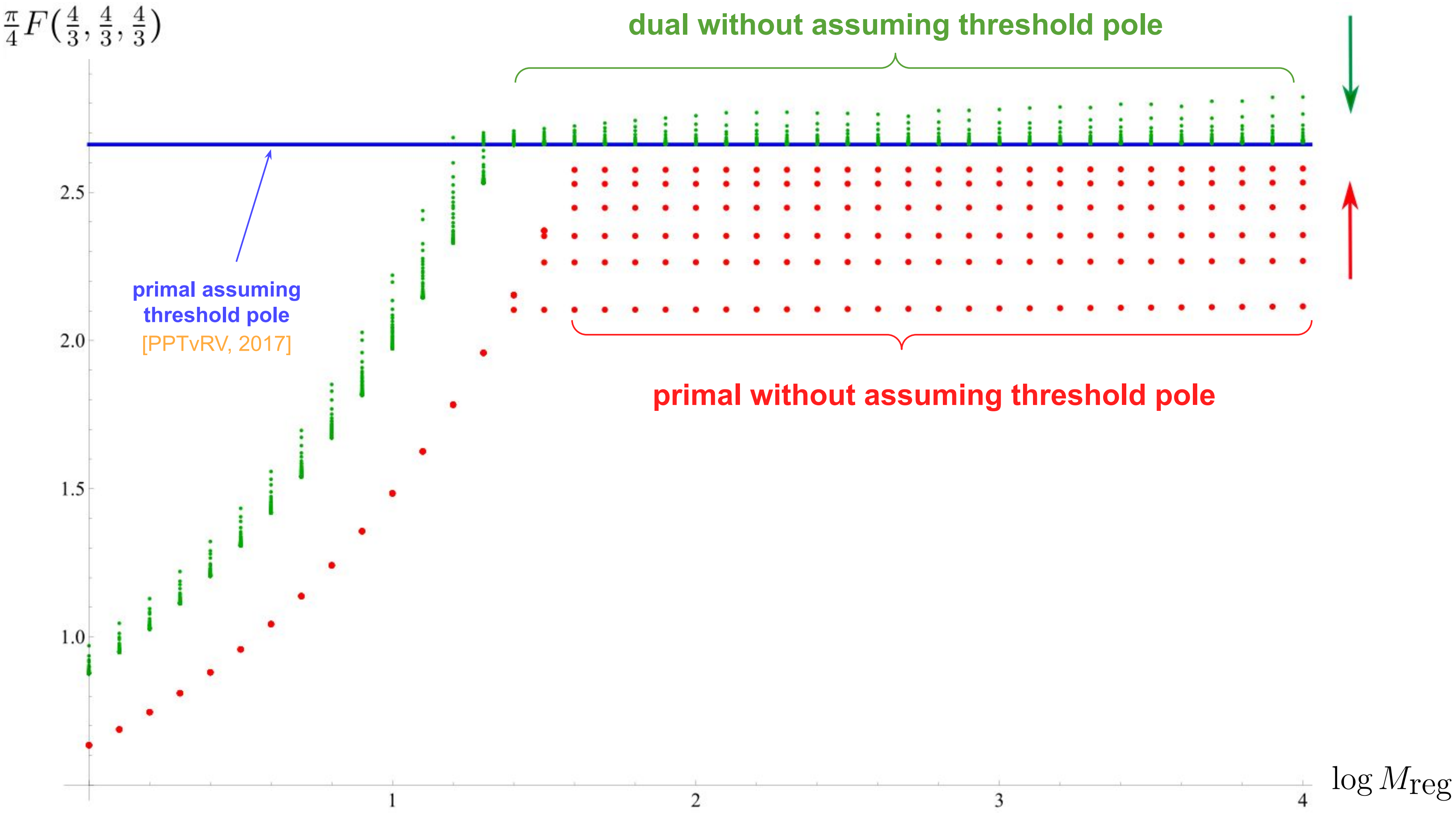
Single pion dual problem

$$\min_{\{k_\ell(s)\}} \mathcal{F}_D = \sum_{\ell \text{ even}} \int_4^\infty ds \Big(|k_\ell(s)| - \mathrm{Re} k_\ell(s) \Big) + M_{\mathrm{reg}} ||\mathrm{Re} K||_*$$

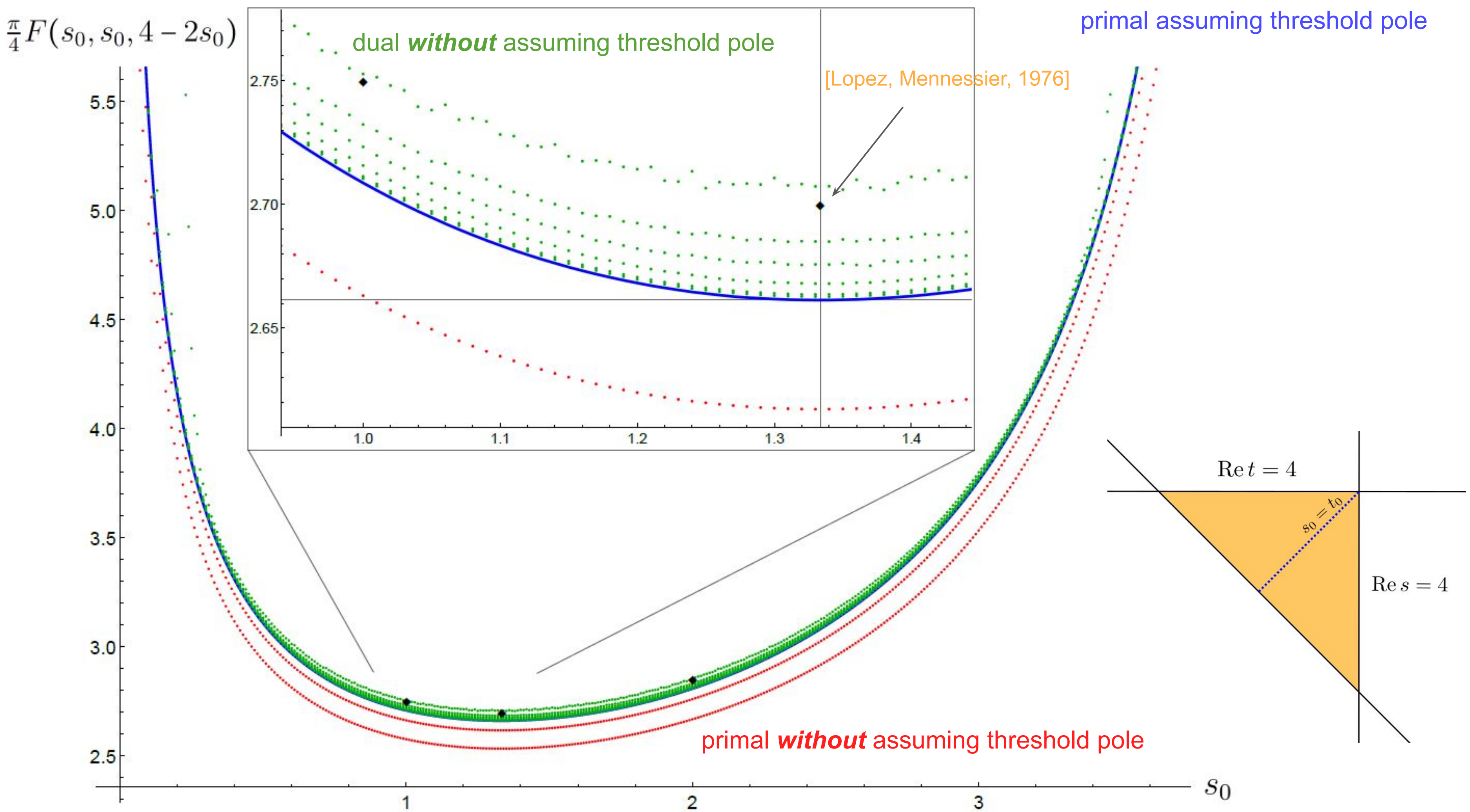
boundary of space of S-matrices

$$\boxed{h_\ell(s) = -i\left(\frac{k_\ell(s)}{|k_\ell(s)|} - 1\right)}$$

dual partial waves have the physical information



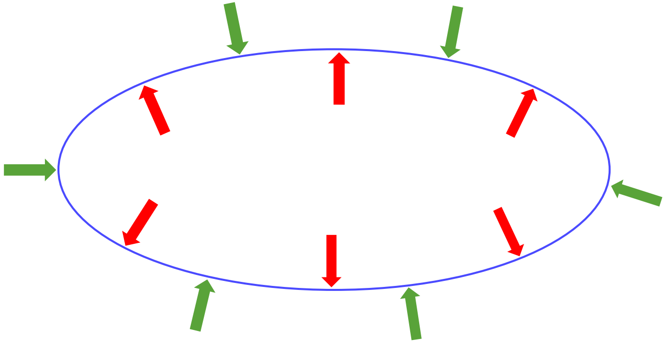
$\frac{\pi}{4}F(\frac{4}{3},\frac{4}{3},\frac{4}{3})$
dual without assuming threshold pole
primal assuming threshold pole
[PPTvRV, 2017]
primal without assuming threshold pole
2.5
2.0
1.5
1.0
1
2
3
4
$\log M_{\mathrm{reg}}$



$\frac{\pi}{4}F(s_0, s_0, 4-2s_0)$
dual ***without*** assuming threshold pole
primal assuming threshold pole
[Lopez, Mennessier, 1976]
primal ***without*** assuming threshold pole
$\mathrm{Re}\, t = 4$
$\mathrm{Re}\, s = 4$
$s_0 = t_0$
s_0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
1
2
3
2.75
2.70
2.65
1.0
1.1
1.2
1.3
1.4

Conclusions

Dual problem has practical advantages for S-matrix bootstrap



Useful tool to explore the space of S-matrices in 3+1 dimensions

Thank you !