Level Splitting of 10d Conformal Symmetry in $AdS_5 \times S^5$

Hynek Paul

based on work with Aprile, Drummond, Santagata:
[2012.12092]

and previous papers
[1907.00992], [2004.07282]

IRN:QFS kick-off meeting
Tours, June 2021

Motivation

- Simplicity and elegance of the VS-amplitude in flat space:

$$\mathcal{V} = \mathcal{R}^4 \times \frac{1}{stu} \frac{\Gamma\left(1-\frac{\alpha' s}{4}\right)\,\Gamma\left(1-\frac{\alpha' t}{4}\right)\Gamma\left(1-\frac{\alpha' u}{4}\right)}{\Gamma\left(1+\frac{\alpha' s}{4}\right)\Gamma\left(1+\frac{\alpha' t}{4}\right)\Gamma\left(1+\frac{\alpha' u}{4}\right)}$$

celebrated central object of early days of string theory

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$\rightarrow$ let us start with tree-level string theory on $AdS_5 \times S^5$

General Setup

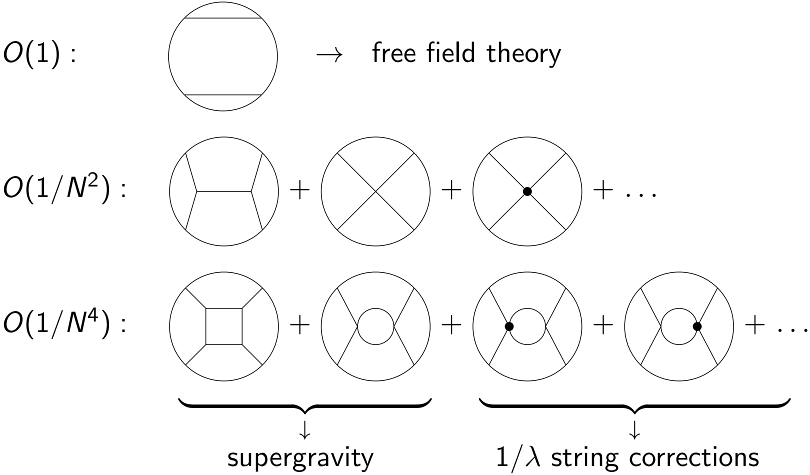
AdS/CFT correspondence

$\mathcal{N}=4$ SYM with gauge group $SU(N)$	$\Longleftrightarrow$	Type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$
4pt correlation functions	$\Longleftrightarrow$	AdS amplitudes (Witten diagrams)
strong coupling limit: $N\to\infty,\ \lambda\to\infty$	$\Longleftrightarrow$	supergravity limit: $g_s\to 0,\ \alpha'\to 0$

- Interested in corrections to supergravity in AdS_5: loop-corrections $(1/N)$ and string-corrections $(1/\lambda)$

General Setup

4pt-functions of single-particle operators $\mathcal{O}_p$: $\mathcal{H}_{p_1 p_2 p_3 p_4}$



Tree-level AdS amplitudes

Tree-level amplitudes are best studied in Mellin space

[Mack,Penedones,Rastelli-Zhou,...]

$$\mathcal{H}_{\vec{p}}(u,v;\sigma,\tau) = \oint ds dt\ u^s v^t \times \Gamma_{\vec{p}}(s,t) \times \mathcal{M}_{\vec{p}}(s,t;\sigma,\tau)$$

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Two main advantages:

1. Simplicity: $\mathcal{M}_{\vec{p}}$ is rational (supergravity)
   polynomial (string corrections)
2. Direct connection to flat-space amplitudes:
   - flat-space limit: $s,t \to \infty$ [Penedones'10]

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Amplitude admits a large N, large λ double-expansion:

$$\mathcal{M}_{\vec{p}} = \frac{1}{N^2}\left(\mathcal{M}^{(0)}_{\vec{p}} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(3)}_{\vec{p}} + \lambda^{-\frac{5}{2}}\mathcal{M}^{(5)}_{\vec{p}} + \ldots\right) + O\left(\frac{1}{N^4}\right)$$

and the set of exchanged double-trace operators $\mathcal{O}_{pq}$

$$\Delta_{pq} = \Delta^{(0)} + \frac{1}{N^2}\left(\eta^{(0)} + \lambda^{-\frac{3}{2}}\eta^{(3)} + \lambda^{-\frac{5}{2}}\eta^{(5)} + \ldots\right) + O\left(\frac{1}{N^4}\right)$$

The supergravity amplitude $\mathcal{M}^{(0)}$

Result for supergravity amplitude [Rastelli-Zhou'16'17]

$$\mathcal{M}^{(0)}_{\vec{p}} = \sum_{i+j+k=d_{34}-2} \frac{a_{ijk}\,\sigma^i \tau^j}{(s-s_0+2k)(t-t_0+2j)(u-u_0+2i)}$$

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1. All correlators descend from the seed-correlator $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$:

$$\mathcal{H}^{(0)}_{p_1p_2p_3p_4} = \mathcal{D}_{p_1p_2p_3p_4} \circ \mathcal{H}^{(0)}_{2222}$$

2. Predicts residual degeneracies for supergravity anomalous dimensions $\eta^{(0)}$

The supergravity amplitude $\mathcal{M}^{(0)}$

Consider more 'democratic' Mellin transform: [Aprile-Vieira'20]

$$\mathcal{H}_{\vec{p}}(u,v;\sigma,\tau) = \oint ds dt \oint d\tilde{s} d\tilde{t} \; u^s v^t \sigma^{\tilde{s}} \tau^{\tilde{t}} \times \Gamma_{\otimes} \times \mathcal{M}_{\vec{p}}(s,t;\tilde{s},\tilde{t})$$

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which makes hidden 10d symmetry manifest:

$$\mathcal{M}^{(0)}_{\vec{p}} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)}$$

with '10-dimensional' variables

$$\mathbf{s} = s + \tilde{s}, \qquad \mathbf{t} = t + \tilde{t}, \qquad \mathbf{s}+\mathbf{t}+\mathbf{u} = -4.$$

The double-trace spectrum

Need to address the mixing of double-trace operators:

$$\mathcal{O}_{pq} = \mathcal{O}_p \Box^{1/2(\tau-p-q)} \partial^\ell \mathcal{O}_q$$

are degenerate for all $(p,q) \in R_{\vec{\tau}}$.

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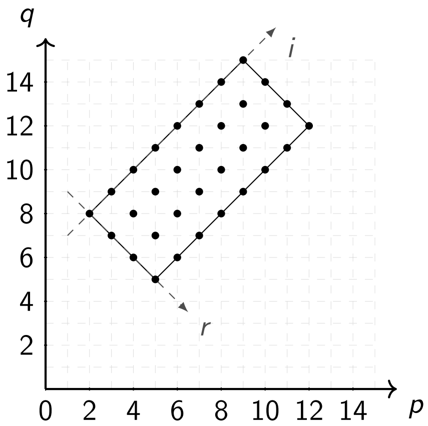

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Tedious computation... leads to a remarkably simple solution!

$$\eta^{(0)} = -2 \, \frac{M_t \, M_{t+\ell+1}}{(\ell_{10}(p)+1)_6}$$

[Aprile-Drummond-Heslop-HP'17'18]

with

- $M_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$
- effective 10d spin $\ell_{10}(p) = \ell + 2p - a - 3 - \frac{1+(-1)^{a+\ell}}{2}$
- idea: many 4d operators descend from the same 10d primary

The double-trace spectrum

Need to address the mixing of double-trace operators:

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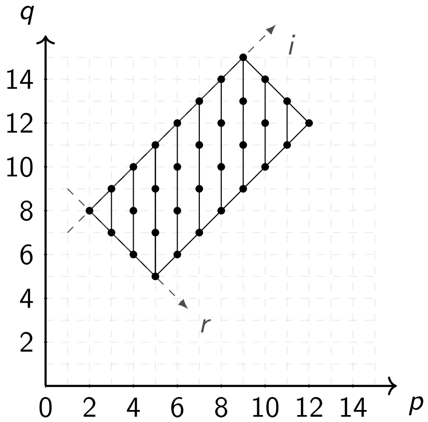
$\rightarrow \eta^{(0)}$ anomalous dimensions exhibit partial residual degeneracy!



Adding string corrections

$$\mathrm{AdS}_5 \times \mathrm{S}^5: \quad \mathcal{M}_{\vec{p}} = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)} + (\alpha')^3 \mathcal{M}^{(3)}_{\vec{p}} + (\alpha')^5 \mathcal{M}^{(5)}_{\vec{p}} + \dots$$

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$$\Downarrow \ \mathbf{s},\mathbf{t}\to\infty$$

$$\text{flat-space:} \qquad \mathcal{V} = \frac{1}{\mathbf{s}\,\mathbf{t}\,\mathbf{u}} + \frac{\zeta_3(\alpha')^3}{32} + \frac{\zeta_5(\alpha')^5}{1024}(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2) + \dots$$

$$\begin{array}{ccc} \Uparrow & \Uparrow & \Uparrow \\ \text{sugra} & \mathcal{R}^4 & \partial^4\mathcal{R}^4 \end{array}$$

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- $\mathcal{M}^{(3)}_{\vec{p}}$ is just a constant
- $\mathcal{M}^{(5)}_{\vec{p}}$ is at most quadratic in $(s,t,\tilde{s},\tilde{t})$
- $\mathcal{M}^{(6)}_{\vec{p}}$ is at most cubic in $(s,t,\tilde{s},\tilde{t})$
- ...

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- $(\alpha')^5$: $\partial^4\mathcal{R}^4 \quad \rightarrow \quad \ell_{10} = 0, 2$
- $(\alpha')^6$: $\partial^6\mathcal{R}^4 \quad \rightarrow \quad \ell_{10} = 0, 2$
- $(\alpha')^7$: $\partial^8\mathcal{R}^4 \quad \rightarrow \quad \ell_{10} = 0, 2, 4$
- $\ldots$

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String corrections successively lift the partial degeneracy:

$\Rightarrow$ level-splitting breaks the 10d conformal symmetry!

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First example: the order $(\alpha')^3$ correction

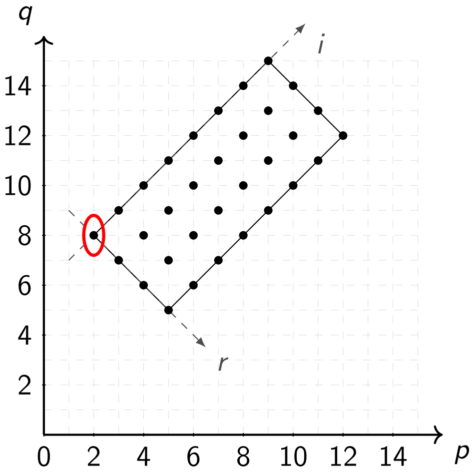
$\rightarrow$ amplitude fully fixed by the flat-space limit:

$$\mathcal{M}^{(3)}_{\vec{p}} = (\Sigma - 1)_3 \, \frac{\zeta_3}{4} \,, \quad \text{where} \;\; \Sigma = \tfrac{1}{2}(p_1 + p_2 + p_3 + p_4)$$

And indeed, one finds this contributes only to states with $\ell_{10} = 0$!

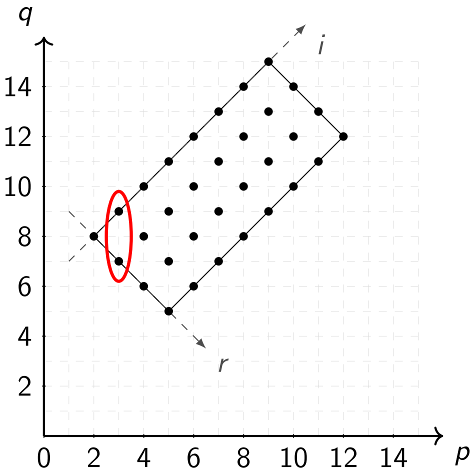
$(\alpha')^3$

$$\mathcal{R}^4 : \quad \ell_{10} = 0$$



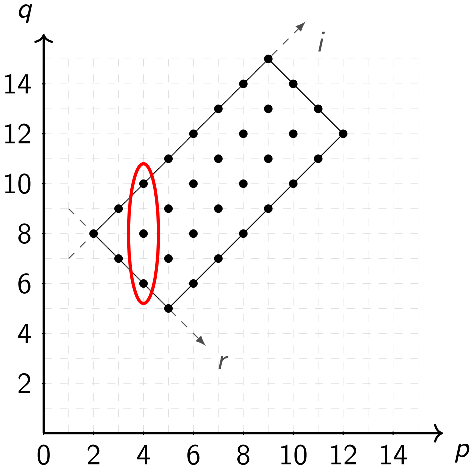
$(\alpha')^5$

$$\partial^4\mathcal{R}^4: \quad \ell_{10} = 0, 2$$



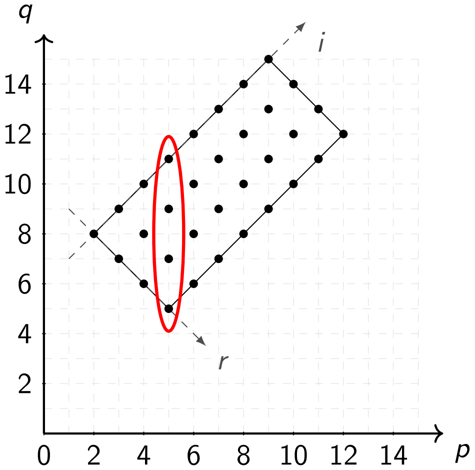
$(\alpha')^7$

$$\partial^8\mathcal{R}^4 : \quad \ell_{10} = 0, 2, 4$$



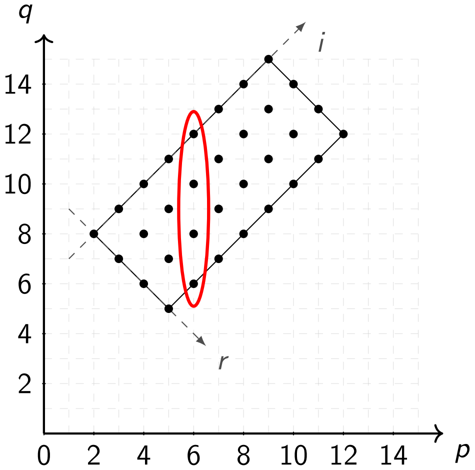
$(\alpha')^9$

$$\partial^{12}\mathcal{R}^4 : \quad \ell_{10} = 0, 2, 4, 6$$



$(\alpha')^{11}$

$$\partial^{16}\mathcal{R}^4 : \quad \ell_{10} = 0, 2, 4, 6, 8$$



Summary and Outlook

- String corrections break the 10d conformal symmetry and induce a level-splitting of the spectrum
- Using constraints on the spectrum (i.e. truncation of ℓ_{10}), we are able to bootstrap amplitudes at orders $(\alpha')^{5,6,7,8,9}$

  see also [Abl-Heslop-Lipstein'20]
- We derive lots of new CFT data by solving the level-splitting at levels 2 (all orders in α') and 3, 4 (for some low orders)

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Outlook and open questions:

- Flat-space amplitude is so simple: how to resum the AdS amplitude? Is there a better choice of variables? ...
- Wealth of new tree-level data can be used to bootstrap one-loop string corrections to higher orders

  [Alday,Drummond-HP,Drummond-Glew-HP]
- Would be interesting to make contact with recent strong coupling results from integrability (octagon function)

Definitions

Mellin space measure:

$$\Gamma_\otimes = \mathfrak{S}\,\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]\Gamma[-s+c_s]\Gamma[-t+c_t]\Gamma[-u+c_u]}{\Gamma[1+\tilde{s}]\Gamma[1+\tilde{t}]\Gamma[1+\tilde{u}]\Gamma[1+\tilde{s}+c_s]\Gamma[1+\tilde{t}+c_t]\Gamma[1+\tilde{u}+c_u]}$$

where

$$\mathfrak{S} = \pi^2 \frac{(-)^{\tilde{t}}(-)^{\tilde{u}}}{\sin(\pi\tilde{t})\sin(\pi\tilde{u})}$$

Further conventions:

$$\mathbf{s} = s + \tilde{s} \qquad \mathbf{t} = t + \tilde{t} \qquad \mathbf{s} + \mathbf{t} + \mathbf{u} = -4$$

$$\tilde{\mathbf{s}} = c_s + 2\tilde{s} \qquad \tilde{\mathbf{t}} = c_t + 2\tilde{t} \qquad \tilde{\mathbf{s}} + \tilde{\mathbf{t}} + \tilde{\mathbf{u}} = \Sigma - 4$$

$$c_s = \frac{p_1+p_2-p_3-p_4}{2} \qquad c_t = \frac{p_1+p_4-p_2-p_3}{2} \qquad c_u = \frac{p_2+p_4-p_3-p_1}{2}$$

$$\Sigma = \frac{p_+ p_2 + p_3 + p_4}{2}$$

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1. Start with a fully crossing symmetric ansatz

$$\mathcal{M}^{(n)}_{\vec{p}} = \# \, (\Sigma - 1)_n \, \mathcal{V}^{(n)}_{\text{flat}}(\mathbf{s}, \mathbf{t}, \mathbf{u}) + \sum_{i=0}^{n-1} (\Sigma - 1)_{i+3} \left\{ \sum_{0 \le d_1 + d_2 \le i} C^{(n,i)}_{d_1,d_2}(\tilde{s}, \tilde{t}, \vec{p}) \, \mathbf{s}^{d_1} \mathbf{t}^{d_2} \right\}$$

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2. Extract partial-wave coefficients and solve rank-constraints
3. Leftover ambiguities do not affect the level-splitting

Results at $(\alpha')^5$

Ansatz:

$$\begin{aligned}\mathcal{M}^{(5)}_{\vec{p}} = \ & (\Sigma-1)_5\ (\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2)\\ &+(\Sigma-1)_4\ k_{4,1}(\mathbf{s}\tilde{\mathbf{s}}+\mathbf{t}\tilde{\mathbf{t}}+\mathbf{u}\tilde{\mathbf{u}})\\ &+(\Sigma-1)_3\ \big(k_{3,1}\Sigma^2+k_{3,2}(c_s^2+c_t^2+c_u^2)+k_{3,3}\left(\tilde{\mathbf{s}}^2+\tilde{\mathbf{t}}^2+\tilde{\mathbf{u}}^2\right)\\ &\qquad\qquad +k_{3,4}\Sigma+k_{3,5}\big)\end{aligned}$$

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& + (\Sigma - 1)_4 \ k_{4,1}(\mathbf{s}\tilde{\mathbf{s}} + \mathbf{t}\tilde{\mathbf{t}} + \mathbf{u}\tilde{\mathbf{u}}) \\
& + (\Sigma - 1)_3 \ \big(k_{3,1}\Sigma^2 + k_{3,2}(c_s^2 + c_t^2 + c_u^2) + k_{3,3}\,(\tilde{\mathbf{s}}^2 + \tilde{\mathbf{t}}^2 + \tilde{\mathbf{u}}^2) \\
& \qquad + k_{3,4}\Sigma + k_{3,5}\big)
\end{aligned}
$$

Rank constraints give

$$
k_{4,1} = -5, \quad k_{3,3} = 5, \quad k_{3,2} - k_{3,1} = 11, \quad k_{3,4} = 0
$$

Leaving two ambiguities:

$k_{3,5}$ (constant) and $k_{3,1}$ $(\Sigma^2 + c_s^2 + c_t^2 + c_u^2)$

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$\rightarrow$ fixed by supersymmetric localisation results:

[Binder-Chester-Pufu-Wang'19,Chester-Pufu'20]

$$k_{3,1}=-\tfrac{27}{2},\quad k_{3,5}=\tfrac{33}{2}$$

Results at $(\alpha')^6$

Ansatz:

$$\begin{aligned}
\mathcal{M}^{(6)}_{\vec{p}} = \; & (\Sigma-1)_6 \; \tfrac{2}{3}(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3) \\
& + (\Sigma-1)_5 \; \Big(k_{5,1}\left(\mathbf{s}^2\tilde{\mathbf{s}}+\mathbf{t}^2\tilde{\mathbf{t}}+\mathbf{u}^2\tilde{\mathbf{u}}\right) + \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\left(\Sigma k_{5,2}+k_{5,3}\right) \Big) \\
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& \qquad + \left(\Sigma k_{4,3}+k_{4,4}\right)\left(\mathbf{s}\tilde{\mathbf{s}}+\mathbf{t}\tilde{\mathbf{t}}+\mathbf{u}\tilde{\mathbf{u}}\right) \Big) \\
& + (\Sigma-1)_3 \; \Big(k_{3,1}(\tilde{\mathbf{s}}^3+\tilde{\mathbf{t}}^3+\tilde{\mathbf{u}}^3) + k_{3,2}\left(c_s^2\tilde{\mathbf{s}}+c_t^2\tilde{\mathbf{t}}+c_u^2\tilde{\mathbf{u}}\right) + \ldots \Big)
\end{aligned}$$

Results at $(\alpha')^6$

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& + (\Sigma-1)_4 \; \Big(k_{4,1} \left(\mathbf{s}\tilde{\mathbf{s}}^2+\mathbf{t}\tilde{\mathbf{t}}^2+\mathbf{u}\tilde{\mathbf{u}}^2\right) + k_{4,2}\left(\mathbf{s}c_s^2+\mathbf{t}c_t^2+\mathbf{u}c_u^2\right) \\
& \qquad\qquad + \left(\Sigma k_{4,3}+k_{4,4}\right)\left(\mathbf{s}\tilde{\mathbf{s}}+\mathbf{t}\tilde{\mathbf{t}}+\mathbf{u}\tilde{\mathbf{u}}\right) \Big) \\
& + (\Sigma-1)_3 \; \Big(k_{3,1}(\tilde{\mathbf{s}}^3+\tilde{\mathbf{t}}^3+\tilde{\mathbf{u}}^3) + k_{3,2}\left(c_s^2\tilde{\mathbf{s}}+c_t^2\tilde{\mathbf{t}}+c_u^2\tilde{\mathbf{u}}\right) + \dots \Big)
\end{aligned}
$$

Rank constraints give

$$
\begin{array}{lll}
k_{5,1} = -6 & k_{3,5} = \frac{55}{3} - \frac{5}{32}k_{3,10} & k_{4,4} = -\frac{4}{3} - 5k_{5,3} - \frac{1}{8}k_{3,10} \\
k_{4,1} = +15 & k_{4,2} = -\frac{7}{3} - \frac{1}{32}k_{3,10} & k_{3,9} = \frac{10}{3} + \frac{5}{16}k_{3,10} + 5k_{5,3}, \\
k_{3,1} = -10 & k_{3,2} = \frac{14}{3} + \frac{1}{16}k_{3,10} & k_{3,4} = -\frac{7}{3} - \frac{1}{32}k_{3,10} \\
k_{5,2} = 4 & k_{4,3} = -\frac{58}{3} + \frac{1}{16}k_{3,10} & k_{3,7} = -\frac{22}{3} - \frac{11}{16}k_{3,10} + k_{3,8} - 11k_{5,3} \\
k_{3,6} = 0 & k_{3,3} = -32 + \frac{1}{8}k_{3,10} &
\end{array}
$$

Leaving three ambiguities

Results at $(\alpha')^6$

Ansatz:

$$\begin{aligned}\mathcal{M}^{(6)}_{\vec{p}} = \ & (\Sigma-1)_6\ \tfrac{2}{3}(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3) \\ & + (\Sigma-1)_5\ \Big(k_{5,1}\left(\mathbf{s}^2\tilde{\mathbf{s}}+\mathbf{t}^2\tilde{\mathbf{t}}+\mathbf{u}^2\tilde{\mathbf{u}}\right)+\left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\left(\Sigma k_{5,2}+k_{5,3}\right)\Big) \\ & + (\Sigma-1)_4\ \Big(k_{4,1}\left(\mathbf{s}\tilde{\mathbf{s}}^2+\mathbf{t}\tilde{\mathbf{t}}^2+\mathbf{u}\tilde{\mathbf{u}}^2\right)+k_{4,2}\left(\mathbf{s}c_s^2+\mathbf{t}c_t^2+\mathbf{u}c_u^2\right) \\ & \qquad\qquad + \left(\Sigma k_{4,3}+k_{4,4}\right)\left(\mathbf{s}\tilde{\mathbf{s}}+\mathbf{t}\tilde{\mathbf{t}}+\mathbf{u}\tilde{\mathbf{u}}\right)\Big) \\ & + (\Sigma-1)_3\ \Big(k_{3,1}(\tilde{\mathbf{s}}^3+\tilde{\mathbf{t}}^3+\tilde{\mathbf{u}}^3)+k_{3,2}\left(c_s^2\tilde{\mathbf{s}}+c_t^2\tilde{\mathbf{t}}+c_u^2\tilde{\mathbf{u}}\right)+\ldots\Big)\end{aligned}$$

and localisation gives

$$k_{3,8}=\tfrac{4}{3}-\tfrac{1}{16}k_{3,10} \qquad k_{3,11}=0 \qquad k_{5,3}=-2$$

leaving one final ambiguity.

Results at higher orders

- Similarly, we obtain results at orders $(\alpha')^7$, $(\alpha')^8$, $(\alpha')^9$

  with a growing number of ambiguities: 10, 16, 33

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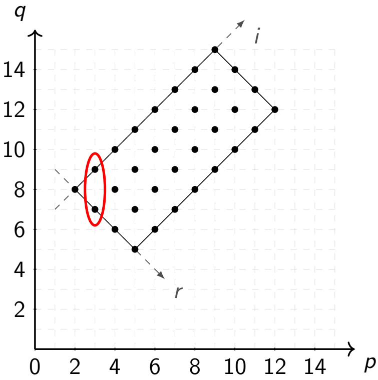

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For example: at level two, we obtain the characteristic polynomial

$$(\tilde{\eta} + r)^2 + (\tilde{\eta} + r)\,\gamma_{2,1} + \gamma_{2,0} = 0$$

where

$$r = (T - B)^2 + B(2 + l) + (2 + a)T$$

$$\gamma_{2,1} = -\tfrac{(n+2)(n+3)}{2n+5}\big(B(2l+5) + (2a+5)T - (a+2)(l+2)\big)$$

$$\gamma_{2,0} = \tfrac{(n+2)^2(n+3)^2}{2n+5}BT$$

which has an interesting $\mathrm{AdS}_5 \leftrightarrow \mathrm{S}^5$ symmetry: $B \leftrightarrow T$ & $a \leftrightarrow \ell$