

Conformal Bootstrap:

Numerical & Analytical

Developments

Slava Rychkov (I.H.E.S., Bures-sur-Yvette
& ENS Paris)

International Research Network "Quantum Fields & Strings"
Inaugural Meeting, Federation Denis Poisson, Tours (virtual)
9.06.2021

Tours



© WorldAtlas.com

MÉMOIRE

Sur le calcul numérique des Intégrales définies.

PAR M. POISSON.

Lu à l'Académie des Sciences, le 11 décembre 1826.

Le calcul des intégrales définies est peut-être la partie de l'analyse dont les applications sont les plus nombreuses et les plus variées. Non-seulement elles comprennent la rectification des courbes, l'évaluation des surfaces et des solides, la détermination des centres de gravité, mais encore, la plupart des problèmes de mécanique ou de physique qui se résolvent par le calcul intégral, conduisent à des expressions des inconnues en intégrales définies. Aussi, depuis Euler et surtout dans ces derniers temps, les géomètres se sont-ils beaucoup occupés d'étendre et de perfectionner cet important calcul. Dans le petit nombre de cas où l'intégrale générale est connue sous forme finie, on en déduit immédiatement l'intégrale définie; dans d'autres cas, beaucoup plus étendus, on parvient à trouver la valeur exacte de l'une sans connaître celle de l'autre; mais le plus souvent on est

Bootstrap = analyze the problem from general principles (not from microscopic equations)

Why ?

-) because micro eqns are hard to solve (strongly coupled, no SUSY, no integrability)
- or ~) because interested in general constraints on all landscape of possible solutions

System

constraints

2d minimal
models

(and other rational CFTs)

'80s -

Liouville

- Virasoro
 - crossing
- $$\text{Y} = \text{Y}$$
- unitarity
 - modular invariance

+ guesses from
micro

2d integrable
massive S-matrices

'70s

$$\cancel{\text{X}} = \cancel{\text{X}}$$

System

constraints

RECENT

3

CFTs in $d > 2$

$$\bullet \text{SO}(d+1,1) \rightarrow \leftarrow = \times$$

- unitarity
- other global symmetries & SUSY (if present)
- gaps

CONFORMAL
BOOTSTRAP

focus today

General 2d CFTs

modular invariance

MODULAR
BOOTSTRAP

General $d > 2$
massive S-matrices

- analyticity
- unitarity

S-MATRIX
BOOTSTRAP

Matrix models

Schwinger-Dyson eqns
+ positivity

Lin 2020

Han-Hartnoll, Kruthoff 2020

CFTs in \mathbb{R}^d , $d \geq 2$

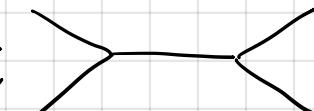
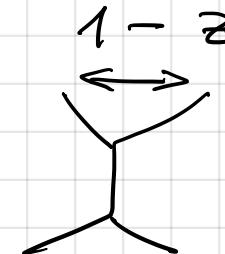
- spectrum of primaries \mathcal{O}_i Δ_i scaling dimension
 \mathfrak{g}_i $SO(d)$ representation
(+ global, SUSY, etc, if any)

- OPE $\mathcal{O}_i \times \mathcal{O}_j \sim \sum_{\text{fixed}} f_{ijk} \times (\text{coord-dep.}) \times \mathcal{O}_k$

- unitarity $\Delta_i \geq \Delta_{\min}(\mathfrak{g}_i)$, $f_{ijk} \in \mathbb{R}$

- crossing $\sum \begin{array}{c} \nearrow \\ \searrow \end{array} \mathcal{O}_K = \sum \begin{array}{c} \searrow \\ \nearrow \end{array} \mathcal{O}_K$

Perfect bootstrap arena

Difficulty : $z \uparrow$  = 

- S - channel converges fast when $z \rightarrow 0$
- to compare with cross - channel need $z \rightarrow 1$
- individual $SO(d+1, 1)$ conf. blocks $\sim \log(1-z)$
 $\hookrightarrow \infty$ - many blocks needed

Strategy 1

Work at mid-point $z = \gamma/2$

6

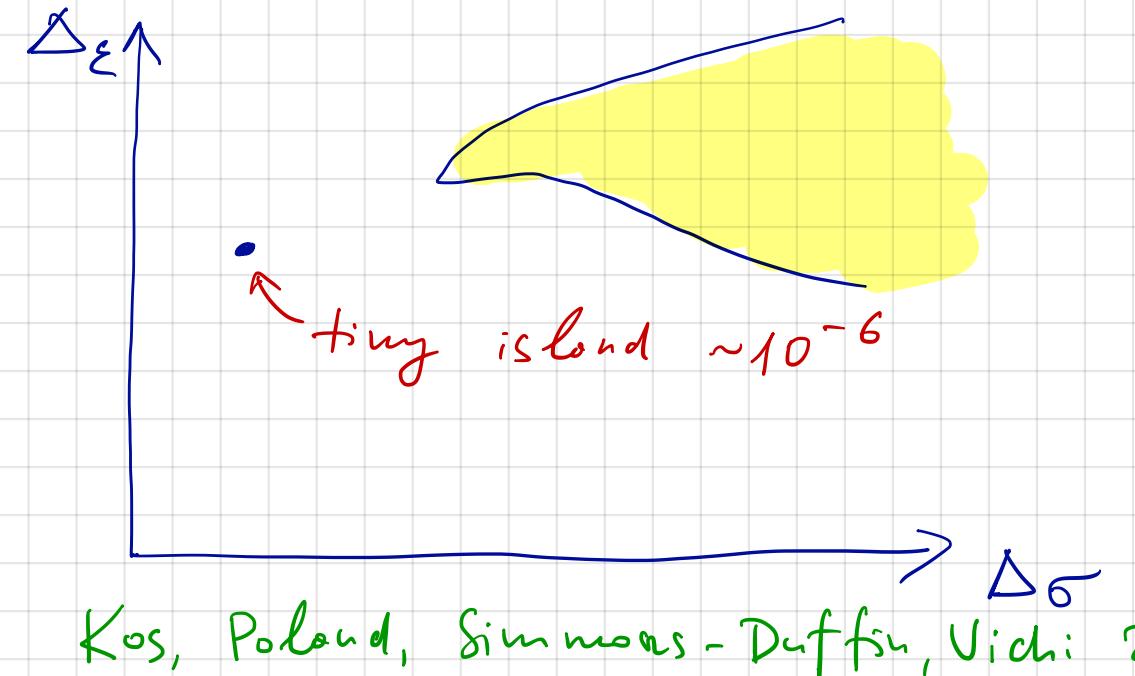
Rattazzi, S.R., Tonni, Vichi 2008

- + Both s, t channels converge exponentially fast
Pappadopulo, S.R., Espil, Rattazzi
- + Sensitive to $\Delta = O(1)$ (pheno most interesting)
- Crossing satisfied by canceling many independent-looking terms
- + Systematically improvable numerics possible
 \rightsquigarrow some miraculous results

Hidden structure?

Example 1

- unitary
- \mathbb{Z}_2 global symmetry
- $\Delta\sigma, \Delta\varepsilon < 3$; all others > 3



3d CFT

\mathbb{Z}_2 global symmetry

$\Delta\sigma, \Delta\varepsilon < 3$; all others > 3

$$\begin{array}{c|cc} & \mathbb{Z}_2 & \\ \hline \sigma & - & \\ \varepsilon & + & \end{array}$$

IR fixed point of
non-SUSY RG flow

$$S(\partial\varphi)^2 + M_C\varphi^2 + \lambda\varphi^4$$

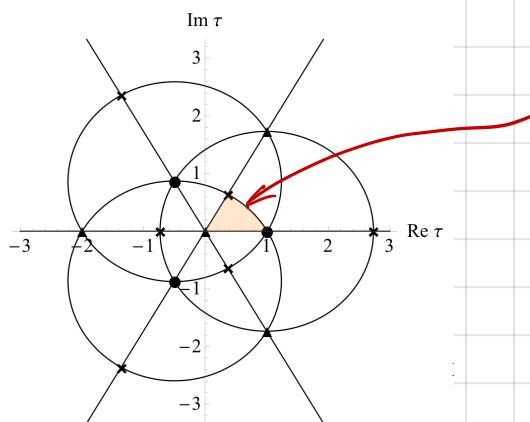
[3D Ising CFT]

[from crossing for $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\varepsilon\varepsilon\rangle$, $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$]

Example 2 | $N=2$ 3d SUSY model

$$W = h_1 X_1 X_2 X_3 + \frac{1}{6} h_2 (X_1^3 + X_2^3 + X_3^3)$$

- $\tau = \frac{h_2}{h_1} \in \mathbb{C}\mathbb{P}^1$ - conformal manifold
 \hookrightarrow use via chiral ring rels $\frac{\tau}{2} X_1^2 + X_2 X_3 = 0$ etc



$\mathbb{C}\mathbb{P}^1/S^1$ (discrete symms)

- Exact SUSY predictions $[X_i] = 2/3$, $[X_i X_j] = 4/3$
- Localization predictions for $\langle T_{\mu\nu} T_{\lambda\sigma} \rangle$
- $X_i \bar{X}_j$ unprotected \rightsquigarrow can be bootstrapped

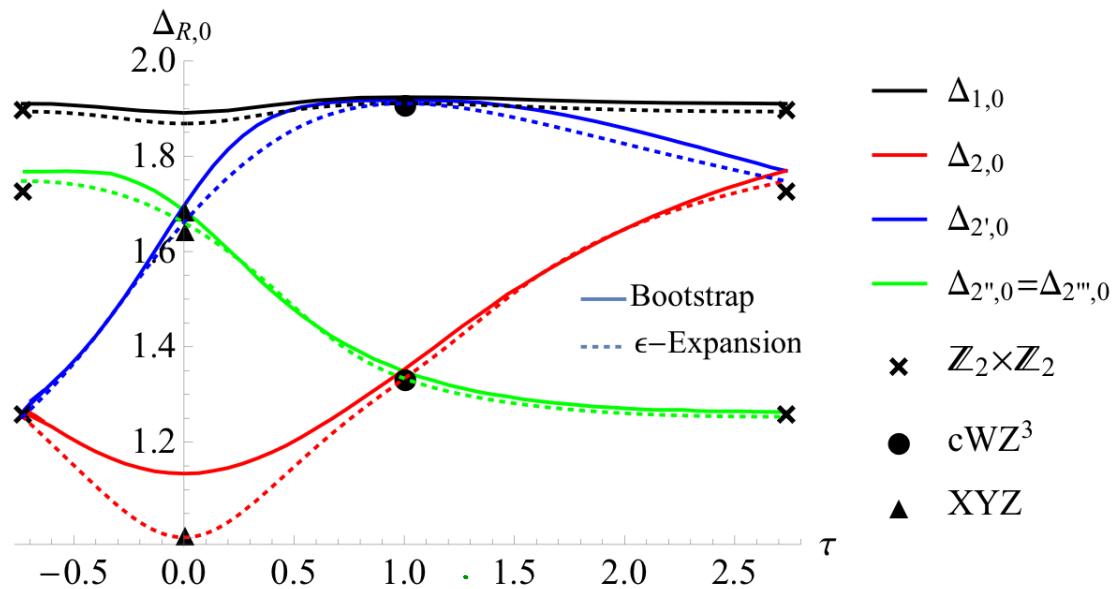
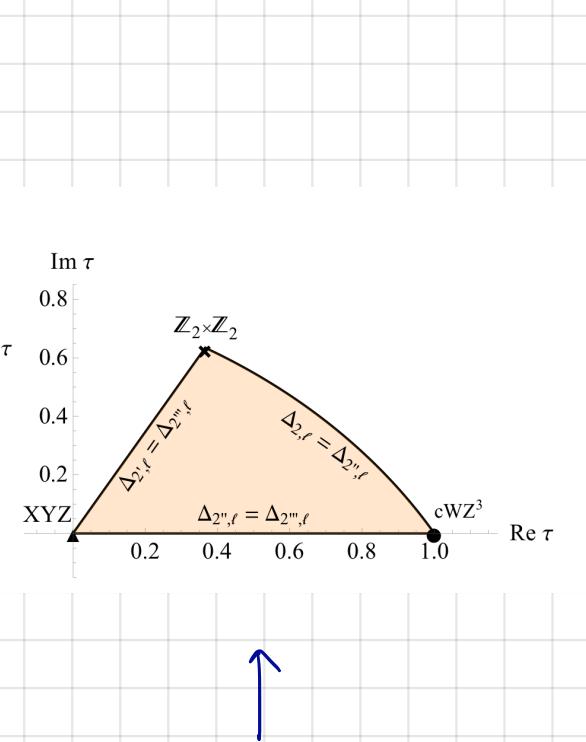


Figure 5: Upper bounds on the unprotected scaling dimensions of the scalar singlet and doublets for real $1 - \sqrt{3} \leq \tau \leq 1 + \sqrt{3}$, computed using the $G \rtimes \mathbb{Z}_2$ flavor symmetry crossing equations. The cross, circle, and triangle denote the results from the previous sections for the enhanced symmetry points $\tau = 1 \pm \sqrt{3}, 1, 0$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$, cWZ³, and XYZ models respectively. (For the XYZ model, the top and bottom triangles correspond to the doublets while the middle one corresponds to the singlet. See also Table 2 in the Discussion section.) The dotted lines show the 3-loop resummed 4- ϵ -expansion results. These bounds were computed with $\Lambda = 19$.



↑
conformal
manifold
(fundamental domain)

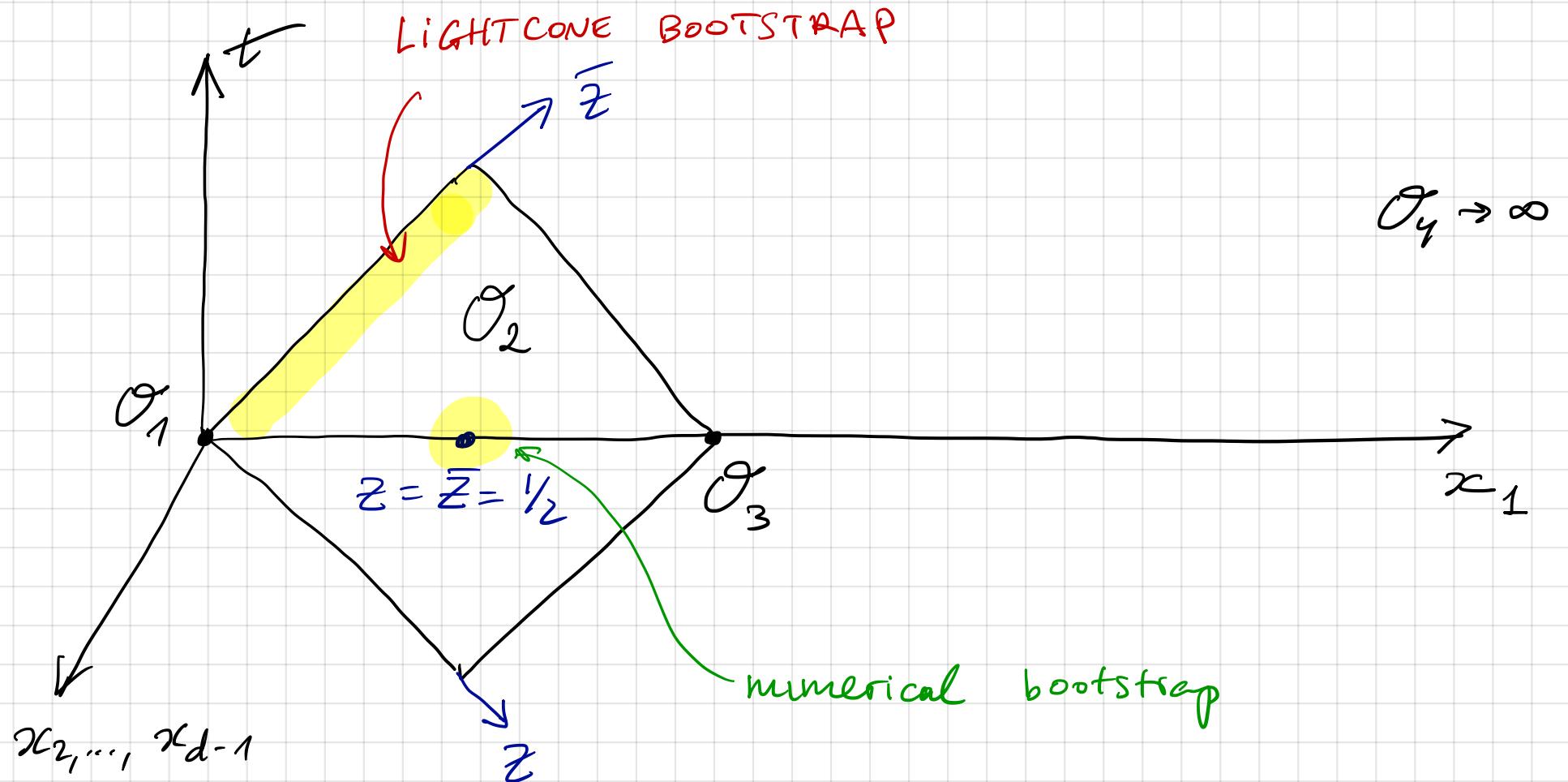
STRATEGY 2

Work at $z \ll 1$

Fitzpatrick, Kaplan, Poland, Simmons-Duffin 2012
 Komargodski, Zhiboedov 2012

- ∞ -many crossed-channel primaries — both a complication and a bonus
- $z \rightarrow 0, \bar{z} \rightarrow \text{const}$ (Lorentzian) **LIGHTCONE BOOTSTRAP**
- Sensitive to twist $\mathcal{T} = \Delta - \ell = O(1)$
 $\ell \rightarrow \infty$ (naively), but works for $\ell = O(1)$ as well
- organizes operators in families (Regge trajectories)
- analytic computations
- error bars nonrigorous, but agrees with numerics pretty well

Idea



Typical result

In any CFT_d, d > 2, OPE

$\varphi \times \varphi$ contains an ∞ family of operators

$$\mathcal{O}_l \quad l = 0, 2, 4, \dots$$

of $\Delta = l + 2\Delta_\varphi + \mathcal{O}\left(\frac{1}{\ell^\#}\right)$

Remark It would be true in AdS,

'double - trace operators' $\sim \varphi \partial^l \varphi$

→ 'double - twist family' [Simmons - Duffin]

- NB:
- Only true in $d > 2$
 - Crucial difference between $d = 2$ and $d > 2$
twist sap

	Δ	ℓ	$\tilde{\tau} = \Delta - \ell$
$T_{\mu\nu}$	0	0	0
	d	2	$d - 2 > 0 \text{ iff } d > 2$

Comparison numerical \leftrightarrow lightcone bootstrap

spins, and verify agrees to for high spins.

14

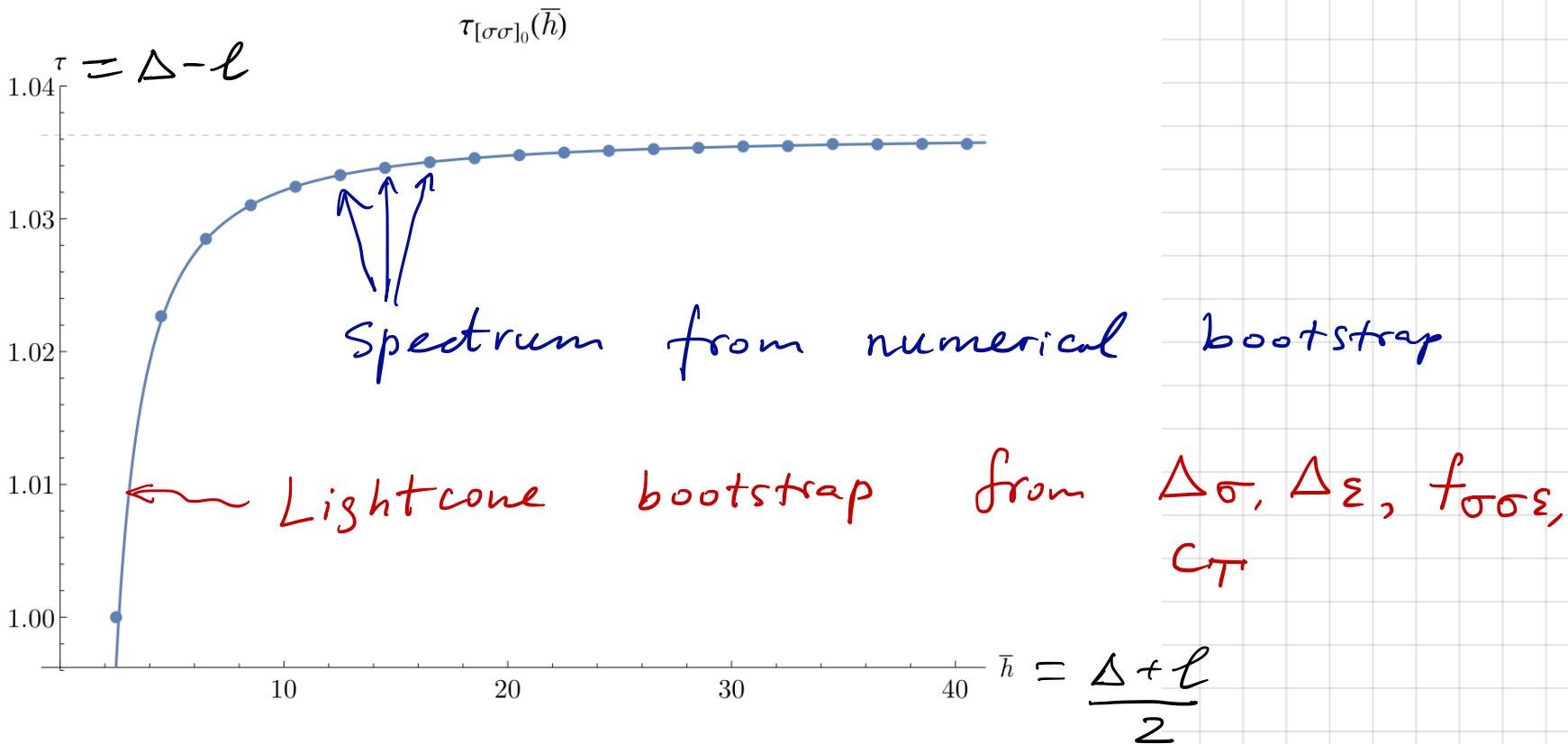


Figure 7: A comparison between the analytical prediction (6.5) (blue curve) and numerical data (blue dots) for $\tau_{[\sigma\sigma]_0}$. The two agree with accuracy 3×10^{-3} and 5×10^{-4} for spins $\ell = 2, 4$, respectively, and $\sim 5 \times 10^{-5}$ for $\ell > 4$. The grey dashed line is the asymptotic value $\tau = 2\Delta_\sigma$. The curve (2.3) from [1] looks essentially the same.

Simmons - Duffin 1612.08471

Lorentzian inversion formula

Caron-Huot 1703.00278 45

$$C(\Delta, \ell) = \int_0^1 dz \int_0^1 d\bar{z} \mathcal{F}(\Delta, \ell, z, \bar{z}) d\text{Disc } G(z, \bar{z})$$



$$d\text{Disc } f(z) = f - \frac{1}{2} (f^\Theta + f^\Lambda)$$

$$C(\Delta, \ell) \sim \frac{f_{ijk}^2}{\Delta - \Delta_K}$$

- "Explains" why operators form Regge trajectories
- Recovers easily lightcone bootstrap results.

Remark

Although lightcone bootstrap & OPE inversion formulas are "analytic", they are not as rigorous as the numerical bootstrap. [Have not been derived from "Euclidean bootstrap axioms"]

Kravchuk, S.R., Qiao 2104.02090

Let's discuss another approach, which is both analytic and rigorous

Analytic functionals

Mazac 1611, 10060
Mazac, Paulos 2018

1d case

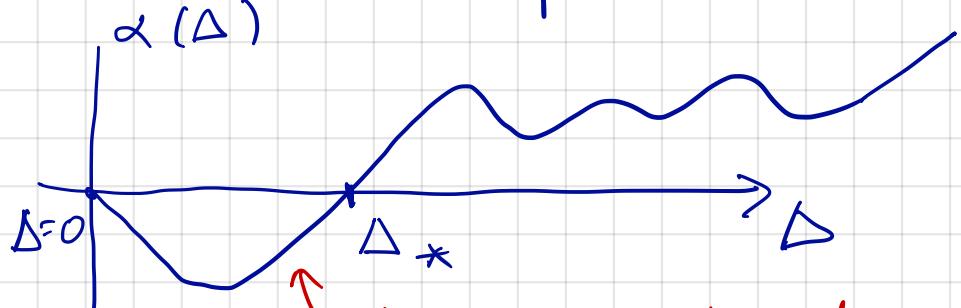
Bootstrap eq: $\sum_{\Delta \geq 0} p_{\Delta} F_{\Delta}(z) = 0 \quad (*)$

← s-channel - t-channel
conformal block.
[known functions]

Functional $\alpha: F(z) \rightarrow \mathbb{R}$

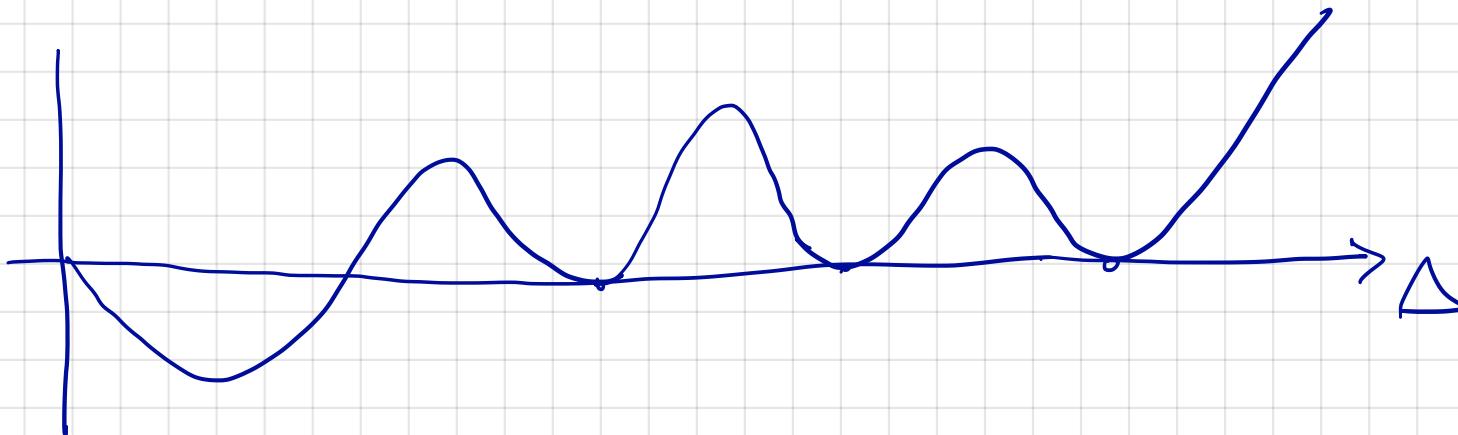
$$(*) \Rightarrow \sum_{\Delta} p_{\Delta} \alpha[F_{\Delta}] = 0$$

Numerical bootstrap $\alpha[F] = \sum_{n=0}^N c_n F^{(n)}(1/2)$



there must be an operator in this range

Better numerical functional:

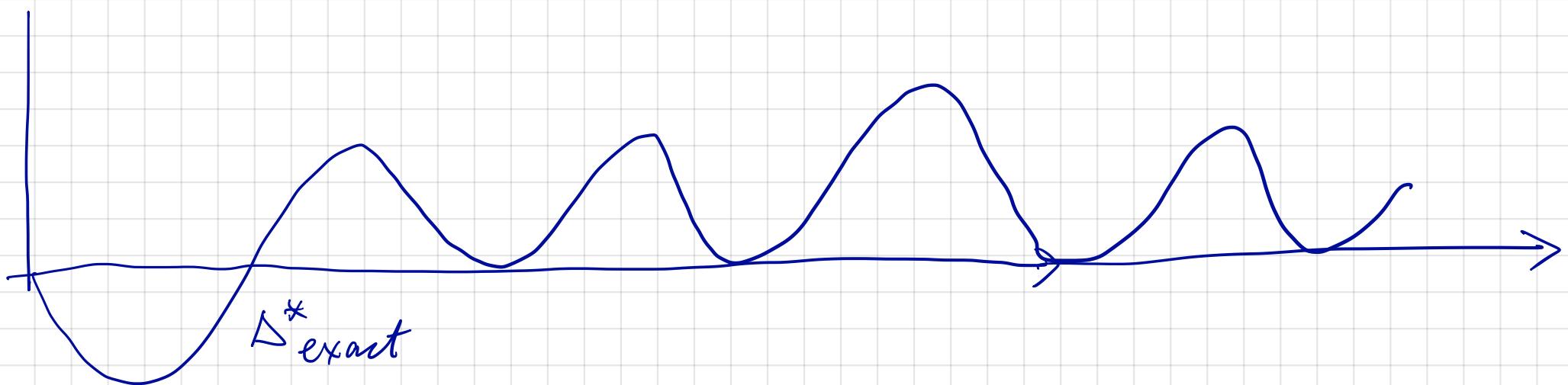


Hypothetical

extremal

functional:

El-Shawak, Paulos
1211, 2810



- Such extremal functionals were constructed (in 1d)¹⁹
with integer spacing of zeroes

Mazac 1611.10060
Mazac, Paulos 2018

- They have the form:

$$\alpha[F] = \oint_{\text{particular contour}} F(z) h(z) dz$$

↑ particular function

\Rightarrow Analytic proof of some numerical bootstrap bounds & new results

Work in $d > 1$ ongoing

-

Paulos 1910.08563
Mazac 1910.12855
Caron-Huot, Mazac, Rastelli, Simmons-Duffin 2008.09931

Topics not mentioned :

- Polyakov - Mellin bootstrap
- Tauberian theory
- AdS / large N
- Boundary/defect CFT in $d > 2$
- Bootstrap on $S^1 \times \mathbb{R}^{d-1}$ ("thermal circle")