Functional Renormalization Group applied to Dark Matter

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Outline

Dark Matter

- At large scale, interactions are dominated by gravitation (general relativity)
- Observed dynamics of large structures do not fit theoretical predictions (85% of matter are missing)
- $\Rightarrow$ hypothesis : there is a non-detected (dark) matter
- assumption : DM is a ideal stochastic gas

Field description

⇒ DM dynamics can be written in term of 2 scalar fields (Fourier mods) :

$$\phi(\mathbf{q},\eta) = \begin{pmatrix} \phi_1(\mathbf{q},\eta) \\ \phi_2(\mathbf{q},\eta) \end{pmatrix} \quad (1)$$

- ϕ_1 : density perturbation
- ϕ_2 : velocity divergence
- $\eta = \log a$ (scale factor) : time (today : $\eta = 0$)

Microscopic action

Action of a stochastic fluid

$$S[\phi,\chi] = \frac{i}{2}\int_{\mathbf{q}} \chi_A P^0_{AB}\chi_B + \int_\eta \chi_A(\delta_{AB}\partial_\eta + \Omega_{AB})\phi_B - \int_{\mathbf{pr}} \gamma_{ABC}\chi_A\phi_B\phi_C \tag{2}$$

- ϕ : "observable" field
- χ : response field (conjugate of ϕ)
- P^0 : (initial) power spectrum

Functional Renormalization group

- RG : studies how varies a theory (i.e. the action) with the scale of observation
- fRG : introduction of an effective action Γ_k that describes the physics between UV-cutoff Λ and IR-cutoff k
- Λ is large such that $\Gamma_\Lambda = S$
- Wetterich equation : flow of Γ_k in function of k.

Schwinger functional

$$W \equiv -i \log Z \tag{3}$$

with $Z = \int_{\phi\chi} e^{iS[\phi,\chi]+J_a\phi_a+K_b\chi_b}$

By taking the derivatives w.r.t. the sources J and K, we obtain the expected values of the fields $\langle\phi_A\rangle$ and $\langle\chi_A\rangle$ and the connected correlation functions $\langle\phi_A\phi_B\rangle_c$, etc...

Effective action

Legendre transform of W in respect to the sources

$$\boxed{\Gamma[\phi,\chi] = \sup_{J,K} \int_{\eta,\mathbf{q}} \left(J_a\langle\phi_a\rangle + K_b\langle\chi_b\rangle\right) - W[J,K]} \tag{4}$$

- Γ takes into the account the initial fluctuations with all wavelengths $\mathbf{q}$
- the dynamic equation is non-linear $\Rightarrow$ large fluctuations are hard to compute
- $\Rightarrow$ we restrict the integration of the fluctuation to $|\mathbf{q}| < k$

$$P^0(\mathbf{q}) \rightarrow P^0_k(\mathbf{q}) \equiv P^0(\mathbf{q})\Theta(|\mathbf{q}| - k) \tag{5}$$

Flow equation

Scale-depending effective action :

$$\Gamma_k = \int_{\eta,\mathbf{q}} J_A\phi_A + K_B\chi_B - W_k + \frac{i}{2}\int_{\mathbf{q}} \chi_A R_{k\ AB}\chi_B \tag{6}$$

Wetterich equation :

$$\boxed{\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial_k P_k^0}{\Gamma_k^{(2)} - iR_k}\right)} \tag{7}$$

$\Rightarrow$ flow of Γ_k in function of the IR-cutoff k
(Regulator $R_k = P_k - P$)

Power spectrum and self-energy

- goal : study the correlations between 2 points in the universe
- the power spectrum P is the Fourier transform of the correlation function ($P = W^{(2,0)}$)
- $\Rightarrow$ P express the correlation between 2 points separated by $\mathbf{q}$ and $\eta - \eta'$
- the self-energy H is the conjugate of P ($H = \Gamma^{(0,2)}$)
- we always can split H into tree-level + correction

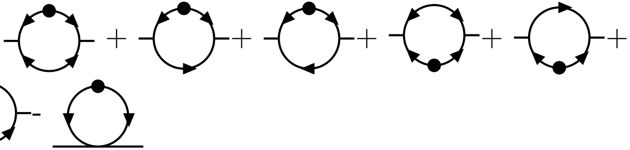
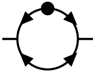
1-loop approximation

- at large scale Λ, $\Gamma_\Lambda = S$
- we integrate the Wetterich eq. at 1-loop

$$\partial_k \Gamma_\Lambda = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial_k P_k^0}{\Gamma_\Lambda^{(2)} - iR_k}\right) \tag{8}$$

- we take the second derivative w.r.t. χ
- we end up with explicit expression of H

Flow equation of the self-energy

- we take the second derivative w.r.t. χ of the Wetterich eq.
- $H \sim$ 
- no $\phi\chi\chi$ term in $\Gamma \Rightarrow \Gamma^{(1,2)} = 0$
- no $\phi\phi\chi\chi$ term in $\Gamma \Rightarrow \Gamma^{(2,2)} = 0$
- 1^{st} and 4^{th} terms are equivalent
- $\Rightarrow -i\partial_k H_{AB} = G^A_{CD}\Gamma^{(2,1)}_{ADE}P_{EF}\Gamma^{(2,1)}_{BFG}G^R_{GH}\partial_k P^0_{HC} =$ 
- finally, we use $P = \int G^R G^R H$ and $H = P^0 + \Pi$

Ansatz

- The flow eq. is hard to solve ⇒ use an ansatz for Π
  - separation of $\mathbf{q}$ and time dependency
  - weak memory effect ⇒ $\Pi \to 0$ when $\eta_1 - \eta_2$ become large

$$\boxed{\Pi_k(\mathbf{q}, \eta_1, \eta_2) = A_k(\mathbf{q}) \exp\Big(f_k^+(\eta_1 + \eta_2) + f_k^-(\eta_1 - \eta_2)^2 \Big)} \tag{9}$$

- and for G

$$G_{k\ ab}(\mathbf{q}, \eta_1, \eta_2) = g_{ab}(\eta_1, \eta_2) \exp\Big(-\mathbf{q}^2 \sigma_k^2 \frac{(e^{\eta_1} - e^{\eta_2})^2}{2} \Big) \tag{10}$$

where σ_k^2 is the velocity dispersion

$$\sigma_k^2 \equiv \frac{1}{3} \int_{|\mathbf{q}|<k} \frac{P^0(\mathbf{q})}{\mathbf{q}^2} \tag{11}$$

Ansatz

- We have 1 equation with 3 unknowns : A_k, f_k^+ and f_k^-
  - $\Rightarrow$ we need 2 other equations
  - $\Rightarrow$ we take derivative w.r.t. η twice
- Then, we evaluate at $\eta_1 = \eta_2 = 0 =$today
- $\Rightarrow$ We obtain a set of 3 self-consistent differential equations
- Next step : numerical method

Discussion

Finally, we obtain the self-energy H that is :

- related to the correlations between 2 points in the universe
- separated by $|\mathbf{q}|$ (in Fourier space) and evaluated both at present time
- with fluctuations with wavelength between k and Λ

Then, we plot H in function of k

Perspectives

- solve for other values of η and η'
- compare to other approaches (other ansatz, PT...)
- compare to observations

Recap

- DM is a ideal stochastic gas
- Power spectrum = Fourier transform of the correlation function
- H is the conjugate of P
- Γ_k captures the fluctuations between k and Λ
- Wetterich eq. gives the flow of Γ_k
- Taking the derivative of the flow, we obtain the flow of H
- We use an ansatz and evaluate the flow for present times

References

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- S. Floerchinger et al., *Renormalization-group flow of the effective action of cosmological large-scale structures*, J. Cosmol. Astropart. Phys 01 (2017) 048
- S. Matarrese and M. Pietroni, *Resumming cosmic perturbations*, J. Cosmol. Astropart. Phys 06 (2007) 026

Dynamics equation

$$\boxed{\begin{aligned}\partial_\eta \phi_a(\mathbf{q}) = & -\Omega_{ab}(\mathbf{q},\eta)\phi_b(\mathbf{q}) + \int_{\mathbf{pr}} \delta(\mathbf{q}-\mathbf{p}-\mathbf{r})\gamma_{abc}(\mathbf{p},\mathbf{r})\phi_b(\mathbf{p})\phi_c(\mathbf{r}) \\ & + \delta(\eta-\eta_0)\phi_a(\mathbf{q},\eta_0)\end{aligned}} \tag{12}$$

with

$$\Omega(\mathbf{q},\eta) \equiv \begin{pmatrix} 0 & 2 \\ -\frac{3}{2}\Omega_m & 1+\frac{\mathcal{H}'}{\mathcal{H}} \end{pmatrix} \tag{13}$$

and the non-zero elements of γ_{abc} are

$$\gamma_{121}(\mathbf{q},\mathbf{p}) = \gamma_{112}(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{q}}{2q^2} \quad \text{and} \quad \gamma_{222}(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2\mathbf{p}\cdot\mathbf{q}}{2p^2q^2} \tag{14}$$

Probability, expected value and correlation

$$\mathcal{P}[\phi(\eta_f)] = \mathcal{N} \int D'\phi D\chi e^{iS[\phi,\chi]} \quad (15)$$

$$\langle \phi_a(\mathbf{q},\eta) \rangle = \mathcal{N} \int_{\phi\chi} \phi_a(\mathbf{q},\eta) e^{iS}$$
$$\langle \phi_a(\mathbf{q},\eta)\phi'_b(\mathbf{q}',\eta') \rangle = \mathcal{N} \int_{\phi\chi} \phi_a(\mathbf{q},\eta)\phi'_b(\mathbf{q}',\eta') e^{iS} \quad (16)$$

Correlation functions

$$\left.\frac{\delta W[\phi,\chi]}{\delta J_a(\mathbf{q},\eta)}\right|_{J,K=0} = \frac{1}{Z[J,K]}\int_{\phi\chi}\phi_a(\mathbf{q},\eta)e^{iS} = \langle\phi_a\rangle(\mathbf{q},\eta) \qquad (17)$$

$$\begin{aligned}
\frac{\delta^2 W}{\delta J_a(\mathbf{q},\eta)\delta J_b(\mathbf{q}',\eta')} &= \langle\phi_a(\mathbf{q},\eta)\phi_b(\mathbf{q}',\eta')\rangle_c \equiv i\delta(\mathbf{q}+\mathbf{q}')P_{ab}(\mathbf{q},\eta,\eta')\\
\frac{\delta^2 W}{\delta J_a(\mathbf{q},\eta)\delta K_b(\mathbf{q}',\eta')} &= \langle\phi_a(\mathbf{q},\eta)\chi_b(\mathbf{q}',\eta')\rangle_c \equiv -\delta(\mathbf{q}+\mathbf{q}')G^R_{ab}(\mathbf{q},\eta,\eta')\\
\frac{\delta^2 W}{\delta K_a(\mathbf{q},\eta)\delta J_b(\mathbf{q}',\eta')} &= \langle\chi_a(\mathbf{q},\eta)\phi_b(\mathbf{q}',\eta')\rangle_c \equiv -\delta(\mathbf{q}+\mathbf{q}')G^A_{ab}(\mathbf{q},\eta,\eta')\\
\frac{\delta^2 W}{\delta K_a(\mathbf{q},\eta)\delta K_b(\mathbf{q}',\eta')} &= 0
\end{aligned} \qquad (18)$$

$$G^A_{ab}(\mathbf{q},\eta,\eta') = G^R_{ba}(-\mathbf{q},\eta',\eta) \qquad (19)$$

$$
\begin{aligned}
\frac{\delta^2\Gamma}{\delta\phi_a(\mathbf{q},\eta)\delta\phi_b(\mathbf{q}',\eta')} &= 0 \\
\frac{\delta^2\Gamma}{\delta\phi_a(\mathbf{q},\eta)\delta\chi_b(\mathbf{q}',\eta')} &= -\delta(\mathbf{q}+\mathbf{q}')D^A_{ab}(\mathbf{q},\eta,\eta') \\
\frac{\delta^2\Gamma}{\delta\chi_a(\mathbf{q},\eta)\delta\phi_b(\mathbf{q}',\eta')} &= -\delta(\mathbf{q}+\mathbf{q}')D^R_{ab}(\mathbf{q},\eta,\eta') \\
\frac{\delta^2\Gamma}{\delta\chi_a(\mathbf{q},\eta)\delta\chi_b(\mathbf{q}',\eta')} &= -i\delta(\mathbf{q}+\mathbf{q}')H_{ab}(\mathbf{q},\eta,\eta')
\end{aligned} \tag{20}
$$

$$
D^A_{ab}(\mathbf{q},\eta,\eta') = D^R_{ba}(-\mathbf{q},\eta',\eta) \tag{21}
$$

Power spectrum and self-energy

The power spectrum P (and the self-energy H) express the correlations of the fluctuations between 2 points in the unvierse, separeted by a "distance" $|\mathbf{q}|$ and time $\eta' - \eta$. They are related by

$$P_{ad}(\mathbf{q}, \eta, \eta''') = \int_{\eta' \eta''} G^R_{ab}(\mathbf{q}, \eta, \eta') H_{bc}(\mathbf{q}, \eta', \eta'') G^A_{cd}(\mathbf{q}, \eta'', \eta''') \quad (22)$$

We can split H in tree-level + correction

$$H_k(\mathbf{q}, \eta, \eta') = P^0_k(\mathbf{q}) \delta(\eta) \delta(\eta') + \Pi_k(\mathbf{q}, \eta, \eta') \quad (23)$$

1-loop approximation

Wetterich eq. :

$$\begin{aligned}\partial_k\Gamma_k =& \frac{i}{2}\tilde{\partial}_k \mathrm{Tr}\Big(\log(\Gamma_k^{(2)} - iR_k)\Big)\\ \Rightarrow \Gamma_k =& \Gamma_\Lambda + \frac{i}{2}\mathrm{Tr}\Big(\log(\Gamma_\Lambda^{(2)} - iR_k) - \log(\Gamma_\Lambda^{(2)} - iR_\Lambda)\Big)\\ \Rightarrow -iH_{k\ AB} =& \Gamma^{(0,2)}_{\Lambda\ AB} + \frac{i}{2}\left(\frac{\delta^2 \mathrm{Tr}\Big(\log(\Gamma_\Lambda^{(2)}[\phi,\chi] - iR_k)\Big)}{\delta\chi_A\delta\chi_B} - \frac{\delta^2 \mathrm{Tr}\Big(\log(\Gamma_\Lambda^{(2)}[\phi,\chi] - iR_\Lambda)\Big)}{\delta\chi_A\delta\chi_B}\right)\\ \equiv& \Gamma^{(0,2)}_{\Lambda\ AB} + \Pi_{k\ AB}\end{aligned} \quad (24)$$

with $A = (a, \mathbf{q}, \eta)$ and $B = (b, \mathbf{q}', \eta')$

At tree-level, the only $\chi\chi$ term in S is the fluctuation term

$$\Gamma^{(0,2)}_{\Lambda\ AB} = -\frac{i}{2}\int_{\mathbf{p}} \frac{\delta^2(\chi_C P^0_{CD}\chi_D)}{\delta\chi_A\delta\chi_B} = -iP^0_{AB} \qquad (25)$$

$$\frac{\delta^2 \mathrm{Tr}\left(\log(\Gamma_\Lambda^{(2)}[\phi,\chi]-iR_k)\right)}{\delta\chi_A\delta\chi_B} = -\mathrm{Tr}\left(W_{\Lambda k}^{(2)}\frac{\delta\Gamma_\Lambda^{(2)}}{\delta\chi_A}W_{\Lambda k}^{(2)}\frac{\delta\Gamma_\Lambda^{(2)}}{\delta\chi_B}\right) + \mathrm{Tr}\left(W_{\Lambda k}^{(2)}\frac{\delta^2\Gamma_\Lambda^{(2)}}{\delta\chi_A\delta\chi_B}\right) \quad (26)$$

with

$$W_{\Lambda k\ AB}^{(2)} = \begin{pmatrix} ig_{AA'}^R P_{k\ A'B'}^0 g_{B'B}^A & -g_{AB}^R \\ -g_{AB}^A & 0 \end{pmatrix} \quad (27)$$

with the linear retarded propagator g^R defined by

$$g_{ab}^R(\eta,\eta') = \frac{e^{\eta-\eta'}}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\Theta(\eta-\eta') - \frac{e^{-3(\eta-\eta')/2}}{5}\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\Theta(\eta-\eta') \quad (28)$$

$$
\begin{aligned}
\frac{\delta^2 \mathrm{Tr}\Big(\log(\Gamma_\Lambda^{(2)}[\phi,\chi] - iR_k)\Big)}{\delta\chi_A\delta\chi_B} &= -\frac{i}{2}\Big(W_{CD}^{(2,0)}\Gamma_{ADE}^{(2,1)}W_{EF}^{(2,0)}\Gamma_{BFC}^{(2,1)}\Big) \\
&= -2g^R_{CC'}P^0_{k\ C'D'}g^A_{D'D}\gamma_{ADE}g^R_{EE'}P^0_{k\ E'F'}g^A_{F'F}\gamma_{BFC} \\
&= -4\int_{\mathbf{p}} g^R_{cc'}(\eta',0)P^0_{k\ c'd'}(\mathbf{p})g^A_{d'd}(0,\eta)\gamma_{ade}(\mathbf{q},-\mathbf{p},\mathbf{p}-\mathbf{q}) \\
&\quad \times g^R_{ee'}(\eta,0)P^0_{k\ e'f'}(\mathbf{p}-\mathbf{q})g^A_{f'f}(0,\eta')\gamma_{bfc}(-\mathbf{q},\mathbf{q}-\mathbf{p},\mathbf{p})
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\Pi_{k\ ab}(\mathbf{q},\eta,\eta') = -4\int_{\mathbf{p}}&\Big(1-\Theta(|\mathbf{p}|-k)\Theta(|\mathbf{p}-\mathbf{q}|-k)\Big) \\
&\times g^R_{cc'}(\eta',0)P^0_{k\ c'd'}(\mathbf{p})g^A_{d'd}(0,\eta)\gamma_{ade}(\mathbf{q},-\mathbf{p},\mathbf{p}-\mathbf{q}) \\
&\times g^R_{ee'}(\eta,0)P^0_{k\ e'f'}(\mathbf{p}-\mathbf{q})g^A_{f'f}(0,\eta')\gamma_{bfc}(-\mathbf{q},\mathbf{q}-\mathbf{p},\mathbf{p})
\end{aligned} \tag{30}
$$

flow equation of the self-energy

$$
\begin{aligned}
-i\partial_k H_{AB} =& \frac{\delta^2 \partial_k \Gamma}{\delta\chi_A \delta\chi_B} \\
=& -\frac{1}{2}\mathrm{Tr}\Big(\frac{\delta}{\delta\chi_A}(W^{(2)}\frac{\delta\Gamma}{\delta\chi_B}W^{(2)}\partial_k P^0)\Big) \\
=& \frac{1}{2}\mathrm{Tr}\Big(W^{(2)}\frac{\delta\Gamma}{\delta\chi_A}W^{(2)}\frac{\delta\Gamma}{\delta\chi_B}W^{(2)}\partial_k P^0\Big) \\
&+ \frac{1}{2}\mathrm{Tr}\Big(W^{(2)}\frac{\delta\Gamma}{\delta\chi_B}W^{(2)}\frac{\delta\Gamma}{\delta\chi_A}W^{(2)}\partial_k P^0\Big) \\
&- \frac{1}{2}\mathrm{Tr}\Big(W^{(2)}\frac{\delta^2\Gamma}{\delta\chi_A\delta\chi_B}W^{(2)}\partial_k P^0\Big) \\
=& -i\partial_k H^{I}_{AB} - i\partial_k H^{II}_{AB} - i\partial_k H^{III}_{AB}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
-i\partial_k H^{I}_{AB} =& \frac{1}{2}\begin{pmatrix} P_{CD} & G^R_{CD} \\ G^A_{DC} & 0 \end{pmatrix}\begin{pmatrix} \Gamma^{(2,1)}_{ADE} & \Gamma^{(1,2)}_{ADE} \\ \Gamma^{(1,2)}_{AED} & \Gamma^{(0,3)}_{ADE} \end{pmatrix}\begin{pmatrix} P_{EF} & G^R_{EF} \\ G^A_{FE} & 0 \end{pmatrix} \\
& \times \begin{pmatrix} \Gamma^{(2,1)}_{BFG} & \Gamma^{(1,2)}_{BFG} \\ \Gamma^{(1,2)}_{BGF} & \Gamma^{(0,3)}_{BFG} \end{pmatrix}\begin{pmatrix} P_{GH} & G^R_{GH} \\ G^A_{HG} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \partial_k P^0_{HC} \end{pmatrix} \\
=& \frac{1}{2}\Big(G^A_{CD}\Gamma^{(2,1)}_{ADE} P_{EF}\Gamma^{(2,1)}_{BFG} G^R_{GH}\partial_k P^0_{HC} \\
& + G^A_{CD}\Gamma^{(2,1)}_{ADE} G^R_{EF}\Gamma^{(1,2)}_{BGF} G^R_{GH}\partial_k P^0_{HC} \\
& + G^A_{CD}\Gamma^{(1,2)}_{ADE} G^A_{EF}\Gamma^{(2,1)}_{BFG} G^R_{GH}\partial_k P^0_{HC}\Big)
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
-i\partial_k H_{AB} =&\frac{1}{2}\Big(G^A_{CD}\Gamma^{(2,1)}_{ADE}P_{EF}\Gamma^{(2,1)}_{BFG}G^R_{GH}\partial_k P^0_{HC} + G^A_{CD}\Gamma^{(2,1)}_{ADE}G^R_{EF}\Gamma^{(1,2)}_{BGF}G^R_{GH}\partial_k P^0_{HC} \\
&+ G^A_{CD}\Gamma^{(1,2)}_{ADE}G^A_{EF}\Gamma^{(2,1)}_{BFG}G^R_{GH}\partial_k P^0_{HC}\Big) + \Big(A \leftrightarrow B\Big) \\
&- \frac{1}{2}\Big(D^R_{CD}\Gamma^{(2,2)}_{ABDE}G^R_{EF}\partial_k P^0_{FC}\Big)
\end{aligned}
\tag{33}
$$

We replace P by $\int G^R G^R H$ and H by $P^0 + \Pi$

$$
\boxed{
\begin{aligned}
-i\partial_k \Pi_{ab}(\mathbf{q},\eta,\eta') =&4\int_{\mathbf{p}} G^R_{dc}(\mathbf{p},0,\eta)\gamma_{ade}(\mathbf{q},-\mathbf{p},\mathbf{p}-\mathbf{q}) \\
&\times \int_{\xi,\xi'} G^R_{ee'}(\mathbf{p}-\mathbf{q},\eta,\xi)G^R_{ff'}(\mathbf{q}-\mathbf{p},\eta,'\xi') \\
&\times \Big(P^0_{e'f'}(\mathbf{p}-\mathbf{q})\delta(\xi)\delta(\xi') + \Pi_{e'f'}(\mathbf{p}-\mathbf{q},\xi,\xi')\Big) \\
&\times \gamma_{bfg}(-\mathbf{q},\mathbf{q}-\mathbf{p},\mathbf{p})G^R_{gh}(\mathbf{p},\eta,0)\partial_k P^0_{hc}(-\mathbf{p})
\end{aligned}
}
\tag{34}
$$

The l.h.s. gives :

$$
\begin{aligned}
\partial_k \Pi =& \partial_k A e^{f^+(\eta_1+\eta_2)+f^-(\eta_1-\eta_2)^2} + \partial_k f^+ (\eta_1+\eta_2)\Pi + \partial_k f^- (\eta_1+\eta_2)^2 \Pi \\
\partial_\eta \partial_k \Pi =& \partial_k A \Big(f^+ + 2f^-(\eta_1-\eta_2)\Big) e^{f^+(\eta_1+\eta_2)+f^-(\eta_1-\eta_2)^2} \\
& + \partial_k f^+ \Big(1 + f^+(\eta_1+\eta_2) + 2f^-(\eta_1^2-\eta_2^2)\Big)\Pi \\
& + \partial_k f^- \Big(2(\eta_1-\eta_2) + f^+(\eta_1-\eta_2)^2 + f^-(\eta_1-\eta_2)^3\Big)\Pi \\
\partial_\eta^2 \partial_k \Pi =& \partial_k A \Big(f^{+\ 2} + 4f^+f^-(\eta_1-\eta_2) + 2f^-\Big) e^{f^+(\eta_1+\eta_2)+f^-(\eta_1-\eta_2)^2} \\
& + \partial_k f^+ \Big(2f^+ + f^{+\ 2}(\eta_1+\eta_2) + 2f^-(\eta_1-\eta_2) + 4f^+f^-(\eta_1^2-\eta_2^2) + 4f^-\eta_1\Big)\Pi \\
& + \partial_k f^- \Big(2 + 4f^+(\eta_1+\eta_2) + f^{+\ 2}(\eta_1-\eta_2)^2 + 4f^+f^-(\eta_1-\eta_2)^3 + 10f^-(\eta_1-\eta_2)^2 \\
& \qquad + 4f^{-\ 2}(\eta_1-\eta_2)^4\Big)\Pi
\end{aligned}
\tag{35}
$$

Ansatz G_k^R

$$\begin{aligned}
\partial_{\eta_i} G_k^R(\mathbf{q},\eta_1,\eta_2) =& \Big(\partial_{\eta_i} g_k^R(\eta_1,\eta_2) - \mathbf{q}^2\sigma_k^2(e^{\eta_i} - e^{\eta_1+\eta_2}) g_k^R(\eta_1,\eta_2)\Big) \\
& \times \exp\Big(-\mathbf{q}^2\sigma_k^2\frac{(e^{\eta_1}-e^{\eta_2})^2}{2}\Big) \\
\partial_{\eta_i}^2 G_k^R(\mathbf{q},\eta_1,\eta_2) =& \Big(\partial_{\eta_i}^2 g_k^R(\eta_1,\eta_2) - \mathbf{q}^2\sigma_k^2(e^{\eta_i} - e^{\eta_1+\eta_2}) \\
& \times \Big(\partial_{\eta_i} g_k^R(\eta_1,\eta_2) - \mathbf{q}^2\sigma_k^2(e^{\eta_i} - e^{\eta_1+\eta_2}) \\
& \times g_k^R(\eta_1,\eta_2) + 1\Big) g_k^R(\eta_1,\eta_2)\Big) \exp\Big(-\mathbf{q}^2\sigma_k^2\frac{(e^{\eta_1}-e^{\eta_2})^2}{2}\Big)
\end{aligned} \tag{36}$$

where $i \in \{1,2\}$. Then, we put it in the r.h.s.

Evaluation at $\eta = \eta' = 0$

To simplify the equations, we solve it for $\eta = \eta' = 0$ (=present time). The l.h.s. gives

$$\begin{aligned}
\partial_k \Pi =& \partial_k A \\
\partial_\eta \partial_k \Pi =& f^+ \partial_k A \\
\partial_\eta^2 \partial_k \Pi =& (f^{+\ 2} + 2f^-)\partial_k A + 2f^+ \Pi \partial_k f^+ + 2\partial_k f^-
\end{aligned} \tag{37}$$