

# Generalised Partons Distributions: Modelling and Lattice Computations

Cédric Mezrag

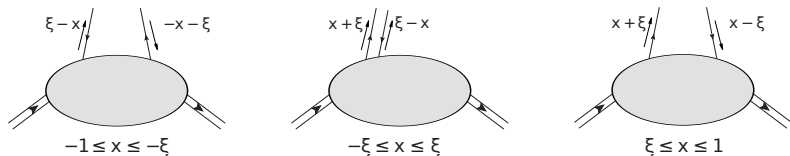
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June 22<sup>nd</sup>, 2021

# A brief introduction to Generalised Partons Distributions (GPDs)

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  - ▶ “hadron-parton” amplitudes which depend on three variables ( $x, \xi, t$ ) and a scale  $\mu$ ,



- ★  $x$ : average momentum fraction carried by the active parton
- ★  $\xi$ : skewness parameter  $\xi \simeq \frac{x_B}{2-x_B}$
- ★  $t$ : the Mandelstam variable

- Generalised Parton Distributions (GPDs):
  - ▶ “hadron-parton” amplitudes which depend on three variables  $(x, \xi, t)$  and a scale  $\mu$ ,
  - ▶ are defined in terms of a non-local matrix elements,

$$\begin{aligned} & \frac{1}{2} \int \frac{e^{ixP^+z^-}}{2\pi} \langle P + \frac{\Delta}{2} | \bar{\psi}^q(-\frac{z}{2}) \gamma^+ \psi^q(\frac{z}{2}) | P - \frac{\Delta}{2} \rangle dz^- |_{z^+=0, z=0} \\ &= \frac{1}{2P^+} \left[ H^q(x, \xi, t) \bar{u} \gamma^+ u + E^q(x, \xi, t) \bar{u} \frac{i\sigma^{+\alpha} \Delta_\alpha}{2M} u \right]. \end{aligned}$$

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D. Müller *et al.*, Fortsch. Phys. 42 101 (1994)

X. Ji, Phys. Rev. Lett. 78, 610 (1997)

A. Radyushkin, Phys. Lett. B380, 417 (1996)

4 GPDs without helicity transfer + 4 helicity flip GPDs

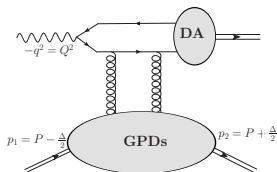
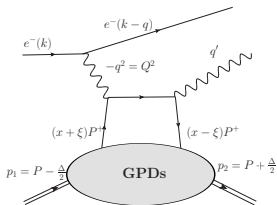
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- ▶ can be split into quark flavour and gluon contributions,
- ▶ are related to PDF in the forward limit  $H(x, \xi = 0, t = 0; \mu) = q(x; \mu)$
- ▶ are universal, *i.e.* are related to the Compton Form Factors (CFFs) of various exclusive processes through convolutions

$$\mathcal{H}(\xi, t) = \int dx C(x, \xi) H(x, \xi, t)$$





- Polynomiality property:

$$\int_{-1}^1 dx x^m H^q(x, \xi, t; \mu) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \xi^{2j} C_{2j}^q(t; \mu) + \text{mod}(m, 2) \xi^{m+1} C_{m+1}^q(t; \mu)$$

X. Ji, J.Phys.G 24 (1998) 1181-1205

A. Radyushkin, Phys.Lett.B 449 (1999) 81-88

Special case :

$$\int_{-1}^1 dx H^q(x, \xi, t; \mu) = F_1^q(t)$$

Lorentz Covariance

- Polynomiality property:
- Positivity property:

Lorentz Covariance

$$\left| H^q(x, \xi, t) - \frac{\xi^2}{1 - \xi^2} E^q(x, \xi, t) \right| \leq \sqrt{\frac{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}{1 - \xi^2}}$$

A. Radysuhkin, Phys. Rev. D59, 014030 (1999)

B. Pire *et al.*, Eur. Phys. J. C8, 103 (1999)

M. Diehl *et al.*, Nucl. Phys. B596, 33 (2001)

P.V. Pobilitza, Phys. Rev. D65, 114015 (2002)

Positivity of Hilbert space norm

- Polynomiality property:
- Positivity property:
- Support property:

Lorentz Covariance

Positivity of Hilbert space norm

$$x \in [-1; 1]$$

M. Diehl and T. Gousset, Phys. Lett. B428, 359 (1998)

Relativistic quantum mechanics

- Polynomiality property:

Lorentz Covariance

- Positivity property:

Positivity of Hilbert space norm

- Support property:

Relativistic quantum mechanics

- Scale evolution property

→ generalization of DGLAP and ERBL evolution equations

D. Müller *et al.*, Fortschr. Phys. 42, 101 (1994)

Renormalization

- Polynomiality property:

Lorentz Covariance

- Positivity property:

Positivity of Hilbert space norm

- Support property:

Relativistic quantum mechanics

- Scale evolution property

Renormalization

## Challenge

Fullfilling all these property is a challenge  
for both modelling and lattice computations

# Modelling GPDs



- GPDs are related to Double Distributions (DDs) through:

$$H(x, \xi, t) = \int_{\Omega} d\beta d\alpha (F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)) \delta(x - \beta - \xi\alpha)$$

The Dirac  $\delta$  insures that the polynomiality is fulfilled, independently of our choice of  $F$  and  $G$



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- They also appear naturally in covariant modelling attempts

Positivity property is not guaranteed, and may be violated.



- On the light front, hadronic states can be expanded on a Fock basis:

$$|P, \pi\rangle \propto \sum_{\beta} \Phi_{\beta}^{q\bar{q}} |q\bar{q}\rangle + \sum_{\beta} \Phi_{\beta}^{q\bar{q}, q\bar{q}} |q\bar{q}, q\bar{q}\rangle + \dots$$

$$|P, N\rangle \propto \sum_{\beta} \Phi_{\beta}^{qqq} |qqq\rangle + \sum_{\beta} \Phi_{\beta}^{qqq, q\bar{q}} |qqq, q\bar{q}\rangle + \dots$$



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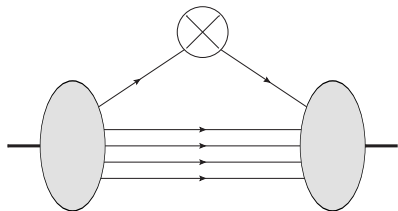
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- Non-perturbative physics is contained within the  $N$ -particle LFWFs  $\Phi^N$
- This formalism allows to recover the probabilistic picture of non-relativistic quantum mechanics

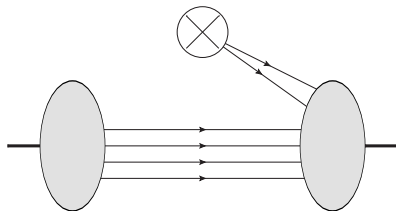
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DGLAP:  $|x| > |\xi|$



- Same  $N$  LFWFs
- No ambiguity

ERBL:  $|x| < |\xi|$

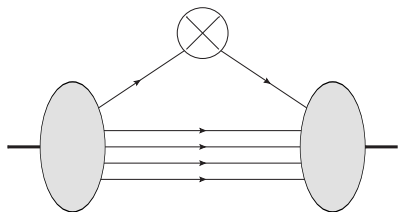


- $N$  and  $N + 2$  partons LFWFs
- Ambiguity

M. Diehl *et al.*, Nucl.Phys. B596 (2001) 33-65

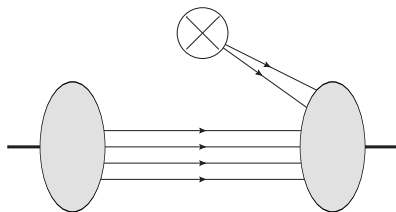
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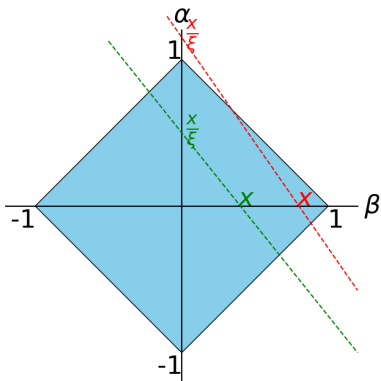


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LFWFs formalism has the positivity property inbuilt but polynomiality is lost by truncating both in DGLAP and ERBL sectors.

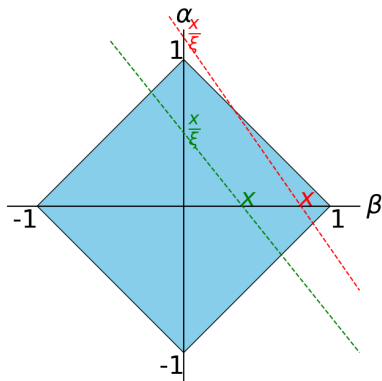
$$H(x, \xi) = \int_{\Omega} d\beta d\alpha \delta(x - \beta - \alpha\xi) [F(\beta, \alpha) + \xi G(\beta, \alpha)]$$



- DGLAP (red) and ERBL (green) lines cut  $\beta = 0$  outside or inside the square
- Every point  $(\beta \neq 0, \alpha)$  contributes **both** to DGLAP and ERBL regions
- For every point  $(\beta \neq 0, \alpha)$  we can draw an infinite number of DGLAP lines.



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Is it possible to recover the DDs from the DGLAP region only?

- Double Distribution representation:

$$H(x, \xi) = D(x/\xi) + \int_{\Omega} d\beta d\alpha \delta(x - \beta - \alpha\xi) F_D(\beta, \alpha)$$

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- Since DD are compactly supported, we can use the **Boman and Todd-Quinto theorem** which tells us

$$H(x, \xi) = 0 \quad \text{for } (x, \xi) \in \text{DGLAP} \Rightarrow F_D(\beta, \alpha) = 0 \quad \text{for all } (\beta \neq 0, \alpha) \in \Omega$$

Boman and Todd-Quinto, Duke Math. J. 55, 943 (1987)

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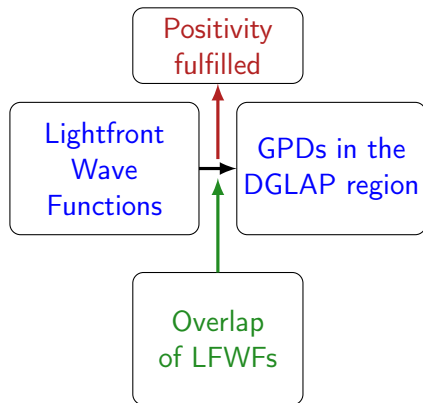
Boman and Todd-Quinto, Duke Math. J. 55, 943 (1987)

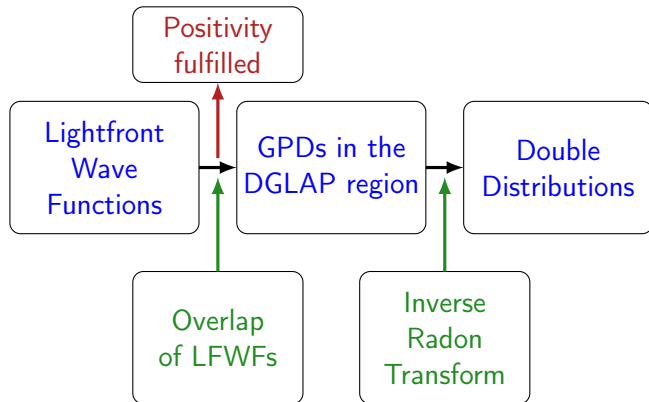
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## New modeling strategy

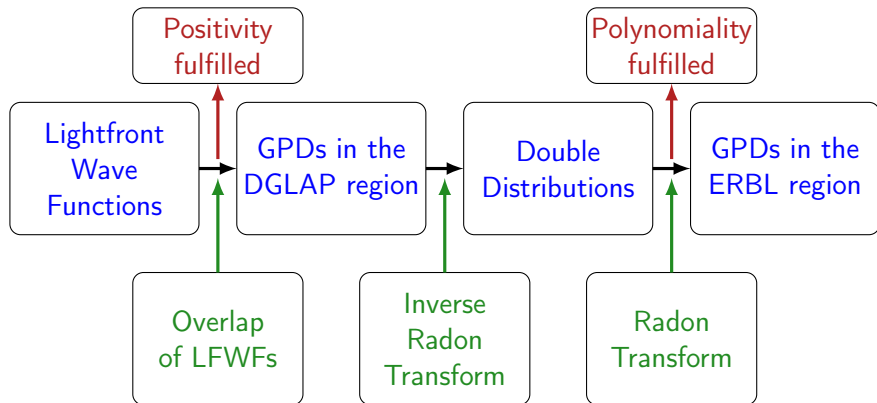
- Compute the DGLAP region through overlap of LFWFs  
 $\Rightarrow$  **fulfilment of the positivity property**
- Extension to the ERBL region using the Radon inverse transform  
 $\Rightarrow$  **fulfilment of the polynomiality property**

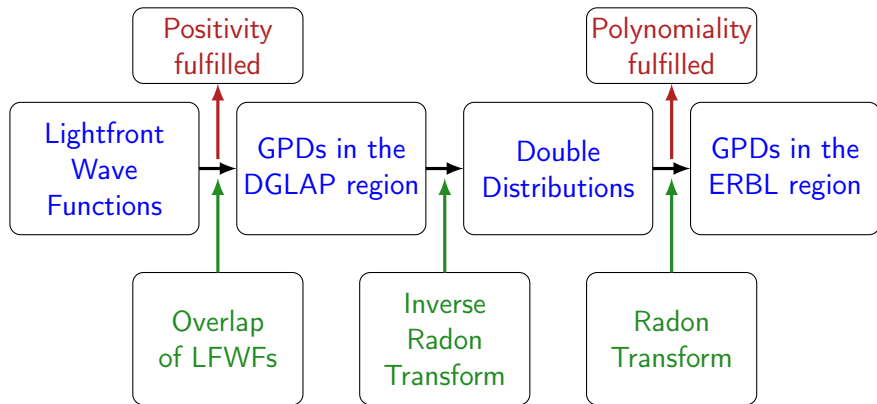
Lightfront  
Wave  
Functions







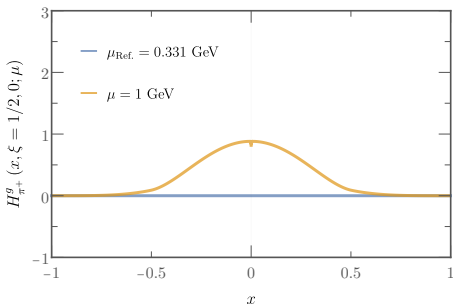
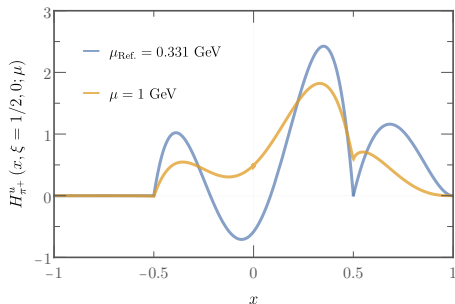


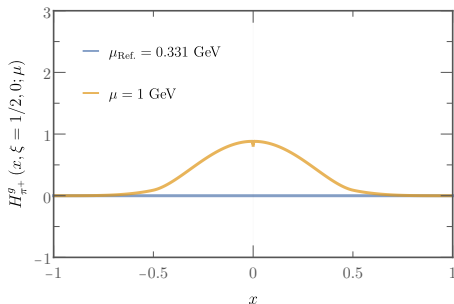
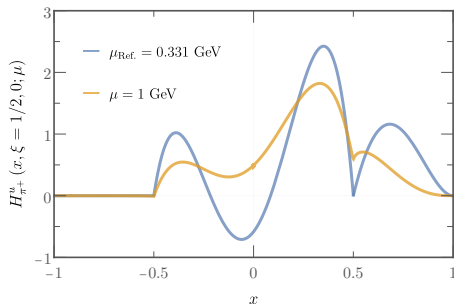


Not necessary to start from LFWFs

→ Fulfilling the positivity and forward limit properties is enough

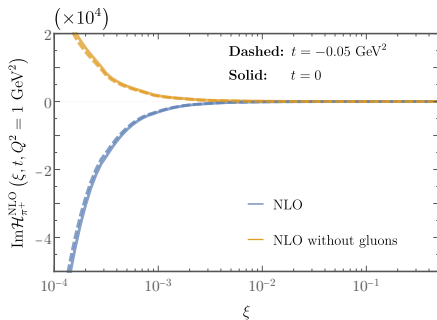
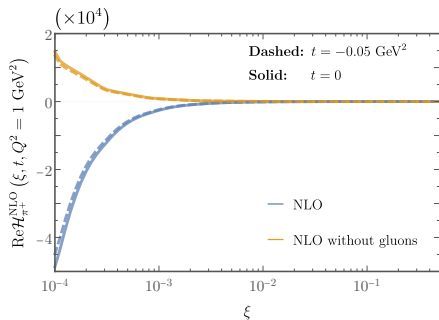
Morgado *et al.*, in preparation

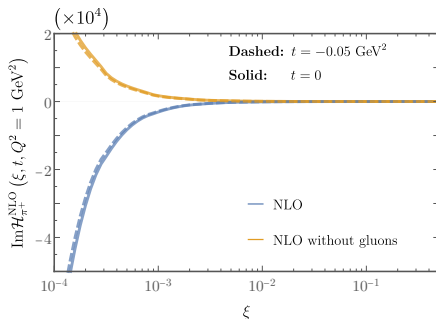
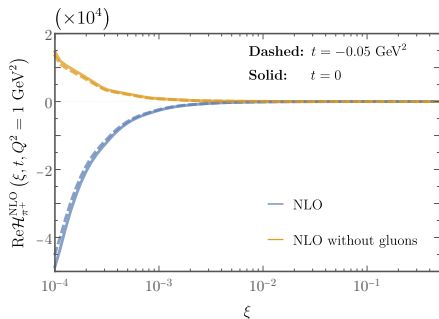




All theoretical constraints are fulfilled by construction !

Morgado *et al.*, in preparation





- The PARTONS framework allows us to study DVCS on virtual pion at LO (Sullivan process)
- Gluon in the pion dominates at EIC kinematics as expected
- Sullivan process measurable at EIC

# Lattice QCD computations

- Lattice computation living in euclidean space “time-like” and “light-like” four vectors are not directly reachable
- Instead of computing matrix element of non-local operators, focus on local operators

$$n_\mu n_{\mu_1} \dots n_{\mu_n} \langle P | \bar{\psi}(0) \gamma^{\{\mu} \overleftrightarrow{D}^{\mu_1} \dots \overleftrightarrow{D}^{\mu_n\}} \psi(0) | P \rangle \propto \int dx x^n q(x)$$

- Results on Mellin moments have been obtained for GPDs
  - ▶ Electromagnetic Form Factors
  - ▶ Axial Form Factors
  - ▶ Gravitational Form Factors



- In principle, one could compute local operator of degree as high as required
- In practice difficulties arise:
  - ▶ above  $n = 3$  operators start mixing among them due to the breaking of rotational invariance  $O(4)$  into a smaller symmetry group  $H(4)$
  - ▶ This introduces power-like singularities in the lattice spacing which need to be treated non-perturbatively for each operator

see e.g. S. Capitani and G. Rossi, Nucl.Phys.B 433 (1995) 351-389

- In practice, only the first Mellin moments ( $n=1,2$ ) are computed on the lattice  
→ this compromises access to the  $x$ -dependence of the distributions through moments

- Some approaches have been developed in the 2000s to try to by-pass this issue

see e.g. K. Cichy and M. Constantinou, Adv.High Energy Phys. 2019 (2019) 3036904

- ▶ Direct computation of the hadronic tensor
- ▶ Use of auxiliary quarks to “tame” divergences

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- ▶ Direct computation of the hadronic tensor
  - ▶ Use of auxiliary quarks to “tame” divergences
- In the last 10 years new approaches arose to compute PDF:
    - ▶ Large-Momentum Effective field theory (LaMET - quasi PDF-)
    - ▶ Ioffe-time pseudo distributions
    - ▶ OPE without OPE
    - ▶ Good Lattice cross section

X. Ji *Phys. Rev. Lett.*, 2013, 110, 262002

Y.-Q. Ma and J.-W. Qiu *Phys. Rev. D*, 2018, 98, 074021

A. Radyushkin, *Phys. Rev. D*, 2017, 96, 034025

A. J. Chambers *et al.*, *PRL* 118 242001 (2017)

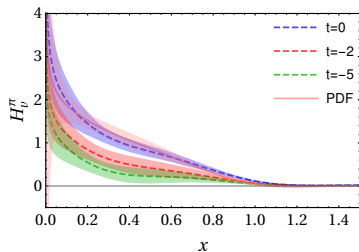
- Define non-local matrix element along a spatial direction:

$$\tilde{q}(\tilde{x}, P_3) = \int_{-\infty}^{\infty} \frac{dz_3}{4\pi} e^{-iz_3 \tilde{x} P_3} \langle p | \bar{\psi}(z_3) \gamma^3 \mathcal{W}(z_3, 0) \psi(0) | p \rangle$$

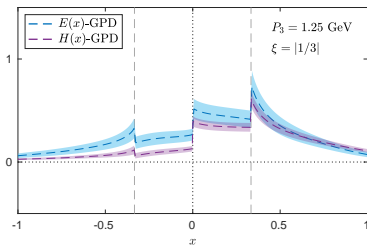
- Boost the hadron in a frame where  $P_3^2 \gg \Lambda_{QCD}^2$  and  $P_3^2 \gg M_H^2$
- Expand the quasi-PDF in terms of the Lightcone one plus corrections:

$$\tilde{q}(\tilde{x}, P_3, \mu^2) = \int_{-1}^1 \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu^2}{P_3^2}\right) q(y, \mu^2) + O\left(\frac{\Lambda_{QCD}^2}{P_3^2}, \frac{M_H^2}{P_3^2}\right)$$

- For  $P_3 \rightarrow \infty$  one expects to recover the PDF through the matching condition
- Technical difficulties : linear divergences needs to be identified and subtracted (cut off in momentum space)
- The same formalism can be applied to GPDs

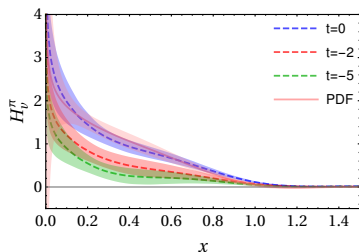


J.-W. Chen Nucl.Phys.B 952 (2020) 114940

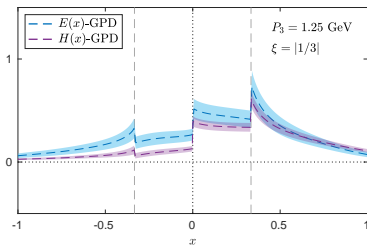


C. Alexandrou *et al.*, PRL 125 (2020) 26, 262001

- Pioneering works

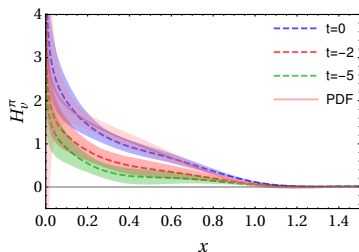


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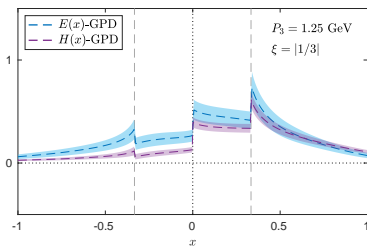


C. Alexandrou *et al.*, PRL 125 (2020) 26, 262001

- Pioneering works
- Issues 1 large- $x$  region
  - ▶ Support property violated (“by construction”)
  - ▶ Positivity property violated



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- Pioneering works
- Issues 1 large- $x$  region
  - ▶ Support property violated (“by construction”)
  - ▶ Positivity property violated
- Issues 2 crossover line  $|x| = |\xi|$ 
  - ▶ Discontinuity  $\rightarrow$  violation of factorisation theorem
  - ▶ Higher-twist pollution?

- Define a distribution with depending of Lorentz scalar quantities:

$$\mathcal{M}(\nu, -z^2) = \langle p | \bar{\psi}(-z) \gamma^\alpha \psi(z) | p \rangle \quad \text{with} \quad \nu = -p \cdot z$$



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- Taking  $\alpha = 0$  and  $z = (0, 0, 0, z_3)$ , one can define de pseudo-distributions with the expected support in  $x$ :

$$\mathcal{P}(x, z_3^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu e^{-ix\nu} \mathcal{M}(\nu, z_3^2) \quad \text{with} \quad \mathcal{P}(x, 0) = q(x)$$

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- Issues with divergencies on the lattice bypass by the definition of RGI reduced distribution  $\mathfrak{M}$  :

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} \quad \rightarrow \quad \text{Good behaviour for } z_3^2 \rightarrow 0$$

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$$\mathcal{M}(\nu, -z^2) = \langle p | \bar{\psi}(-z) \gamma^\alpha \psi(z) | p \rangle \quad \text{with} \quad \nu = -p \cdot z$$

- Taking  $\alpha = 0$  and  $z = (0, 0, 0, z_3)$ , one can define de pseudo-distributions with the expected support in  $x$ :

$$\mathcal{P}(x, z_3^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu e^{-ix\nu} \mathcal{M}(\nu, z_3^2) \quad \text{with} \quad \mathcal{P}(x, 0) = q(x)$$

- Issues with divergencies on the lattice bypass by the definition of RGI reduced distribution  $\mathfrak{M}$  :

$$\mathfrak{M}(\nu, z^2) = \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)} \quad \rightarrow \quad \text{Good behaviour for } z_3^2 \rightarrow 0$$

- Matching condition given on  $\mathfrak{M}$

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Not being an expert, I remain neutral here

## Summary

- After 25 years, GPDs formalism is well established . . .
- . . . but the GPDs themselves remain poorly known
- Modelling GPDs is challenging because of all theoretical constraints
- Lattice QCD starts going beyond few moments but difficulties remain

## Perspectives

- Significant efforts in phenomenology remain to be done (CFF and GPD)
- Exploiting models built to fulfill all theoretical constraints
- What lattice QCD could bring remains to be clarified

In the perspective of EIC and EICC, a lot of work remains to be done to exploit the forthcoming data.



Thank you for your attention