Leading hadronic contribution to the muon magnetic moment from lattice QCD

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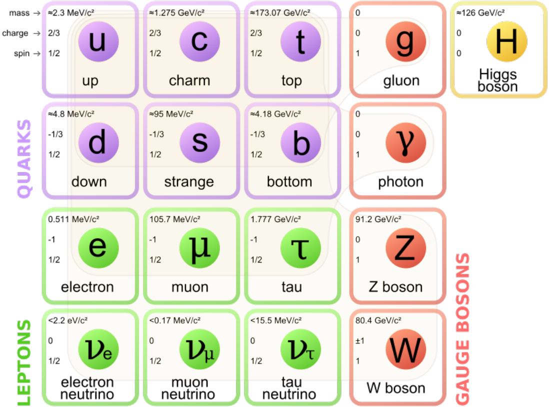
Budapest-Marseille-Wuppertal collaboration [BMWc]

Borsanyi, Fodor, Guenther, Hoelbling, Katz, LL, Lippert, Miura, Szabo, Parato, Stokes, Toth, Torok, Varnhorst

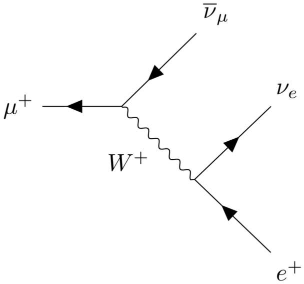
Nature, 7 April 2021 $\rightarrow$ BMWc '20
PRL 121 (2018) (Editors' Selection)
& Aoyama et al., Phys. Rept. (2020) 887, 1-166 $\rightarrow$ WP '20

The Standard Model on a page

Describes all known elementary particles and three of the four fundamental interactions



(Wikimedia)



- muon (μ) $\sim$ electron (e): same interactions w/ gauge bosons, not with the Higgs
- But: $m_\mu \simeq 207 \times m_e$ & $\tau_\mu \simeq 2 \times 10^{-6}$ sec

Lepton magnetic moments and BSM physics

Interaction with an external EM field: QM

Dirac eq. w/ minimal coupling (1928):

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e_\ell\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi$$

nonrelativistic limit $\downarrow$ (Pauli eq.)

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2}{2m_\ell} - \underbrace{\frac{e_\ell\hbar}{2m_\ell}\vec{\sigma}\cdot\vec{B}}_{\vec{\mu}_\ell\cdot\vec{B}} + e_\ell A_0\right]\phi$$

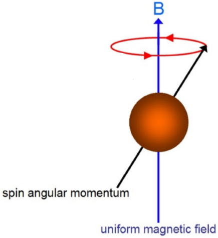
with

$$\vec{\mu}_\ell = g_\ell\left(\frac{e_\ell}{2m_\ell}\right)\vec{S}, \qquad \vec{S} = \hbar\frac{\vec{\sigma}}{2}$$

and

$$g_\ell\big|_{\text{Dirac}} = 2$$

"That was really an unexpected bonus for me, completely unexpected." (P.A.M. Dirac)



Interaction with an external EM field: QFT

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

$$\left. + \gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

$F_1(q^2) \rightarrow$ Dirac form factor: $F_1(0) = 1$

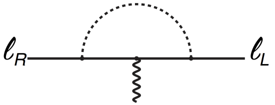
$F_2(q^2) \rightarrow$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \dfrac{g_\ell - \overbrace{2}^{g_\ell|_{\text{Dirac}}}}{2}$

$F_3(q^2) \rightarrow$ $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell / e_\ell$

$F_4(q^2) \rightarrow$ $\not{P}$, anapole moment: $\vec{\sigma} \cdot (\vec{\nabla} \times \vec{B})$

- $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once theory's couplings are renormalized (in a renormalizable theory)
- a_ℓ dimensionless
  - $\Rightarrow$ corrections including only ℓ and γ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell/M)^{2p} \times \ln^q(m_\ell^2/M^2)$
  - $\rightarrow$ contributions from particles w/ $m \ll m_\ell$ are e.g. $\propto \ln^2(m_\ell^2/m^2)$

Why are a_ℓ special?



$$\longrightarrow \quad \frac{a_\ell}{2m_\ell} e F^{\mu\nu} [\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- Loop induced $\Rightarrow$ sensitive to new dofs
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, s and b decays, LHC direct searches, ...
- Chirality flipping $\Rightarrow$ a_ℓ related to mass generation (Czarnecki et al '01) ($M \gg m_\ell$)

$$a_\ell^{\mathrm{M}} = O(1) \left(\frac{\Delta m_\ell}{m_\ell} \right) \left(\frac{m_\ell}{M} \right)^2$$

- In EW theory, $M = M_W$ and

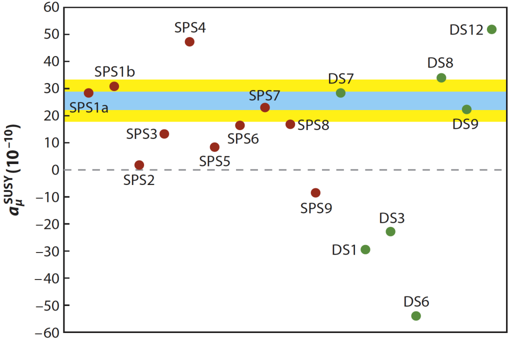
$$\frac{\Delta m_\ell}{m_\ell} \sim \frac{\alpha}{4\pi}$$

- BSM can be very different, e.g. SUSY $M = M_{\mathrm{SUSY}}$ and $(\Delta m_\ell / m_\ell) \sim (\alpha/4\pi) \times \mathrm{sign}\mu \, \tan\beta$

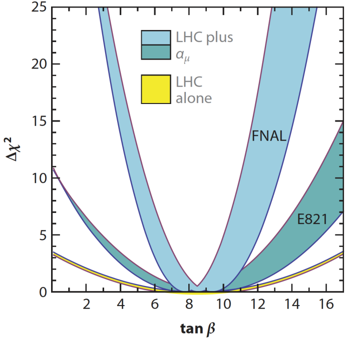
Why is a_μ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\text{GeV}$ $\tau_e : \tau_\mu : \tau_\tau =$ "∞" $: 2.\cdot 10^{-6} : 3.\cdot 10^{-15}\,\text{s}$

- a_μ is $(m_\mu/m_e)^2 \sim 4. \times 10^4$ times more sensitive to new Φ than a_e
- a_τ is even more sensitive to new Φ, but is too shortly lived



(Miller et al '12)



(SPS1a)

Big > 0 effect $\rightarrow \text{sign}\mu = 1$, $M_{\text{N}\phi} \sim 100 \div 500\,\text{GeV}$, $\tan\beta \sim 3 \div 40$

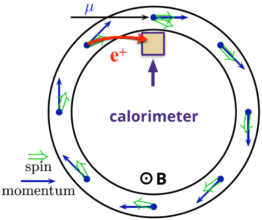
Big question

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, what is the new Φ and can it be seen elswhere?

Experimental measurement of a_μ

Measurement principle for a_μ



Precession determined by

$$\vec{\mu}_\mu = 2(1+a_\mu)\frac{Qe}{2m_\mu}\vec{S}$$

$$\vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c}\vec{S}$$

$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} + \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] - \eta_\mu \frac{Qe}{2m_\mu}\left[\frac{\vec{E}}{c} + \vec{\beta}\times\vec{B}\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &

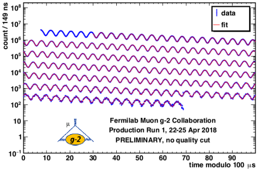
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9)\times 10^{-19} e\cdot\mathrm{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

$$\rightarrow \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$

Fermilab E989 @ magic γ: measurement (simplified)



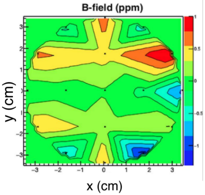
ω_a

Extract from decay positron time spectra

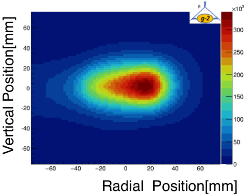
$$N(t) = N_0 e^{-t/\tau_\mu}[1 + A\cos(\omega_a t + \phi)]$$

$$a_\mu = \left(\frac{g_e}{2}\right)\left(\frac{\omega_a}{\langle\omega_p\rangle}\right)\left(\frac{\mu_p}{\mu_e}\right)\left(\frac{m_\mu}{m_e}\right)$$

0.26 ppt 3 ppb 22 ppb ⇨ 2017 CODATA



Map the magnetic field



Obtain muon distribution In the storage ring

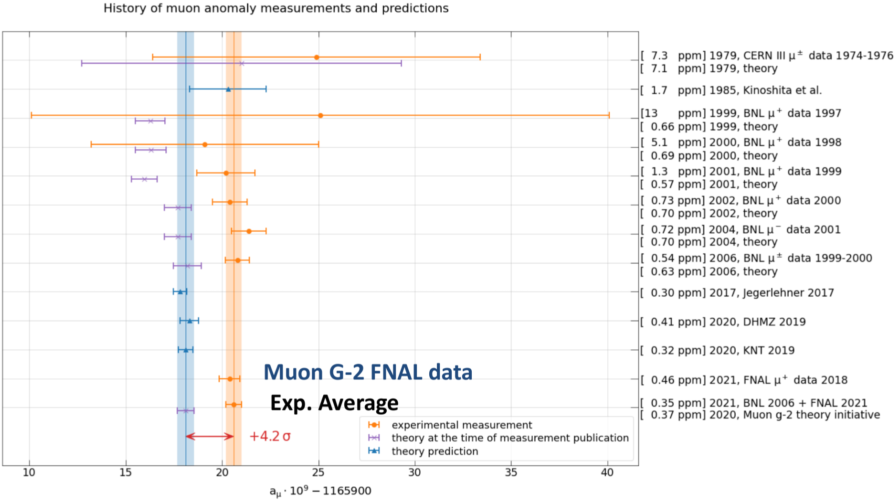
$$\langle\omega_p\rangle \approx \omega_p \otimes \rho(r)$$

Average magnetic field weighted by muon distribution

ω_p: free proton precession frequency

Using proton NMR $\hbar\omega_p = 2\mu_p B$

$g-2$ updated experimental history (8 April 2021)



$$a_\mu(\text{AVG}) = 116\,592\,061(41) \times 10^{-11} \quad (0.35\,\text{ppm}).$$

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G. Venanzoni, CERN Seminar, 8 April 2021

Based on only 6% of expected FNAL data!

Standard model calculation of a_μ

At needed precision: all three interactions and most SM particles

$$
\begin{aligned}
a_\mu^{\rm SM} &= a_\mu^{\rm QED} + a_\mu^{\rm had} + a_\mu^{\rm EW} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi\sin\theta_W}\right)^2\left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

QED contributions to a_ℓ

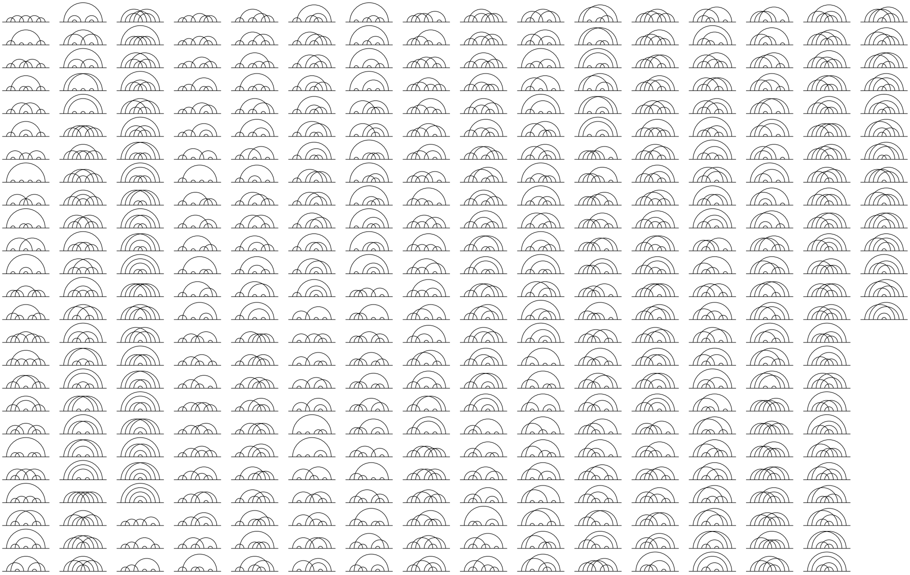
Loops with only photons and leptons

$$a_\ell^{\text{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}$, $A_1^{(4)}$, $A_1^{(6)}$, $A_2^{(4)}$, $A_2^{(6)}$, $A_3^{(6)}$ known analytically (Schwinger '48; Sommerfield '57, '58; Petermann '57; ...)
- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta '95; Aguilar et al '08; Aoyama, Kinoshita, Nio '96-'18)
    - Automated generation of diagrams
    - Numerical evaluation of loop integrals
    - Only some diagrams are known analytically
    - Not all contributions are fully, independently checked

5-loop QED diagrams



(Aoyama et al '15)

QED contribution to a_μ

$$a_\mu^{\text{QED}}(Cs) = 1\,165\,847\,189.31(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(23)_{\alpha(Cs)} \times 10^{-12}\ [0.9\ \text{ppb}]$$

$$a_\mu^{\text{QED}}(a_e) = 1\,165\,847\,188.42(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(28)_{\alpha(a_e)} \times 10^{-12}[0.9\ \text{ppb}]$$

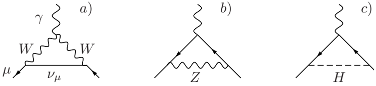
(Aoyama et al '12, '18, '19)

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} = 734.2(4.1) \times 10^{-10}$$

$$\stackrel{?}{=} a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$

Electroweak contributions to a_μ: Z, W, H, etc. loops

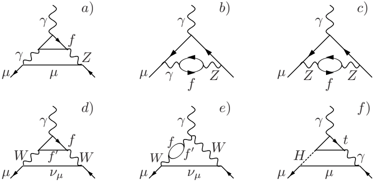
1-loop



$$a_\mu^{\mathrm{EW},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right)$$
$$= 19.479(1) \times 10^{-10}$$

(Gnendiger et al '15, Aoyama et al '20 and refs therein)

2-loop



$$a_\mu^{\mathrm{EW},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right)$$
$$= -4.12(10) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_\mu^{\mathrm{EW}} = 15.36(10) \times 10^{-10}$$

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

Clearly right order of magnitude:

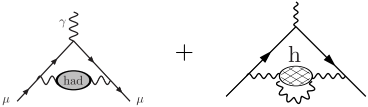
$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

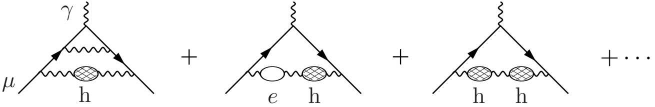
Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

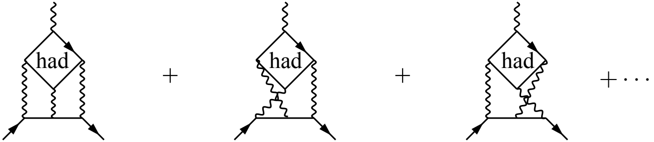
Hadronic contributions to a_μ: diagrams



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$

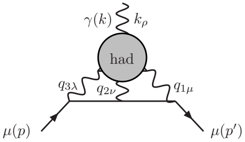


$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

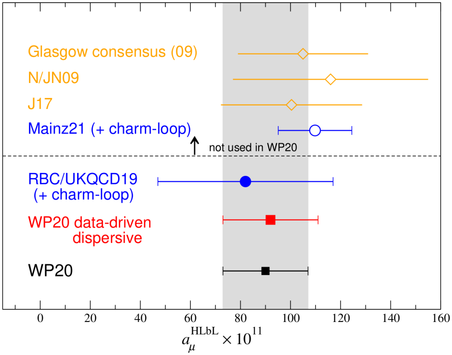


$\rightarrow a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

Hadronic light-by-light



- HLbL much more complicated than HVP, but ultimate precision needed is $\simeq$ 10% instead of $\simeq$ 0.2%
- For many years, only accessible to models of QCD w/ difficult to estimate systematics (Prades et al '09): $a_\mu^{\text{HLbL}} = 10.5(2.6) \times 10^{-10}$
- Also, lattice QCD calculations were exploratory and incomplete
- Tremendous progress in past 5 years:
  - → Phenomenology: rigorous data driven approach [Colangelo, Hoferichter, Kubis, Procura, Stoffer,. . . '15-'20]
  - → Lattice: first two solid lattice calculations
- All agree w/ older model results but error estimate much solid and will improve
- Agreed upon average w/ NLO HLbL and conservative error estimates [WP '20]
- $a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} = 709.7(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP}}$



[Colangelo '21]

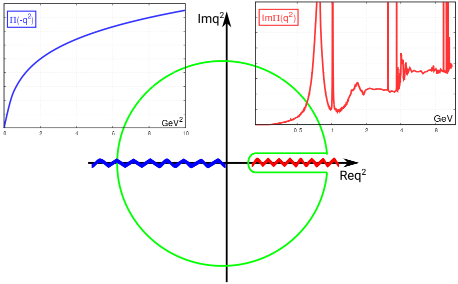
Hadronic vacuum polarization (HVP)

- $\Pi_{\mu\nu}(q) =$ 

$= (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi(q^2)$

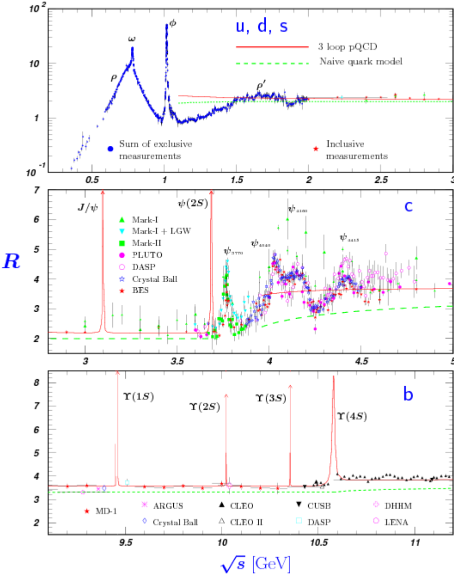
- For $a_\mu^{\text{LO-HVP}}$ need $\hat\Pi(q^2) \equiv \Pi(q^2) - \Pi(0)$ for spacelike $q^2 = -Q^2$ and $Q^2 \in [0, \infty[$

- $\hat\Pi(q^2)$ is real and analytic except for cut along real, positive q^2 axis



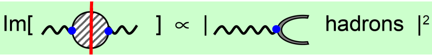
- Can get $\hat\Pi(q^2)$ for spacelike q^2 from contour integral $\rightarrow$ dispersion relation

HVP from $e^+e^- \to$ had (or $\tau \to \nu_\tau +$ had)



(PDG compilation)

Use (Bouchiat et al 61) optical theorem (unitarity)



$$\mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \to \mathrm{had})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

and a once subtracted dispersion relation (analyticity)

$$\begin{aligned}\hat{\Pi}(Q^2) &= \int_0^\infty ds\, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) \\ &= \frac{Q^2}{12\pi^2} \int_0^\infty ds\, \frac{1}{s(s+Q^2)} R(s)\end{aligned}$$

$\Rightarrow \hat{\Pi}(Q^2)$ & $a_\mu^{\text{LO-HVP}}$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

Can also use $I(J^{PC}) = 1(1^{--})$ part of $\tau \to \nu_\tau +$ had and isospin symmetry + corrections

Standard model prediction and comparison to experiment

SM prediction vs experiment on April 7, 2021 (v1)

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	693.9 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
	693.1 ± 4.0	[WP '20]
HVP LO (lattice<2021)	711.6 ± 18.4	[WP '20]
HVP NLO	-9.83 ± 0.07	
		[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice<2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
HLbyL NLO (pheno)	0.2 ± 0.1	[WP '20]
QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (R-ratio)	684.5 ± 4.0	[WP '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.37 ppm]	11659181.0 ± 4.3	[WP '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp $-$ SM	25.1 ± 5.9 [4.2σ]	

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} - a_\mu^{\text{HVP NLO+NNLO}} = 718.2(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP LO}}$$

SM prediction vs experiment on April 7, 2021 (v2)

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
HVP LO (R-ratio)	692.8 ± 2.4	[KNT '19]
	693.9 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
	693.1 ± 4.0	[WP '20]
HVP LO (lattice)	707.5 ± 5.5	[BMWc '20]
HVP NLO	-9.83 ± 0.07	[Kurz et al '14, Jegerlehner '16, WP '20]
HVP NNLO	1.24 ± 0.01	[Kurz '14, Jeger. '16]
HLbyL LO (pheno)	9.2 ± 1.9	[WP '20]
HLbyL LO (lattice<2021)	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
HLbyL LO (avg)	9.0 ± 1.7	[WP '20]
HLbyL NLO (pheno)	0.2 ± 0.1	[WP '20]
QED [5 loops]	11658471.8931 ± 0.0104	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[Gnendiger '15, WP '20]
HVP Tot. (lat. + R-ratio)	698.9 ± 5.5	[WP '20, BMWc '20]
HLbL Tot.	9.2 ± 1.8	[WP '20]
SM [0.49 ppm]	11659195.4 ± 5.7	[WP '20 + BMWc '20]
Exp [0.35 ppm]	11659206.1 ± 4.1	[BNL '06 + FNAL '21]
Exp − SM	10.7 ± 7.0 $[1.5\sigma]$	

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} - a_\mu^{\text{HLbL}} - a_\mu^{\text{HVP NLO+NNLO}} = 718.2(4.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{HVP LO}}$$

A brief introduction to lattice QCD

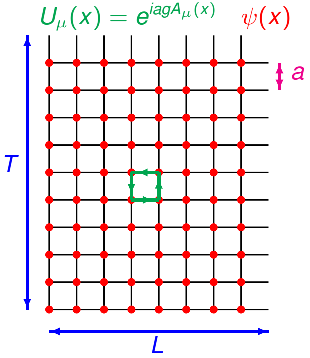
What is lattice QCD (LQCD)?

To describe matter w/ sub-% precision, QCD requires $\geq$ 104 numbers at every spacetime point
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G - \int \bar{\psi} D[M] \psi}\, O[U, \psi, \bar{\psi}]$$
$$= \int \mathcal{D}U\, e^{-S_G}\, \mathrm{det}(D[M])\, O[U]_{\mathrm{Wick}}$$

- $\mathcal{D}U e^{-S_G}\, \mathrm{det}(D[M]) \geq 0$ & finite # of dofs
  $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

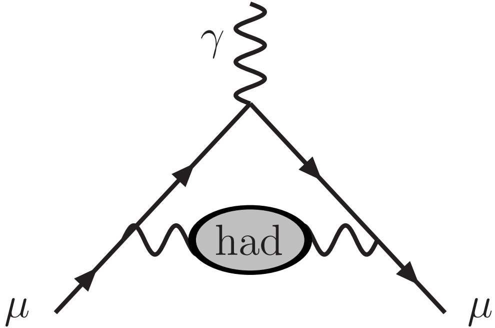
Our “accelerators”

Such computations require some of the world’s most powerful supercomputers

- 1 year on supercomputer
  ∼ 100 000 years on laptop

- In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).

Lattice QCD calculation of a_μ^{HVP}



All quantities related to a_μ will be given in units of 10^{-10}

HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ (Blum '02)

$$\Pi_{\mu\nu}(Q) =$$

$$= \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then (Lautrup et al '69, Blum '02)

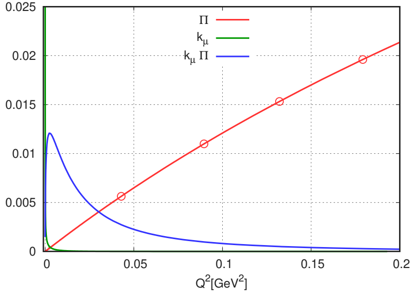
$$a_\ell^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} k(Q^2/m_\ell^2) \hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$ and

$$k(r) = \left[r + 2 - \sqrt{r(r+4)}\right]^2 / \sqrt{r(r+4)}$$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\text{MeV}$ for μ

However, $Q_{\text{min}} \equiv \frac{2\pi}{T} \sim 135\,\text{MeV}$ for lattice w/ $T = \frac{3}{2}L \sim 9\,\text{fm}$



$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

Low-Q^2 challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q=0)\neq 0$ in FV that contaminates $\Pi(Q^2)\sim\Pi_{\mu\nu}(Q)/Q^2$ for $Q^2\to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\rm min}=2\pi/T\sim 135\,\mathrm{MeV}>\frac{m_\mu}{2}\sim 50\,\mathrm{MeV}$ for $T\sim 9\,\mathrm{fm}$

$$\Downarrow$$

- Compute on $T\times L^3$ lattice in $N_f=2+1+1$ QCD

$$C^{\rm iso}_{TL}(t)=\frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(x)J_i(0)\rangle$$

- Decompose ($C^{I=1}_{TL}=\frac{9}{10}C^{ud}_{TL}$)

$$C^{\rm iso}_{TL}(t)=C^{ud}_{TL}(t)+C^{s}_{TL}(t)+C^{c}_{TL}(t)+C^{\rm disc}_{TL}(t)=C^{I=1}_{TL}(t)+C^{I=0}_{TL}(t)$$

- Define $\forall Q_0\in\mathbb{R}$ (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C) (see also Charles et al '17)

$$\hat{\Pi}^f_{TL}(Q^2)\equiv\Pi^f_{TL}(Q^2)-\Pi^f_{TL}(0)=\frac{1}{3}\sum_{i=1}^{3}\frac{\Pi^f_{ii,TL}(0)-\Pi^f_{ii,TL}(Q)}{Q^2}-\Pi^f_{TL}(0)=a\sum_{t=0}^{T-a}\ \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2}+\frac{t^2}{2}\right]\ \mathrm{Re}\,C^f_{TL}(t)$$

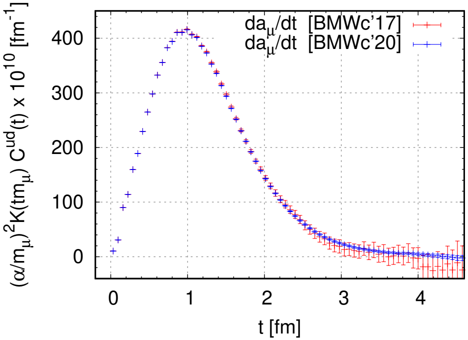
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_{TL}^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \alpha^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2}{}' K(tm_\ell, Q^2_{\max}/m_\ell^2)\ \mathrm{Re}C_{TL}^f(t)$$

where

$$K(\tau, r_{\max}) = \int_0^{r_{\max}} dr\, k(r) \left(\tau^2 - \frac{4}{r}\sin^2\frac{\tau\sqrt{r}}{2}\right)$$

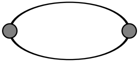


$(144 \times 96^3$, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

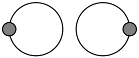
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light π

E. Two types of contributions



quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

F. $\langle J_\mu^{ud}(x)J_\nu^{ud}(0)\rangle_{qc}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ many algorithmic improvements + very high statistics + rigorous bounds

G. Must control $\langle J_\mu(x)J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_\mu$ $\rightarrow L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: 0.134 $\rightarrow$ 0.064 fm

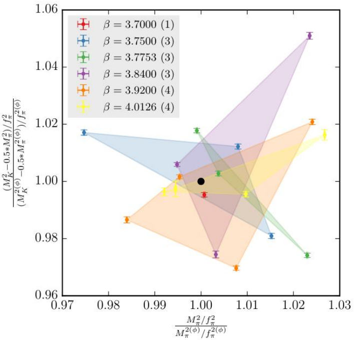
$\rightarrow$ improve approach to continuum limit w/ phenomenological models (SRHO, SMLLGS) w/ 2-loop SU(2) SχPT for systematic error

Simulation details: ad D - H

27 high-statistics simulations w/ N_f=2+1+1 flavors of 4-stout staggered quarks:

- Bracketing physical m_{ud}, m_s, m_c
- 6 a's: 0.134 → 0.064 fm
- $L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$
- Conserved EM current

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	64 × 48	904
3.7500	0.1191	96 × 56	2072
3.7753	0.1116	84 × 56	1907
3.8400	0.0952	96 × 64	3139
3.9200	0.0787	128 × 80	4296
4.0126	0.0640	144 × 96	6980



For sea-quark QED corrections

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	24 × 48	716
		48 × 64	300
3.7753	0.1116	28 × 56	887
3.8400	0.0952	32 × 64	4253

- State-of-the-art techniques:
  - EigCG
  - Low mode averaging [Neff et al '01, Giusti et al '04, . . .]
  - All mode averaging [Blum et al '13]
  - Solver truncation [Bali et al '09]

⇒ Nearly 20,000 gauge configurations

⇒ 10's of millions of measurements

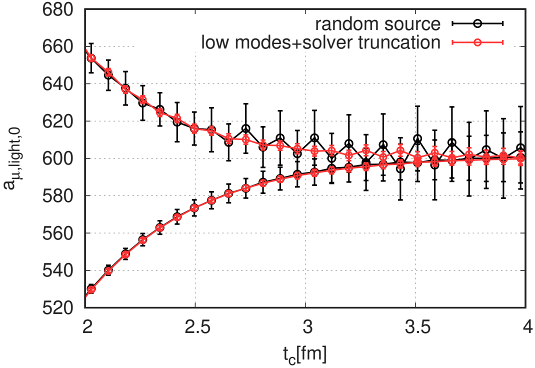
Noise reduction: ad F-G

N/S in $C_L^{ud}(t)$ grows like $e^{(M_\rho - M_\pi)t}$

- LMA: use exact (all-to-all) quark propagators in IR and stochastic in UV [Giusti et al '04]

- Decrease noise by replacing $C_L^{ud}(t)$ by average of rigorous upper/lower bounds above $t_c = 4\,\mathrm{fm}$ [Lehner '16, BMWc '17]

  $$0 \le C_L^{ud}(t) \le C_L^{ud}(t_c)\, e^{-E_{2\pi}(t-t_c)}$$



$\Rightarrow \times 5$ in precision: few pemil accuracy on each ensemble

More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0,+\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$

$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\,\hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

using $O(\alpha_s^4)$ results from `rhad` package [Harlander et al '03]

J. Include c quark for higher precision and good matching onto perturbation theory $\rightarrow$ done

K. Even in our large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects can be significant

$\rightarrow$ 1-loop SU(2) χPT [Aubin et al '16] suggests 2% even in our large volumes

$\rightarrow$ perform dedicated FV study w/ even larger volumes $(\sim 11\,\mathrm{fm})^4$

$\rightarrow$ check and supplement w/ 2-loop χPT [Bijnens et al '99, BMWc '20], ρ-π-γ EFT (RHO) [Sakurai '60, Jegerlehner et al '11, Chakraborty et al '17], Gounaris-Sakurai inspired model (MLLGS) [GS '68, Lellouch & Lüscher '01, Meyer '11, Francis et al '13], Hansen-Patella (HP) [Hansen et al '19, '29]

L. Our $N_f = 2+1+1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at permil-level precision

$\rightarrow$ perform lattice calculation of ALL $O(\alpha)$ and $O(\delta m = m_d - m_u)$ effects

Yet more challenges

M. Need permil determination of QCD scale in our simulations

$\Rightarrow$ 2‰ calculation of Ω^- baryon mass

$\Rightarrow$ Calculate and use Wilson-flow scale [Lüscher '10, BMWc '12] $w_0 = 0.17236(29)(66)$ for defining isospin limit

N. Need thorough and robust determination of statistical and systematic errors

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Each one is weighted by AIC weight

$$\text{AIC} \sim \exp\left[-\frac{1}{2}(\chi^2 + 2n_{\text{par}} - n_{\text{data}})\right]$$

  - Simplify w/ importance sampling
  - Use median of distribution for central values
  - Use $16 \div 84\%$ confidence interval to get total error

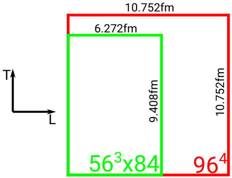
(Nature paper has 95 pp. Supplementary information detailing methods)

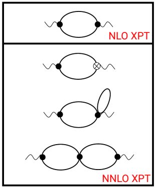
Finite-volume corrections: ad K

Early estimate of these e^{-LM_π} effects [Aubin et al '16]: 2% on $a_\mu^{\text{LO-HVP}}$ in our $L = 6$ simulations

$\rightarrow$ Perform dedicated lattice study

- 4 very-high statistics $N_f = 2+1$, super-smeared (4HEX) simulations
- Tuned so that staggered M_π^{HMS} brackets physical M_π
- L up to 11 fm ($a \simeq 0.112$ fm)!





$\rightarrow$ Check w/ EFTs and models: dominated by long-distance $\pi\pi$ effects

- NNLO (2-loop) χPT [Bijnens '99, Aubin et al '19, BMWc '20]
- Lellouch-Lüscher formalism w/ Gounaris-Sakurai model (LLGS) [Lellouch & Lüscher '01,Meyer '11, Francis '13, Giusti et al '18, BMWc '20]
- QFT relation to Compton scattering (HP) [Hansen et al '19-'20]
- ρ-π-γ EFT (RHO) [Sakurai '60, Jegerlehner & Szafron '11, HPQCD '17]

$[\times 10^{-10}]$	lattice	NLO	NNLO	MLLGS	HP	RHO
$a_\mu^{\text{LO-HVP}}(\text{big}) - a_\mu^{\text{LO-HVP}}(\text{ref})$	18.1(2.0)(1.4)	11.6	15.7	17.8	16.7	15.2

Model validation $\Rightarrow$ $a_\mu^{\text{LO-HVP}}(\infty) - a_\mu^{\text{LO-HVP}}(\text{big}) = 0.6(3) \times 10^{-10}$ from NLO & NNLO χPT

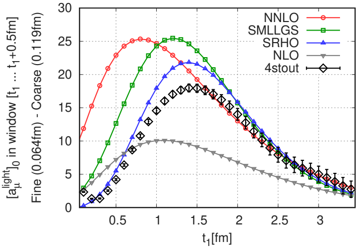
$$a_\mu^{\text{LO-HVP}}(\infty) - a_\mu^{\text{LO-HVP}}(\text{ref}) = 18.7(2.0)_{\text{stat}}(1.4)_{\text{cont}}(0.3)_{\text{big}}(0.6)_{I=0}(0.1)_{\text{qed}}[2.5]$$

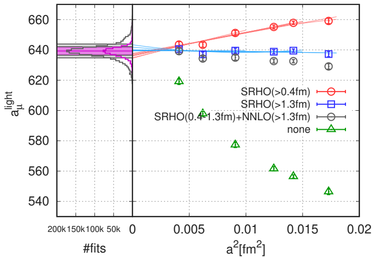
Continuum extrapolation: ad H

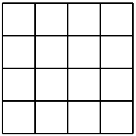
Long-distance discretization effects in $a_{\mu,ud}^{\text{LO-HVP}}$ due to taste violations in $\pi\pi$ states [HPQCD '17]

Correct w/ SMLLGS [BMWc '20] or SRHO [HPQCD '17]

- Parameters fixed w/ experiment
- Reproduces observed discretization effects well
- Corrections vanish in continuum limit
- 6 a's $\rightarrow$ full control over continuum limit







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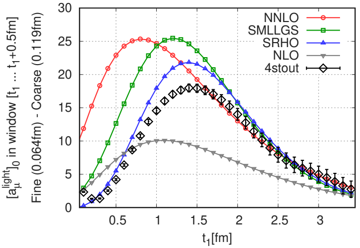

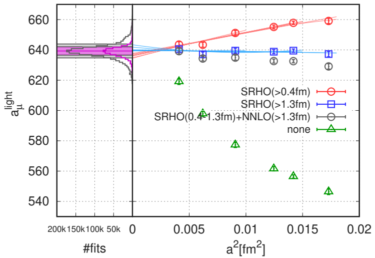

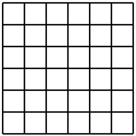

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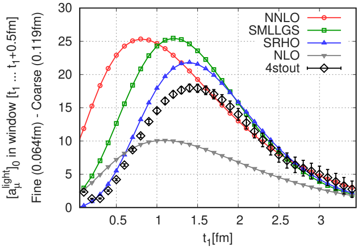

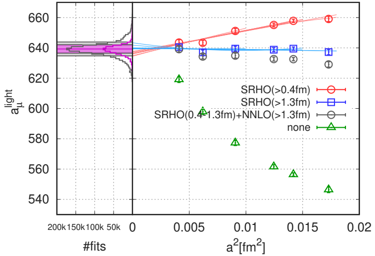

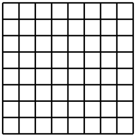

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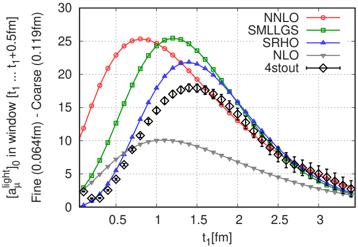

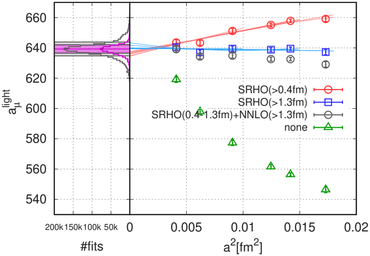

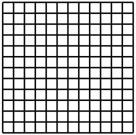

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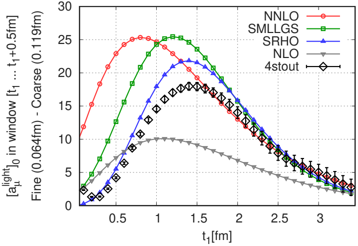

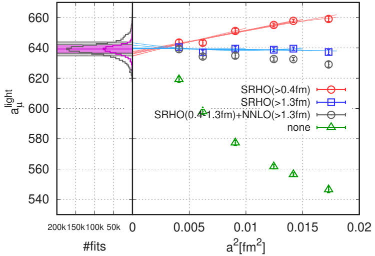

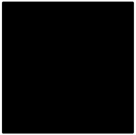

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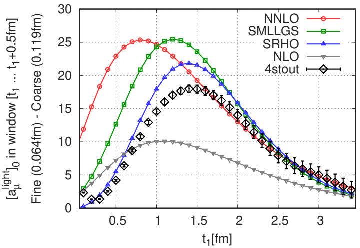
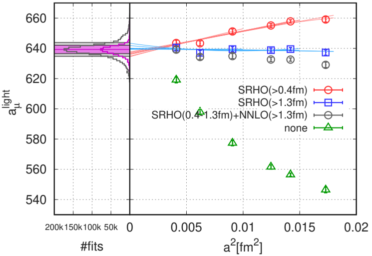




- Allow different improvements in different time windows
- Improves approach to continuum limit $\Rightarrow$ reduced uncertainties
- Does NOT modify this limit $\Rightarrow$ NO model dependence of result
- Systematics: • cuts on a; • SRHO vs SLLGS + NNLO SχPT; • different window boundaries

Including isospin breaking on the lattice: ad I

$$S_{\mathrm{QCD+QED}} = S_{\mathrm{QCD}}^{\mathrm{iso}} + \frac{1}{2}\delta m \int (\bar{d}d - \bar{u}u) + ie \int A_\mu j_\mu, \qquad j_\mu{=}\bar{q}Q\gamma_\mu q, \qquad \delta m{=}m_d{-}m_u$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle j_\mu j_\nu \rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and QCD scale

(1) operator insertion method [RM123 '12, '13, ...]

$$\begin{aligned}\langle \mathcal{O} \rangle_{\mathrm{QCD+QED}} \quad = \quad & \langle \mathcal{O}_{\mathrm{Wick}} \rangle_{G_\mu}^{\mathrm{iso}} - \frac{\delta m}{2} \langle [\mathcal{O} \int (\bar{d}d - \bar{u}u)]_{\mathrm{Wick}} \rangle_{G_\mu}^{\mathrm{iso}} - \frac{e^2}{2} \langle [\mathcal{O} \int_{xy} j_\mu(x) D_{\mu\nu}(x-y) j_\nu(y)]_{\mathrm{Wick}} \rangle_{G_\mu}^{\mathrm{iso}} \\ & + e^2 \langle\langle \left[\mathcal{O} \partial_e \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \int_x j_\mu(x) A_\mu(x) - \frac{1}{2} \mathcal{O} \partial_e^2 \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \right]_{\mathrm{Wick}} \rangle_{A_\mu} \rangle_{G_\mu}^{\mathrm{iso}}\end{aligned}$$

(2) direct method [Eichten et al '97, BMWc '14, ...]

Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

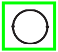
(3) combinations of (1) & (2) [BMWc '20]

We include ALL $O(e^2)$ and $O(\delta m)$ effects

For valence e^2 effects use easier (2), and for δm and e^2 sea effects, (1)

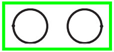
Summary of contributions to $a_\mu^{\text{LO-HVP}}$

Isospin symmetric


connected light
633.7(2.1)(4.2)


connected strange
53.393(89)(68)

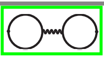
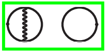

connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

QED isospin-breaking: valence

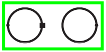


connected -1.23(40)(31)

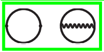


disconnected -0.55(15)(10)

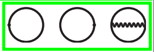
Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

QED isospin-breaking: sea



connected 0.37(21)(24)

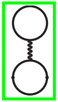


disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

Finite-size effects

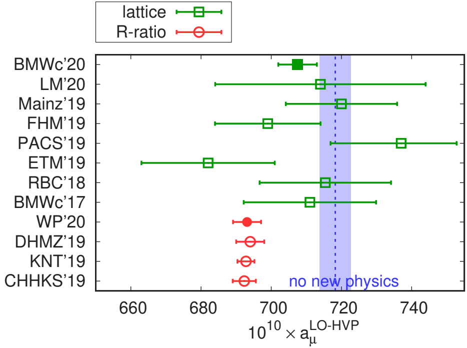
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

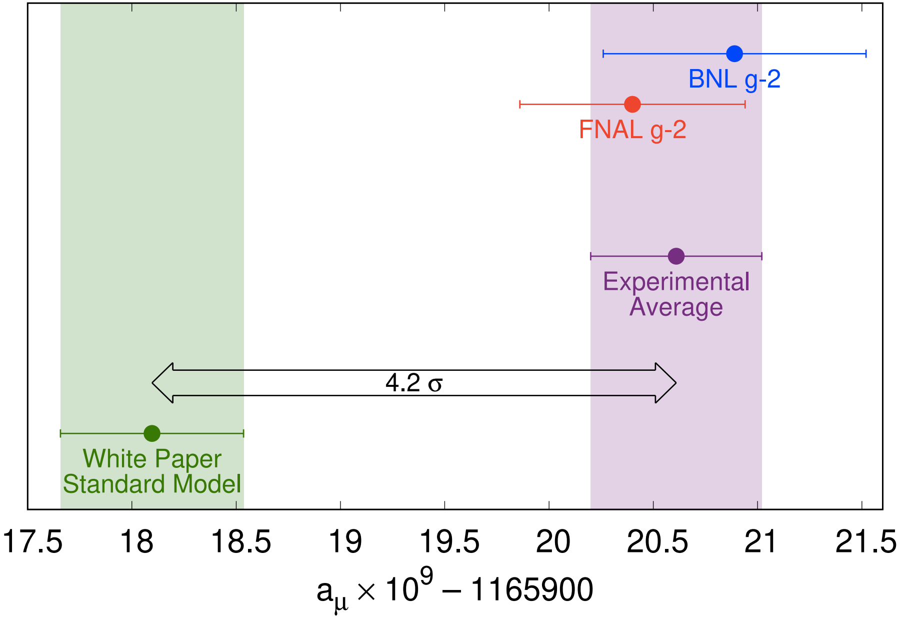
Comparison and outlook

Comparison

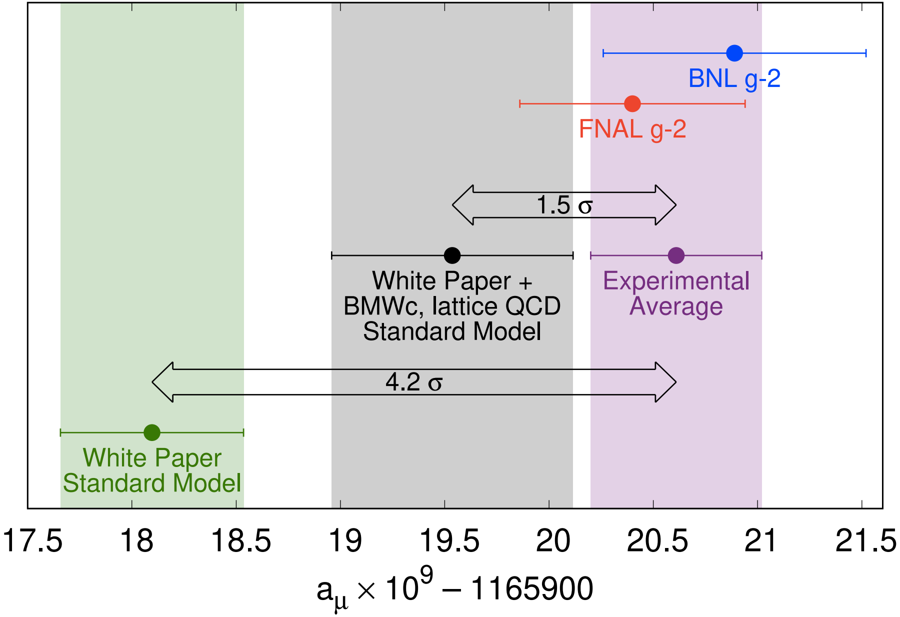


- Consistent with other lattice results
- Total uncertainty is $\sim \div 3$ ...
- ... and comparable to R-ratio and experiment
- Consistent w/ experiment @ 1.5σ ("no new physics" scenario) !
- 2.1σ larger than R-ratio average value [WP '20]

Fermilab plot, April 7 2021, v1

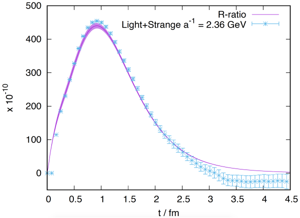


Fermilab plot, April 7 2021, v2

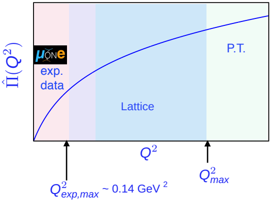


What next?

- FNAL E989 to reduce error by factor of 2.5 in coming years
- HLbL error must be reduced by factor of 2
- Must reduce ours by factor of 4 !
- Will experiment still agree with our prediction ?
- Must be confirmed by other lattice groups
- If confirmed, must understand why lattice doesn't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty [RBC/UKQCD '18]
- Important to pursue $e^+e^- \to$ hadrons measurements [CMD-3, Belle III, . . .]
- $\mu e \to \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD
- Important to build J-PARC $g_\mu - 2$ and pursue a_e experiments



[RBC/UKQCD '18]



[Marinkovic et al '19]

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BACKUP

Do our results imply NP @ EW scale?

- Passera et al '08: first exploration of connection $a_\mu^{\text{LO-HVP}} \leftrightarrow \Delta^{(5)}_{\text{had}}\alpha(M_Z^2)$
- Crivellin et al '20, most aggressive scenario (see also Keshavarzi et al '20): our results suggest a 4.2σ overshoot in $\Delta^{(5)}_{\text{had}}\alpha(M_Z^2)$ compared to result of fit to EWPO
- Assume same 2.8% relative deviation in R-ratio as we found in $a_\mu^{\text{LO-HVP}}$
- Hypothesis is not consistent w/ BMWc '17 nor new result