Strong and Electro-Weak Matter 2021

$\mathcal{N}=4$ supersymmetric Yang-Mills thermodynamics to order λ^2

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Outline:

- Background and Motivation
- Free energy up to λ^2 of the $\mathcal{N}=4$ supersymmetric Yang-Mills in 4-dimensions($SYM_{4,4}$)
- Large-N_c generalized Padé approximant
- Comparison for scaled entropy density
- Summary and outlook

Background and Motivation

The perturbative expansion of the free energy of the $\text{SYM}_{4,4}$ at high temperature(T) can be written in the form

$$F(\lambda \to 0) \sim T^4 \left[a_0\lambda^0 + a_2\lambda^1 + a_3\lambda^{3/2} + (a_4 + a_4' \log\lambda)\lambda^2 + \mathcal{O}\left(\lambda^{5/2}\right)\right], \quad (1)$$

where $\lambda = g^2 N_c$, is the 't Hooft coupling, N_c and g is the color and the coupling constant in QCD, respectively.

$\sim a_0\lambda^0$ ⟹ Free energy of the ideal $\text{SYM}_{4,4}$ plasma

$\sim a_2\lambda^1$ ⟹ The 2-loop vacuum Feynman diagrams

$\sim \mathcal{O}(\lambda^{n>1})$ ⟹ Reorganizing the perturbation theory[1] to account for the thermal mass of the gauge bosons and scalar fields

⟹ non-analytic, e.g. $\mathcal{O}(\lambda^{3/2})$

⟹ non-analytic, e.g. $\mathcal{O}(\lambda^2 \log\lambda)$

D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys.53 (1981) 43

P.B. Arnold and C.-X. Zhai, hep-ph/9408276; hep-ph/9410360

Like QCD[2], there are uncanceled infrared divergences at the three-loop level

Background and Motivation

Full $\mathcal{O}(\lambda^2)$ $\Longrightarrow$ Require 3-loop calculation $\Longrightarrow$ Generate $\mathcal{O}(\lambda^{5/2})$

$\Uparrow$

Consider the dressed propagators

In the weak-coupling limit the $\mathrm{SYM}_{4,4}$ free energy has been calculated through order $\lambda^{3/2}$ giving[3] A.Fotopoulos and T.R. Taylor, hep-th/9811224 M.A. Vazquez-Mozo, hep-th/9905030 C.-j. Kim and S.-J. Rey, hep-th/9905205

$$\frac{\mathcal{F}}{\mathcal{F}_{\mathrm{ideal}}} = \frac{\mathcal{S}}{\mathcal{S}_{\mathrm{ideal}}} = 1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} + \cdots \ , \quad (2)$$

where: $\mathcal{F}_{\mathrm{ideal}} = -d_A \pi^2 T^4/6$ is the ideal or Stefan-Boltzmann limit of the free energy,

$\mathcal{S}_{\mathrm{ideal}} = 2 d_A \pi^2 T^3/3$ is the entropy density,

$d_A = N_c^2 - 1$ is the dimension of the adjoint representation.

The aim of our work was to get the 4th term ~$(a_4 + a_4' \log\lambda)\lambda^2$ in eq.(1)

Background and Motivation

➤ **Under the scheme called**
regularization by dimensional reduction **(RDR)**[4]

W.Siegel, Phys. Lett. B 84(1979) 193
D.Capper, D.Jones and P.Van Nieuwenhuizen, Nuclear Physics B 167(1980) 479
L.V. Avdeev and A.A. Vladimirov, Nucl. Phys. B 219 (1983) 262
P.Howe, A.Parkes and P.West, Physics Letters B 147 (1984) 409

RDR $\Longrightarrow$ A modified version of the dimensional regularization based on dimensional reduction which manifestly preserves gauge invariance, unitarity, and supersymmetry

$\Downarrow$

Applied to pure Yang-Mills theory; Yang-Mills theory coupled to scalars and fermions; supersymmetric QED and $\mathcal{N} = 1$ SYM

$SYM_{4,4}$ can be obtained by dimensional reduction from the $\mathcal{N}$=1 SYM theory in 10-dimensions ($SYM_{1,10}$) without thermal mass contributions.

Background and Motivation

➢ **Lagrangian density for $\mathrm{SYM}_{1,10}$ and $\mathrm{SYM}_{4,4}$**

The Lagrangian density for the $\mathrm{SYM}_{1,10}$ is

$$\mathcal{L}_{\mathrm{SYM}_{1,10}} = \mathrm{Tr}\left[-\frac{1}{2}G_{MN}^2 + 2i\bar{\psi}\Gamma^M D_M\psi\right], \qquad (3)$$

where $M, N = 0, \cdots, 9$, the field strength tensor is $G_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N]$, and $D_M = \partial_M - ig[A_M, \cdot]$ is the covariant derivative in the adjoint representation of $SU(N_c)$.

The Lagrangian density for the $\mathrm{SYM}_{4,4}$ can be obtained by the dimensional reduction of eq.(3), which is

$$\mathcal{L}_{\mathrm{SYM}_{4,4}} = \mathrm{Tr}\left[-\frac{1}{2}G_{\mu\nu}^2 + (D_\mu\Phi_A)^2 + i\bar{\psi}_i \not{D}\psi_i - \frac{1}{2}g^2(i[\Phi_A, \Phi_B])^2 - ig\bar{\psi}_i\left[\alpha_{ij}^{\mathtt{p}}X_{\mathtt{p}} + i\beta_{ij}^{\mathtt{q}}\gamma_5 Y_{\mathtt{q}}, \psi_j\right]\right], \qquad (4)$$

where $\mu, \nu = 0, \cdots, 3$. There are four Majorana fermions, ψ_i with $i = 1, \cdots, 4$, and six independent real scalar fields, $\Phi \equiv (X_1, Y_1, X_2, Y_2, X_3, Y_3)$.

Background and Motivation

➢ The resummed Lagrangian density

The resummed Lagrangian density can be written as

$$\mathcal{L}^{\rm resum}_{{\rm SYM}_{4,4}} = \{\mathcal{L}_{{\rm SYM}_{4,4}} + {\rm Tr}[m_D^2 A_0^2 \delta_{p0} - M^2 \Phi_A^2 \delta_{p0}]\} - {\rm Tr}[m_D^2 A_0^2 \delta_{p0} - M^2 \Phi_A^2 \delta_{p0}] \quad (5)$$

Contribute to the gluon and scalar propagators

Contribute to the gluon and scalar counterterms

where

1) m_D is the thermal gluon mass and M is the thermal scalar mass, only contribute to the zero Matsubara modes of the two fields;

2) δ_{p_0} is shorthand for the Kronecker delta function $\delta_{p_0,0}$.

Background and Motivation

The calculation of the thermodynamics up to $\mathcal{O}(\lambda^2)$

Feynman diagrams up to 2-loop order

⇓

Compute directly in $\mathrm{SYM}_{4,4}$ under the RDR scheme , since we must dress the gluon and scalar propagators differently

⇓

Feynman diagrams for the 3-loop order

3-loop gluonic and scalar counterterm with bare propagators

⇓

Compute directly in $\mathrm{SYM}_{4,4}$

⇓

$g_M^M = 4$, $\mathrm{Tr}I_4 = 4$, the dimension for the gluon field $D = 4$ and for all momentum $d = 4 - 2\epsilon$

Massless 3-loop vacuum diagrams

⇓

Compute in $\mathrm{SYM}_{1,10}$ under the RDR scheme

⇓

$g_M^M = 10$, $\mathrm{Tr}I_{32} = 8$, the dimension for the gluon field $\mathcal{D} = 10$ and for all momentum $d = 4 - 2\epsilon$

Background and Motivation

Feynman diagrams up to the three-loop level,

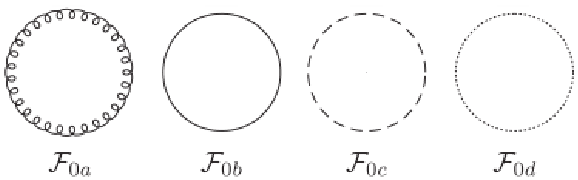

Fig.1 The 1-loop diagrams for $SYM_{4,4}$

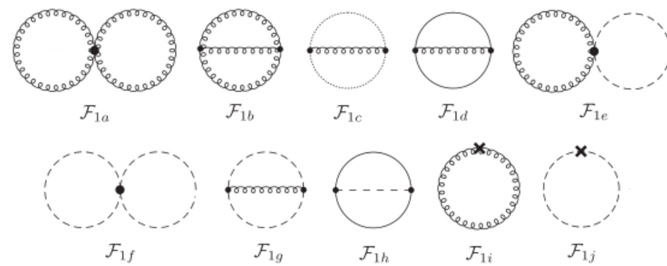

Fig.2 The 2-loop diagrams for $SYM_{4,4}$

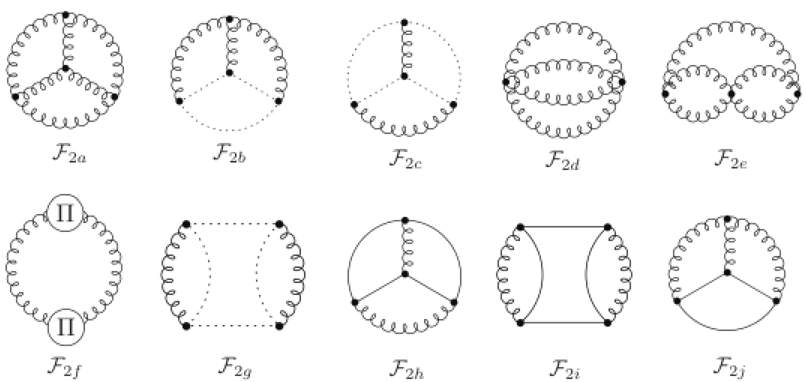

Fig.3 The 3-loop diagrams with bare propagators for $SYM_{1,10}$

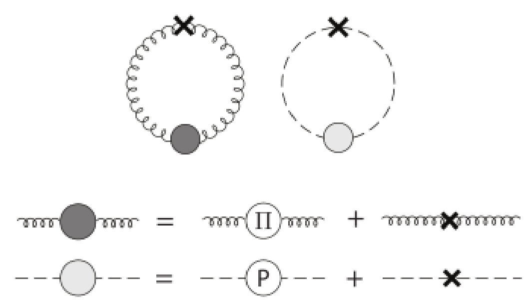

Fig.4 The 3-loop gluonic and scalar counterterm diagrams with bare propagators for $SYM_{4,4}$

Dashed lines indicate a scalar field and dotted lines indicate a ghost field. The crosses are the thermal counterterms.

Free energy up to λ^2 of $\text{SYM}_{4,4}$

- **The resumed one-loop free energy**

$$F_{\text{1-loop}}^{\text{resum}} = d_A \mathcal{F}_{0a} + d_F \mathcal{F}_{0b} + d_S \mathcal{F}_{0c} + d_A \mathcal{F}_{0d} \ , \qquad (6)$$

with $d_F = 4d_A$ and $d_S = 6d_A$. By using the resummed gluon and scalar propagators, one obtains

$$F_{\text{1-loop}}^{\text{resum}} = d_A \left[\frac{D+4}{2} b_0 - 4 f_0 - \frac{T}{12\pi} (m_D^3 + 6M^3) \right] , \qquad (7)$$

where $b_0 \equiv \sum\!\!\!\!\!\!\!\int_P \log P^2 = -\frac{\pi^2}{45} T^4$ and $f_0 \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \log P^2 = \frac{7\pi^2}{360} T^4$.

By imposing $D = 4$, $m_D^2 = 2\lambda T^2$, $M^2 = \lambda T^2$ and truncating at $\mathcal{O}(\epsilon^0)$, we obtain

$$F_{\text{1-loop}}^{\text{resum}} = -d_A \left(\frac{\pi^2 T^4}{6} \right) \left[1 + \frac{3+\sqrt{2}}{\pi^3} \lambda^{3/2} \right] . \qquad (8)$$

Free energy up to λ^2 of $\text{SYM}_{4,4}$

➢ **The resummed two-loop free energy**

$$F_{\text{2-loop}}^{\text{resum}} = d_A\Big\{\lambda[\mathcal{F}_{1a}+\mathcal{F}_{1b}+\mathcal{F}_{1c}+\mathcal{F}_{1d}+\mathcal{F}_{1e}+\mathcal{F}_{1f}+\mathcal{F}_{1g}+\mathcal{F}_{1h}] + \mathcal{F}_{1i}+\mathcal{F}_{1j}\Big\}\,, \tag{9}$$

by using the resummed gluon and scalar propagators, one obtains

$$F_{\text{2-loop}}^{\text{resum}} = \lambda d_A\left\{\frac{D+4}{4}\left[(D+4)b_1^2 - 16b_1f_1 + 8f_1^2\right] + 6\frac{M^2T^2}{(4\pi)^2}\left(\frac{3}{2} - \log\frac{M}{T} + 2\log 2\right) + \frac{m_D^2T^2}{(4\pi)^2}\left(\frac{3}{4} + \frac{D}{8} - \log\frac{m_D}{T} + 2\log 2\right) + 3\frac{m_D M T^2}{(4\pi)^2}\right\}\,, \tag{10}$$

where $b_n \equiv \sum\!\!\!\!\!\!\int_P \frac{1}{P^{2n}}\,, \qquad f_n \equiv \sum\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^{2n}} = (2^{2n+1-d}-1)b_n\,, \quad n \geq 1$.

By imposing $D=4,\ m_D^2 = 2\lambda T^2,\ M^2 = \lambda T^2$ and truncating at $\mathcal{O}(\epsilon^0)$, we obtain

$$F_{\text{2-loop}}^{\text{resum}} = -d_A\left(\frac{\pi^2T^4}{6}\right)\left[-\frac{3}{2\pi^2}\lambda - \frac{3}{2\pi^4}\left(\frac{23}{8} + \frac{3\sqrt{2}}{4} + \frac{15\log 2}{4} - \log\lambda\right)\lambda^2\right]\,. \tag{11}$$

Free energy up to λ^2 of SYM$_{4,4}$

➢ The resummed three-loop free energy

$$F^{\text{resum}}_{\text{3-loop}} = \mathcal{F}^{\text{vacuum}}_{\text{3-loop}} + \mathcal{F}^{\text{sct}}_{\text{3-loop}} + \mathcal{F}^{\text{bct}}_{\text{3-loop}} \,, \qquad (12)$$

where

$$F^{\text{vacuum}}_{\text{3-loop}} = d_A \lambda^2 [\mathcal{F}_{2a} + \mathcal{F}_{2b} + \mathcal{F}_{2c} + \mathcal{F}_{2d} + \mathcal{F}_{2e} + \mathcal{F}_{2f} + \mathcal{F}_{2g} + \mathcal{F}_{2h} + \mathcal{F}_{2i} + \mathcal{F}_{2j}]\Big|^{\mathcal{D}=10}_{d=4-2\epsilon} \,. \qquad (13)$$

Infrared divergences will be generated from eq.(13) due to three-momentum integrations. These divergences will be canceled by the thermal mass counterterm diagrams in Fig.4.

The shaded blob can be expressed as

$$\Delta\Pi_{\mu\nu}(P) \equiv \Pi_{\mu\nu}(P) - \Pi^{\rho\rho}(0)\delta_{\mu 0}\delta_{\nu 0}\delta_{P_0} \,,$$

and

$$\Delta\mathcal{P}(P) \equiv \mathcal{P}(P) - \mathcal{P}(0)\delta_{P_0} \,,$$

where $\Pi_{\mu\nu}(P)$ and $\mathcal{P}(P)$ are the self energy of bosons and scalars.

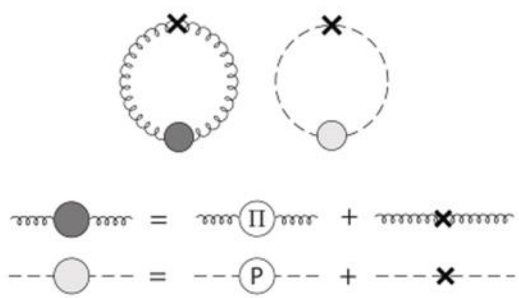

Fig.4

Free energy up to λ^2 of SYM$_{4,4}$

We obtain

$$\mathcal{F}^{\text{sct}}_{\text{3-loop}} = d_A\lambda^2 6[(D+4)b_1 - 8f_1]\left[\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} - 2\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2}\right], \quad (14)$$

$$\mathcal{F}^{\text{bct}}_{\text{3-loop}} = d_A\lambda^2(D-2)[(D+4)b_1 - 8f_1]\left[\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} - 2\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2} - \frac{1}{8}(D+4)\frac{T^2}{(4\pi)^2}\right], \quad (15)$$

where

$$\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} = \frac{T^2}{(4\pi)^2}\left[\frac{1}{4\epsilon} + \log\frac{\bar{\mu}}{4\pi T} + \log 2\pi + \frac{1}{2}\right] + \mathcal{O}(\epsilon),$$

$$\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2} = \frac{T^2}{(4\pi)^2}\log 2 + \mathcal{O}(\epsilon).$$

By imposing $D=4, \mathrm{Tr}I_{32}=8, m_D^2 = 2\lambda T^2,\ M^2 = \lambda T^2$, truncating at $\mathcal{O}(\epsilon^0)$, and adding eq.(13), we obtain

$$F^{\text{resum}}_{\text{3-loop}} = -d_A\left(\frac{\pi^2 T^4}{6}\right)\frac{\lambda^2}{2\pi^4}\left[3 + 3\gamma + 3\frac{\zeta'(-1)}{\zeta(-1)} + 5\log 2 - 6\log\pi\right]. \quad (16)$$

Free energy up to λ^2 of $\mathrm{SYM}_{4,4}$

➢ Final result of the free energy up to λ^2 of $\mathcal{N}$=4 SYM

By combining eqs.(8) (11) and (16), the final result for the resummed free energy up to 3-loop level for $\mathrm{SYM}_{4,4}$ in the RDR scheme is

$$F = -d_A\left(\frac{\pi^2 T^4}{6}\right)\left\{1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{\frac{3}{2}} + \frac{1}{\pi^4}\left[-\frac{45}{16} - \frac{9\sqrt{2}}{8} + \frac{3}{2}\gamma_E + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{25}{8}\log 2 - 3\log\pi + \frac{3}{2}\log\lambda\right]\lambda^2\right\}. \quad (17)$$

1) This result holds for all N_c , and is independent of the momentum scale $\bar{\mu}$;
2) Infrared divergences generated at three-loop level are canceled by considering the thermal mass contribution of gauge bosons and scalars;
3) No coupling constant renormalization counterterm is required, since the coupling does not run in $\mathrm{SYM}_{4,4}$

Large-N_c generalized Padé approximant

With new perturbative coefficients in eq.(17), one can produce an updated Padè approximant[5]

C.-j. Kim and S.-J. Rey, hep-th/9905205
J.P. Blaizot, E.Iancu, U.Kraemmer and A.Rebhan, hep-ph/0611393

Padè approximant is constructed by interpolating between the weak- and strong-coupling limits

Based on the large-N_c structure of the strong-coupling expansion, we find the following form can reconstruct all known coefficients in both the weak- and strong-coupling limits

$$\frac{S}{S_{\mathrm{ideal}}} = \frac{1 + a\lambda^{1/2} + b\lambda + c\lambda^{3/2} + d\lambda^2 + e\lambda^{5/2}}{1 + a\lambda^{1/2} + \bar{b}\lambda + \frac{4}{3}c\lambda^{3/2} + \frac{4}{3}d\lambda^2 + \frac{4}{3}e\lambda^{5/2}} . \qquad (18)$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

To ensure that in the strong-coupling limit
(a) one obtains the correct asymptotic limit of 3/4
(b) terms of the form $\lambda^{-1/2}$, $\lambda^{-1/2}\log\lambda$, λ^{-1}, and $\lambda^{-1}\log\lambda$ do not appear in the strong-coupling expansion.

Large-N_c generalized Padé approximant

To fix the remaining coefficients in eq. (18)

- In the weak-coupling limit, eq.(18) reproduces the perturbative result in eq.(17) through $\mathcal{O}(\lambda^2, \lambda^2 \log \lambda)$
- In the strong-coupling limit, eq.(18) reproduces the eq.(19) through $\mathcal{O}(\lambda^{-3/2})$

The strong coupling behavior of the free energy has been computed using the anti-de Sitter space/CFT (AdS/CFT) correspondence[6] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, hep-th/9805156

$$\frac{\mathcal{F}}{\mathcal{F}_{\rm ideal}} = \frac{\mathcal{S}}{\mathcal{S}_{\rm ideal}} = \frac{3}{4}\left[1 + \frac{15}{8}\zeta(3)\lambda^{-3/2} + \mathcal{O}(\lambda^{-3})\right], \quad (19)$$

then we obtain

$$a = \frac{4\pi^2}{135\zeta(3)} + \frac{2\left(3+\sqrt{2}\right)}{3\pi},$$

$$b = \frac{1}{\pi^2}\log\left(\frac{\lambda}{\pi^2}\right) + \frac{16\pi\left[45\left(3+\sqrt{2}\right)\zeta(3)+\pi^3\right]}{18225\zeta^2(3)} + \frac{72\left(\gamma_E + \frac{\zeta'(-1)}{\zeta(-1)}\right) + 138\sqrt{2} + 109 - 150\log(2)}{72\pi^2},$$

$$\bar{b} = b + \frac{3}{2\pi^2}, \qquad c = \frac{2}{15\zeta(3)}, \qquad d = \frac{180\left(3+\sqrt{2}\right)\zeta(3) + 8\pi^3}{2025\pi\zeta^2(3)}, \qquad e = \frac{2b}{15\zeta(3)} - \frac{3}{5\pi^2\zeta(3)}.$$

Comparison for scaled entropy density

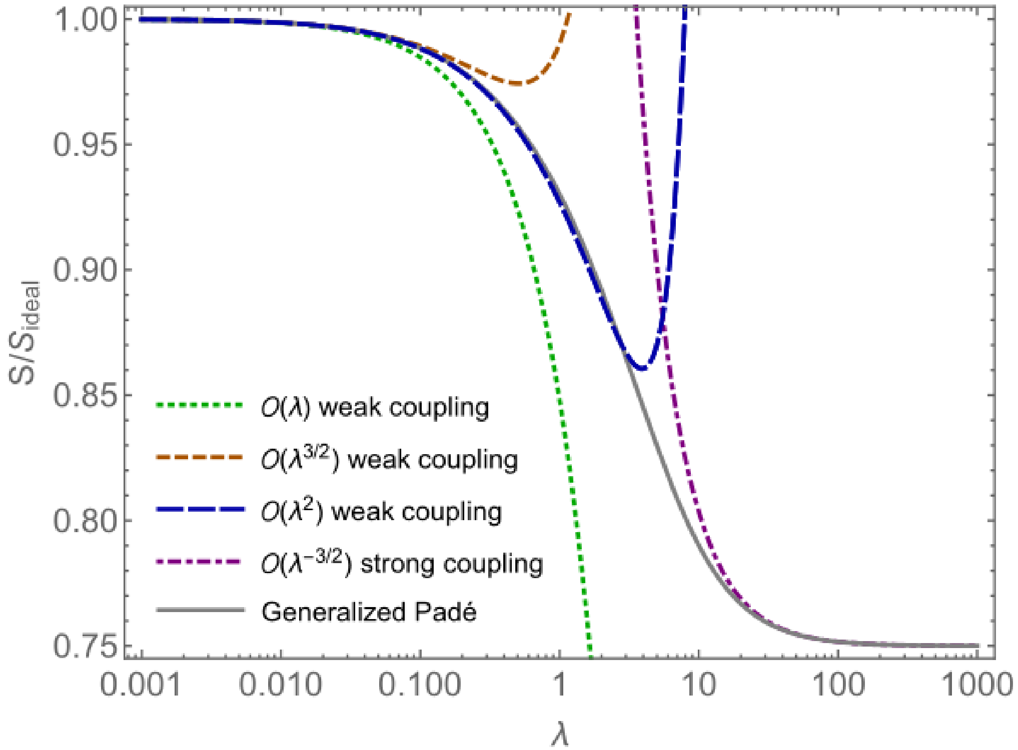

Fig.5 $\mathrm{SYM}_{4,4}$ scaled entropy density $\mathcal{S}/\mathcal{S}_{\text{ideal}}$ as a function of λ. The green dotted, red dashed, and blue long-dashed curves correspond to the perturbative result truncated at $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^{3/2})$, and $\mathcal{O}(\lambda^2)$, respectively. The purple dot-dashed curve corresponds to the large-N_c strong-coupling result truncated at $\mathcal{O}(\lambda^{-3/2})$. The solid gray line is the updated Padè approximant.

Adding each perturbative order extends the estimated range of validity in λ by an order of magnitude.

Summary and outlook

- Calculate the resummed free energy up to order λ^2 for $SYM_{4,4}$ under the RDR scheme.
- Construct a new large-N_c Padè approximant based on our result.
- Compare our final result for the scaled entropy density to the updated Padè approximants.
- In the near future we plan to compute the coefficient of $\lambda^{5/2}$ in the $SYM_{4,4}$ free energy.
- We also plan to extend our prior two-loop HTLpt calculation of $SYM_{4,4}$ thermodynamics to three-loop order.