Phase diagram of linear sigma model with helically imbalanced quarks

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Outline

Motivation

Polarization of QCD matter

- STAR collaboration: Λ polarization in HIC[1]
- Anomalous transport: Chiral + helical vortical effects[2]
- Chirality works at $m = 0$, helicity works for any m.

[1] STAR Collaboration, Nature **548** (2017) 62–65.
[2] VEA, M. N. Chernodub, arXiv:1912.11034 [hep-th].
[3] B. V. Jacak, B. Muller, Science **337** (2012) 310.

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QCD phase transition

- At high temperatures, quarks & gluons become deconfined $\Rightarrow$ QGP.[3]
- Effective models: quarks acquire dynamical medium-dependent mass.
- In the linear σ model, $m_* = g \langle \sigma \rangle \neq 0$ in the confined state.

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Persistent (thermodynamic) polarization

- $\partial_\mu J^\mu = 0 \Rightarrow \widehat{Q}$ which can be modelled thermodynamically via μ.
- μ_A inconsistent with $m_* \neq 0$.
- Is μ_H compatible with QCD models?
- How does μ_H affect QCD phase transition?

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Chirality (γ^5) vs. helicity (h)

- For $m = 0$, γ^5 and $h = \dfrac{\boldsymbol{S} \cdot \boldsymbol{P}}{p}$ share the eigenmodes U_j and $V_j = i\gamma^2 U_j^*$:

$$\begin{pmatrix} 2h \\ \gamma^5 \end{pmatrix} U_j = 2\lambda_j U_j, \qquad \begin{pmatrix} 2h \\ -\gamma^5 \end{pmatrix} V_j = 2\lambda_j V_j, \tag{1}$$

- $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ satisfies $\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi$.
- $J_h^\mu = \bar{\psi}\gamma^\mu h\psi + \overline{h\psi}\gamma^\mu\psi$ satisfies $\partial_\mu J_H^\mu = 0$ (for all m).

- Why is chirality good? Chiral vortical / magnetic / separation / etc. effects
- Why is chirality bad? $m \neq 0$; Axial anomaly ($\partial_\mu J_A^\mu = -\frac{e^2}{8\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}$)

- Why is helicity good? Works at $m \neq 0$; Helical vortical effects
- Why is helicity bad? interactions; anomaly?; ambiguous when $m \neq 0$

CPT symmetries

	Q_V	Q_A	Q_H	$\boldsymbol{J}_V$	$\boldsymbol{J}_A$	$\boldsymbol{J}_H$	$\boldsymbol{\omega}$
C	$-$	$+$	$-$	$-$	$+$	$-$	$+$
P	$+$	$-$	$-$	$-$	$+$	$+$	$+$
T	$+$	$+$	$+$	$-$	$-$	$-$	$-$

$$: \widehat{Q}_V := \sum_j (\hat{b}_j^\dagger \hat{b}_j - \hat{d}_j^\dagger \hat{d}_j), \qquad \sigma_V \simeq \frac{2\mu_H T}{\pi^2} \ln 2 + \frac{\mu_V \mu_A}{\pi^2},$$

$$: \widehat{Q}_A := \sum_j 2\lambda_j (\hat{b}_j^\dagger \hat{b}_j + \hat{d}_j^\dagger \hat{d}_j), \qquad \sigma_A \simeq \frac{T^2}{6} + \frac{\boldsymbol{\mu}^2}{2\pi^2},$$

$$: \widehat{Q}_H := \sum_j 2\lambda_j (\hat{b}_j^\dagger \hat{b}_j - \hat{d}_j^\dagger \hat{d}_j), \qquad \sigma_H \simeq \frac{2\mu_V T}{\pi^2} \ln 2 + \frac{\mu_H \mu_A}{\pi^2}.$$

- J_V^μ, J_A^μ and J_H^μ form a triad: same T, different C and P.
- New vortical effects $\boldsymbol{J}_\ell = \sigma_\ell \boldsymbol{\omega}$ allowed by CPT symmetries.
- H and V share a duality – could h be important in QCD?

Thermodynamics of free fermions with $\mu_{V/A/H}$

- The Lagrangian for free, massive fermions at finite $\mu_{V/A/H}$ is[4]

$$\mathcal{L}_{\rm free} = \overline{\psi}(i\partial\!\!\!/ + \mu_V \gamma^0 + \mu_A \gamma^0\gamma^5 + 2\mu_H \gamma^0 h - m)\psi. \tag{2}$$

- Performing the Fourier transform $\psi \to \chi_p e^{-ip_\mu x^\mu}$, the Dirac Eq. reduces to

$$\mathcal{M}(p)\chi_p = 0. \tag{3}$$

- $\det \mathcal{M} = \prod_{s,\varkappa}[p_0 - p_{0,\varkappa}^{(s)}] = 0$ reveals energy branches:

$$p_{0,\varkappa}^{(s)}(\boldsymbol{p}) = -\mu_V - \varkappa\mu_H + s\sqrt{m^2 + (|\boldsymbol{p}| - \varkappa\mu_A)^2}. \tag{4}$$

- $s = \pm 1$ corresponds to particles / anti-particles.
- $\varkappa = \pm 1$ corresponds to right- / left- handed fermions.

[4] M. Laine, A. Vuorinen, *Basics of Thermal Field Theory* (Springer, 2016).

Free energy

- Using the path-integral formalism in Euclidean time $\tau = it$, the partition function $\mathcal{Z}$ can be expressed as

$$\mathcal{Z} \sim \prod_{\{P\}} \det \mathcal{M}.$$

- The free energy $\Omega = -T \ln \mathcal{Z} = \Omega_{\rm ZP} + \Omega_T$ consists of the zero-point (vacuum) and thermal parts:

$$\Omega_{\rm ZP} = -\frac{1}{2} \sum_{\varkappa = \pm 1} \sum_{s = \pm 1} \int \frac{d^3p}{(2\pi)^3} s\, p^{(s)}_{0,\varkappa},$$

$$\Omega_T = -\sum_{\varkappa = \pm 1} \sum_{s = \pm 1} \int \frac{d^3p}{(2\pi)^3} T \ln\left[1 + e^{-s\, p^{(s)}_{0,\varkappa}/T}\right].$$

- The charge densities can be obtained via $n_\ell = \langle J^0_\ell \rangle = -\dfrac{\partial \Omega}{\partial \mu_\ell}$.

(μ_V, μ_H) duality

- For $\mu_A = 0$ but $\mu_V, \mu_H \neq 0$, the "energy branches" are

$$p_{0,\varkappa}^{(s)}(\boldsymbol{p}) = -\mu_V - \varkappa\mu_H + s\omega_{\mathbf{p}}, \qquad \omega_{\mathbf{p}} = \sqrt{m^2 + \boldsymbol{p}^2}.$$

- $\Omega_{ZP}^{VH} = -2\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}$ is independent of μ_ℓ and T.

- The thermal part,

$$\Omega_T^{VH}(\mu_V, \mu_H) = -T\sum_{s,\varkappa}\int \frac{d^3p}{(2\pi)^3} \ln\left[1 + \exp\left(-\frac{\omega_{\mathbf{p}} - s(\mu_V + \varkappa\mu_H)}{T}\right)\right],$$

  exhibits the symmetry $\Omega_T^{VH}(\mu_V, \mu_H) = \Omega_T^{VH}(\mu_H, \mu_V)$.
- Also, Ω_T^{VH} is invariant under $\mu_V \to -\mu_V$, and/or $\mu_H \to -\mu_H$.

- The charge densities can be computed exactly in the small mass limit:

$$n_V = \frac{\mu_V T^2}{3} + \frac{\mu_V(\mu_V^2 + 3\mu_H^2)}{3\pi^2} - \frac{\mu_V m^2}{2\pi^2}, \qquad n_H = \frac{\mu_H T^2}{3} + \frac{\mu_H(\mu_H^2 + 3\mu_V^2)}{3\pi^2} - \frac{\mu_H m^2}{2\pi^2}.$$

Incompatibility between $\mu_A \neq 0$ and $m \neq 0$

- For $\mu_V = \mu_H = 0$ but $\mu_A \neq 0$, the "energy branches" are

$$p_{0,\varkappa}^{(s)}(\boldsymbol{p}) = s\sqrt{(|\boldsymbol{p}| - \varkappa\mu_A)^2 + m^2}.$$

- In this case, $\Omega_{ZP}^A = \Omega_{\rm ZP}^{VH} + \Omega_{\rm dens}^A$ contains also a density-dependent part:

$$\Omega_{\rm dens}^A = -\sum_{\varkappa=\pm 1} \int \frac{d^3p}{(2\pi)^3}[\sqrt{(|\boldsymbol{p}| - \varkappa\mu_A)^2 + m^2} - \sqrt{\boldsymbol{p}^2 + m^2}].$$

- While everything works at $m = 0$, the $T = 0$ limit of n_A at finite m diverges:

$$n_A(\mu_A)\rfloor_{T=0, m\ll|\mu_A|} = \frac{\mu_A^3}{3\pi^2} + \frac{m^2\mu_A}{\pi^2}\ln\frac{\Lambda_{\rm UV}}{m} + \dots, \qquad (5)$$

  where $\Lambda_{\rm UV}$ is an UV cutoff.
- Hence, $\mu_A \neq 0$ cannot be used when $m \neq 0$.[5]

[5] M. Ruggieri, M. N. Chernodub, Z.-Y. Lu, Phys. Rev. D **102** (2020) 014031.

Linear σ model with quarks (LSM_q)

- LSM_q is a low-energy effective model of QCD, exhibiting the chiral phase transition.
- Considering the two-flavour model, $\psi = (u, d)^T$, the LSM_q Lagrangian density is

$$\begin{aligned}
\mathcal{L} =& \mathcal{L}_q + \mathcal{L}_\sigma, \\
\mathcal{L}_q =& \overline{\psi}[i\partial\!\!\!/ - g(\sigma + i\gamma^5 \boldsymbol{\tau} \cdot \boldsymbol{\pi})]\psi, \\
\mathcal{L}_\sigma =& \frac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi^0 \partial^\mu \pi^0) + \partial_\mu \pi^+ \partial^\mu \pi^- - V(\sigma, \boldsymbol{\pi}),
\end{aligned}$$

  where $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$ and $\pi^0 = \pi^3$ correspond to the isotriplet of the pseudoscalar pions, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$, while σ is the pseudoscalar field.
- The potential,

$$V(\sigma, \boldsymbol{\pi}) = \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2 - v^2)^2 - h\sigma,$$

  reaches a minimum when $\langle \boldsymbol{\pi} \rangle = 0$ and $\langle \sigma \rangle^3 - v^2 \langle \sigma \rangle - \frac{h}{\lambda} = 0$.

Mean field approximation

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- In the mean field approximation, the quantum fluctuations of σ and $\boldsymbol{\pi}$ are neglected, such that

$$\mathcal{L}_{\rm MF} = \overline{\psi}[i\partial\!\!\!/ - M(\sigma)]\psi - V(\sigma),$$

where $V(\sigma) \equiv V(\sigma, 0)$ and $M(\sigma) = g\sigma$ is the dynamical quark mass.

- The model parameters are taken as[6]

$$g = 3.3, \qquad \lambda = 19.7, \qquad v = 87.7 \text{ MeV}, \qquad h = (121 \text{ MeV})^3,$$

such that
  - $\langle\sigma\rangle = f_\pi = 93$ MeV $\equiv$ pion decay constant;
  - $M(\langle\sigma\rangle) = 307 \text{ MeV} \simeq \frac{1}{3} m_{\rm nucleon}$;
  - $m_\pi = \sqrt{\lambda(\langle\sigma\rangle^2 - v^2)} = 138$ MeV matches the pion mass.

[6] O. Scavenius, A. Mocsy, I. N. Mishustin, D. H. Rischke, PRC 64 (2001) 045202.

Thermodynamics of the LSM_q

- We now add $\mu_V = \mu_q = \mu_B/3$ $(N_c = 3)$ and μ_H, such that

$$\Omega(\sigma;\mu_V,\mu_H) = V(\sigma) + \Omega_q(\sigma;\mu_V,\mu_H), \tag{6}$$

where $\Omega_q(\sigma;\mu_V,\mu_H) = \Omega_{\rm vac}(\sigma) + \Omega_T(\sigma;\mu_V,\mu_H)$ and

$$\begin{aligned} V(\sigma) =& \frac{\lambda}{4}(\sigma^2 - v^2)^2 - h\sigma, \\ \Omega_T(\sigma;\mu_V,\mu_H) =& -N_f N_c T \sum_{s,\varkappa} \int \frac{d^3p}{(2\pi)^3} \\ & \times \ln\left\{1 + \exp\left[-\frac{1}{T}(\omega_{\mathbf{p}}(\sigma) - s(\mu_V + \varkappa\mu_H))\right]\right\}, \end{aligned}$$

where $\omega_{\mathbf{p}}(\sigma) = \sqrt{\boldsymbol{p}^2 + g^2\sigma^2}$ depends on σ.

- $\Omega_{\rm vac}$ is ignored in order to avoid UV cutoff-dependence.
- For given T, μ_V and μ_H, σ is obtained by minimising $\Omega(\sigma;\mu_V,\mu_H)$.

Dense matter at $T = 0$

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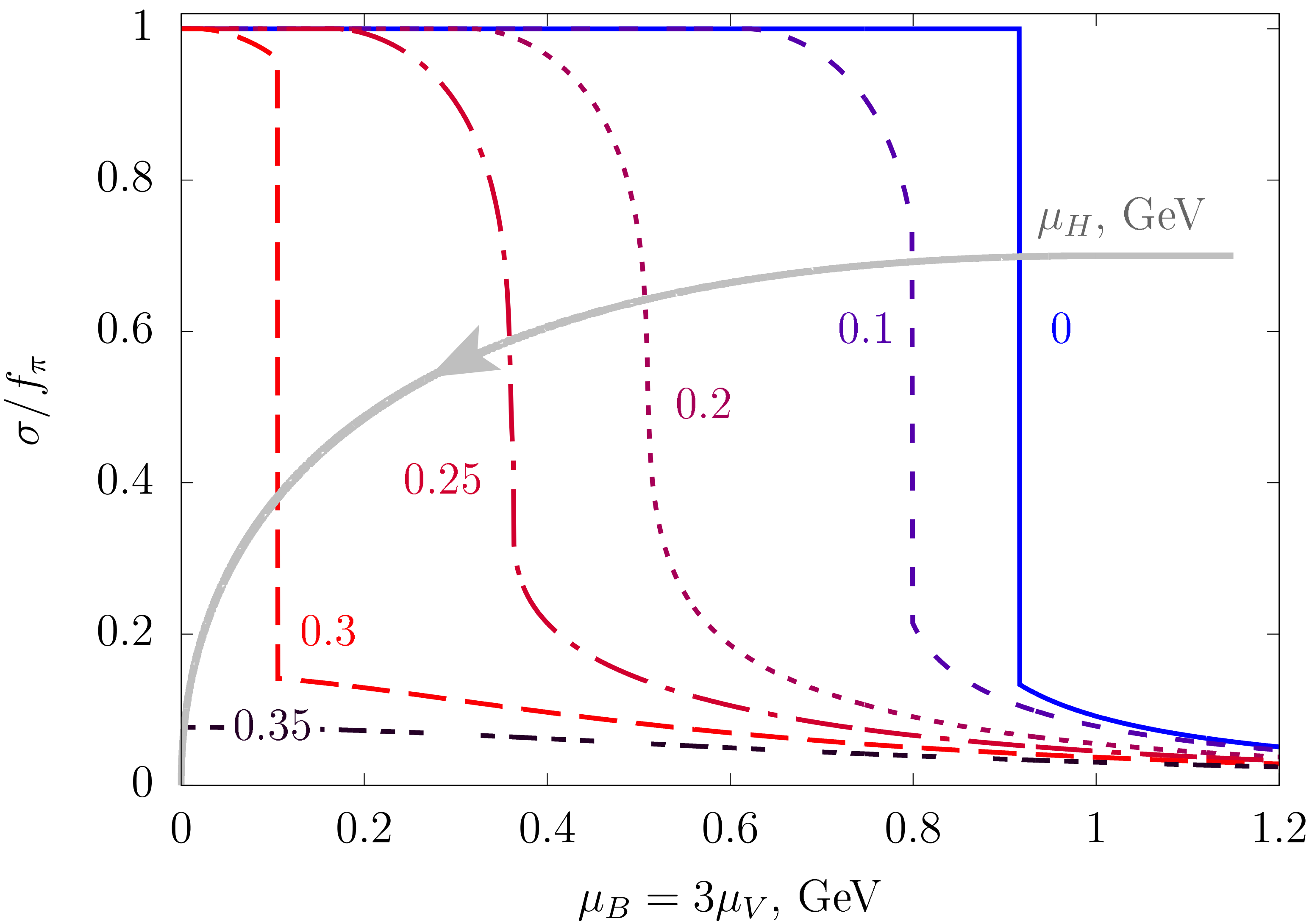


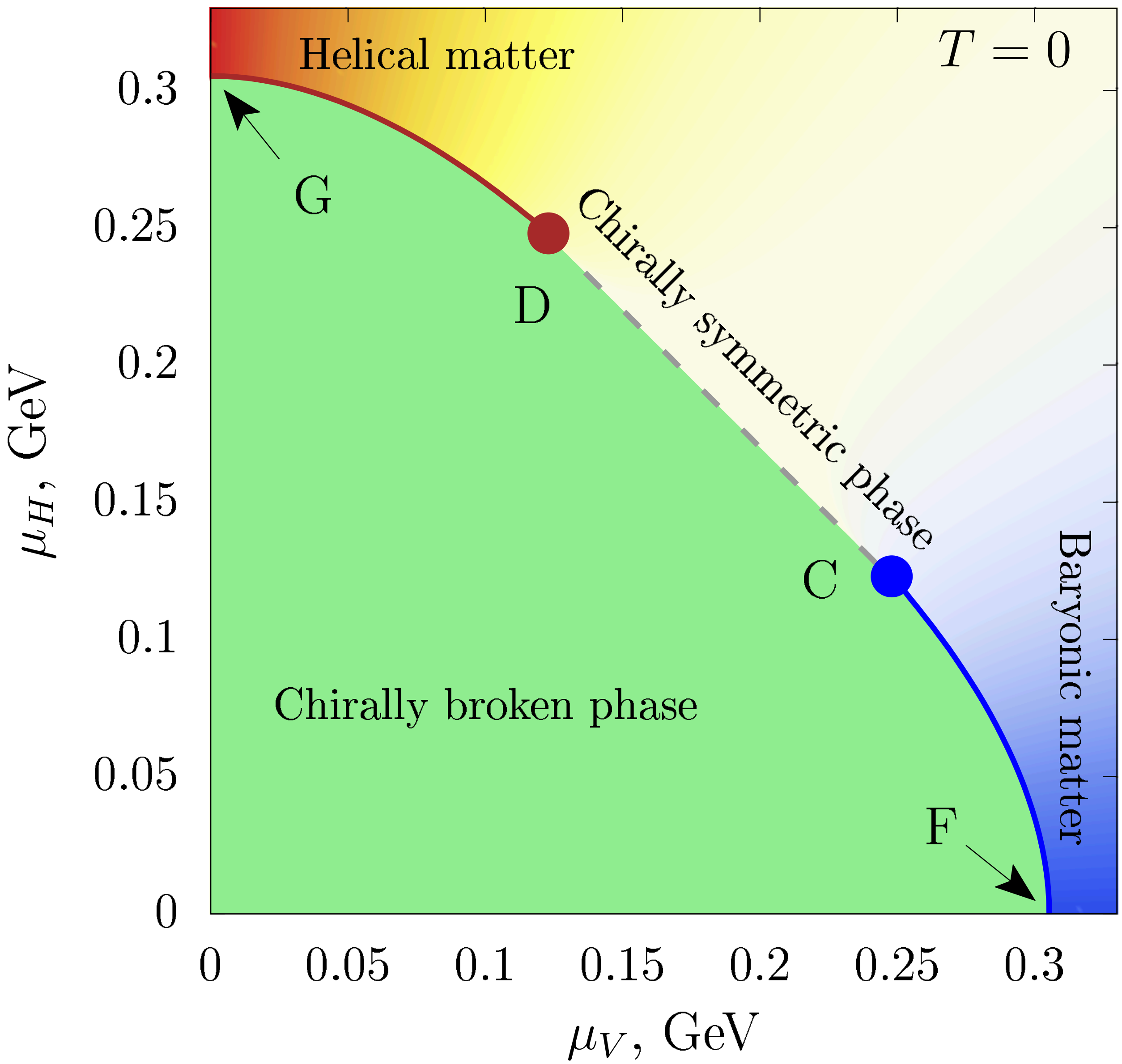
- At $\mu_H = 0$, 1st order PT occurs at $\mu_V = \mu_c = 305$ MeV.
- Increasing μ_H leads to crossover and then again to 1st order PT.

Dense matter at $T = 0$

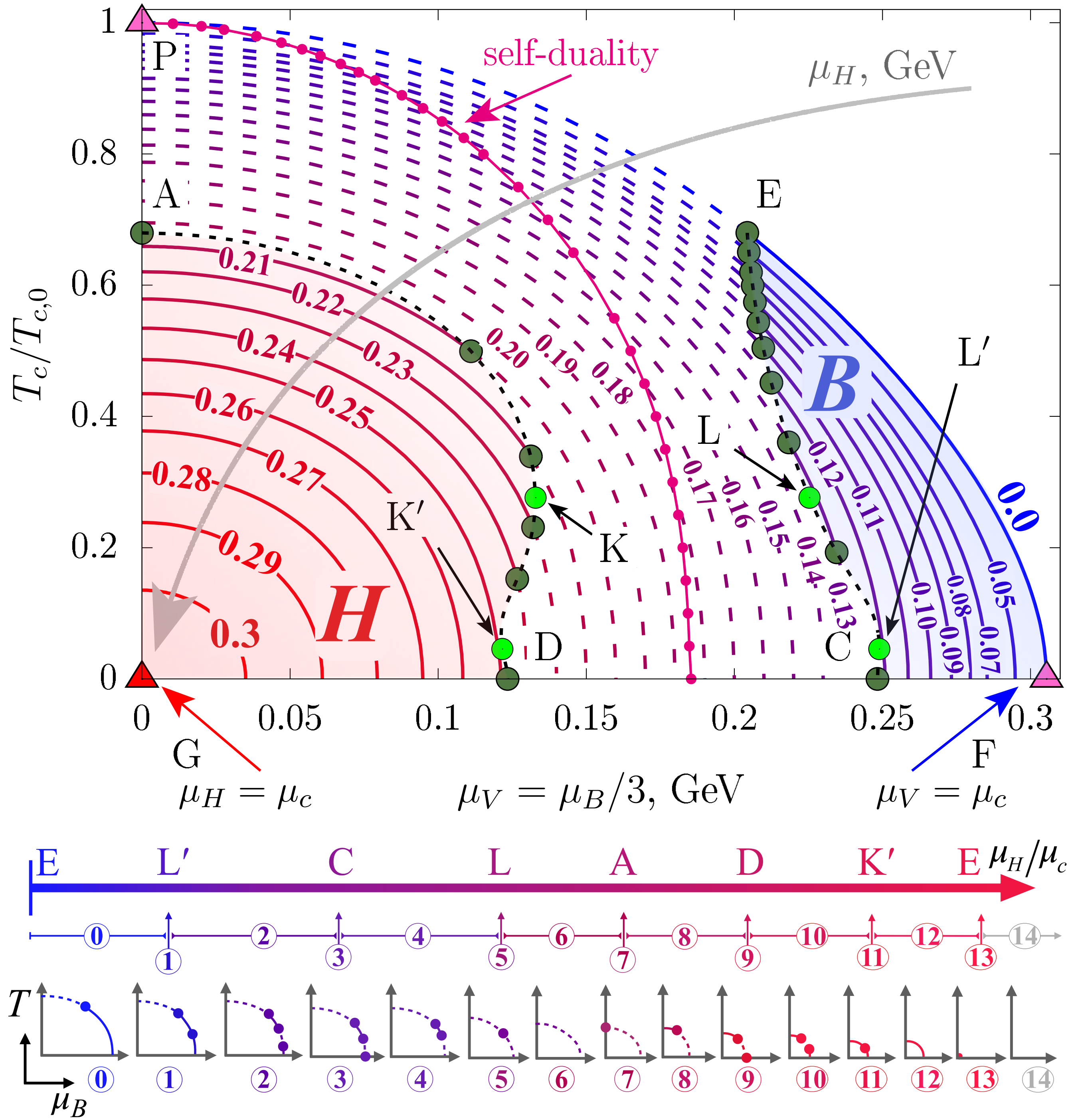
- PT1 along FC and GD.
- Crossover along CD.
- $\mu_H \leftrightarrow \mu_V$ duality.

- $\mu_V^F = \mu_H^G = \mu_c$.
- $\mu_H^F = \mu_V^G = 0$.

- $\mu_V^C = \mu_H^D = 0.81\mu_c$.
- $\mu_H^C = \mu_V^D = 0.40\mu_c$.



Finite-temperature phase diagram



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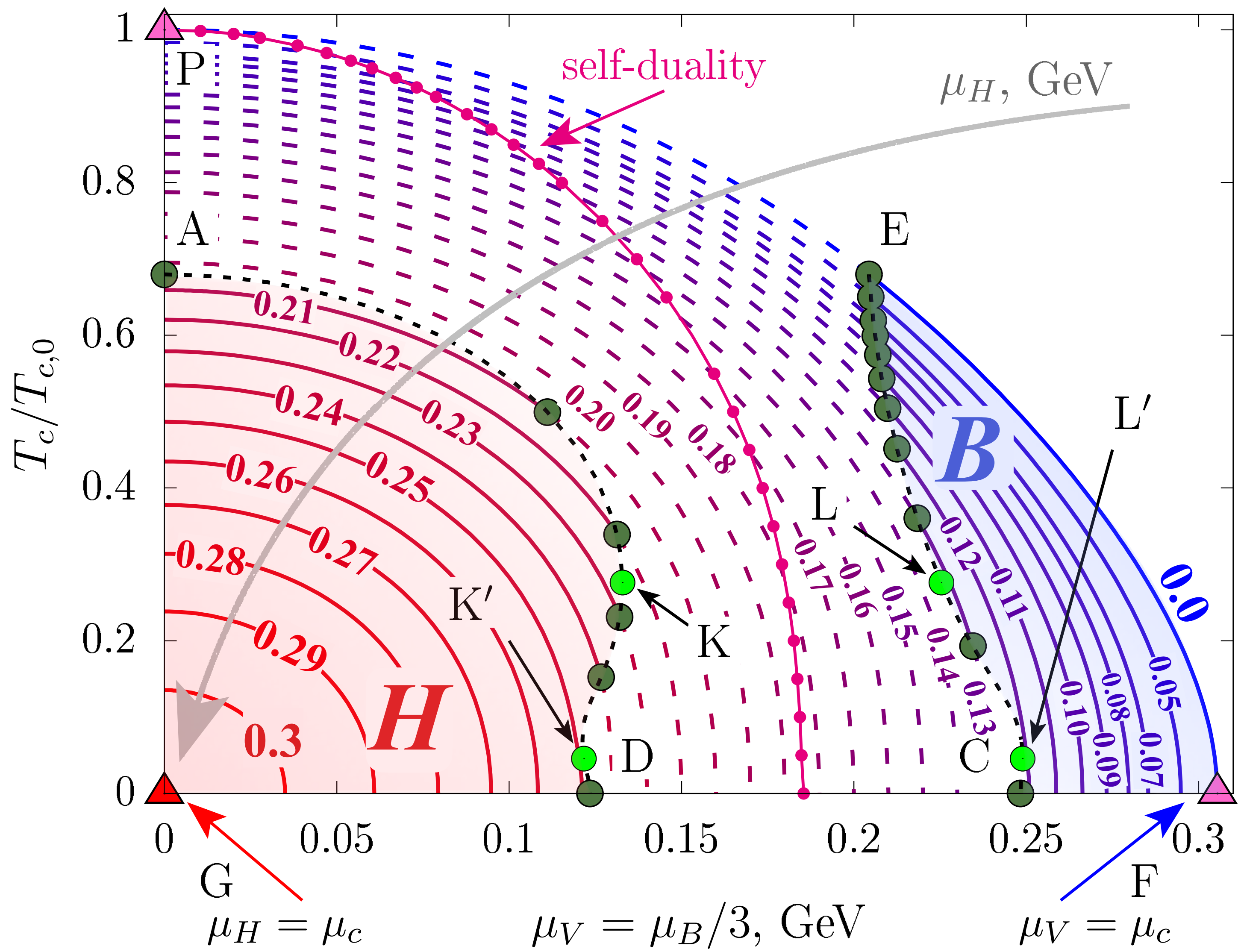
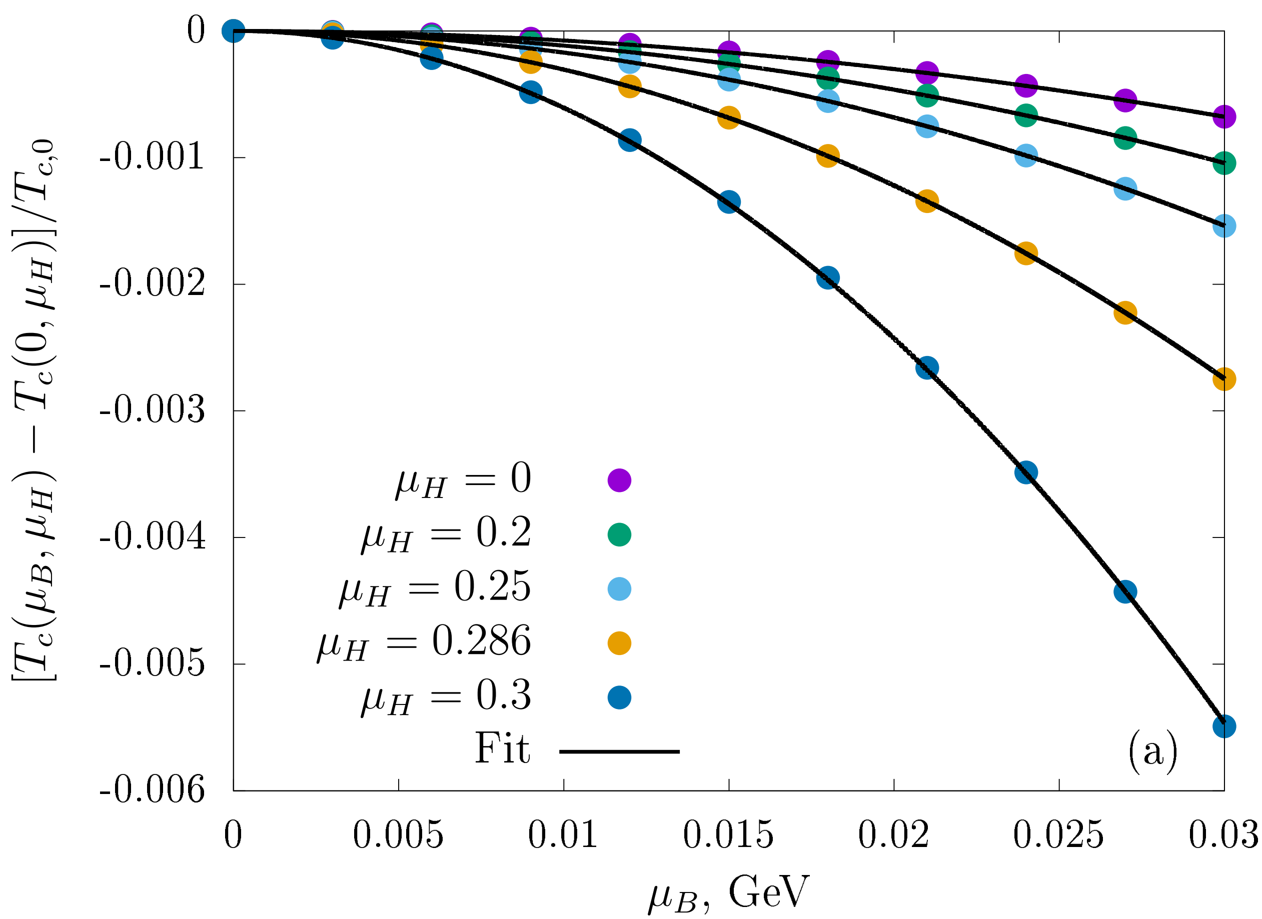


	F	E	P	L'	C	L	A	K	D	K'	G
μ_V (MeV)	305	204	0	249	248	225	0	133	123	122	0
μ_H (MeV)	0	0	0	122	123	133	204	225	248	249	305
T (MeV)	0	0	100	6.7	0	41	100	41	0	6.7	0

Quadratic ansatz

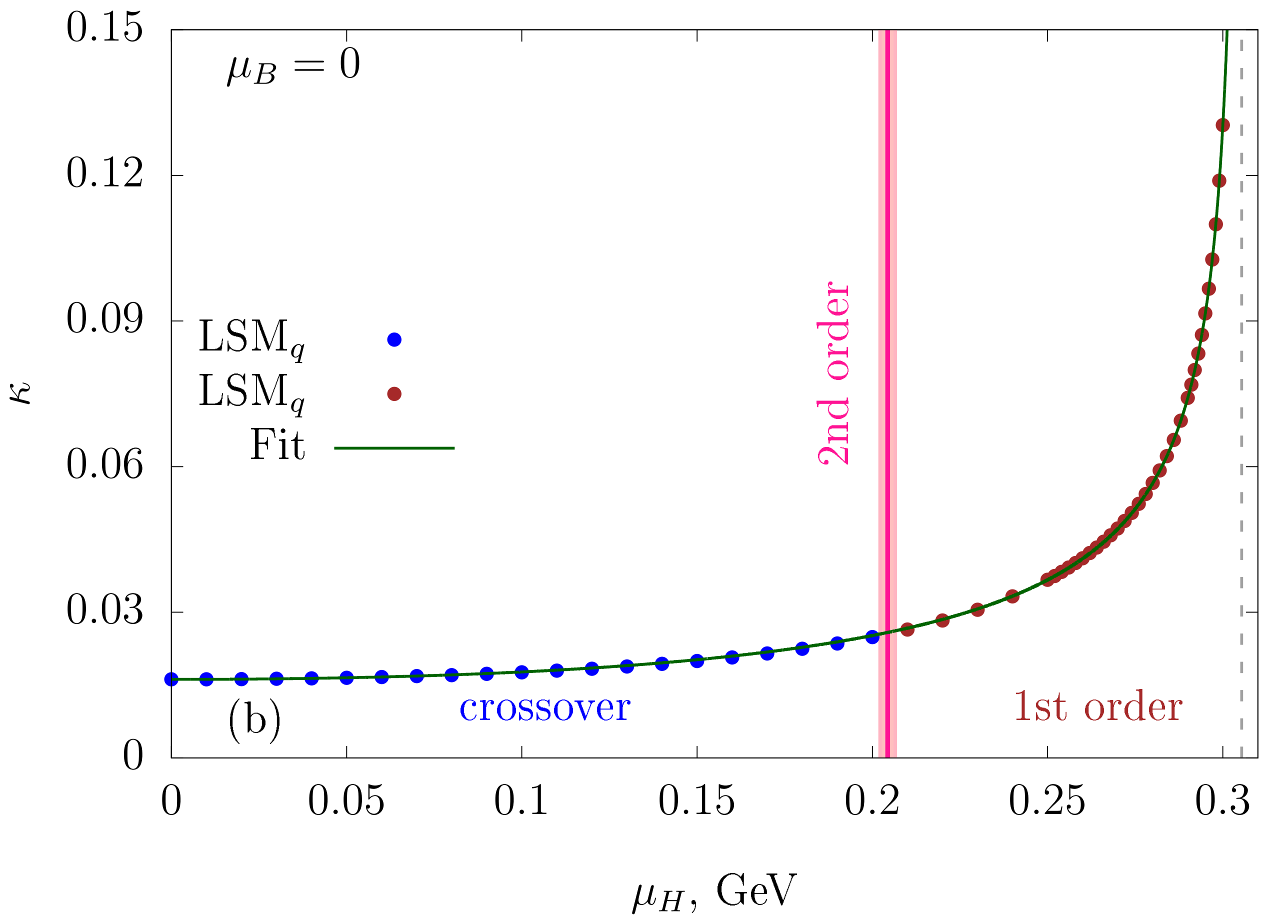


- The PT temperature at small μ_B and finite μ_H can be approximated by

$$\frac{T_c(\mu_B,\mu_H)}{T_{c,0}} = \frac{T_c(\mu_H)}{T_{c,0}} - \kappa(\mu_H)\left(\frac{\mu_B}{T_{c,0}}\right)^2 + \dots, \qquad (7)$$

where $\kappa(\mu_H)$ is the curvature.

μ_H dependence of κ



$$\kappa^{\rm fit}(\mu_H) = \kappa_0 \left[1 + \alpha \left(\frac{\mu_H}{\mu_{H,c}}\right)^2 \left(1 - \frac{\mu_H}{\mu_c}\right)^{-\gamma}\right], \tag{8}$$

- $\kappa(0) \simeq 0.016$ is the curvature at $\mu_H = 0$.
- Best fit for $\alpha = 0.70$, $\gamma = 0.58$.

Conclusion

- J_H^μ is classically conserved for free fields, even when $m \neq 0$.
- μ_H can account for helicity imbalance.
- μ_A incompatible with LSM_q, while (μ_H, μ_V) form a dual pair.

- Non-trivial changes to the chiral phase diagram can be seen when $\mu_H \neq 0$:
  - New critical points at $T = 0$: C and D.
  - At $T > 0$, the critical point E becomes a critical line: $E - L - L' - C$.
  - New critical line (dual to $ELL'C$): $A - K - K' - D$.
  - New region of 1st order phase transition $\equiv H$ matter (dual to B matter).
  - Curvature κ of transition temperature at vanishing μ_B depends on μ_H.