IS OVERLAPPING DOUBLE-GLUON BREMSSTRAHLUNG IN A QUARK-GLUON PLASMA ACCURATELY DESCRIBED BY THE $N_{color} = \infty$ LIMIT?

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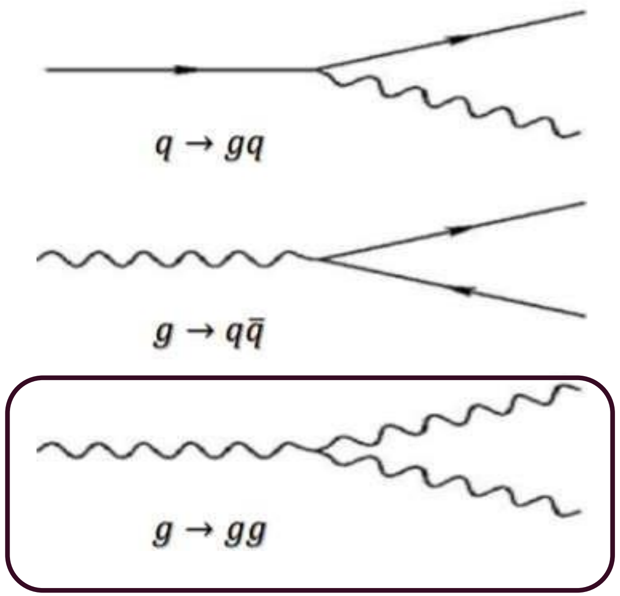
INTRODUCTION

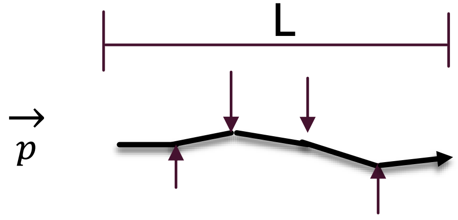
- High energy particles traveling through QGP lose their energy through splitting processes like bremsstrahlung or pair production



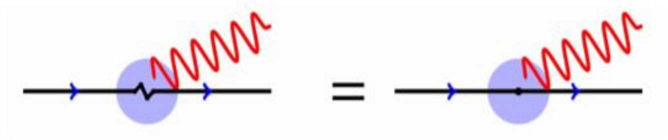
OUTLINE

- LPM effect
- Problem set-up
- From SU(3) to SU(N)
- Correction to the $N_c = \infty$ result for double gluon bremsstrahlung

THE LANDAU-POMERANCHUK-MIGDAL (LPM) EFFECT IN QED

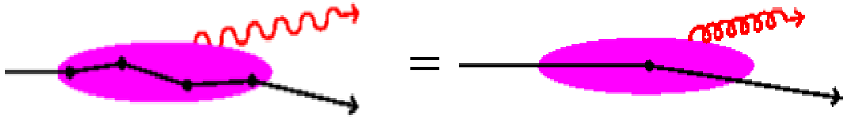


- Imagine a non-relativistic electron going through the atmosphere
- Each collision with the medium offers a chance for bremsstrahlung
- Prob of splitting $\sim \boldsymbol{\alpha\ per\ collision}$
- But, the photon cannot resolve details that are smaller than its wavelength. This will create a region of fuzziness, drawn as the shaded region.



- Lorentz boosting this process, similar effect happens, but the circular region of fuzziness will become elongated like an ellipse.
- The duration of this ellipse is called the formation time of

  the bremsstrahlung photon.



- So at high energy,

  prob of splitting $\sim \boldsymbol{\alpha\ per\ formation\ length}$

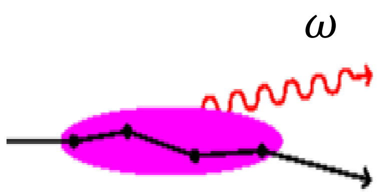
THE LPM EFFECT IN QCD

- Same effect happens in QCD

  prob of splitting $\sim \boldsymbol{\alpha\ per\ formation\ length}$

- For a high energy parton in the collinear limit, the formation time grows with energy. So high energy gluons will suffer more suppression due to the LPM effect.

$$t_{form} \sim \sqrt{\frac{\omega}{\hat{q}}}$$



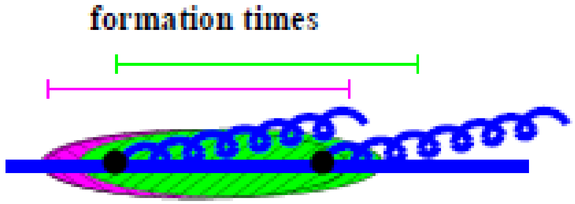
- This was figured out in the 1950's for QED by LPM and for QCD in the 1990's by BDMPS-Z.

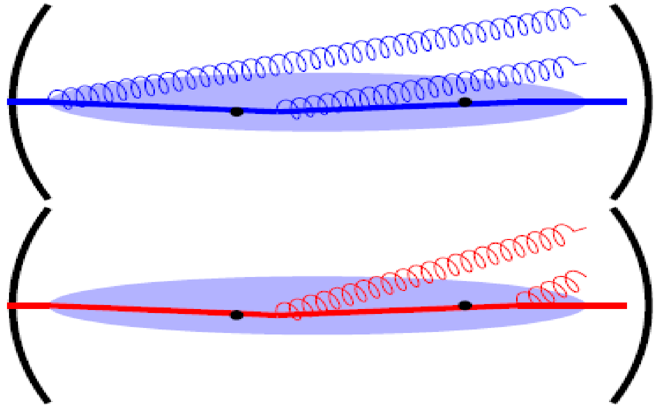
CONSECUTIVE SPLITTING

- What if two gluons are radiated within a single formation time?
- We need to include double gluon bremsstrahlung.
- Previously calculated first in the leading-log approximation.

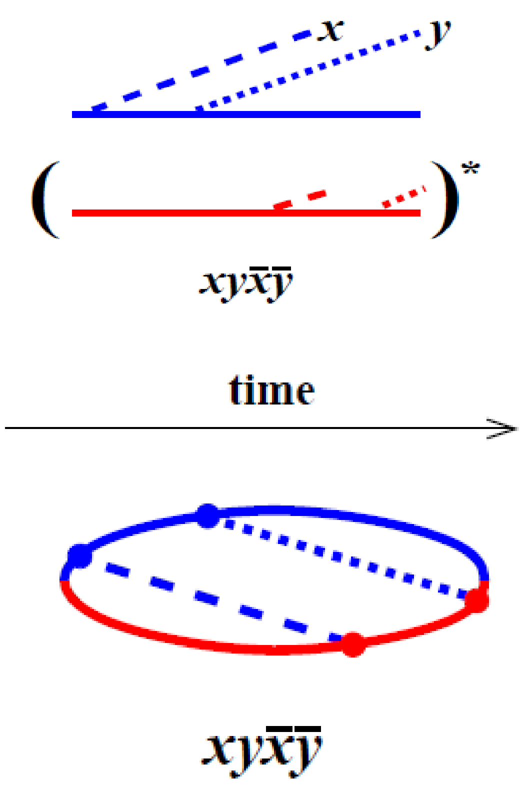
  [Blaizot,Mehtar-Tani/Iancu/Wu (2014)]
- Then beyond leading log in the Large N_c-limit ($N_c = \infty$) .

  [P.Arnold and S.Iqbal (2015)] and [P.Arnold,H.Chang,S.Iqbal (2016)]
- My work: How large is the correction for $N_c = 3$ (QCD) ?





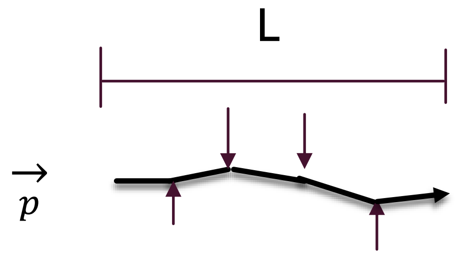
SET-UP



- The top diagram represents double gluon splitting

  (gluons are represented by straight lines).
- The bottom diagram represents the Amplitude + Conjugate Amplitude as a single process. [Adopted by Zakharov]
- The dotted and dashed lines represent the emitted gluons
- This is an example of what we call a "*sequential diagram*"
- Interactions with the medium are implicit
- At each moment in time, particles in the (Amplitude+Conjugate Amplitude) have to always form a *color singlet*.

MEDIUM INTERACTIONS

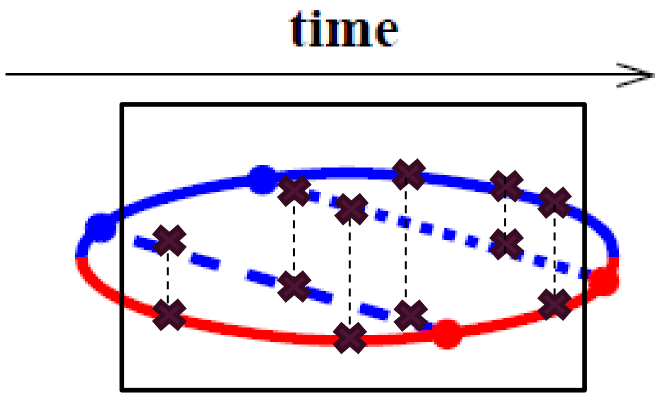


- $\hat{q}_R$ is the only characteristic of the medium that is relevant to our problem
- It is defined as the average squared transverse momentum transferred from the medium per unit length, and the subscript R refers to the color representation.

$$\hat{q}_R = \langle p_\perp^2 \rangle / L$$

- The color representation *combinations* that give the color singlets will affect how the particles interact with the medium, and can change every time there is an interaction with the medium. (more on this later)

MEDIUM INTERACTIONS



- Medium averaged evolution can be treated as a non-Hermitian two-dim QM problem in the transverse plane.
- Our QM potential, $V(\boldsymbol{x})$ where $\boldsymbol{x}$ is a 2-dim position vector in the transverse plane, accounts for medium-averaged correlations of interactions in the plasma and depends on $\hat{q}_R$.
- Splitting vertices can be obtained from the DGLAP splitting functions.

ASSUMPTIONS

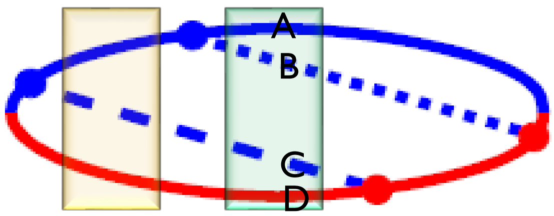
- The medium is infinite (QGP is homogonous on scale of formation time).
- The High energy limit $\varepsilon_{\boldsymbol{p}} = \sqrt{p_z^2 + \boldsymbol{p}_\perp^2} \simeq p_z + \frac{\boldsymbol{p}_\perp^2}{2p_z} \simeq \frac{\boldsymbol{p}_\perp^2}{2p_z} + \text{constant}$

  (the kinetic energy in a non-relativistic two-dimensional quantum mechanics problem)
- The Harmonic approximation (it is difficult to deflect high energy particles), also know as the $\hat{q}_R$ or multiple scattering approximation. As a result, the effective Hamiltonian governing the evolution of these particles becomes that of coupled harmonic oscillators with complex frequencies.

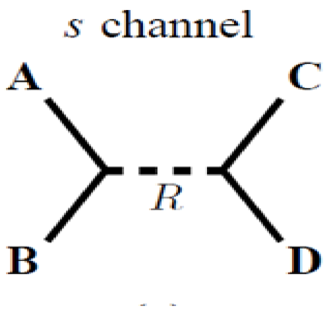
SU(3)

- Knowing the color representation combinations for these singlets affects how the particles interact with the medium at any given moment
- For 3 gluons, there's effectively only one way to do that.
- For 4 gluons, color conservation is not enough.
- We need to find a basis of color singlet states from 4 particles
- Combining two gluons gives $8 \otimes 8 = 1_s \oplus 8_a \oplus 8_s \oplus 10_a \oplus \overline{10}_a \oplus 27_s$
- For example, the "s-channel" basis for the four gluons $|s_1\rangle, |s_{8_{aa}}\rangle, |s_{8_{as}}\rangle, |s_{8_{sa}}\rangle, |s_{8_{ss}}\rangle, |s_{10_a}\rangle, |s_{\overline{10}_a}\rangle, |s_{27_a}\rangle$

3-particle singlet



4-particle singlet



Warning: this is not a scattering diagram

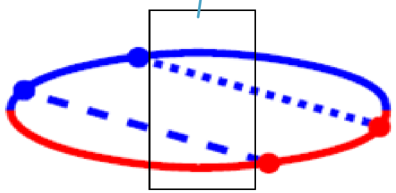
SU(N)

- One more irreducible representation appears in the tensor product Adj⊗Adj for SU(N) than SU(3).
- In a notation similar to SU(3), our states become "8" ⊗ "8" = $1_s \oplus$ "8_a" $\oplus$"8_s"$\oplus$"10_a"$\oplus$ "$\overline{10}_a$" $\oplus$ "27_s"$\oplus$ "0_s"

Its dimension → 0 as N_c → 3

CORRECTION TO THE $N_c = \infty$ RESULT FOR (DOUBLE GLUON BREMSSTRAHLUNG)

4-particle evolution ≡ 2 coupled harmonic oscillators



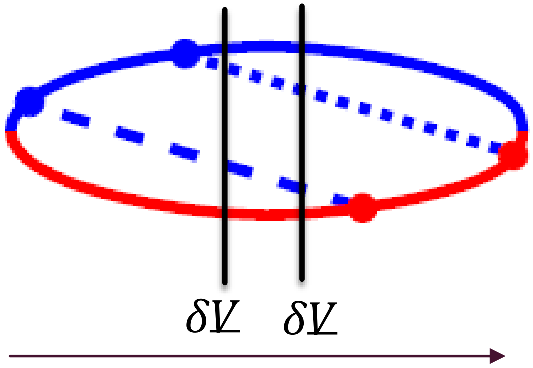
- Corrections to the $N_c = \infty$ result only appear with the 4-particle evolution.
- In the harmonic approximation, this corresponds to finding the correction to the propagator of two coupled harmonic oscillators.
- For a pure gluonic diagram, the next order correction is of $O\left(\frac{1}{{N_c}^2}\right)$.

Only 6 of the color-singlet states paly a role

- $V(\boldsymbol{x})$is a 6x6 matrix in the color-singlet space, and a 2x2 matrix in the position space.

accounts for medium-averaged correlations of interactions in the plasma

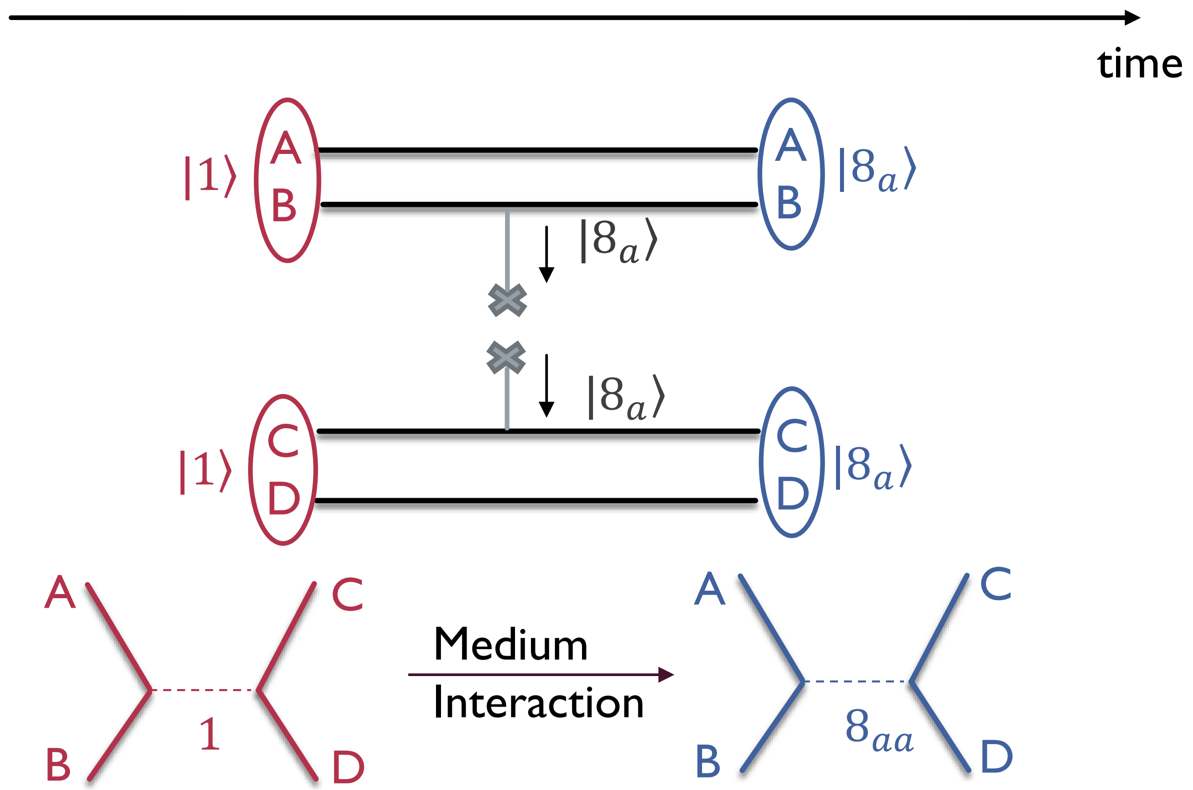
CORRECTION TO THE $N_c = \infty$ RESULT FOR (DOUBLE GLUON BREMSSTRAHLUNG)



- $\mathbb{V} = \mathbb{V}_0 + \delta\mathbb{V} + \delta^2\mathbb{V} + \cdots$, where $\mathbb{V}_0$ is the $N_c = \infty$ potential, $\delta\mathbb{V}$ is the $\frac{1}{N_c}$ correction and $\delta^2\mathbb{V}$ is the $\frac{1}{{N_c}^2}$ correction.
- Then, write everything in the basis where $\mathbb{V}_0$ is diagonal.
- We call the relevant $N_c = \infty$ eigenstates: $|1\rangle, |+\rangle, |-\rangle, |\mathcal{T}_0\rangle, |\mathcal{T}_+\rangle$ and $|\mathcal{T}_-\rangle$, which can be written in terms of e.g. the previous "s-channel basis".

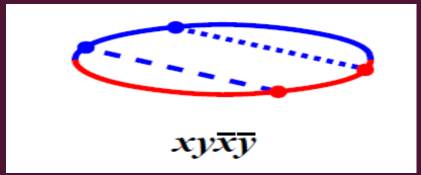
COLOR TRANSITIONS IN A QUALITATIVE WAY

- Black straight lines are the high energy gluons
- The grey line ending with a cross represents gluonic interactions with the medium.

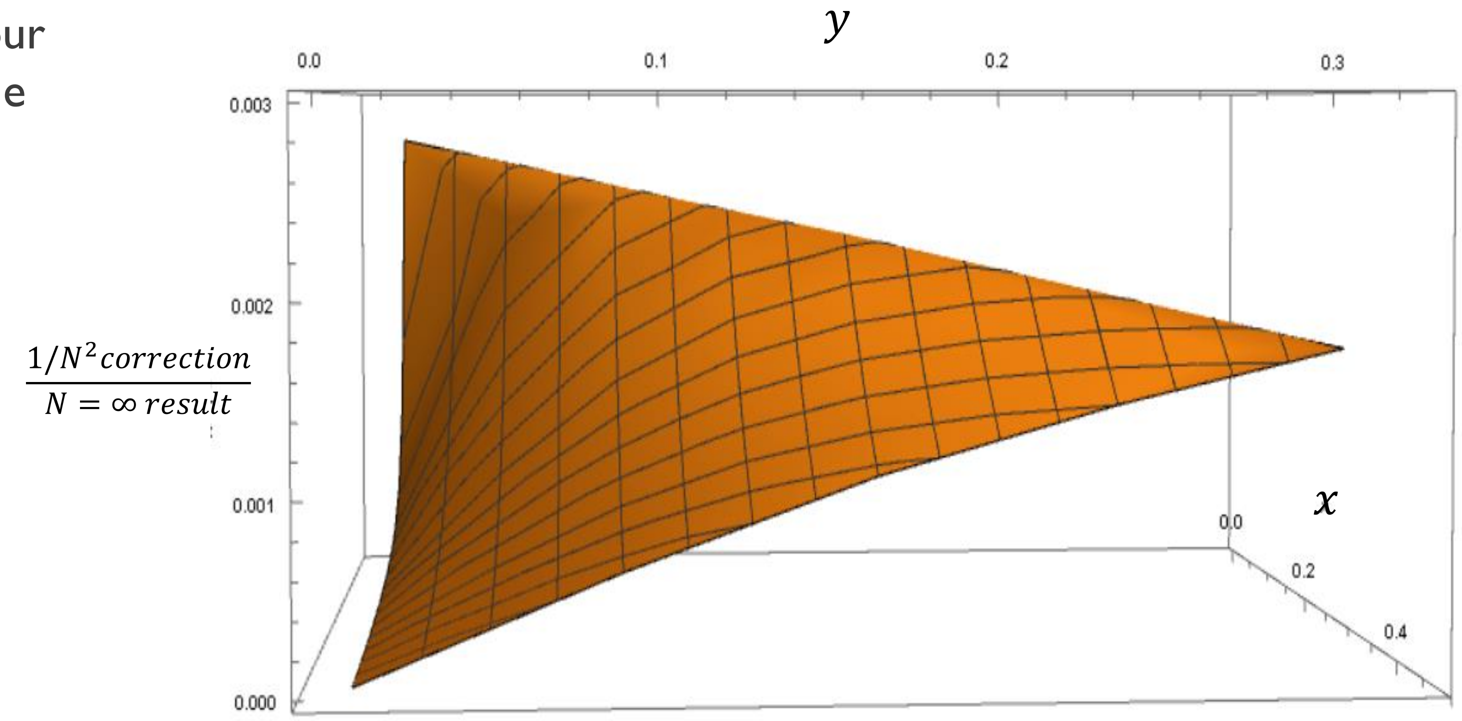


FIRST RESULTS : SEQUENTIAL DIAGRAMS

e.g.

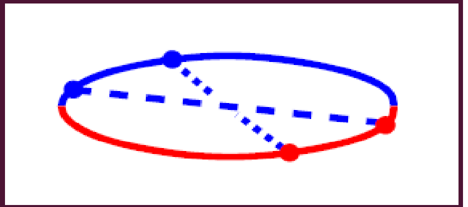


- This vertical axis shows the relative size of our $1/N_c^2$ correction for $N_c = 3$ compared to the $N_c = \infty$ result , and restricted to the region $y < x < 1 - x - y$
- The correction was at most 0.3%

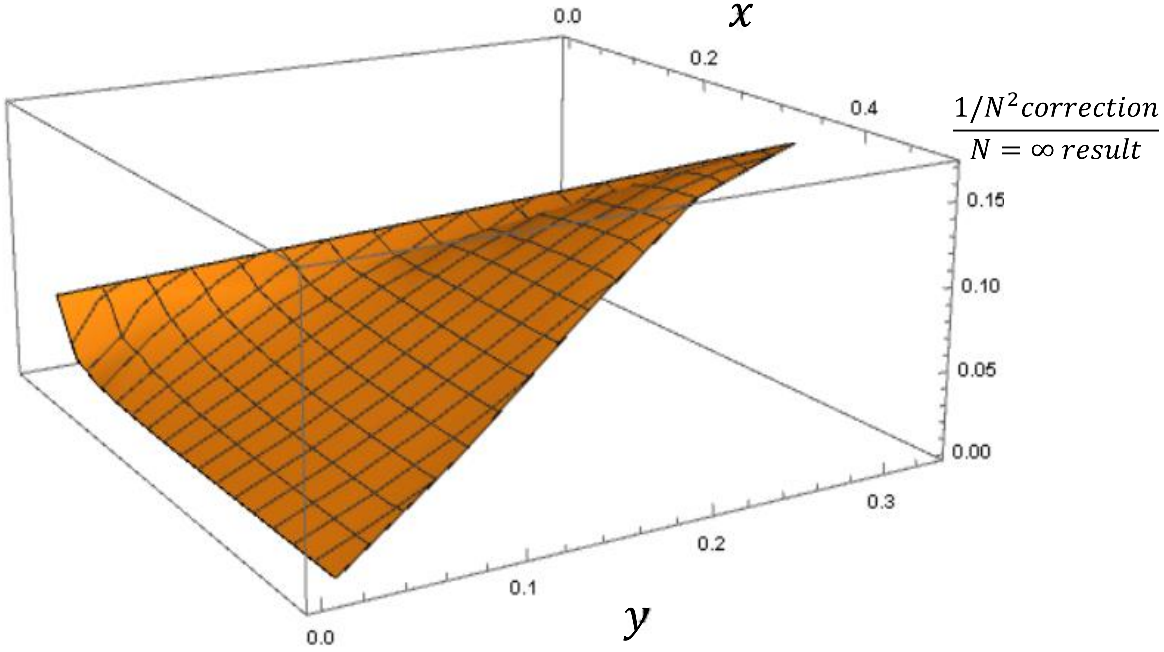


FIRST RESULTS : CROSSED DIAGRAMS

e.g.

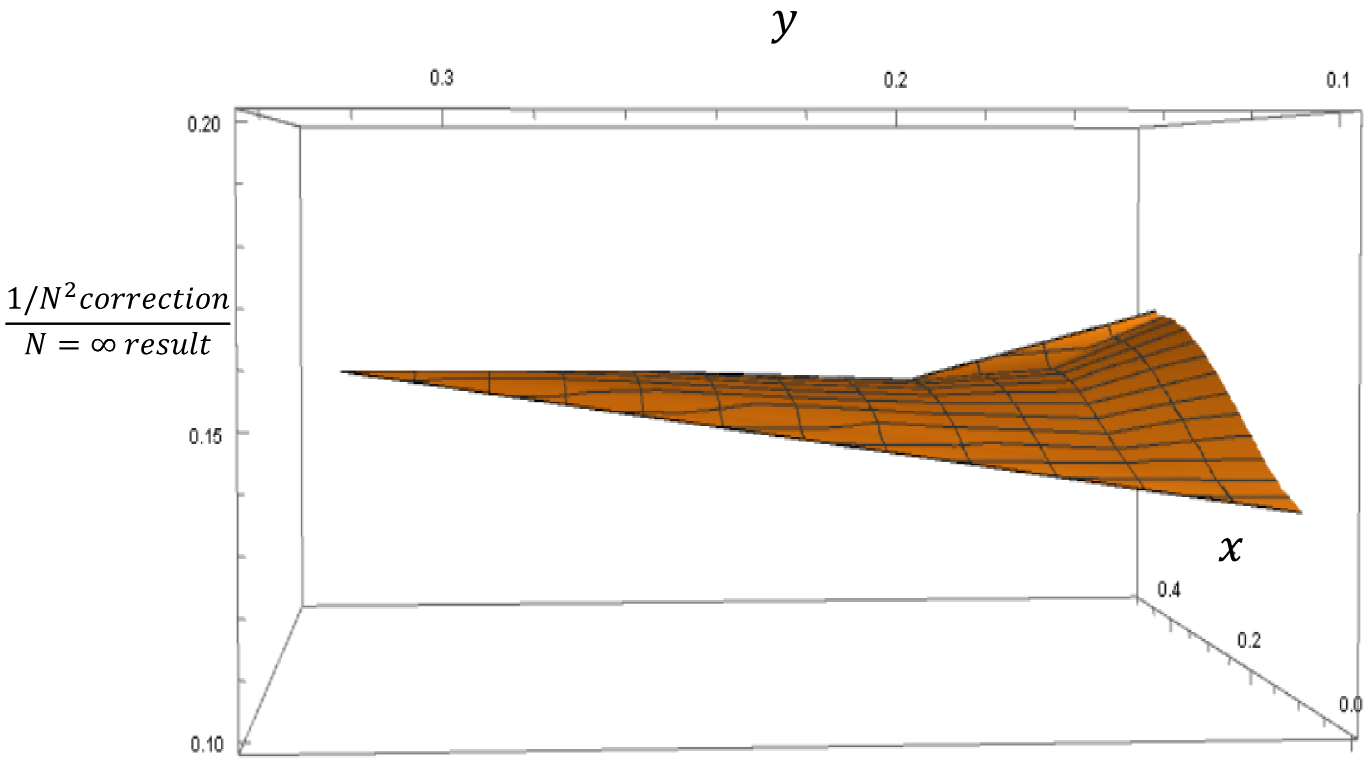


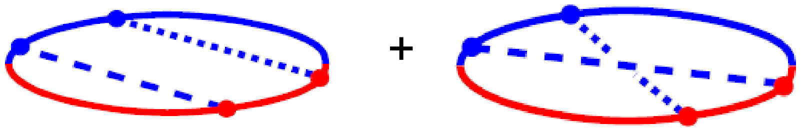
- This vertical axis shows the relative size of our $1/N_c^2$ correction for $N_c = 3$ compared to the $N_c = \infty$ result , and restricted to the region $y < x < 1 - x - y$
- The correction was at most 15%



HOW LARGE IS THE $1/N_c^2$ CORRECTION? (SEQUENTIAL +CROSSED)

- The correction is around 15% and increases as y becomes smaller.
- This result shows that the correction for $N_c = 3$ for double gluon bremsstrahlung is close to the naive expectation ($\frac{1}{N_c^2} \simeq 10\%$) .





Disclosure: small-y behavior (y < 0.1) not shown for sake of not having to explain a complicated story ☺

THANKS

THE SMALL $-y$ BEHAVIOUR (SEQUENTIAL + CROSSED)

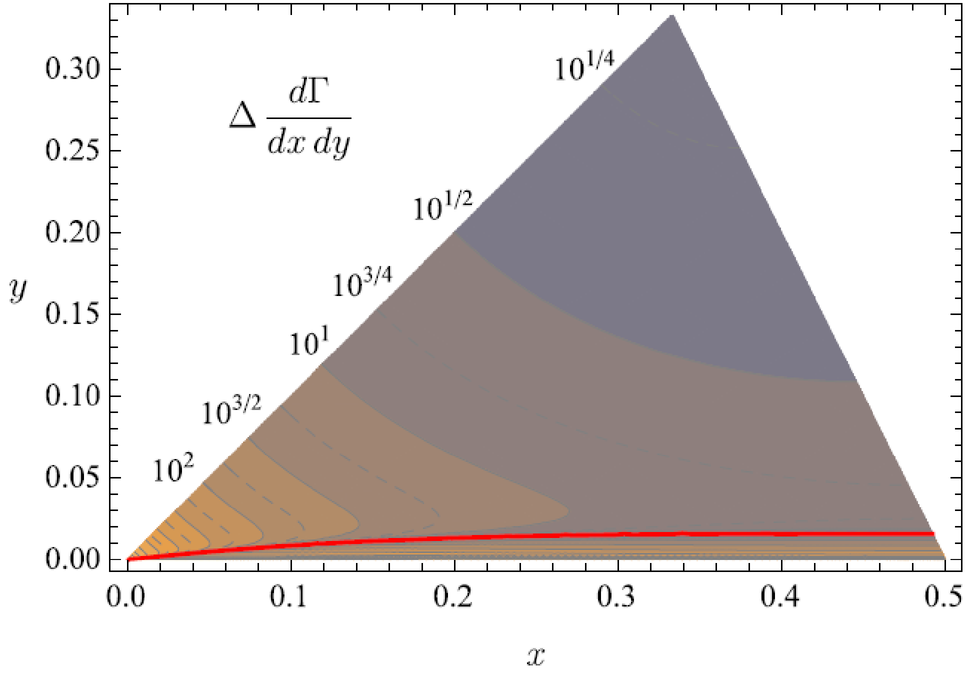
- The $N_c = \infty$ result

  [P.Arnold,H.Chang,S.Iqbal (2016)]

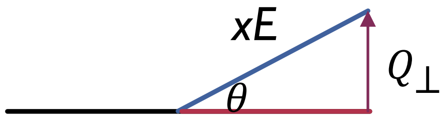
$$\left[\frac{\Delta d\Gamma}{dx\,dy}\right]_{seq} \approx -\frac{3}{2}\frac{C_A^2\,\alpha_s^2}{\pi^2 x y^{\frac{3}{2}}}\sqrt{\frac{\hat{q}}{E}}\ln\left(\frac{x}{y}\right)$$

$$\left[\frac{\Delta d\Gamma}{dx\,dy}\right]_{cross} \approx +\frac{3}{2}\frac{C_A^2\,\alpha_s^2}{\pi^2 x y^{\frac{3}{2}}}\sqrt{\frac{\hat{q}}{E}}\ln\left(\frac{x}{y}\right)$$

- One needs to calculate the $\frac{1}{N_c^2}$ correction to the virtual diagrams



FORMATION TIME

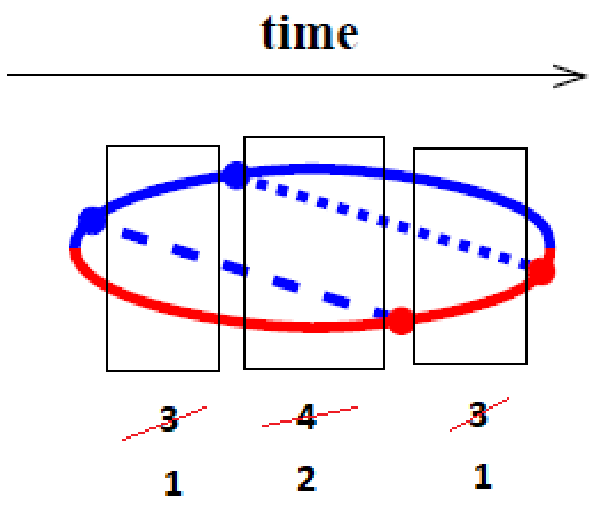


- The least energetic gluon is easily deflected
- If it picks up a transverse momentum $Q_{\perp}$, the angular separation between the two gluons is $\theta \sim \frac{Q_{\perp}}{x\,E}$
- Over time, the transverse separation $b \sim \theta\, t \sim \frac{Q_{\perp}}{x\,E}\, t$
- This process becomes decoherent when $b \sim \frac{1}{Q_{\perp}}$, this condition defines the formation time.
- By definition, $Q_{\perp}^2 = \hat{q}\, t$. Combining the above results with this equation gives $t_{form} \sim \sqrt{\frac{xE}{\hat{q}}}$

SET-UP
HIGH ENERGY PARTICLE APPROXIMATION

- Large p_z approximation $$\varepsilon_{\boldsymbol{p}} = \sqrt{p_z^2 + \boldsymbol{p}_\perp^2 + \mathrm{M}^2} \simeq p_z + \frac{p_\perp^2 + \mathrm{M}^2}{2p_z} \simeq \frac{\boldsymbol{p}_\perp^2}{2p_z} + \text{constant}$$
- Compare to the kinetic energy in a non-relativistic two-dimensional quantum mechanics problem
- We see here that the longitudinal momentum plays the role of the "mass" (m) , however, these "masses" must add up to zero for all the particles in the amplitude (evolve in time with e^{-iHt}) and conjugate amplitude(evolve in time with e^{+iHt}).

SET-UP
From n particles to n-2 particles



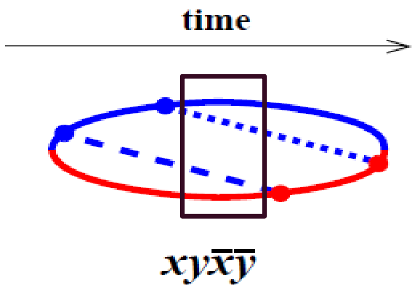
- In addition, we have translation symmetry in the transverse plane to the initial high energy gluon, which leads to conservation of transverse momentum

- Using the constraints that all "masses" must add up to zero and working in the Center of Mass frame, give us a set of coordinates which reduce the problem to n-2 particle evolution

SEQUENTIAL & CROSSED DIAGRAMS

Sequential Diagram

(+ more diagrams)



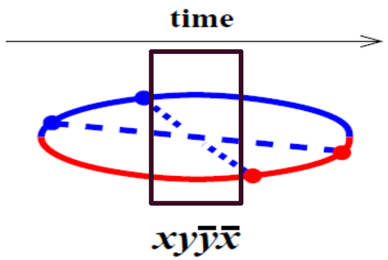
- The four gluons starts and ends in the

$$|s_{8_{aa}}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

- Corrections only come from going to 2nd-order perturbation theory in the $1/N_c$ corrections to δV.

Crossed Diagram

(+more diagrams)



- The four gluons starts in the $|s_{8_{aa}}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$ and ends in the

$$|u_{8_{aa}}\rangle = \frac{-1}{\sqrt{2}}\left(|\mathcal{T}_0\rangle + |-\rangle\right)$$

$$+\frac{1}{2N_c}\left(2|1\rangle - |\mathcal{T}_-\rangle - |\mathcal{T}_+\rangle\right) + \frac{3}{2\sqrt{2}N_c^2}|\mathcal{T}_0\rangle + \ldots$$

- Corrections come from going to 2nd -order perturbation theory in the $1/N_c$ corrections to δV, 1st order in the $\delta^2 V$, and 1st order in the δV times a $1/N_c$ contribution from the final state $|u_{8_{aa}}\rangle$.

the $N_c = \infty$ eigenstates:
$|1\rangle, |+\rangle, |-\rangle, |\mathcal{T}_0\rangle, |\mathcal{T}_+\rangle, |\mathcal{T}_-\rangle$

CORRECTION TO THE $N_c = \infty$ RESULT FOR (DOUBLE GLUON BREMSSTRAHLUNG)*

- For the sequential diagram, $\delta V'$ gives 6 different color transitions

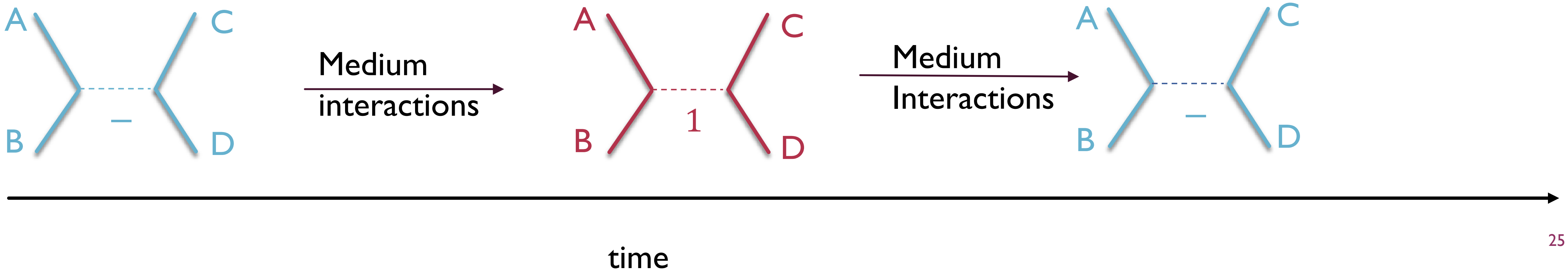
$$|-\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |-\rangle \;,\quad |-\rangle \xrightarrow{\delta V'} |\mathcal{T}_-\rangle \xrightarrow{\delta V'} |-\rangle \;,$$
$$|-\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |+\rangle \,,$$

$$|+\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |+\rangle \;,\quad |+\rangle \xrightarrow{\delta V'} |\mathcal{T}_+\rangle \xrightarrow{\delta V'} |+\rangle \;,$$
$$|+\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |-\rangle .$$

- For the crossed diagram, $\delta V'$ gives 5 different color transitions

$$|-\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |-\rangle \;,\quad |-\rangle \xrightarrow{\delta V'} |\mathcal{T}_-\rangle \xrightarrow{\delta V'} |-\rangle \;,$$
$$|-\rangle \xrightarrow{\delta V'} |\mathcal{T}_-\rangle \xrightarrow{\delta V'} |\mathcal{T}_0\rangle ,$$

$$|+\rangle \xrightarrow{\delta V'} |1\rangle \xrightarrow{\delta V'} |-\rangle \,, |+\rangle \xrightarrow{\delta V'} |\mathcal{T}_+\rangle \xrightarrow{\delta V'} |\mathcal{T}_0\rangle \;.$$



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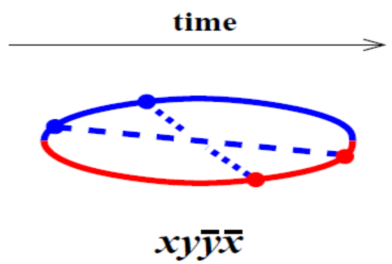


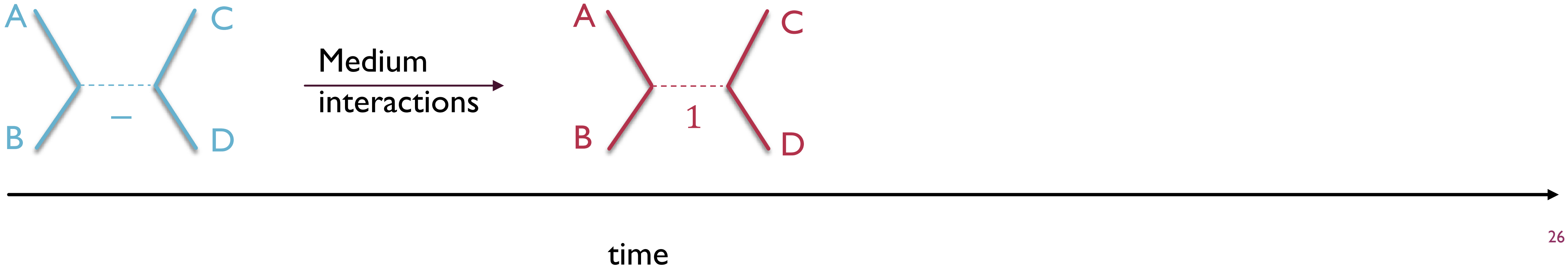
Additional transitions for the crossed diagram:

- $\delta \underline{V}'$ gives 4 different color transitions through a $1/ N_c$ overlapping correction

$$|-\rangle \xrightarrow{\delta \underline{V}'} |1\rangle, \quad |-\rangle \xrightarrow{\delta \underline{V}'} |\mathcal{T}_-\rangle, \quad |+\rangle \xrightarrow{\delta \underline{V}'} |1\rangle, \quad |+\rangle \xrightarrow{\delta \underline{V}'} |\mathcal{T}_+\rangle$$

- $\delta^2 \underline{V}'$ gives 2 different color transition through 1st order perturbation

$$|-\rangle \xrightarrow{\delta^2 \underline{V}'} |\mathcal{T}_0\rangle, |+\rangle \xrightarrow{\delta^2 \underline{V}'} |\mathcal{T}_0\rangle$$



WHY $1/N^2$ NOT $1/N$?

- A graph (with no external legs) is proportional to N^{F+V-E}
- Every 2d orientable surface is topologically equivalent to a 2-sphere with holes and handles
- $\chi = 2 - 2H - B$ where H is the numbers of handles required to draw graph without crossing lines and B is the number of fermion loops.

- A result of this representation is: *The leading connected vaccum-to-vaccum graphs are of order* N_c^2*.They are planar diagrams made up of only gluons.*
- For a diagram with no quarks (B=0), the expansion is in powers of $1/N_c^2$

SU(3) DIFFICULTIES*

- Using the constraints of the problem and the harmonic approximation, the $2D$ Hamiltonian is

$$\underline{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^T \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{b} & \underline{c} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

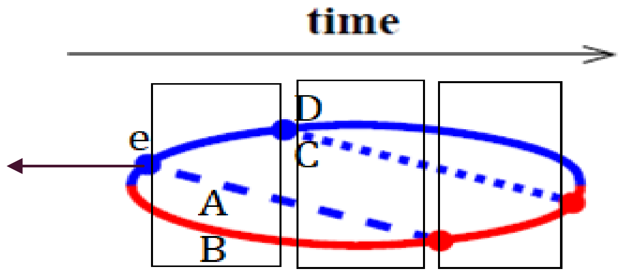
  where a,b and c are 5x5 matrices in the color space and p_1, p_2, q_1, q_2 are transverse –plane vectors

- However, the a,b and c matrices don't commute outside of the Large- N_c limit.
- One can solve for the propagator numerically by solving the corresponding Schrodinger equation numerically, but we think it's complicated.

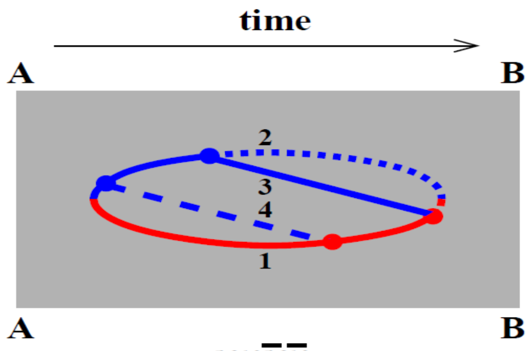
FROM 8-DIM TO 6-DIM

- Using permutation symmetries of ABCD (rectangular symmetries) which are two reflections and 180◦ rotation symmetry, our application lies within the 5 dimensional subspace : $|s_1\rangle, |s_{8_{aa}}\rangle, |s_{8_{ss}}\rangle$, $|s_{10+\overline{10}}\rangle, |s_{27}\rangle$
- The three gluons during the first phase of evolution are in the color singlet state $f_{ABe}|ABe\rangle$, then this vertex (e) splits into CD with a color factor f_{CDe}
- This means that the intermediate region start in the $f_{ABe}\, f_{CDe}|ABCD\rangle$
- The initial state and final state are the $|s_{8_{aa}}\rangle$

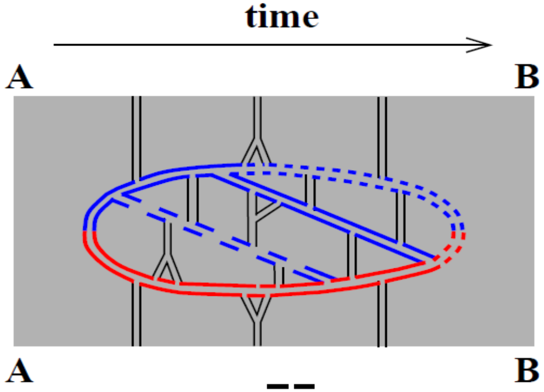
$f_{ABe}|ABe\rangle$



THE N=∞ POTENTIAL



$xy\bar{x}\bar{y}_2$

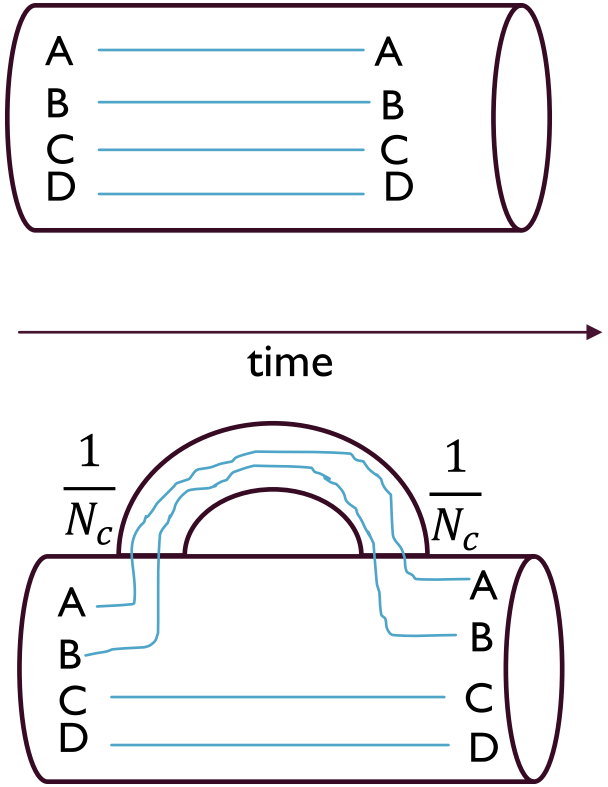


$xy\bar{x}\bar{y}_2$

- $V(b_1, \ldots, b_N; t) = V_F(b_2 - b_1; t) + V_F(b_3 - b_2; t) + \cdots + V_F(b_N - b_{N-1}; t) + V_F(b_1 - b_N; t)$
- The requirement of planar diagrams means that high energy gluon only interacts (via medium correlations) with one other, and those interactions are independent of the interactions of other gluons.

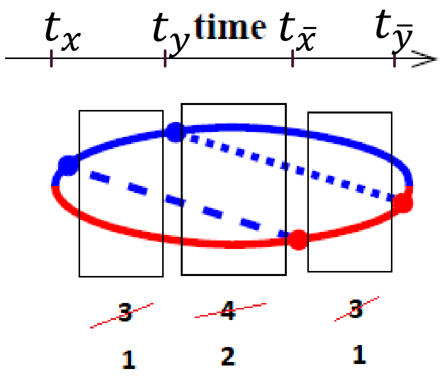
- For double gluon splitting, the expansion is in powers of $\frac{1}{N_c^2}$ $(\mathrm{H} = 1)$

- For example, this color transition $|-\rangle \overset{\delta V}{\rightarrow} |1\rangle \overset{\delta V}{\rightarrow} |-\rangle$ would look like that.



THE RESULT FOR $N_c = \infty$ FOR DOUBLE GLUON BREMSSTRAHLUNG

- Using these approximation and constraints, the differential rate looks like



$$\frac{dI}{dx\,dy} = \int_{B's,C's\,,t's} \sum_{pol} \langle |i\,\delta H|B^x\rangle\langle B^x, t_x|B^y, t_y\rangle\langle B^y|i\,\delta H|C_{41}^y, C_{23}^y\rangle$$

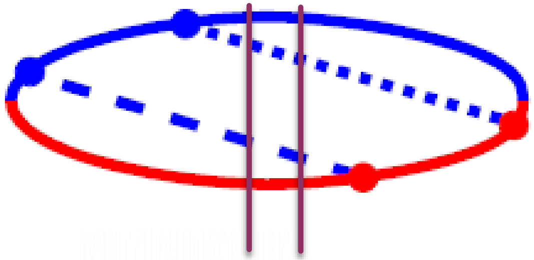
$$\langle C_{41}^y, C_{23}^y, t_y|C_{41}^{\bar{x}}, C_{23}^{\bar{x}}\rangle\langle C_{41}^{\bar{x}}, C_{23}^{\bar{x}}|i\,\overline{\delta H}|B^{\bar{x}}\rangle\langle B^{\bar{x}}, t_{\bar{x}}|B^{\bar{y}}, t_{\bar{y}}\rangle\langle B^{\bar{y}}|i\,\overline{\delta H}|\rangle$$

where the B's and C's are the transverse positions for the 1-particle and 2-particles respectively.

- Eventually, this formula is reduced to an integral over Δt (duration of the double splitting) that can be solved numerically

CORRECTION TO THE $N_c = \infty$ RESULT FOR (DOUBLE GLUON BREMSSTRAHLUNG)

- Use 2nd order perturbation theory to find the correction to the 4-particle propagator, then use that in the result for the rate (the $N_c = \infty$ result).
- In the harmonic approximation, this corresponds to finding the correction to the propagator of two coupled harmonic oscillators



- For the sequential diagram,

$$\delta^2 G \propto \int_{\boldsymbol{C}_1,\boldsymbol{C}_2 t_1,t_2,\Delta t} \sum_{\lambda_i} \langle \pm | G_0(0 \to 1) | \pm \rangle \langle \pm | \delta V' | \lambda_i \rangle \langle \lambda_i | G_0(1 \to 2) | \lambda_i \rangle \langle \lambda_i | \delta V' | \pm \rangle \langle \pm | G_0(2 \to 3) | \pm \rangle$$

where $|\lambda_i\rangle$ is $|1\rangle, |\mathcal{T}_+\rangle$ or $|\mathcal{T}_-\rangle$, $\boldsymbol{C}_1, \boldsymbol{C}_2$ are the intermediate positions and t_1 , $t_2, \Delta t$ the intermediate time.

- For the sequential diagram, those corrections only come from going to second-order perturbation theory in the $1/\ N_c$ corrections to δV.
- The four gluons starts and ends in $f_{ABe}\, f_{CDe}|ABCD\rangle$ for the sequential diagram, written as $|s_{8_{aa}}\rangle = \frac{1}{\sqrt{2}}\ (\ |+\rangle + |-\rangle\ )$ (the 2x2 subspace when we take $N_c \to \infty$).
- One consequence is that only $\delta T'$ (prime means $N_c = \infty$ eigenstates) contribute to the $1/N_c$ correction to δV for the sequential diagram
- So $\delta V'$=$\delta T'$ *(coupled harmonic oscillator potential)
- All $\delta T'$ non-vanishing elements are equal to $\frac{1}{\sqrt{2}N_c}$

LARGE N_c LIMIT

- A Gluon field carries two indices in the color space, but a quark field carries one index.
- 't Hooft represented a gluon line by a double line.
- A vertex $\propto N$, A propagator $\propto 1/N$, color loop $\propto N$
- A graph (with no external legs) is proportional to N^{F+V-E}
- Every 2d orientable surface is topologically equivalent to a 2-sphere with holes and handles
- $\chi = 2 - 2H - B$ where H is the numbers of handles required to draw graph without crossing lines and B is the number of fermion loops.