The hydrodynamic gradient expansion beyond Bjorken flow

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Based on arXiv:2007.05524 & arXiv:2107.xxxxx in collaboration with Michal Heller, Michal Spalinski, Viktor Svensson & Ben Withers

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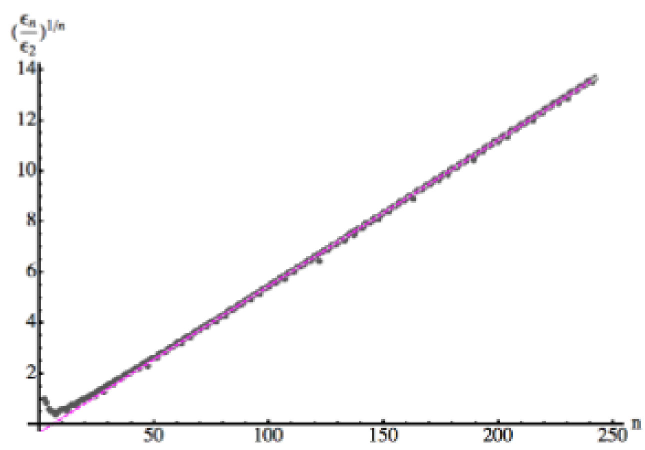
Introduction & motivation

- Hydrodynamics is an indispensable framework to model real-world phenomena: heavy-ion collisions, cosmological phase transitions…
- It provides an effective description of the late-time, long-distance behavior of any medium endowed with conserved currents.
- Despite its scientific importance and long history, there are still fundamental aspects waiting to be fully understood.
- In this talk, we are going to focus on one of these open problems: <u>the large-order behavior of the gradient expansion</u> in classical relativistic hydrodynamics (no stochastic fluctuations).
- From a physical standpoint, improving our knowledge of hydrodynamic gradient expansions is crucial to understand better the applicability regime of low-order truncated relativistic hydrodynamics.

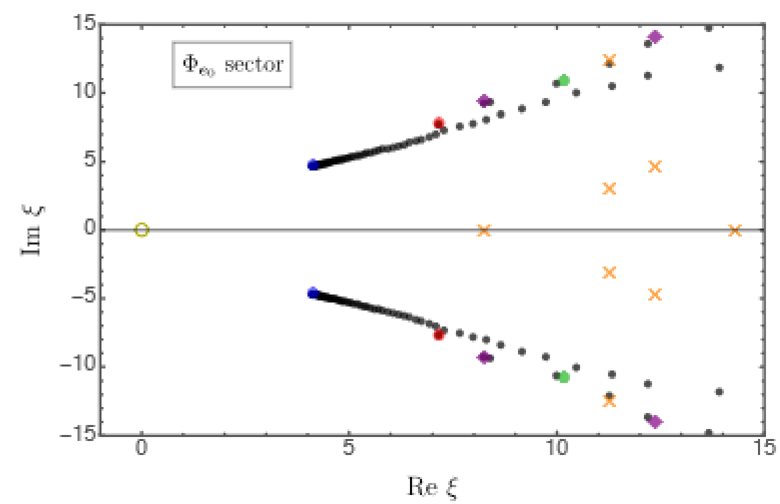
By now, we have a well-developed understanding of hydrodynamic gradient expansions for highly symmetric fluid flows -in particular, Bjorken flow- across several models of physical relevance: Muller-Israel-Stewart theories, kinetic theory and holography.

Among other facts, these studies have brought to the table:

- The explicit demonstration that hydrodynamic gradient expansions can be <u>factorially divergent</u> asymptotic series.
- A detailed understating of how the <u>large-order behavior</u> of these gradient expansions <u>encodes</u> information about the <u>transient nonhydrodynamic contributions</u> to the full energy-momentum tensor.



From [Heller, Janik & Witaszczyk, 2013]



From [Aniceto, Jankowski, Meiring & Spalinski, 2018]

In this talk, we will discuss whether these observations can be generalized to fluid flows with less symmetry restrictions

Classical hydrodynamics

- We will assume that a local rest-frame exists and work in the Landau frame

$$\langle T_{\mu\nu}(X)\rangle = \mathcal{E} U_\mu U_\nu + P(\mathcal{E})(\eta_{\mu\nu} + U_\mu U_\nu) + \Pi_{\mu\nu} \qquad \langle T^\mu_\nu \rangle U^\nu = -\mathcal{E} U^\mu, \quad U^\mu U_\mu = -1$$

- We will only consider conformal fluids,

$$\langle T^\mu_\mu \rangle = 0 \longrightarrow P(\mathcal{E}) = \frac{1}{3}\mathcal{E} \quad \& \quad \Pi^\mu_\mu = 0$$

- Our main object of interest will be the <u>constitutive relations</u>

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}[\mathcal{E}, U_\alpha] = -\eta \sigma_{\mu\nu} + \ldots$$

  which express the dissipative tensor as a <u>gradient expansion</u> in terms of the hydrodynamic fields: energy density and fluid velocity.

- The question we will focus on is the <u>large-order behavior of the gradient-expanded constitutive relations</u> when evaluated on a particular fluid flow. <u>We will not discuss the initial value problem</u>.
- We will discuss first (very briefly) this problem in the linear response regime, considering later the inclusion of nonlinear effects.
- When nonlinearities are included, we will mostly work in BRSSS theory, commenting on generalizations of our results as we go along.

The gradient expansion in the linear response regime

See arXiv:2007.05524 with M. Heller, M. Spalinski, V. Svensson and B. Withers

- We considered the most general purely spatial formulation of the gradient expansion,

$$\left(\partial_j u_l + \partial_l u_j - \frac{2}{d-1}\delta_{jl}\partial_r u^r\right) \qquad \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\partial_r u^r \qquad \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\epsilon$$

$$\Pi_{jl} = -A(\partial^2)\,\sigma_{jl} - B(\partial^2)\,\pi^u_{jl} - C(\partial^2)\,\pi^\epsilon_{jl}$$

$$\sum_{n=0}^{\infty} a_n(-\partial^2)^n \qquad \sum_{n=0}^{\infty} b_n(-\partial^2)^n \qquad \sum_{n=0}^{\infty} c_n(-\partial^2)^n$$

$a_n,\ b_n\ c_n$: transport coefficients

- The transport coefficients are related to the hydrodynamic mode frequencies of the microscopic theory,

$$a_n = [k^{2n+2}]\,(i\,s\,T\,\omega_\perp)\,,$$

$$b_n = [k^{2n+4}]\left(-i\,\frac{d-1}{d-2}\,s\,T\left(\omega_\parallel^+ + \omega_\parallel^-\right) + 2\,i\,s\,T\omega_\perp\right),$$

$$c_n = [k^{2n+4}]\left(-\frac{k^2}{d-2} - \frac{d-1}{d-2}\,\omega_\parallel^+\,\omega_\parallel^-\right).$$

$\omega_\perp(k)$: shear channel

$\omega_\parallel^\pm(k)$: sound channel

[Withers, 2018] [Grozdanov, Kovtun, Starinets & Tadic, 2019] [Abbasi & Tahery, 2020] [Jansen & Pantelidou, 2020] [Arean, Davison, Gouteraux & Suzuki, 2020] [Heller, AS, Spalinski, Svensson & Withers, 2020] ...

- Their large-order behavior is controlled by critical momenta related to the singularities of the hydrodynamic mode frequencies in the complex k-plane,

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = |k_*^{(A)}|^{-2}$$

singularity of $isT\omega_\perp$ closest to the origin:
critical momentum (microscopic theory quantity)

...

- The large-order behavior of the spatial derivatives is controlled by the support of the hydrodynamic fields in momentum space, $|k_{\max}|$

- Final convergence criterion for the purely spatial gradient expansion

$$|k_{\max}| < \min(|k_*^{(A)}|, |k_*^{(B)}|, |k_*^{(C)}|) \longrightarrow \text{The gradient expansion converges}$$

$$\min(|k_*^{(A)}|, |k_*^{(B)}|, |k_*^{(C)}|) < |k_{\max}| < \infty \longrightarrow \text{The gradient expansion diverges geometrically}$$

$$|k_{\max}| \to \infty \longrightarrow \underline{\text{The gradient expansion diverges factorially}}$$

Main lesson: <u>nonlinearities are not needed in order to get a factorially divergent gradient expansion</u>

Including nonlinearities: the case of conformal BRSSS theory

Our focus will be BRSSS theory [Baier, Romatschke, Son, Starinets & Stephanov, 2007]. This is a phenomenological model that, besides the energy density and the fluid velocity, treats $\Pi_{\mu\nu}$ as independent dynamical degrees of freedom.

$$\nabla^\mu T_{\mu\nu} = 0$$

Weyl-covariant derivative

$$T_{\mu\nu} = \mathcal{E} U_\mu U_\nu + P(\mathcal{E})(g_{\mu\nu} + U_\mu U_\nu) + \Pi_{\mu\nu} \qquad (\tau_\pi U^\alpha \mathcal{D}_\alpha + 1)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \ldots,$$

$\propto T^4$ $\quad \frac{1}{3}\mathcal{E}$ $\quad \propto \frac{1}{T}$ $\quad \propto T^3$

Our fluid will live in four-dimensional Minkowski space.

We will focus on 1+1-dimensional longitudinal flows, but we will not demand invariance under longitudinal boosts

$$U^\mu = (\cosh u(t,z), \sinh u(t,z), 0, 0)$$

The dissipative tensor can be expressed in terms of a single d.o.f.

$$\Pi^{\mu\nu} = \begin{pmatrix} -2\sinh^2(u)\Pi_\perp & \sinh(2u)\Pi_\perp & 0 & 0 \\ \sinh(2u)\Pi_\perp & -2\cosh^2(u)\Pi_\perp & 0 & 0 \\ 0 & 0 & \Pi_\perp & 0 \\ 0 & 0 & 0 & \Pi_\perp \end{pmatrix}$$

The gradient expansion

To construct our gradient expansion, we introduce a formal parameter ϵ by means of a homogeneous rescaling of the spacetime coordinates,

$$t \to \frac{t}{\epsilon}, \quad z \to \frac{z}{\epsilon}$$

and write the formal power series

$$\Pi_\perp = \sum_{n=1}^{\infty} \Pi_{\perp,n}(t,z)\epsilon^n$$

Nonlinear terms

The dynamical constitutive relation are transformed into the recursion relation given by

$$\Pi_{\perp,1} = \frac{2}{3}\eta\partial_\mu U^\mu \qquad \Pi_{\perp,n+1} = -\frac{4}{3}\tau_\pi(\partial_\mu U^\mu)\Pi_{\perp,n} - \tau_\pi U^\mu \partial_\mu \Pi_{\perp,n} + \dots$$

Given $\mathcal{E}$ and U^μ, what is the large-order behavior of the $\Pi_{\perp,n}$ coefficients?

Our approach will rely on analytic arguments and numerical results that complement each other

The gradient expansion at large order: analytic results

Factorial-over-power large-order ansatz, $\Pi_{\perp.n} = \frac{\Gamma(n+\beta)}{\chi(t,z)^{n+\beta}}(A(t,z)+\ldots)$ → singulant

The singulant equation of motion can be found analytically,

$$U^{\mu}\partial_{\mu}\chi(t,z) = \frac{1}{\tau_{\pi}(T(t,z))} \longrightarrow U^{\mu}\partial_{\mu}\chi(t,z) = i\omega_{NH}(k=0)|_{T=T(t,z)}$$

Nonhydrodynamic sound mode of conformal BRSSS at zero momentum

<u>Lessons</u>:

1. The gradient expansion can be factorially divergent.

2. The large-order behavior of the gradient expansion contains information about the nonhydrodynamic modes of the underlying theory.

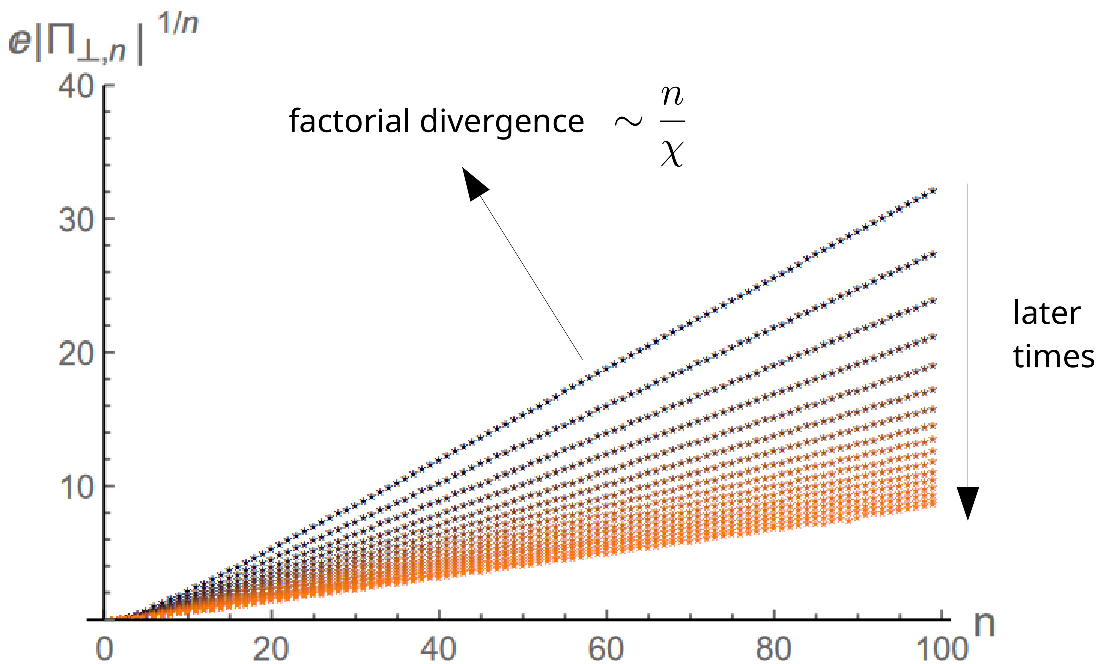
3. Nonlinearities do not affect the leading & subleading large-order behavior (they first enter in A).

Are these predictions realized in practice?

The gradient expansion at large order: numerical results

Strategy: $\begin{matrix} T(0,z) \\ u(0,z) \\ \Pi_\perp(0,z) \end{matrix} \xrightarrow[\text{time evolve numerically}]{\text{Discretize \&}} \begin{matrix} T(t,z) \\ u(t,z) \\ \Pi_\perp(t,z) \end{matrix} \xrightarrow[\text{gradient expansion}]{\text{Evaluate the}} \{\Pi_{\perp,n}(t,z),\, n = 1,\ldots,N\}$

When the dust settles, we find that our numerical analysis confirms the analytic prediction!



The best strategy to extract the singulant from the large-order behavior of the gradient expansion is to consider the Borel transform

$$\Pi_\perp = \sum_{n=1}^{\infty} \Pi_{\perp,n}(t,z)\epsilon^n \xrightarrow{\text{Borel transform}} \Pi_\perp^{(B)} = \sum_{n=1}^{\infty} \frac{\Pi_{\perp,n}(t,z)}{n!}\epsilon^n \xrightarrow{\text{analytic continuation}} \tilde{\Pi}_\perp^{(B)}$$

factorially divergent

finite convergence radius

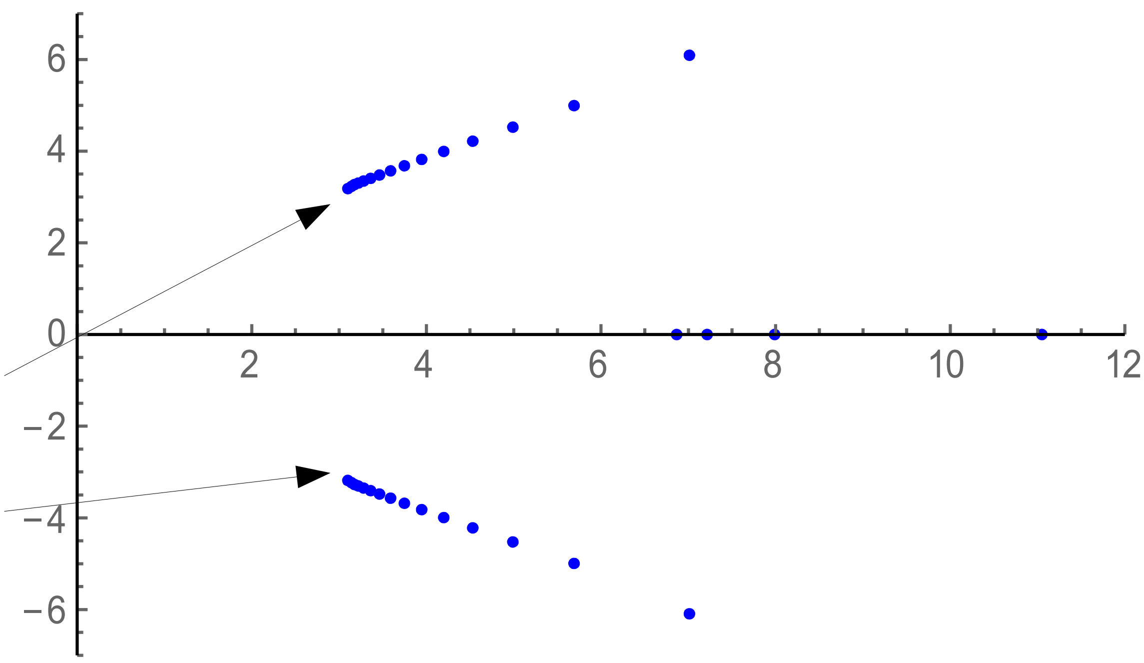
defined in the whole complex plane

Main idea: <u>the singulants are the singularities of the analytically continued Borel transform</u> $\tilde{\Pi}_\perp^{(B)}$

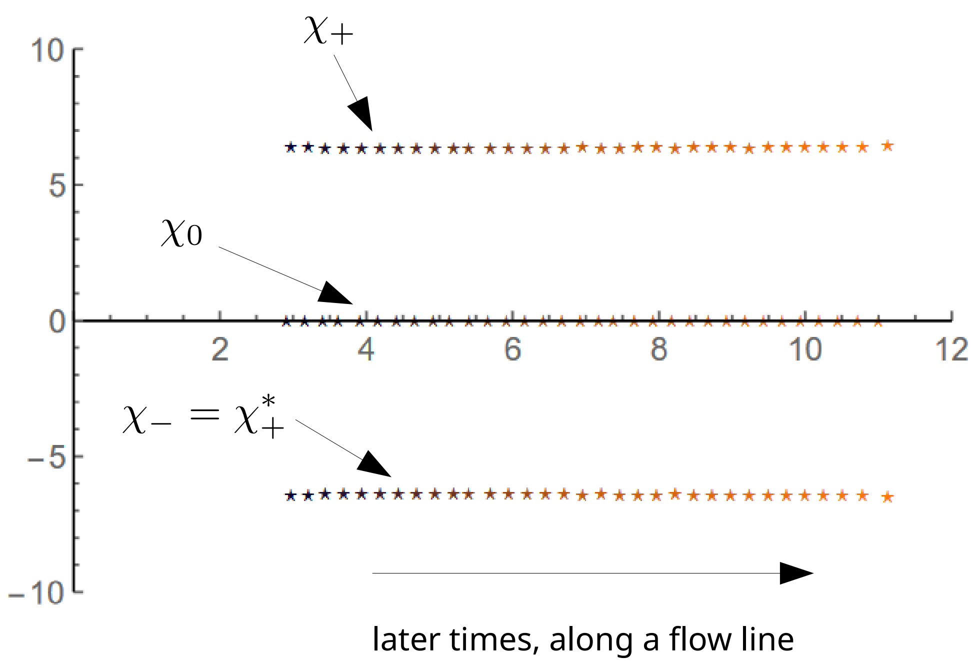
In practice: the analytic continuation of the Borel transform is done by means of Padé approximants,

$$f(z) = \sum_{n=0}^{N} f_n z^n \to \frac{\sum_{n=0}^{N_1} p_n z^n}{1+\sum_{n=0}^{N_2} q_n z^n}$$

$$N_1 + N_2 = N$$



A branch-point of $\tilde{\Pi}_\perp^{(B)}$ appears as a point of pole accumulation of its Padé approximant



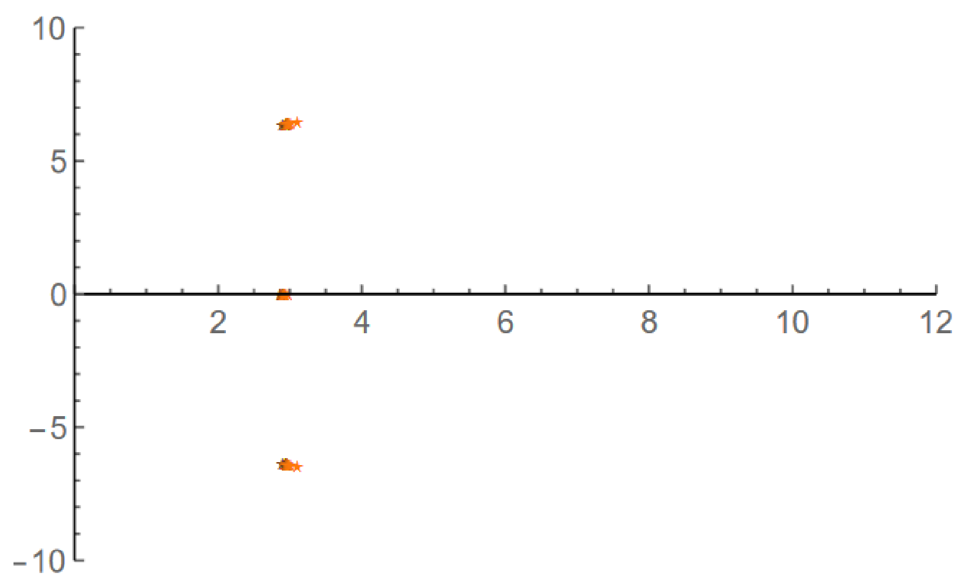
There is a dominant purely real singulant
and two subdominant complex-conjugated ones

The singulants real parts increase with time,
the imaginary parts remain constant

If we compute the difference

$$\chi(\tau,\sigma) - \int_0^\tau \frac{d\tau'}{\tau_\pi(T(\tau',\sigma))}$$

the singulant trajectories collapse to a single point
as predicted by the singulant equation of motion



The gradient expansion in other phenomenological models

We have found that the factorial-over-power ansatz describes correctly the large-order behavior of the gradient expansion computed numerically in other models as well

Heller-Janik-Spalinski-Witaszczyk model (HJSW)

[arXiv:1409.5087]

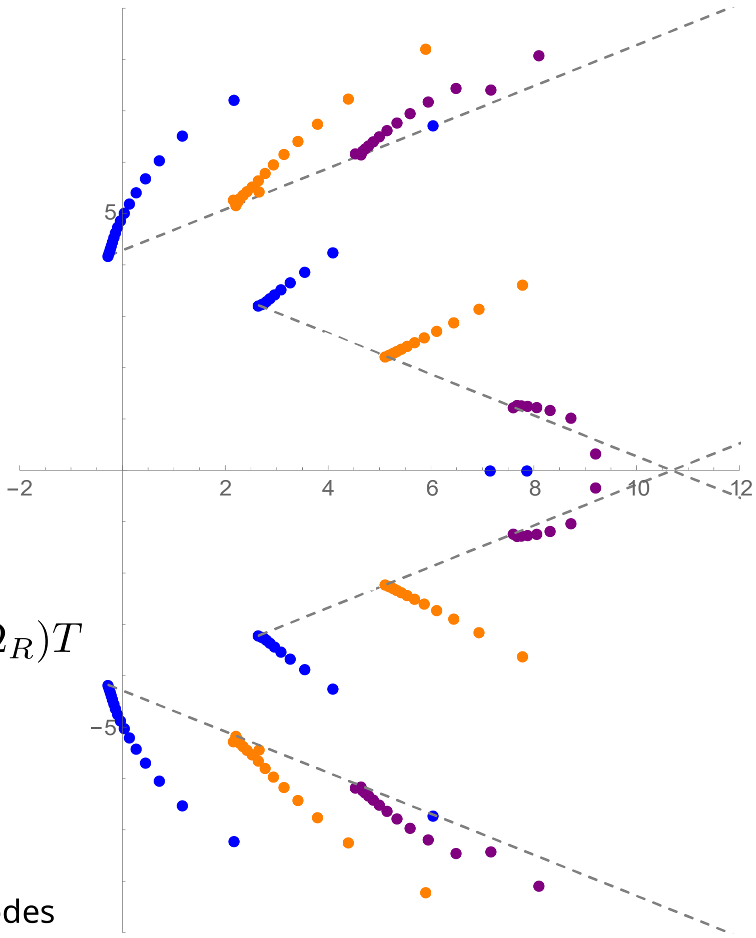
$$\nabla^\mu T_{\mu\nu} = 0$$

$$T_{\mu\nu} = \mathcal{E} U_\mu U_\nu + P(\mathcal{E})(g_{\mu\nu} + U_\mu U_\nu) + \Pi_{\mu\nu}$$

$$\left(\left(\frac{U^\alpha}{T} \mathcal{D}_\alpha \right)^2 + 2\Omega_I \left(\frac{U^\alpha}{T} \mathcal{D}_\alpha \right) + |\Omega|^2 \right) \Pi_{\mu\nu} = -\eta |\Omega|^2 \sigma_{\mu\nu} + \ldots$$

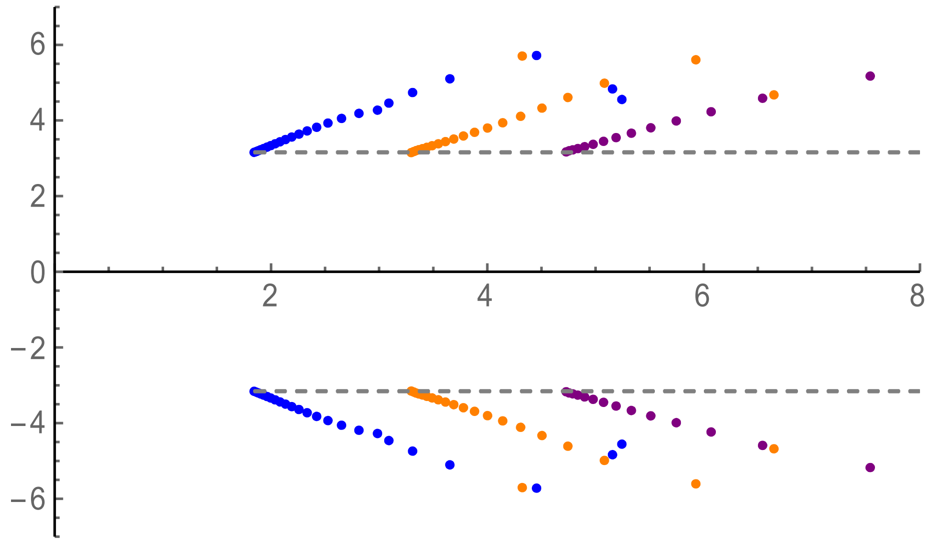
$$U^\mu \partial_\mu \chi = (\Omega_I \pm i\Omega_R) T$$

nonhydrodynamic sound modes
at zero momentum



Denicol-Noronha model (DN) [arXiv:1908.09957]

$$\nabla^\mu T_{\mu\nu} = 0 \qquad \nabla^\mu J_\mu = 0$$
$$T_{\mu\nu} = \mathcal{E} U_\mu U_\nu + P(\mathcal{E})(g_{\mu\nu} + U_\mu U_\nu) + \Pi_{\mu\nu} \quad J_\mu = n U_\mu$$
$$(\tau_\pi U^\alpha \mathcal{D}_\alpha + 1)\Pi_{\mu\nu} = -\eta \sigma_{\mu\nu} + \dots$$



$$U^\mu \partial_\mu \chi = \frac{1}{\tau_\pi(n)}$$

nonhydrodynamic sound mode
at zero momentum

In conformal BRSSS, HJSW and DN the singulant equation of motion only depends on the nonhydrodynamic modes at zero spatial momentum.

This is not generic: it follows from the fact that the constitutive relation only involves derivatives along the flow,

$$U^\mu \mathcal{D}_\mu$$

To explore more general scenarios, we constructed a new model whose constitutive relation also features second-order spacetime derivatives in the direction orthogonal to U^μ: generalized HJSW model

$$\left(\left(\frac{1}{T}\mathcal{D}\right)^2 + 2\Omega_I\left(\frac{1}{T}\mathcal{D}\right) \boxed{-\frac{\hat{c}_L^2}{T^2}\frac{1}{2}\left[\Delta_\mu^\alpha\Delta_\nu^\beta + \Delta_\nu^\alpha\Delta_\mu^\beta - \frac{2}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}\right](\Delta^{\rho\sigma}\mathcal{D}_\rho\mathcal{D}_\sigma)} + |\Omega|^2\right)\Pi_{\mu\nu} = -\eta|\Omega|^2\sigma_{\mu\nu} + \ldots$$

$$\Big\downarrow \text{ factorial-over-power ansatz}$$

Singulant e.o.m.

$$(U^\mu\partial_\mu\chi)^2 + 2\Omega_I T(U^\mu\partial_\mu\chi) \boxed{-\hat{c}_L^2(e^\mu\partial_\mu\chi)^2} + |\Omega|^2T^2 = 0$$

e^μ unit normalized, orthogonal to U^μ

Lessons:

1. The singulant equation of motion is not generically related to the nonhydrodynamic modes at $k=0$

2. It is also not directly related to the nonhydrodynamic modes at finite momentum, but to the dispersion relations of infinitesimal plane-wave fluctuations of $\Pi_\perp$ when the derivatives of the hydrodynamic fields are taken to be zero.

$$U^\mu\partial_\mu \longleftrightarrow i\omega = i\Omega(k) \qquad e^\mu\partial_\mu \longleftrightarrow ik$$

Final remarks:

1. A preliminary analysis suggests that lesson 2. also holds in Holography.

2. For Bjorken flow, $e^\mu\partial_\mu\chi = 0$ and we recover the result that the singulant time-evolution is controlled by the nonhydrodynamic modes at zero momentum.

Conclusions & open problems

Take-home messages

- We have shown that factorially divergent hydrodynamic gradient expansions are ubiquitous: their existence does not rely essentially on the fact that the fluid flow under consideration enjoys a high-degree of symmetry.
- We have found that this fact holds at the fully nonlinear level in different models of physical relevance. It also holds at the linearized level in full generality.
- Even if nonlinearities are included, we have shown that the leading & subleading contributions to the large-order behavior of the gradient expansion for a general fluid flow can be partially understood in terms of linear response.

Open questions

- Can we extend our nonlinear results to other formulations of the gradient expansion, like a purely spatial one?
- Can we cross-check the analytic predictions with a numerical computation of the gradient expansion in RTA kinetic theory? In Holography?
- Can we relate the singulants at $t = 0$ to the initial data?
- Can we employ this relation, together with the known singulant evolution equations, to place constraints on when the low-order truncated gradient expansion will provide a good description of the energy-momentum tensor, i.e., the hydrodynamization time?

Many thanks for your time!