False Vacuum Decay in Real Time

Wen-Yuan Ai

CP3, UC Louvain

Based on: JHEP 12 (2019) 095

In collaboration with:
Björn Garbrecht, Carlos Tamarit

June 29, SEWM 2021, online

Outline

- Introduction
- Optical theorem for FVD and the complex bounce
- The Picard-Lefschetz theory
- Summary

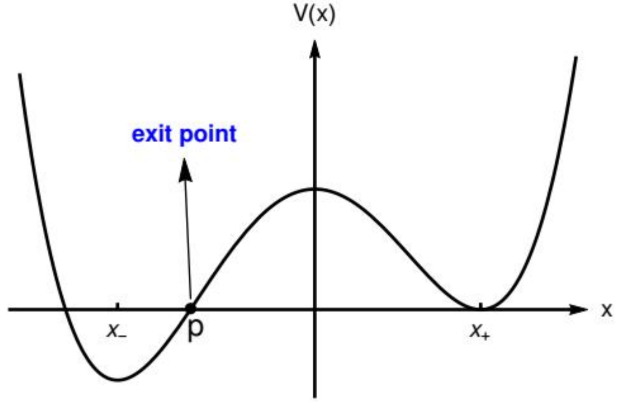
Outline

- Introduction
- Optical theorem for FVD and the complex bounce
- The Picard-Lefschetz theory
- Summary

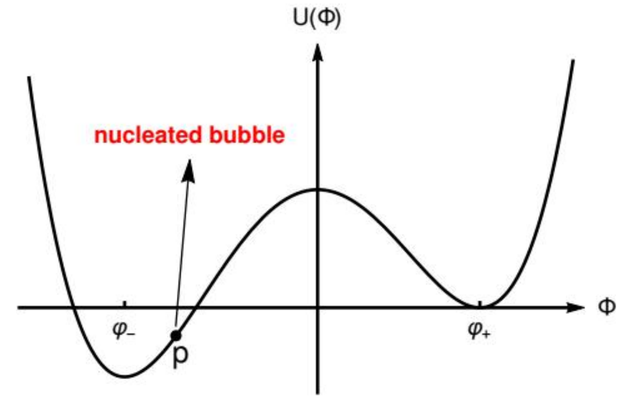
False vacuum decay

What is FVD?

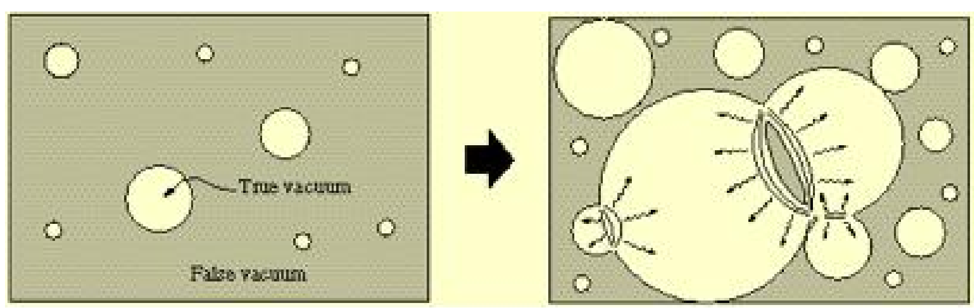
Quantum Mechanics:



Quantum Field Theory:



Relevance to high-energy phenomenology: Electroweak Metastability, Cosmological Phase Transitions (typically at finite temperature)



What we did in our work?

What we know about FVD

- ❑ The decay rate from the Euclidean/imaginary-time method
- ❑ Bubble configuration at the time of nucleation

However

- ❑ Check for the Euclidean method in quantum field theory
- ❑ The real-time picture has never been understood

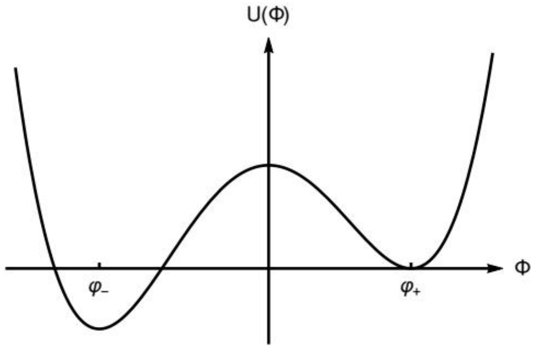
Our work

- ❑ Relate the real-time tunneling calculations to the optical theorem
- ❑ Directly compute the Feynman path integral for quantum tunneling/FVD
- ❑ Develop techniques for carrying out the calculation
- ❑ New insights

The imaginary-time formalism on FVD

Central idea: Callan & Coleman, 1977

False Vacuum ⟶ Unstable ⟶ Complex Energy



$$E_0 = \mathrm{Re}E_0 + i\mathrm{Im}E_0$$

Decay rate: $\Gamma = -2\mathrm{Im}E_0$

Consider the Euclidean transition amplitude $\langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle$. Insert a complete basis of energy eigenstates

$$\langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle = \sum_n e^{-E_n\mathcal{T}}|\langle\varphi_+|n\rangle|^2$$

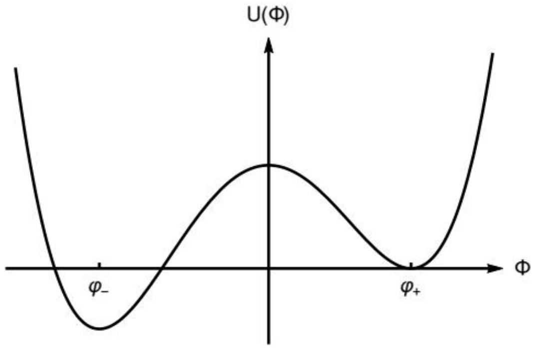
Taking $\mathcal{T} \to +\infty$,

$$\langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle \overset{\mathcal{T}\to+\infty}{=\!=\!=} e^{-E_0\mathcal{T}}|\langle\varphi_+|0\rangle|^2$$

The imaginary-time formalism on FVD

Central idea: Callan & Coleman, 1977

False Vacuum ⟶ Unstable ⟶ Complex Energy



$$E_0 = \mathrm{Re}E_0 + i\mathrm{Im}E_0$$

Decay rate: $\mathit{\Gamma} = -2\mathrm{Im}E_0$

Consider the Euclidean transition amplitude $\langle \varphi_+ | e^{-H\mathcal{T}} | \varphi_+ \rangle$. Insert a complete basis of energy eigenstates

$$\langle \varphi_+ | e^{-H\mathcal{T}} | \varphi_+ \rangle = \sum_n e^{-E_n \mathcal{T}} |\langle \varphi_+ | n \rangle|^2$$

Taking $\mathcal{T} \to +\infty$,

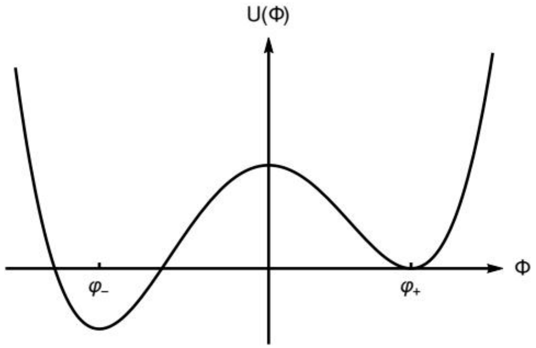
$$\langle \varphi_+ | e^{-H\mathcal{T}} | \varphi_+ \rangle \overset{\mathcal{T} \to +\infty}{=\!=} e^{-E_0 \mathcal{T}} |\langle \varphi_+ | 0 \rangle|^2$$

$$\langle \varphi_+ | e^{-H\mathcal{T}} | \varphi_+ \rangle = \int \mathcal{D}\Phi \; e^{-S_E[\Phi]} \equiv Z^E$$

The imaginary-time formalism on FVD

Central idea: Callan & Coleman, 1977

False Vacuum ⟶ Unstable ⟶ Complex Energy



$$E_0 = \mathrm{Re}E_0 + i\mathrm{Im}E_0$$

Decay rate: $\Gamma = -2\mathrm{Im}E_0$

Consider the Euclidean transition amplitude $\langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle$. Insert a complete basis of energy eigenstates

$$\langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle = \sum_n e^{-E_n\mathcal{T}}|\langle\varphi_+|n\rangle|^2$$

Taking $\mathcal{T} \to +\infty$,

$$\left.\begin{aligned} \langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle &\overset{\mathcal{T}\to+\infty}{=\!=\!=} e^{-E_0\mathcal{T}}|\langle\varphi_+|0\rangle|^2 \\ \langle\varphi_+|e^{-H\mathcal{T}}|\varphi_+\rangle &= \int \mathcal{D}\Phi\, e^{-S_E[\Phi]} \equiv Z^E \end{aligned}\right\} \Rightarrow \boxed{\Gamma = -2\mathrm{Im}E_0 = \lim_{\mathcal{T}\to+\infty} \frac{2}{\mathcal{T}} \mathrm{Im}(\ln Z^E)}$$

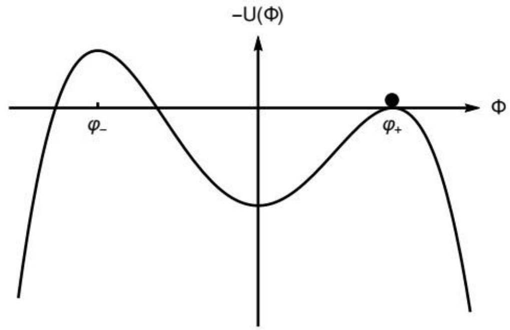
Compute the partition function

Method of steepest descent:

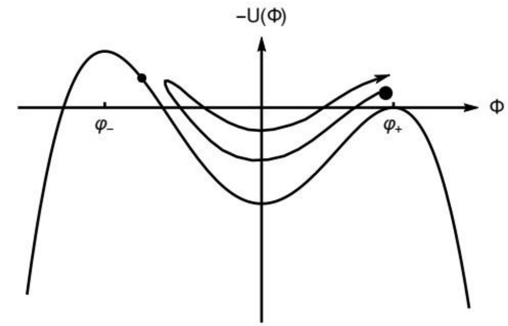
1. Find stationary points: $\delta S_E|_{\varphi_a} = 0$
2. Expand about the stationary points: $\Phi = \varphi_a + \Delta\Phi_a$

$$\int \mathcal{D}\Phi \; e^{-S_E[\Phi]} = \sum_a e^{-S_E[\varphi_a]} \int \mathcal{D}\Delta\Phi_a \; e^{-\frac{1}{2}\Delta\Phi_a \left(\frac{\delta^2 S_E[\Phi]}{\delta\Phi^2}\Big|_{\varphi_a} \right) \Delta\Phi_a + \dots}$$

In Euclidean space, the potential is upside-down. There are two types of sationary points.



Trivial false vacuum φ_F



Bounce φ_B

Callan-Coleman:
$$\Gamma/V = e^{-S_E[\varphi_B]} \left(\frac{S_E[\varphi_B]}{2\pi} \right)^2 \left| \frac{\det'[-\partial^2 + U''(\varphi_B)]}{\det[-\partial^2 + U''(\varphi_F)]} \right|^{-1/2}$$

Outline

- Introduction
- Optical theorem for FVD and the complex bounce
- The Picard-Lefschetz theory
- Summary

Optical theorem for FVD

The transition amplitude corresponding to false vacuum decay is

$$\int \mathcal{D}\varphi_{\rm out}\ \langle\varphi_{\rm out}|e^{-iHT}|{\rm FV}\rangle$$

Optical theorem for FVD

The transition amplitude corresponding to false vacuum decay is

$$\int \mathcal{D}\varphi_{\rm out}\ \langle \varphi_{\rm out}|e^{-iHT}|{\rm FV}\rangle$$

Instead, we consider the false-vacuum-to-false-vacuum transition amplitude

$$\langle {\rm FV}|e^{-iHT}|{\rm FV}\rangle = 1 + iM$$

Then unitarity gives

$$\Gamma T = \int \mathcal{D}\varphi_{\rm out}\ |\langle \varphi_{\rm out}|e^{-iHT}|{\rm FV}\rangle|^2 = 2\ {\rm Im}M$$

We refer to this as the optical theorem for FVD.

Optical theorem for FVD

The transition amplitude corresponding to false vacuum decay is

$$\int \mathcal{D}\varphi_{\rm out}\ \langle \varphi_{\rm out}|e^{-iHT}|{\rm FV}\rangle$$

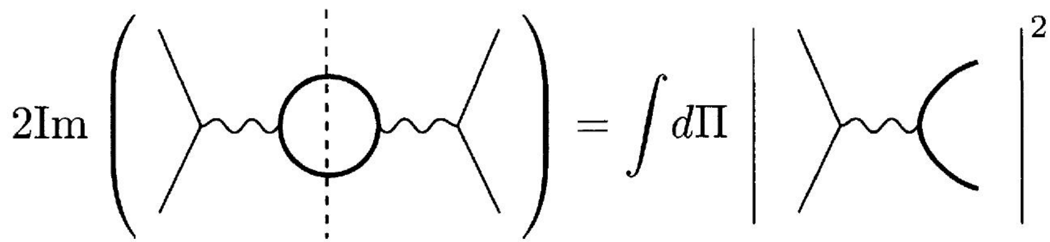
Instead, we consider the false-vacuum-to-false-vacuum transition amplitude

$$\langle {\rm FV}|e^{-iHT}|{\rm FV}\rangle = 1 + iM$$

Then unitarity gives

$$\Gamma T = \int \mathcal{D}\varphi_{\rm out}\ |\langle \varphi_{\rm out}|e^{-iHT}|{\rm FV}\rangle|^2 = 2\ {\rm Im}M$$

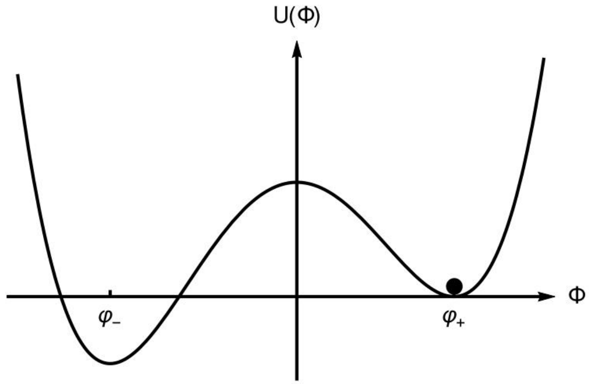
We refer to this as the optical theorem for FVD.



Complex analysis for path integral

Again, the false-vacuum-to-false-vacuum transition amplitude can be calculated by the path integral

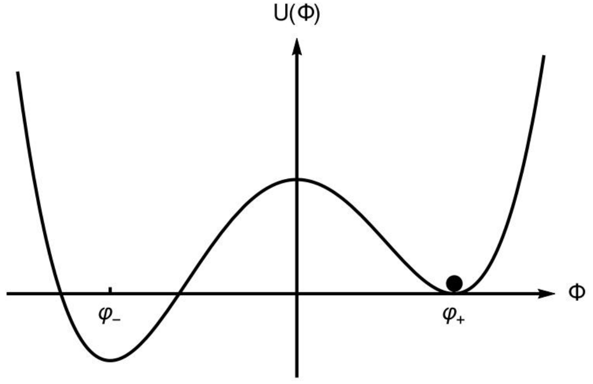
$$\langle \mathrm{FV}| e^{-iHT} |\mathrm{FV}\rangle = \mathcal{N}^2 \int \mathcal{D}\Phi \; e^{iS_M[\Phi]}$$



Complex analysis for path integral

Again, the false-vacuum-to-false-vacuum transition amplitude can be calculated by the path integral

$$\langle \mathrm{FV}| e^{-iHT} |\mathrm{FV}\rangle = \mathcal{N}^2 \int \mathcal{D}\Phi \, e^{iS_M[\Phi]}$$



Complex analysis to path integral!

Complexify field configurations, deform the contour

↓

Complex saddle points

↓

The method of the steepest descent

Complex bounce

We can immediately identify a complex stationary point

$$\phi_B(t;\mathbf{x}) = \varphi_B(\tau \to ie^{-i\epsilon}t, \mathbf{x})$$

Can the complex bounce give the decay rate in the Minkowkian path integral?

Complex bounce

We can immediately identify a complex stationary point

$$\phi_B(t;\mathbf{x}) = \varphi_B(\tau \to ie^{-i\epsilon}t, \mathbf{x})$$

Can the complex bounce give the decay rate in the Minkowkian path integral?

Cherman & Unsal, arXiv:1408.0012

Complex bounce

We can immediately identify a complex stationary point

$$\phi_B(t;\mathbf{x}) = \varphi_B(\tau \to ie^{-i\epsilon}t,\mathbf{x})$$

Can the complex bounce give the decay rate in the Minkowkian path integral?

Cherman & Unsal, arXiv:1408.0012

Our work: fully recover the Callan-Coleman result!

Complex bounce

We can immediately identify a complex stationary point

$$\phi_B(t;\mathbf{x}) = \varphi_B(\tau \to ie^{-i\epsilon}t, \mathbf{x})$$

Can the complex bounce give the decay rate in the Minkowkian path integral?

Cherman & Unsal, arXiv:1408.0012

Our work: fully recover the Callan-Coleman result!

With the complex bounce, one can decompose the path integral into

$$\mathcal{N}^2 \int \mathcal{D}\Phi \; e^{iS_M[\Phi]} = \mathcal{N}^2 Z_F^M + \mathcal{N}^2 Z_B^M$$

Together with $\langle \mathrm{FV}|e^{-iHT}|\mathrm{FV}\rangle = 1 + iM$ and $\Gamma T = 2\,\mathrm{Im} M$, one obtains

$$\Gamma = -\frac{2}{T}\,\mathrm{Re}\left(\frac{Z_B^M}{Z_F^M}\right)$$

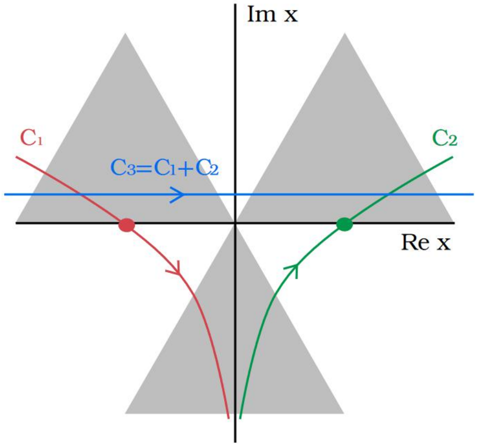
Outline

- Introduction
- Optical theorem for FVD and the complex bounce
- The Picard-Lefschetz theory
- Summary

The Picard-Lefschetz theory

An example: Airy function Witten, 2011

$$Ai(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{i\lambda\left(\frac{x^3}{3} - x\right)}$$



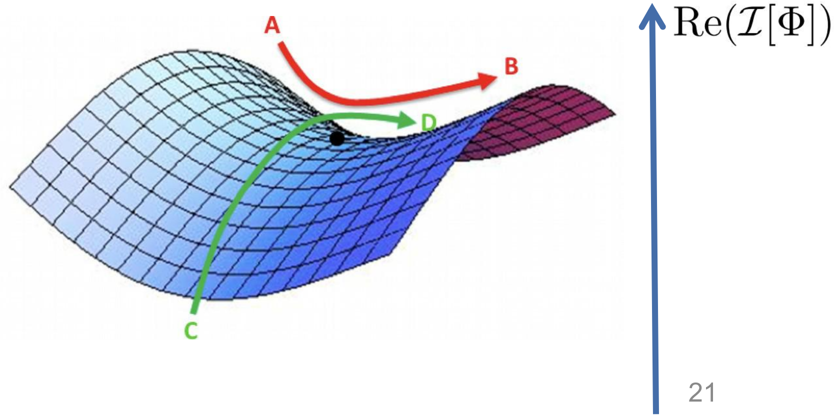
C_1, C_2 are called Lefschetz thimbles, giving steepest-descent paths from the stationary points

In our case, we have

$$\int \mathcal{D}\Phi \; e^{\mathcal{I}[\Phi]}, \quad \text{where } \mathcal{I}[\Phi] = iS_M[\Phi]$$

One can define the "height" function

$$h[\Phi] \equiv \mathrm{Re}(\mathcal{I}[\Phi])$$



The flow equation

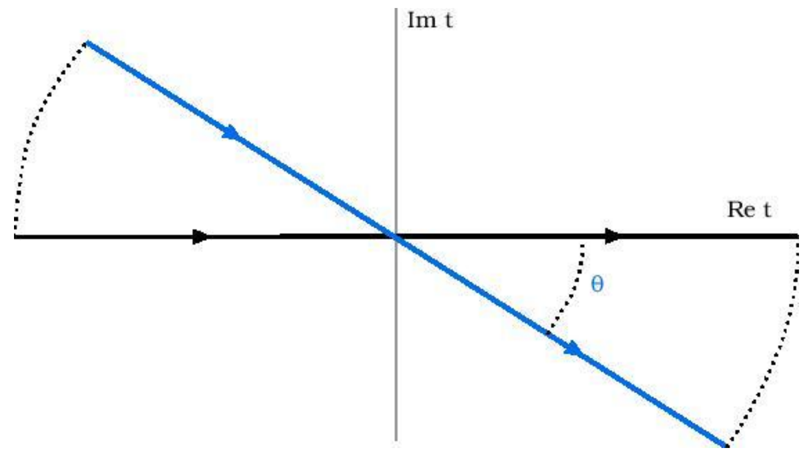
The Lefschetz thimble is given by the flow equations

$$\frac{\partial \Phi(x;u)}{\partial u} = -\overline{\left(\frac{\delta \mathcal{I}[\Phi(x;u)]}{\delta \Phi(x;u)}\right)}; \quad \frac{\partial \overline{\Phi}(x;u)}{\partial u} = -\frac{\delta \mathcal{I}[\Phi(x;u)]}{\delta \Phi(x;u)}$$

with boundary condition $\Phi(x; u = -\infty) = \phi_a$. It is easy to check that

$$\frac{\partial h}{\partial u} = \frac{1}{2}\left(\frac{\delta \mathcal{I}}{\delta \Phi} \cdot \frac{\partial \Phi}{\partial u} + \frac{\delta \overline{\mathcal{I}}}{\delta \overline{\Phi}} \cdot \frac{\partial \overline{\Phi}}{\partial u}\right) = -\left|\frac{\partial \Phi(x;u)}{\partial u}\right|^2 \leq 0$$

To be more general, we consider arbitrarily rotated time



$\theta = \epsilon$: Minkowskian $\qquad$ $\theta = \pi/2$: Euclidean

The flow eigenequation

Solving the flow equations can be transferred to solving the flow eigenequations

Tanizaki & Koike, Annals Phys. 2014

$$(\mathcal{M}_a^\theta)^* \overline{\chi_n^a}(x) = \kappa_n^a \chi_n^a(x)$$

where the fluctuation operator is

$$\mathcal{M}_a^\theta = e^{2i\theta} \frac{\partial^2}{\partial t^2} - \nabla^2 + U''(\phi_a^\theta)$$

The path integral is given as

$$Z_a = \int \mathcal{D}\Delta\Phi_a \, e^{I[\Phi]} \approx J_a e^{I[\phi_a]} \prod_n \frac{1}{\sqrt{\kappa_n^a}}$$

The flow eigenequation

Solving the flow equations can be transferred to solving the flow eigenequations Tanizaki & Koike, Annals Phys. 2014

$$(\mathcal{M}_a^\theta)^* \overline{\chi_n^a}(x) = \kappa_n^a \chi_n^a(x)$$

where the fluctuation operator is

$$\mathcal{M}_a^\theta = e^{2i\theta} \frac{\partial^2}{\partial t^2} - \nabla^2 + U''(\phi_a^\theta)$$

The path integral is given as

$$Z_a = \int \mathcal{D}\Delta\Phi_a \, e^{I[\Phi]} \approx J_a e^{I[\phi_a]} \prod_n \frac{1}{\sqrt{\kappa_n^a}}$$

In our work WA, B.Garbrecht, C. Tamarit, 2019

- ❑ Transform the flow eigenequations to the proper eigenequations

$$\mathcal{M}_a^\theta f_n^a(x) = \lambda_n^a f_n^a(x)$$

- ❑ Prove that the above proper eigenequations can be solved by analytic continuation of the Euclidean eigenfunctions
- ❑ Carefully work out the Jacobian

New insights from the real-time picture?

Implications of the optical-theorem: a wave function after quantum tunneling/FVD?

Questions remain to be addressed:

- ❑ 1. What is the wave function?
- ❑ 2. Will the wave function collapse immediately? Any effects on the gravitational-wave signals?

Outline

- Introduction
- Optical theorem for FVD and the complex bounce
- The Picard-Lefschetz theory
- Summary

Summary

- A real-time picture based on the optical theorem has been built.

- Have confirmed the Callan-Coleman result in real-time calculations

- Theoretical techniques related to the Picard-Lefschetz theory are developed

Backup: Compute the partition function

Method of steepest descent:

1. Find stationary points: $\delta S_E|_{\varphi_a} = 0$
2. Expand about the stationary points: $\Phi = \varphi_a + \Delta\Phi_a$

$$\int \mathcal{D}\Phi \; e^{-S_E[\Phi]} = \sum_a e^{-S_E[\varphi_a]} \int \mathcal{D}\Delta\Phi_a \; e^{-\frac{1}{2}\Delta\Phi_a \left(\left.\frac{\delta^2 S_E[\Phi]}{\delta\Phi^2}\right|_{\varphi_a} \right) \Delta\Phi_a + \dots}$$

The fluctuation operators: $\left.\frac{\delta^2 S_E[\Phi]}{\delta\Phi^2}\right|_{\varphi_a} = -\partial^2 + U''(\varphi_a(x))$

The Gaussian integral can be calculated by studying the eigenequations for the fluctuation operators

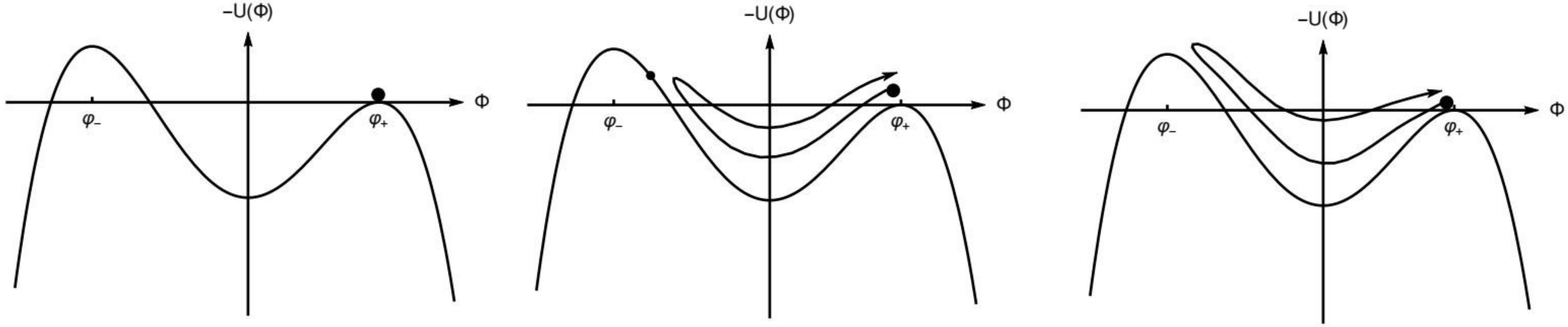
$$\left[-\partial^2 + U''(\varphi_a)\right] \phi_n^a = \lambda_n^a \phi_n^a$$

Decomposing the fluctuation fields $\Delta\Phi_a = \sum_n c_n^a \phi_n^a$. The path integral measure becomes $\mathcal{D}\Delta\Phi_a = \prod_n \frac{\mathrm{d}c_n^a}{\sqrt{2\pi}}$, giving

$$\int \mathcal{D}\Delta\Phi_a \, e^{-\frac{1}{2}\Delta\Phi_a \left(\left.\frac{\delta^2 S_E}{\delta\Phi^2}\right|_{\varphi_a} \right) \Delta\Phi_a} = \int \prod_n \frac{\mathrm{d}c_n^a}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda_n^a (c_n^a)^2} = \prod_n \sqrt{\frac{1}{\lambda_n^a}} = \det\left[-\partial^2 + U''(\varphi_a)\right]^{-1/2}$$

Stationary points

In Euclidean space, the potential is upside-down. There are three types of sationary points.



Trivial false vacuum φ_F Bounce φ_B Shot φ_S

irrelevant for tunneling Andreassen etl. 2017

Finally, the decay rate can be written as Callan & Coleman, 1977

$$\Gamma/V = e^{-S_E[\varphi_B]} \left(\frac{S_E[\varphi_B]}{2\pi} \right)^2 \left| \frac{\det'[-\partial^2 + U''(\varphi_B)]}{\det[-\partial^2 + U''(\varphi_F)]} \right|^{-1/2}$$

The flow eigenequation

Substituting the expansion $\Phi = \phi_a + \Delta\Phi_a$ into the flow equation, we obtain

$$\frac{\partial \Delta\Phi_a(x;u)}{\partial u} = -ie^{i\theta}(\mathcal{M}_a^\theta)^* \overline{\Delta\Phi_a}(x;u)$$

where

$$\mathcal{M}_a^\theta = e^{2i\theta}\frac{\partial^2}{\partial t^2} - \nabla^2 + U''(\phi_a^\theta)$$

Making the Ansatz

Tanizaki & Koike, Annals Phys. 2014

$$\Delta\Phi_a(x;u) = \sum_n \sqrt{-i}e^{i\theta/2} g_n^a(u)\chi_n^a(x)$$

where $g_n^a(u) = a_n^a e^{\kappa_n^a u}$, one obtains the flow eigenequation

$$(\mathcal{M}_a^\theta)^* \overline{\chi_n^a}(x) = \kappa_n^a \chi_n^a(x)$$

The path integral can be computed as

$$Z_a = \int \mathcal{D}\Delta\Phi_a\, e^{I[\Phi]} \approx J_a e^{I[\phi_a]} \int \prod_n \frac{\mathrm{d}g_n^a}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_n \kappa_n^a (g_n^a)^2} = J_a e^{I[\phi_a]} \prod_n \frac{1}{\sqrt{\kappa_n^a}}$$

The block form of the flow eigenequations

It is difficult to solve the flow eigenequations $(\mathcal{M}_a^\theta)^* \overline{\chi_n^a}(x) = \kappa_n^a \chi_n^a(x)$ directly! We write

$$\begin{pmatrix} \mathbf{0} & (\mathcal{M}_a^\theta)^* \\ \mathcal{M}_a^\theta & \mathbf{0} \end{pmatrix} \begin{pmatrix} \chi_n^a(x) \\ \overline{\chi_n^a}(x) \end{pmatrix} = \kappa_n^a \begin{pmatrix} \chi_n^a(x) \\ \overline{\chi_n^a}(x) \end{pmatrix}$$

One can check that there is an associated equation

$$\begin{pmatrix} \mathbf{0} & (\mathcal{M}_a^\theta)^* \\ \mathcal{M}_a^\theta & \mathbf{0} \end{pmatrix} \begin{pmatrix} i\chi_n^a(x) \\ -i\overline{\chi_n^a}(x) \end{pmatrix} = -\kappa_n^a \begin{pmatrix} i\chi_n^a(x) \\ -i\overline{\chi_n^a}(x) \end{pmatrix}$$

The above equations can be viewed as normal eigenequations! Then we have

$$\prod_n \left[-(\kappa_n^a)^2\right] = \det \begin{pmatrix} \mathbf{0} & (\mathcal{M}_a^\theta)^* \\ \mathcal{M}_a^\theta & \mathbf{0} \end{pmatrix} \Rightarrow \prod_n \frac{1}{\sqrt{\kappa_n^a}} = \frac{1}{\sqrt{|\det \mathcal{M}_a^\theta|}}$$

Further, we also carefully calculated the Jacobian $J_a = e^{-\frac{1}{2}\mathrm{Arg}\det \mathcal{M}_a^\theta}$. We finally obtain

WA, B.Garbrecht, C. Tamarit, 2019

$$Z_a \approx e^{I[\phi_a]} \frac{1}{\sqrt{\det \mathcal{M}_a^\theta}}$$

Analytic continuaion

Equivalently, we need to solve the normal eigenequation

$$\mathcal{M}_a^\theta f_n^a(x) = \lambda_n^a f_n^a(x)$$

This can be solved by analytic continuation from the Euclidean eigenequations and we prove that WA, B.Garbrecht, C. Tamarit, 2019

$$\det \mathcal{M}_a^\theta = \det \left(-\partial^2 + U''(\varphi_a)\right)\Big|_{\mathcal{T}\to ie^{-i\theta}T}$$

Nontrivial! Need to examine orthonormality and completeness of the analytically continued eigenfunctions.

Substituting the above equation into

$$Z_a \approx e^{I[\phi_a]} \frac{1}{\sqrt{\det \mathcal{M}_a^\theta}}$$

and

$$\varGamma = -\frac{2}{T}\,\mathrm{Re}\left(\frac{Z_B^M}{Z_F^M}\right)$$

we can finally recover the Callan-Coleman result!