Lattice QCD at finite density with the complex Langevin method

A status report

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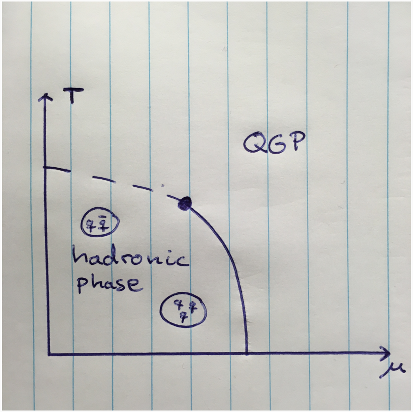
Table of contents

QCD at finite density

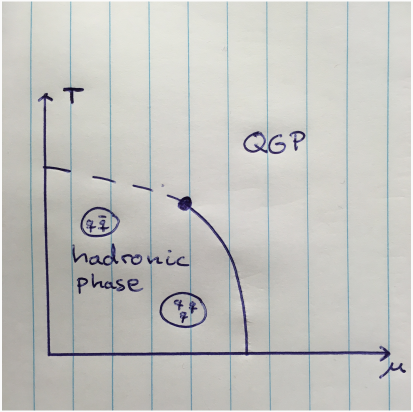
Phases of Quantum Chromodynamics (QCD)

- largely unknown, in particular phase transitions
- heavy-ion collision experiments (LHC, RHIC, FAIR)
- insight of large interest also beyond particle physics



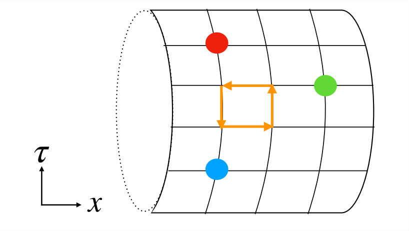
Lattice QCD

- strong force non-perturbative
- compute quantities from first principles ⇒ lattice QCD
- Challenge: sign problem



Formulating QCD on the lattice

- IR cutoff: box of volume V
- UV cutoff: lattice spacing a
- quark fields $\psi(x), \bar{\psi}(x)$
- gluon fields (links) $U_\nu(x) \in \mathrm{SU}(3)$
- time extent $\leftrightarrow$ temperature T



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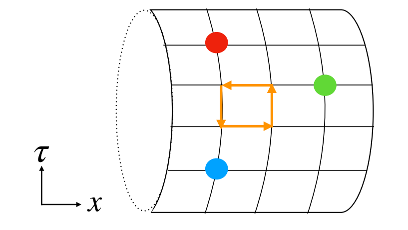


$$D_{x,y} = m\mathbb{1}\delta_{x,y} + \frac{1}{2}\sum_\nu (\mathbb{1} - \gamma_\nu)\ e^{\mu\delta_{0,\nu}}\ U_{x,\nu}\delta_{x+\hat{\nu},y} + (\mathbb{1} + \gamma_\nu)\ e^{-\mu\delta_{0,\nu}}\ U^\dagger_{x-\hat{\nu},\nu}\delta_{x-\hat{\nu},y}$$

Simulating QCD on the lattice

- path-integral quantization
- compute observables using sampling methods
- large scale numerical simulations

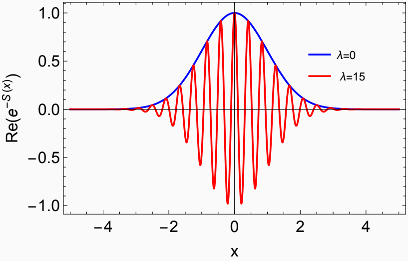
$$Z = \int_{\mathrm{SU}(3)^{4\Omega}} \mathrm{d}U \exp(-S_G[U]) \det D(U)\,,$$

$$\Omega = N_\tau \times N_s^3$$

Challenges

Challenge 1 - The Sign Problem

- $\mu > 0$ makes the action complex
- simulation costs grow **exponentially** with $\Omega = N_\tau \times N_s^3$
- this get's worse with lowering T and increasing μ



Toy model $S(x) = \frac{x^2}{2} - i\lambda x$

Ways around the sign problem

- Reweighting, Taylor expansions, imaginary mu
- dual methods, strong coupling expansions
- **complexification** methods (complex Langevin, Lefschetz thimbles, flowed manifolds, ...)
- Quantum Computing

For QCD at $\mu \neq 0$ we work with the **complex Langevin method** $\rightarrow$ stochastic quantization

Stochastic quantization in a nutshell

- evolve fields def. on Ω in fictitious fifth time called θ using the **Langevin equation**
- stationary distribution of the stochastic process is Boltzmann factor, i.e. the path-integral weight
- requirement: $S \in \mathbb{R}, S \geq c \in \mathbb{R}$

$$\frac{\partial A_i}{\partial \theta} = -\frac{\delta S}{\delta A_i} + \eta_i\,, \quad i = (x, \nu, a) \tag{1}$$

- equivalent to path-integral quantization

From Stochastic quantization to complex Langevin

- extension to complex actions
- requirement: meromorphic drift ✓
- complexification $\mathrm{SU}(3) \to \mathrm{SL}(3, \mathbb{C})$
- replace $U^{\dagger}_{x,\nu} \to U^{-1}_{x,\nu}$

Complex Langevin simulations

Euler-Maruyama update scheme

$$U_{x,\nu}^{n+1} = \exp[-it^a(-D_{x,\nu,a}S_G[U] + \eta_{x,\nu,a})]U_{x,\nu}^n$$

$\langle \eta_{x,\nu,a} \rangle = 0\,, \quad \langle \eta_{x,\nu,a}\eta_{y,\rho,b} \rangle = 2\delta_{x,y}\delta_{\nu,\rho}\delta_{a,b}$ where $a = 0, \ldots, N_c^2 - 1$

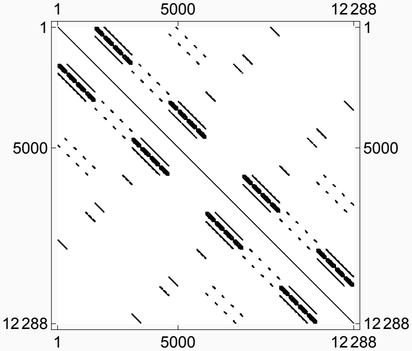
A technical note for lattice practitioners

- $SL(3, \mathbb{C})$ is non-compact
- need for stabilizing the complex Langevin process to control run away trajectories and to guarantee for correct results
- How?
    - Adaptive step size
    - Gauge Cooling
    - Dynamic Stabilization

A closer look at the drift force

Fermionic drift force

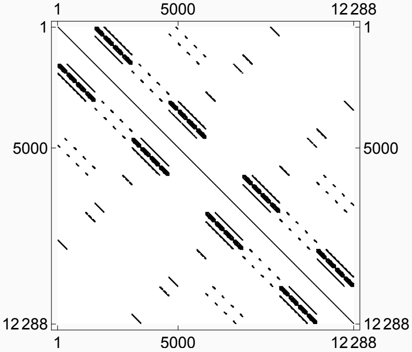
$$-D_i S_F[U] \propto \mathrm{tr}\left(D^{-1}\frac{\partial D}{\partial U_i}\right)$$



$$D_{x,y} = m\mathbb{1}\delta_{x,y} + \frac{1}{2}\sum_{\nu}(\mathbb{1}-\gamma_\nu)\, e^{\mu\delta_{0,\nu}}\; U_{x,\nu}\delta_{x+\hat{\nu},y} + (\mathbb{1}+\gamma_\nu)\, e^{-\mu\delta_{0,\nu}}\; U^{-1}_{x-\hat{\nu},\nu}\delta_{x-\hat{\nu},y}$$

Challenge 2 - Invert the Wilson-Dirac matrix

- size $= (12\Omega)^2 = 2473901162496$ for $N_\tau = 32$ and $N_s = 16$
- condition number worsens with increasing μ
- **There is hope**
  - sparse matrix
  - inversion is massively parallelizable
  - work with preconditioners



Our plan

Our plan

(1) handle the fermionic force (99 % of the simulation time)

(2) map the phase structure (confinement and chiral transition) at high and intermediate T

Results

Setup

- $N_f = 2$ mass-degenerate quarks
- $N_s = 16$, $a = 0.08$fm
- Wilson plaquette action, Wilson-Dirac fermions (tree-level)
- $\beta = 5.6$, $\kappa = 0.1580$, see JHEP 02 (2006) 011 by Lüscher et. al.
- $m_\pi = 550$ MeV, $m_N = 1.5$ GeV

Setup

- $T \in \{20, ..., 640\}$MeV
- $\mu \in \{0, ..., 5\}$GeV
- fermionic force: even-odd preconditioned conjugate gradient algorithm

Stable Langevin trajectories require Dynamic Stabilization.

Observables

Physics

- Polyakov loop

$$P(\vec{x}) = \mathrm{tr}\left(\prod_{x_0=0}^{N_\tau-1} U_0(\vec{x}, x_0)\right)$$

- $\langle P \rangle = 0 \leftrightarrow$ confinement
- $\langle P \rangle \neq 0 \leftrightarrow$ deconfinement

- chiral condensate

$$\langle \bar{\psi}\psi \rangle = \frac{\partial \log(Z)}{\partial m}$$

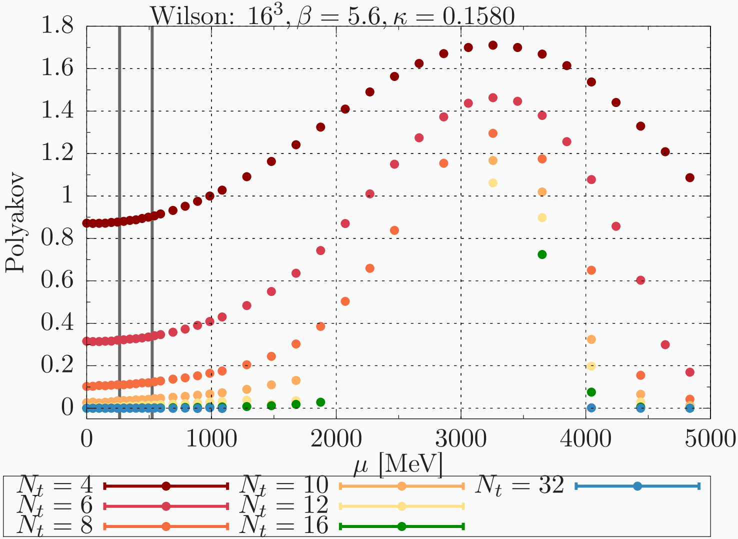
- density

$$\langle n \rangle = \frac{1}{\Omega}\frac{\partial \log(Z)}{\partial \mu}$$

Numerics

- unitarity norm
- iteration number

PRELIMINARY - deconfinement transition



grey solid lines: $m_\pi/2$ (left) and $m_N/3$ (right)

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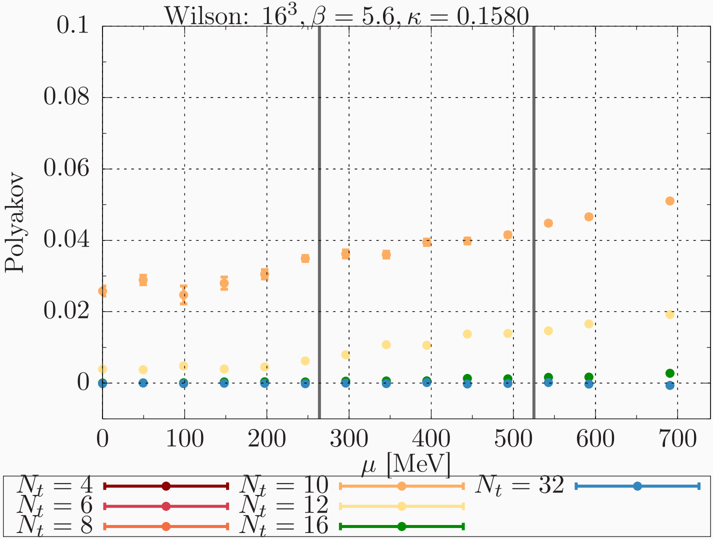

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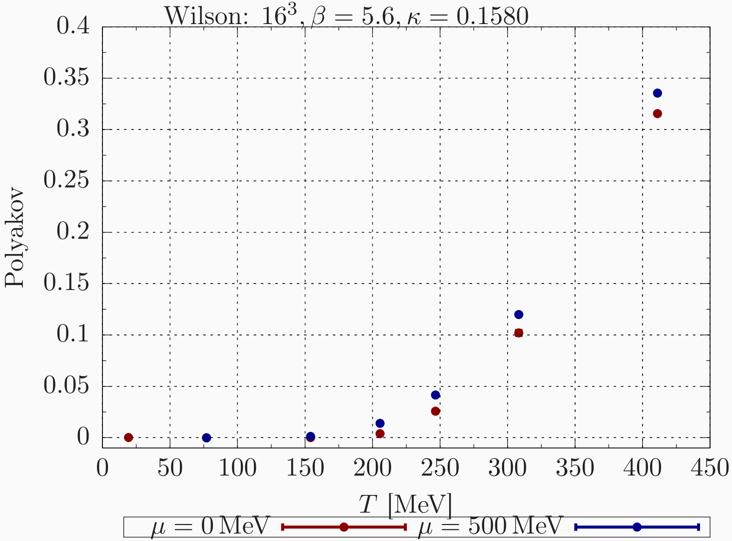
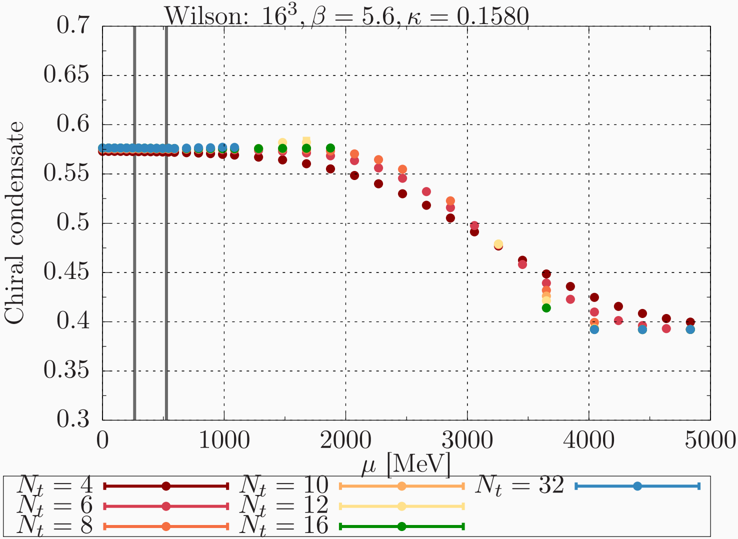


PRELIMINARY - chiral transition



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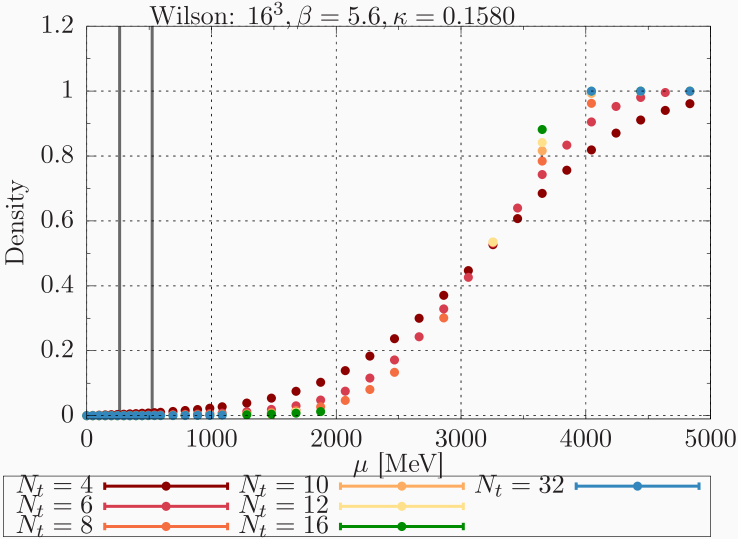

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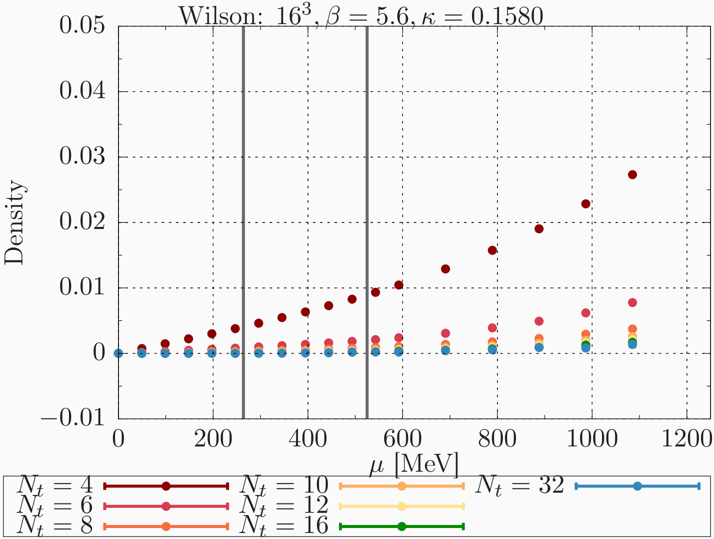

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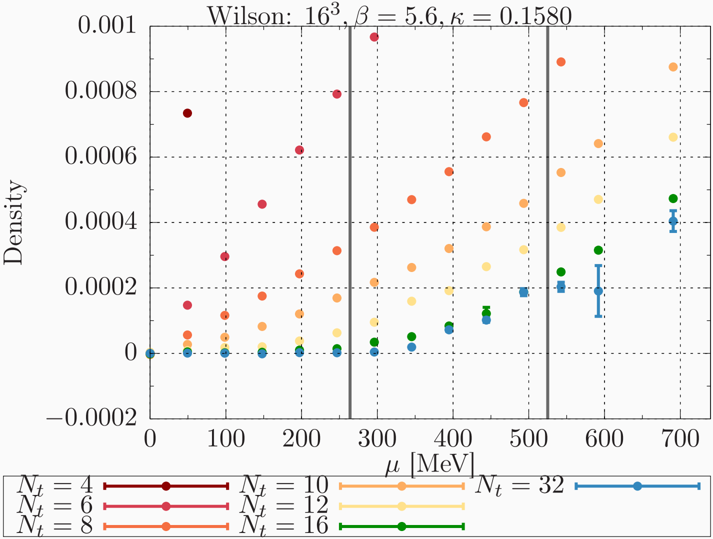

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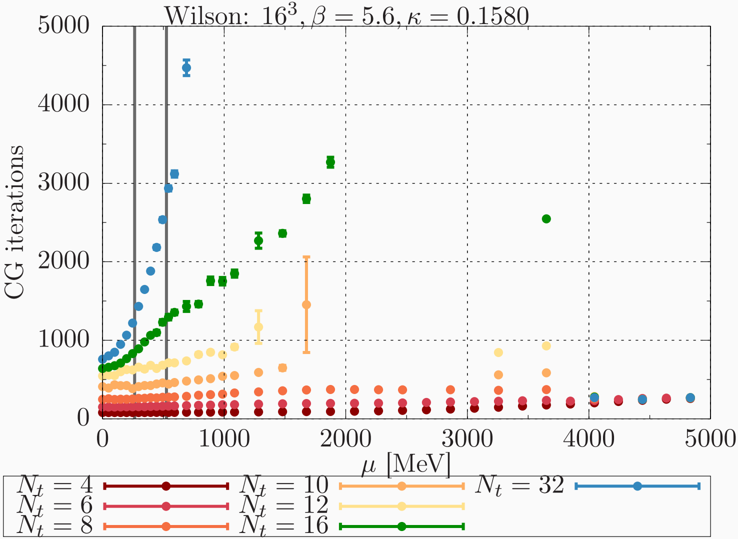
Matrix inversion - conjugate gradient iterations



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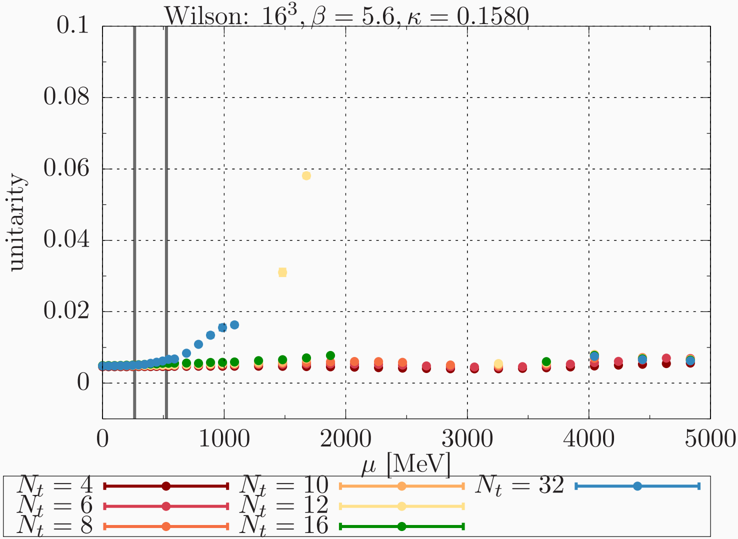
Unitarity norm - "distance" to SU(3)



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Outlook and perspectives

Conclusions and outlook

- simulation software for $N_f = 2$ QCD at finite μ ready, approaching physical pion masses ✓
- ToDo list
  - CL extrapolations
  - improve inversion of the Dirac matrix $\rightarrow$ lower T
  - phase transitions and volume scaling
  - improved actions
  - anisotropy
- complexification as a blessing and curse

Thanks a lot!