Implementing gauge symmetry in machine learning models

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Strong and Electro-Weak Matter 2021

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Based on

M. Favoni, A. Ipp, D. I. Müller, D. Schuh,
"Lattice gauge equivariant convolutional neural networks"
Preprint (2020) [arXiv:2012.12901]

Code: gitlab.com/openpixi/lge-cnn Group: openpixi.org

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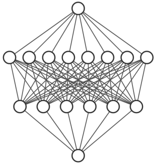
Der Wissenschaftsfonds.

Introduction

Applications of machine learning (ML):
Computer vision, natural language processing, medicine and (high energy) physics

Artificial neural networks (ANNs or NNs)

- Highly expressive basis for function approximation
- Universal approximators for non-linear functions
- Typically high number of free parameters, "black boxes"



Neural networks applied to physical data (e.g. field theory)

- High expressivity: NNs a priori do not know about symmetry
- Symmetries in data have to be learned (approximated)
- This work: NNs which respect (non-Abelian) gauge symmetry

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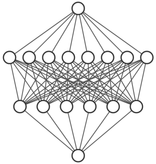

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- Symmetries in data have to be learned (approximated)
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Related works

Incomplete list of recent related works (from other authors, chronologically)

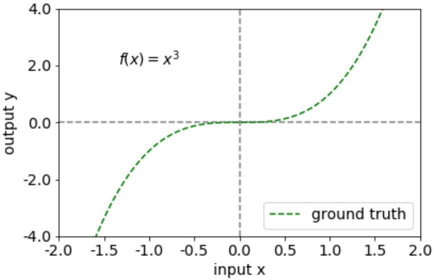
- P. E. Shanahan, A. Trewartha, W. Detmold
  "Machine learning action parameters in lattice quantum chromodynamics"
  PRD 97 (2018) [arXiv:1801.05784]
- T. S. Cohen, M. Weiler, B. Kicanaoglu, M. Welling
  "Gauge Equivariant Convolutional Networks and the Icosahedral CNN"
  ICML 2019 (2019) [arXiv:1902.04615]
- G. Kanwar, M. S. Albergo, D. Boyda, K. Cranmer, D. C. Hackett, S. Racanière, D. J. Rezende, P. E. Shanahan
  "Equivariant flow-based sampling for lattice gauge theory"
  PRL 125 (2020) [arXiv:2003.06413]
- D. Boyda, G. Kanwar, M. S. Albergo, S. Racanière, D. J. Rezende, M. S. Albergo, K. Cranmer, D. C. Hackett, P. E. Shanahan
  "Sampling using SU(N) gauge equivariant flows"
  PRD 103 (2021) [arXiv:2008.05456]
- D. L. Boyda, M. N. Chernodub, N. V. Gerasimeniuk, V. A. Goy, S. D. Liubimov, A. V. Molochkov
  "Machine-learning physics from unphysics: Finding deconfinement temperature in lattice Yang-Mills theories from outside the scaling window"
  PRD 103 (2021) [arXiv:2009.10971]
- A. Tomiya, Y. Nagai
  "Gauge covariant neural network for 4 dimensional non-abelian gauge theory"
  Preprint (2021) [arXiv:2103.11965]

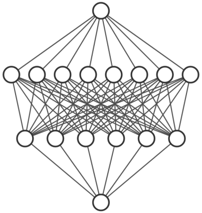
Symmetries and neural networks

Toy model: non-linear regression for $f(x) = x^3$

- Training data: input data $\mathbf{x} \in \mathbb{R}^N$ and output data (labels) $\mathbf{y} \in \mathbb{R}^N$
- Neural network $h_\theta : \mathbb{R} \to \mathbb{R}$ with unknown parameters $\theta \in \mathbb{R}^M$

  $\Rightarrow$ Composition of affine transformations and non-linear activation functions
- Minimize loss function: $L(\theta) = \frac{1}{N} \sum_{i=1}^{N} (h_\theta(x_i) - y_i)^2$ (training)





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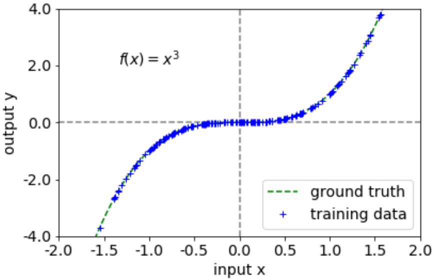

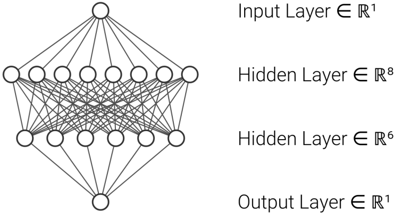

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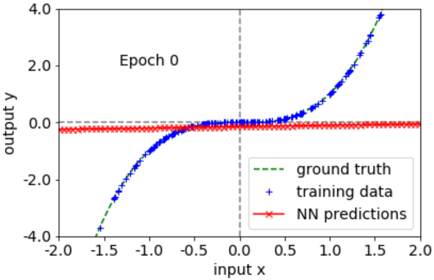

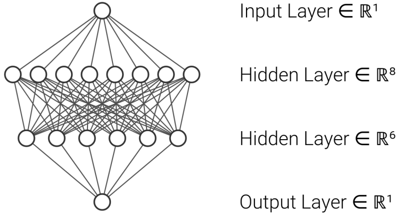

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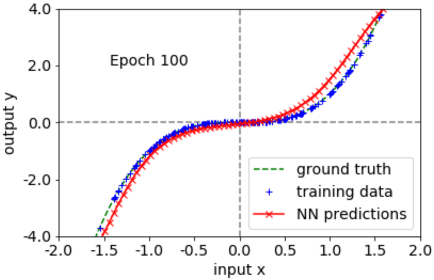

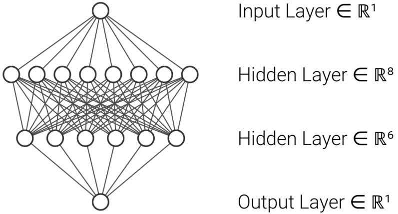

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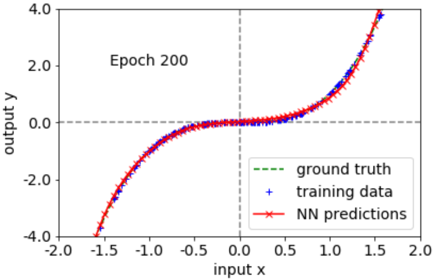

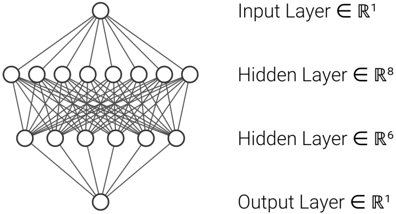

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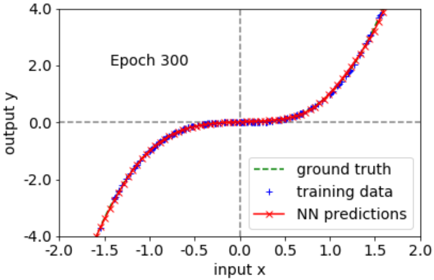

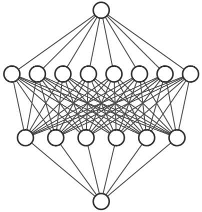

Input Layer ∈ $\mathbb{R}^1$

Hidden Layer ∈ $\mathbb{R}^8$

Hidden Layer ∈ $\mathbb{R}^6$

Output Layer ∈ $\mathbb{R}^1$

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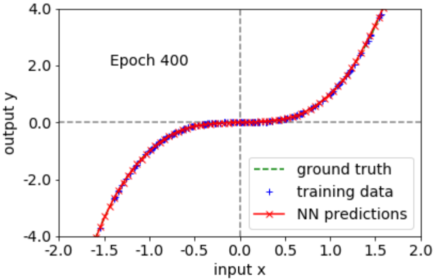

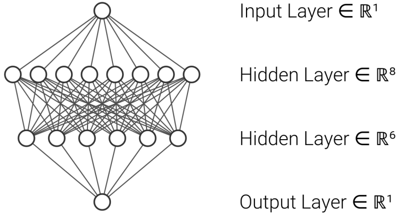

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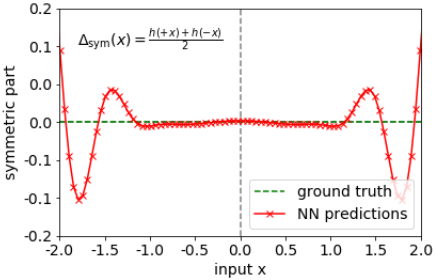

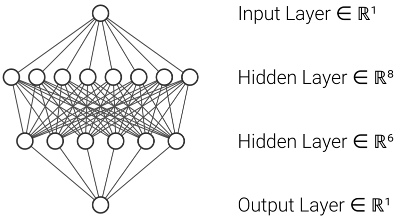

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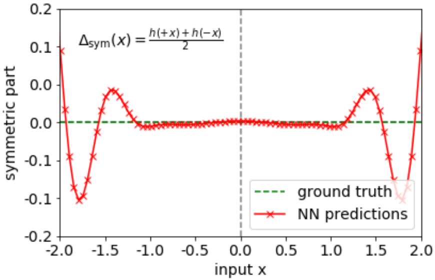

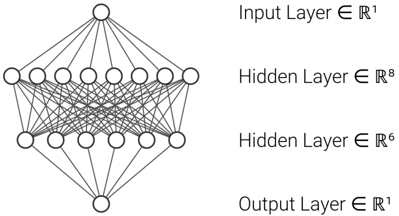

Symmetries in NNs are generally only approximate!

$$f(-x) = -f(x) \qquad h_\theta(-x) \neq -h_\theta(x)$$

Convolutional neural networks

Convolutional NNs (CNNs) use translational equivariance for data on lattices (e.g. images)

- Restrict parameters θ to enforce translational equivariance
- Compact kernels (locality)
  $\rightarrow$ only consider compact neighborhoods
- Weight sharing (homogeneity)
  $\rightarrow$ same operation at every point
- Translational equivariance
  "Translations on input induce translations on output"
- More general: G-CNNs (rotations, reflections, ...)
- Symmetry is not learned, but implemented
- Applications in lattice field theories e.g. ϕ^4

S. Bulusu, M. Favoni, A. Ipp, D. I. Müller, D. Schuh,
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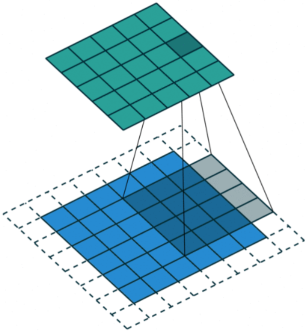

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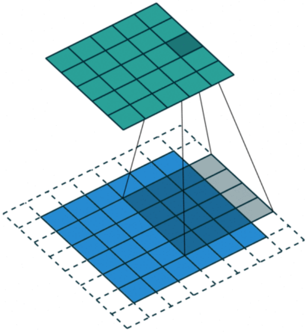

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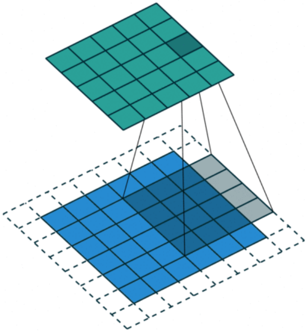

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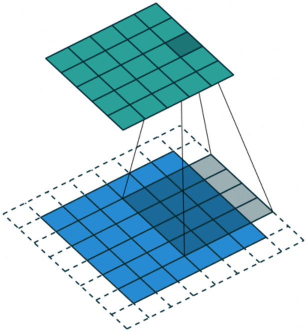

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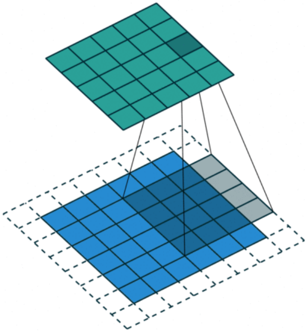

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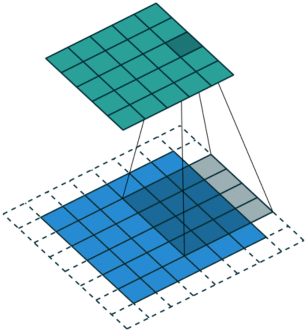

Lattice gauge theory

Lattice gauge theory: gauge theories on lattices with exact gauge invariance

- $\mathcal{U}$: Gauge links (parallel transport)

$$U_{\mathbf{x},\mu} \simeq \exp\left(iga^\mu A_\mu(\mathbf{x}+\mathbf{a}^\mu/2)\right) \in \mathrm{SU}(N_c)$$

- $\mathcal{W}$: 1×1 Wilson loops (plaquettes)

$$W_{\mathbf{x},\mu\nu} = U_{\mathbf{x},\mu}U_{\mathbf{x}+\mu,\nu}U_{\mathbf{x}+\mu+\nu,-\mu}U_{\mathbf{x}+\nu,-\nu}$$

- Lattice gauge transformations for $\mathcal{U}$ and $\mathcal{W}$

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- Wilson action

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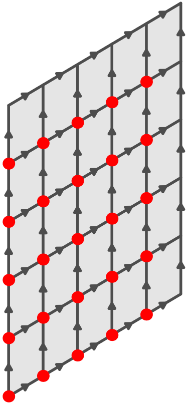

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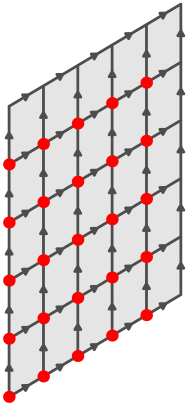

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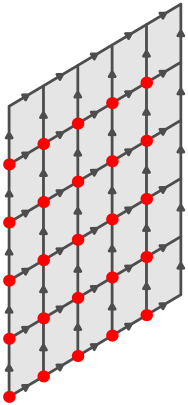

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$$W_{\mathbf{x},\mu\nu} = U_{\mathbf{x},\mu} U_{\mathbf{x}+\mu,\nu} U_{\mathbf{x}+\mu+\nu,-\mu} U_{\mathbf{x}+\nu,-\nu}$$

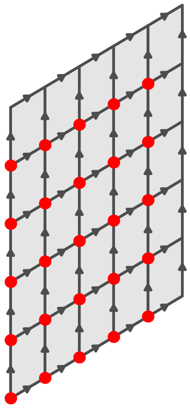
- Lattice gauge transformations for $\mathcal{U}$ and $\mathcal{W}$

$$T_\Omega U_{\mathbf{x},\mu} = \Omega_{\mathbf{x}} U_{\mathbf{x},\mu} \Omega^\dagger_{\mathbf{x}+\mu} \qquad \Omega_{\mathbf{x}} \in \mathrm{SU}(N_c)$$

$$T_\Omega W_{\mathbf{x},\mu\nu} = \Omega_{\mathbf{x}} W_{\mathbf{x},\mu\nu} \Omega^\dagger_{\mathbf{x}}$$

- Wilson action

$$S_W[T_\Omega \mathcal{U}] = S_W[\mathcal{U}] = \frac{\beta}{N_c} \sum_{\mathbf{x},\mu,\nu} \mathrm{Re}\,\mathrm{Tr}\left(\mathbf{1} - W_{\mathbf{x},\mu\nu}\right)$$



$\mathcal{U} = \{U_{\mathbf{x},\mu}\}$

$\mathcal{W} = \{W_{\mathbf{x},\mu\nu}\}$

Gauge equivariance and invariance

- Lattice gauge transformations for $\mathcal{U}$ and $\mathcal{W}$

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$$T_\Omega W_{\mathbf{x},\mu\nu} = \Omega_{\mathbf{x}} W_{\mathbf{x},\mu\nu} \Omega^\dagger_{\mathbf{x}}$$

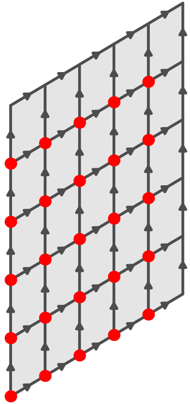
- Gauge equivariant function

$$g(T_\Omega \mathcal{U}, T_\Omega \mathcal{W}) = T'_\Omega g(\mathcal{U}, \mathcal{W})$$

- Gauge invariant function

$$g(T_\Omega \mathcal{U}, T_\Omega \mathcal{W}) = g(\mathcal{U}, \mathcal{W})$$

- Physical observables are gauge invariant functions
- Neural network layers should be gauge equivariant!



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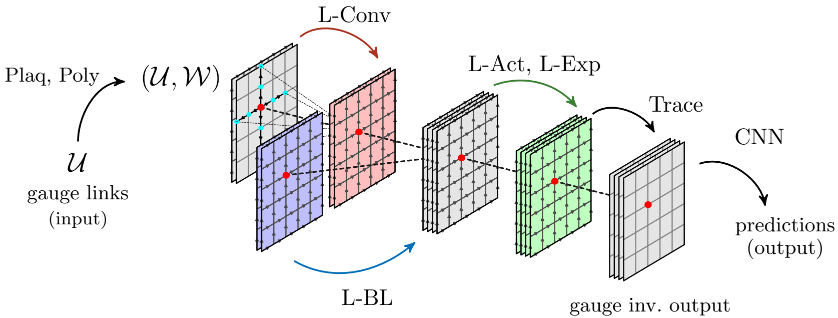
Lattice gauge equivariant convolutional neural networks

L-CNNs: A collection of gauge equivariant layers for lattice gauge configurations



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  [no trainable parameters]
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  $(\mathcal{U}, \mathcal{W}) \to (\mathcal{U}, \mathcal{W}')$
  [Conv + parallel transport]
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- Postprocessing: **Trace**
  Invariant output
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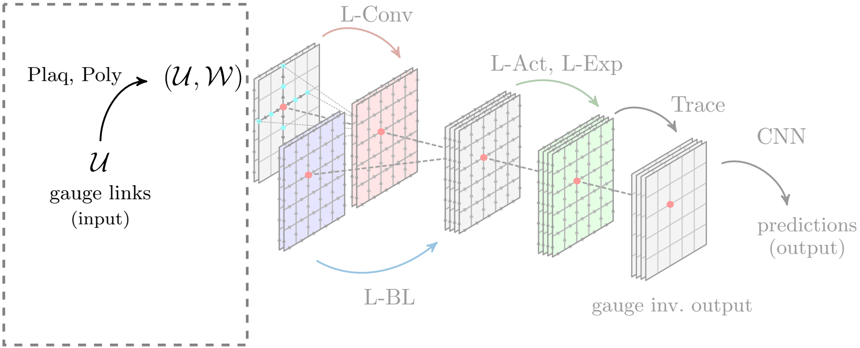
Lattice gauge equivariant convolutional neural networks

L-CNNs: A collection of gauge equivariant layers for lattice gauge configurations



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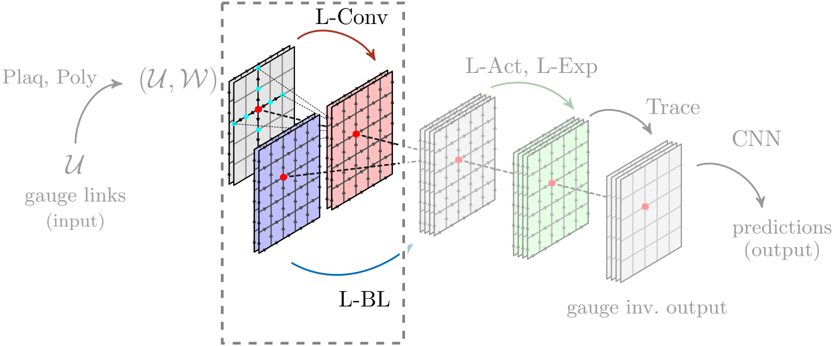

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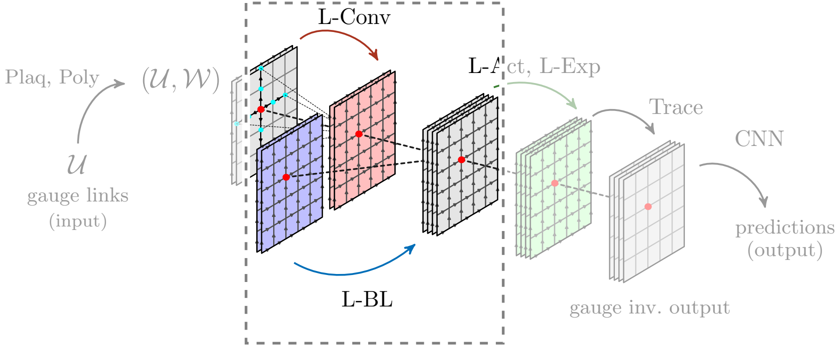

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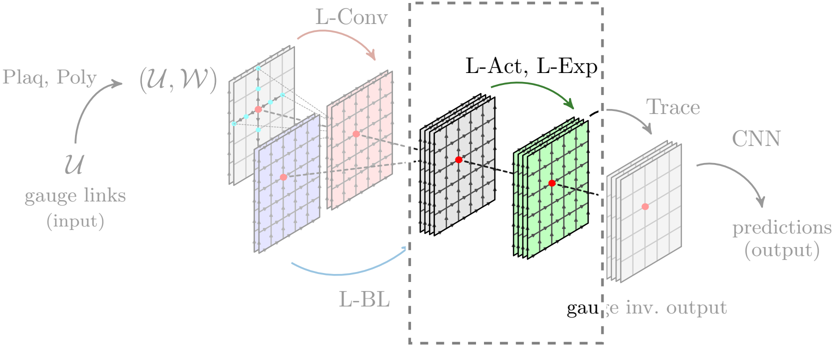

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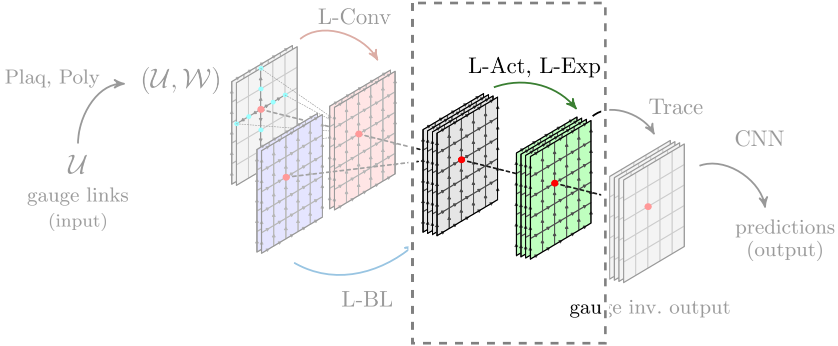

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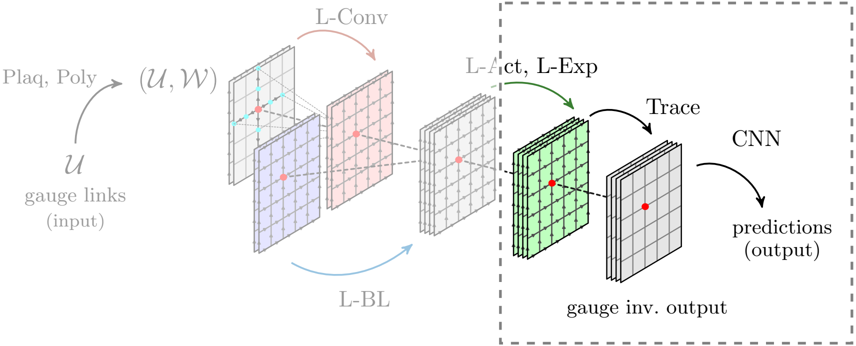

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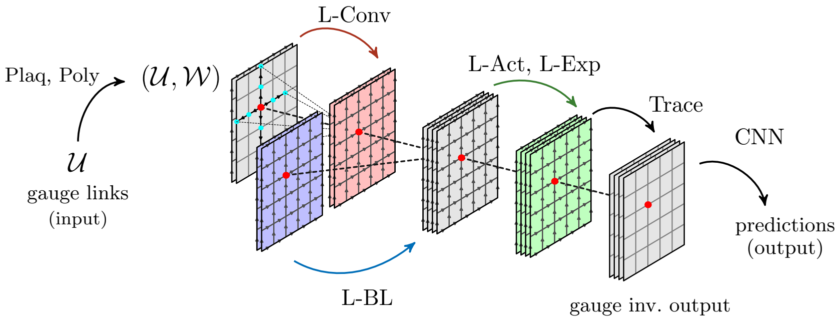

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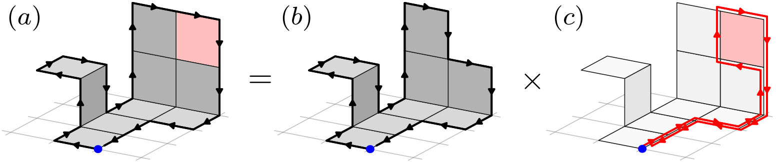
Arbitrary Wilson loops using L-CNNs

- Repeated applications of **L-Conv** and **L-BL** operations can be used to generate arbitrarily sized Wilson loops if input $\mathcal{W}$ consists of plaquettes (preprocessing layer **Plaq**)



- Non-contractible loops can also be generated by including Polyakov loops in the input $\mathcal{W}$ (preprocessing layer **Poly**)
- Non-linear functions of Wilson loops are possible through **L-Act**, **Trace** and passing gauge invariant output to traditional CNNs
- L-CNNs are universal approximators for gauge invariant functions on the lattice

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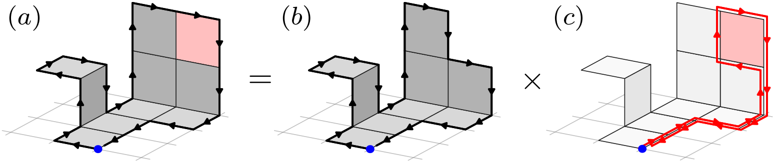

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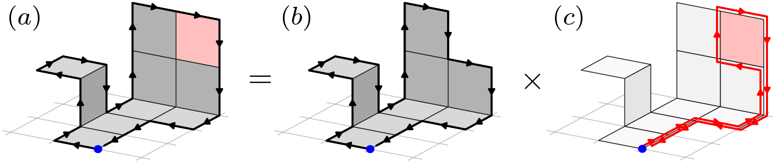

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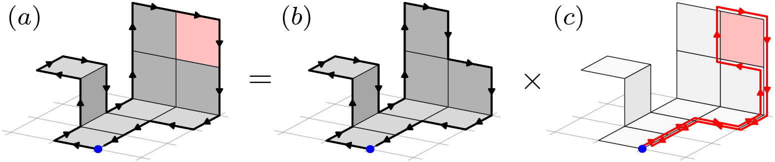

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Benchmarks and testing I

Benchmark problem: regression of Wilson loops from 1×1 to 4×4 on 2D lattice

- Data set: SU(2) gauge field configurations $\mathcal{U}$ from MC simulation, $\beta \in \{0.1, \dots, 6.0\}$
- Input ("x"): gauge field configuration $\mathcal{U} \in \mathbb{C}^{N_x \times N_y \times 2 \times N_c \times N_c}$
- Output ("y"): $\mathrm{Re}\,\mathrm{Tr}\left[W^{(n\times m)}\right]/N_c \in \mathbb{R}$
- Metric: mean squared error (MSE)
- Training on small lattice: 8×8 (10^4 samples)
- Testing on larger lattices: $8 \times 8, 16 \times 16$, 32×32, 64×64 (10^3 samples)

Comparison study

- L-CNN models: 1 – 4 **L-Conv** + **L-BL** layers

  $\mathcal{O}(10)$ – $\mathcal{O}(10^4)$ trainable parameters, **90** individual models
- Baseline models: traditional CNNs, 1 – 6 layers, 1 – 512 channels, activation functions

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- Both architectures get same information (links $\mathcal{U}$, plaquettes $\mathcal{W}$)

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Benchmarks and testing II

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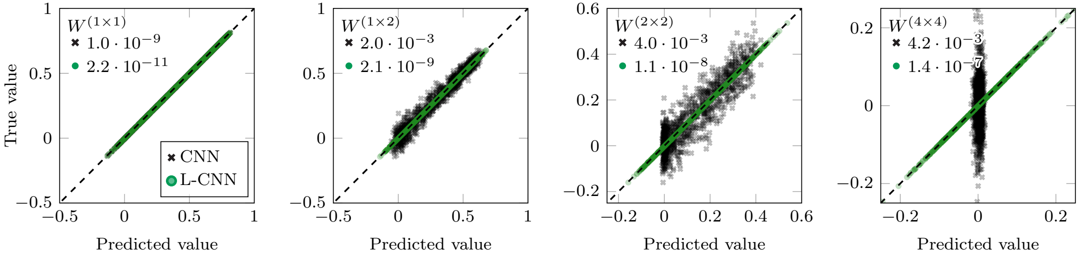
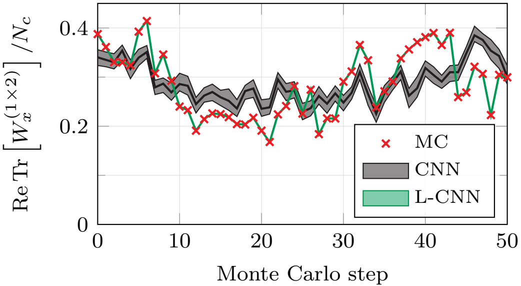


Plot: True vs. predicted values for CNNs and L-CNNs for $n \times m$ Wilson loops (best models)

- From left to right: increase in loop size $\rightarrow$ more difficult task
- Deteriorating performance of baseline CNNs with increased loop size
- Best L-CNN **always** beats best baseline CNN
- Consistent performance of L-CNNs across all loop and lattice sizes

Benchmarks and testing III

Benchmark problem: regression of Wilson loops from 1×1 to 4×4 on 2D lattice



Plot: Sensitivity to gauge transformations for 1×2 Wilson loop

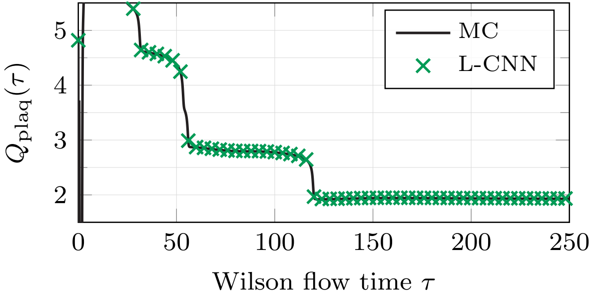
- Test for sensitivity to random gauge transformations

$$T_\Omega U_{\mathbf{x},\mu} = \Omega_{\mathbf{x}} U_{\mathbf{x},\mu} \Omega^\dagger_{\mathbf{x}+\mu}, \qquad T_\Omega W_{\mathbf{x},\mu\nu} = \Omega_{\mathbf{x}} W_{\mathbf{x},\mu\nu} \Omega^\dagger_{\mathbf{x}}$$

- Baseline CNNs are sensitive to gauge transformations
- L-CNNs are invariant by construction (and more accurate)

Benchmarks and testing IV

L-CNNs also work in higher dimensions!



Plot: L-CNN predictions vs. true values (MC) for Q_{plaq} on a 8×24^3 configuration

- L-CNN model for topological charge prediction
- Trained on MC configurations
- Tested on "cooled" configurations (Wilson flow)

Summary and outlook

L-CNNs: framework for lattice gauge equivariant convolutional neural networks

- Fully respects SU(N_c) gauge symmetry
- Universal approximators for gauge invariant functions
- Automated extraction of physical information from lattice configurations
- Better performance than traditional CNNs in presented regression tasks
- Open source (based on *PyTorch*)

  Repository: gitlab.com/openpixi/lge-cnn
  Our group: openpixi.org

What's next?

- Improvements to code (more modules, performance, memory consumption, . . .)
- More complicated observables
- Application to normalizing flows?

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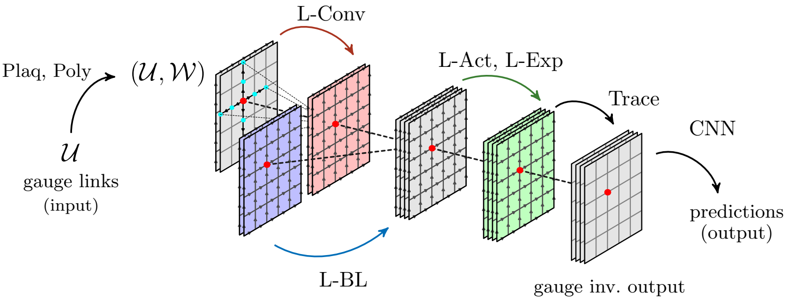
Thank you for your attention!



Code: gitlab.com/openpixi/lge-cnn Group: openpixi.org

E-Mail: dmueller@hep.itp.tuwien.ac.at

Backup

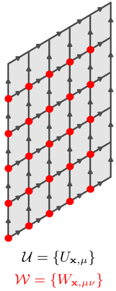
Individual layers

Preprocessing and equivariant convolutions

Preprocessing layers (**Plaq, Poly**)

- Operations defined for tuples $(\mathcal{U}, \mathcal{W})$ with gauge links $\mathcal{U}$ and locally transforming matrices $\mathcal{W}$
- Preprocess input $\mathcal{U}$ to generate $\mathcal{W}$

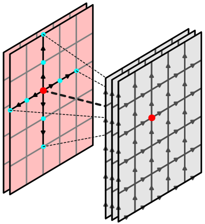
$$\textbf{Plaq, Poly} : \mathcal{U} \to (\mathcal{U}, \mathcal{W})$$



Lattice gauge equivariant convolutions (**L-Conv**)

- Similar to CNN layers: compact kernels, weight sharing
- Parallel transport of data $\mathcal{W}$ to common point using $\mathcal{U}$
- Path (in)dependence, implementation for D dimensions

$$\textbf{L-Conv} : W'_{\mathbf{x},i} = \sum_{j,\mu,k} \omega_{i,j,\mu,k} U_{\mathbf{x},k\cdot\mu} W_{\mathbf{x}+k\cdot\mu,j} U^\dagger_{\mathbf{x},k\cdot\mu}$$



- Equivariant convolutions: $(\mathcal{U}, \mathcal{W}) \to (\mathcal{U}, \mathcal{W}')$

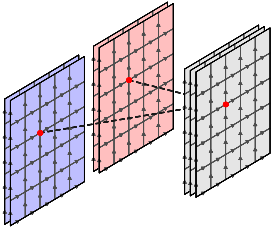
Bilinear layers and activation functions

Equivariant bilinear layers (**L-BL**):

- Multiplication of $\mathcal{W}$'s at same lattice point is equivariant

$$\textbf{L-BL}: W''_{\mathbf{x},i} = \sum_{j,k} \alpha_{ijk} W_{\mathbf{x},j} W'_{\mathbf{x},k}$$

- Bilinear layers: $(\mathcal{U}, \mathcal{W}) \times (\mathcal{U}, \mathcal{W}') \rightarrow (\mathcal{U}, \mathcal{W}'')$

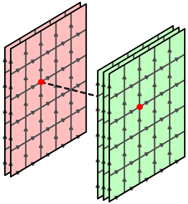


Gauge equivariant activation functions (**L-Act**):

- Multiplication of $\mathcal{W}$ with gauge invariant scalar functions a

$$\textbf{L-Act}: W'_{\mathbf{x}} = a(\mathrm{Tr} W_{\mathbf{x}})\, W_{\mathbf{x}}$$

- Activation functions: $(\mathcal{U}, \mathcal{W}) \rightarrow (\mathcal{U}, \mathcal{W}')$



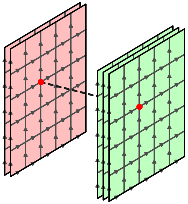
Exponential layers and the trace operation

Equivariant exponential layers (**L-Exp**):

- Equivariant method to modify links $\mathcal{U} \to \mathcal{U}'$
- Multiplication of $\mathcal{U}$ with locally transforming SU(N_c)

$$U'_{\mathbf{x},\mu} = \exp\left(i \sum_i \beta_{\mu,i} \left[W_{\mathbf{x},i}\right]_{\mathrm{ah}}\right) U_{\mathbf{x},\mu}$$

- Equivariant exponential layer: $(\mathcal{U}, \mathcal{W}) \to (\mathcal{U}', \mathcal{W})$

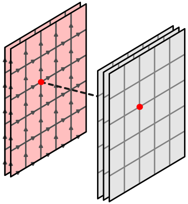


Generate gauge invariant output (**Trace**)

- Compute traces of $\mathcal{W}$'s: gauge invariant complex numbers

$$\textbf{Trace} : w_{\mathbf{x},i} = \mathrm{Tr}\, W_{\mathbf{x},i} \in \mathbb{C}$$

- No trainable parameters ("postprocessing")
- Gauge invariant output can be passed to traditional CNN



L-CNN and CNN architectures

Baseline CNNs for 1×1 and 1×2 loops

- Various input data: $\mathcal{U}$, $\mathcal{U}+\mathcal{W}$, $\mathcal{U}+\mathcal{W}+\mathcal{W}^\dagger$ (later just $\mathcal{U}+\mathcal{W}+\mathcal{W}^\dagger$)
- Activation functions: *ReLU*, *LeakyReLU*, *tanh*, *sigmoid*
- Architecture sizes: "small", "medium", "large", "wide"

$W^{(1\times1)}$, $W^{(1\times2)}$			
Small	Architecture 1	Architecture 2	Architecture 3
	C2D(2, N_{in}, 4) C2D(1, 4, 8) GAP $Linear$(8, 4) $Linear$(4, 1)	C2D(2, N_{in}, 4) C2D(2, 4, 4) GAP $Linear$(4, 4) $Linear$(4, 1)	C2D(1, N_{in}, 8) C2D(2, 8, 4) GAP $Linear$(4, 1) -
$N_{\text{param}}^{(U)}$	341	353	273
$N_{\text{param}}^{(U,W)}$	469	481	337
$N_{\text{param}}^{(U,W,W^\dagger)}$	597	609	401
Medium	Architecture 1	Architecture 2	Architecture 3
	C2D(2, N_{in}, 8) C2D(2, 8, 8) C2D(2, 8, 8) GAP $Linear$(8, 4) $Linear$(4, 1)	C2D(2, N_{in}, 8) C2D(2, 8, 8) - GAP $Linear$(8, 4) $Linear$(4, 1)	C2D(3, N_{in}, 4) C2D(2, 4, 8) - GAP $Linear$(8, 4) $Linear$(4, 1)
$N_{\text{param}}^{(U)}$	1089	825	757
$N_{\text{param}}^{(U,W)}$	1345	1081	1045
$N_{\text{param}}^{(U,W,W^\dagger)}$	1601	1337	1333

Large	Architecture 1	Architecture 2	Architecture 3
	C2D(2, N_{in}, 16) C2D(2, 16, 16) C2D(2, 16, 16) GAP $Linear$(16, 8) $Linear$(8, 1)	C2D(3, N_{in}, 16) C2D(3, 16, 8) - GAP $Linear$(8, 8) $Linear$(8, 1)	C2D(3, N_{in}, 16) C2D(1, 16, 8) C2D(3, 8, 16) GAP $Linear$(16, 8) $Linear$(8, 1)
$N_{\text{param}}^{(U)}$	3265	3561	3769
$N_{\text{param}}^{(U,W)}$	3777	4713	4921
$N_{\text{param}}^{(U,W,W^\dagger)}$	4289	5865	6073
Wide	Architecture 1	Architecture 2	Architecture 3
	C2D(2, N_{in}, 128) - GAP $Linear$(128, 1) -	C2D(2, N_{in}, 256) C2D(3, 256, 32) GAP $Linear$(32, 1) -	C2D(2, N_{in}, 512) - GAP $Linear$(512, 64) $Linear$(64, 1)
$N_{\text{param}}^{(U)}$	8449	90433	66177
$N_{\text{param}}^{(U,W)}$	12545	98625	82561
$N_{\text{param}}^{(U,W,W^\dagger)}$	16641	106817	98945

Figure: CNN architectures for 1×1 and 1×2 loops in 2D

Baseline CNNs for 2×2 and 4×4 loops

$W^{(2\times2)}$			
Small	Architecture 1	Architecture 2	Architecture 3
	C2D(2, 32, 4)	C2D(2, 32, 2)	C2D(2, 32, 4)
	C2D(2, 4, 4)	C2D(1, 2, 4)	C2D(2, 4, 2)
	GAP	GAP	GAP
	Linear(4, 4)	*Linear*(4, 1)	*Linear*(2, 1)
	Linear(4, 1)	-	
N_{param}	609	275	553
Medium	Architecture 1	Architecture 2	Architecture 3
	C2D(2, 32, 4)	C2D(2, 32, 8)	C2D(3, 32, 4)
	C2D(2, 4, 8)	C2D(2, 8, 8)	C2D(2, 4, 8)
	C2D(2, 8, 8)	C2D(2, 8, 8)	C2D(3, 8, 8)
	C2D(2, 8, 8)	C2D(2, 8, 8)	C2D(2, 8, 8)
	GAP	GAP	GAP
	Linear(8, 16)	*Linear*(8, 8)	*Linear*(8, 4)
	Linear(16, 1)	*Linear*(8, 1)	*Linear*(4, 1)
N_{param}	1341	1905	2181
Large	Architecture 1	Architecture 2	Architecture 3
	C2D(2, 32, 8)	C2D(2, 32, 8)	C2D(3, 32, 8)
	C2D(2, 8, 16)	C2D(2, 8, 16)	C2D(3, 8, 16)
	C2D(2, 16, 32)	C2D(2, 16, 32)	C2D(3, 16, 32)
	C2D(2, 32, 64)	C2D(2, 32, 64)	C2D(3, 32, 16)
	-	C2D(2, 64, 32)	-
	GAP	GAP	GAP
	Linear(64, 16)	*Linear*(32, 8)	*Linear*(16, 8)
	Linear(16, 1)	*Linear*(8, 1)	*Linear*(8, 1)
N_{param}	12953	20393	12889

$W^{(4\times4)}$			
Small	Architecture 1	Architecture 2	Architecture 3
	C2D(2, 32, 4)	C2D(2, 32, 4)	C2D(2, 32, 4)
	C2D(2, 4, 4)	C2D(1, 4, 8)	C2D(2, 4, 2)
	GAP	GAP	GAP
	Linear(4, 4)	*Linear*(8, 4)	*Linear*(2, 1)
	Linear(4, 1)	*Linear*(4, 1)	-
N_{param}	609	597	553
Medium	Architecture 1	Architecture 2	Architecture 3
	C2D(3, 32, 16)	C2D(2, 32, 16)	C2D(3, 32, 8)
	C2D(1, 16, 8)	C2D(2, 16, 24)	C2D(2, 8, 16)
	C2D(3, 8, 16)	C2D(2, 24, 16)	C2D(1, 16, 32)
	-	-	C2D(2, 32, 16)
	-	-	C2D(2, 16, 8)
	GAP	GAP	GAP
	Linear(16, 8)	*Linear*(16, 8)	*Linear*(8, 8)
	Linear(8, 1)	*Linear*(8, 1)	*Linear*(8, 1)
N_{param}	6073	5321	6049
Large	Architecture 1	Architecture 2	Architecture 3
	C2D(3, 32, 16)	C2D(2, 32, 16)	C2D(4, 32, 16)
	C2D(3, 16, 32)	C2D(2, 16, 32)	C2D(4, 16, 32)
	C2D(3, 32, 64)	C2D(2, 32, 64)	C2D(4, 32, 32)
	C2D(3, 64, 32)	C2D(2, 64, 64)	C2D(4, 32, 16)
	-	C2D(2, 64, 32)	-
	-	C2D(2, 32, 16)	-
	GAP	GAP	GAP
	Linear(32, 16)	*Linear*(16, 16)	*Linear*(16, 8)
	Linear(16, 1)	*Linear*(16, 8)	*Linear*(8, 8)
	-	*Linear*(8, 1)	*Linear*(8, 1)
N_{param}	46769	39553	41273

Figure: CNN architectures for 2×2 and 4×4 loops in 2D

L-CNNs for 1×2, 2×2 and 4×4 in 2D

$W^{(1\times2)}$			
	Small	Medium	Large
	$L\text{-}CBL(2, 1, 2)$ $Trace$ $Linear(4, 1)$	$L\text{-}CBL(3, 1, 4)$ $Trace$ $Linear(8, 1)$	$L\text{-}CBL(4, 1, 8)$ $Trace$ $Linear(16, 1)$
N_{param}	35	117	329
$W^{(2\times2)}$			
	Small	Medium	Large
	$L\text{-}CBL(2, 1, 2)$ $L\text{-}CBL(2, 2, 2)$ $Trace$ $Linear(4, 1)$	$L\text{-}CBL(3, 1, 4)$ $L\text{-}CBL(3, 4, 4)$ $Trace$ $Linear(8, 1)$	$L\text{-}CBL(4, 1, 8)$ $L\text{-}CBL(4, 8, 8)$ $Trace$ $Linear(16, 1)$
N_{param}	125	1305	13521
$W^{(4\times4)}$			
	Small	Medium	Large
	$L\text{-}CBL(2, 1, 2)$ $L\text{-}CBL(2, 2, 2)$ $L\text{-}CBL(3, 2, 2)$ $L\text{-}CBL(3, 2, 2)$ $Trace$ $Linear(4, 1)$	$L\text{-}CBL(3, 1, 4)$ $L\text{-}CBL(3, 4, 4)$ $L\text{-}CBL(4, 4, 4)$ $L\text{-}CBL(4, 4, 4)$ $Trace$ $Linear(8, 1)$	$L\text{-}CBL(4, 1, 8)$ $L\text{-}CBL(4, 8, 8)$ $L\text{-}CBL(4, 8, 8)$ $L\text{-}CBL(4, 8, 8)$ $Trace$ $Linear(16, 1)$
N_{param}	465	4833	39905

Figure: L-CNN architectures for Wilson loops in 2D

L-CNNs for 2×2, 4×4 and Q_{plaq} in 4D

$W^{(2\times2)}$		
	Small	Medium
	L-CBL(2, 6, 2) *L-CBL*(2, 2, 2) *Trace* *Linear*(4, 1)	*L-CBL*(3, 6, 4) *L-CBL*(3, 4, 4) *Trace* *Linear*(8, 1)
N_{param}	1801	8305
$W^{(4\times4)}$		
	Small	Medium
	L-CBL(2, 6, 2) *L-CBL*(2, 2, 2) *L-CBL*(3, 2, 2) *L-CBL*(3, 2, 2) *Trace* *Linear*(4, 1)	*L-CBL*(3, 6, 4) *L-CBL*(3, 4, 4) *L-CBL*(4, 4, 4) *L-CBL*(4, 4, 4) *Trace* *Linear*(8, 1)
N_{param}	2109	14377
q^{plaq}		
	Small	
	L-CBL(2, 6, 4) *Trace* *Linear*(8, 1)	
N_{param}	3181	

Figure: L-CNN architectures for Wilson loops in 4D