Effective Field Theories for Heavy Quarks:

Heavy Quark Effective Theory and Heavy Quark Expansion

Thomas Mannel
Theoretische Elementarteilchenphysik, Naturwissensch.- techn. Fakultät,
Universität Siegen, 57068 Siegen, Germany

OXFORD
UNIVERSITY PRESS

Acknowledgements

I thank the organizers of the Les Houches school for the invitation; it was a real pleasure to lecture for highly motivated students in such an inspiring environment. I also thank Keri Vos for a careful reading of the write-up of the lectures. These lecture notes are printed in the proceedings of the 2017 school.

Contents

1 Introduction

Effective Field Theory (EFT) and Renormalization Group (RG) methods have developed into quite universal tools that can be applied in various fields of physics. Most efficient use of EFT methods can be made in systems, in which vastly different mass scales appear and appropriate ratios of these mass scales define small parameters one aims to expand in.

In nuclear and particle physics, the obvious scales are the masses of the fundamental constituents. For the purpose of these lectures we will not discuss the Higgs mechanism which is assumed to give masses to the quarks and leptons, we rather assume that the masses are fundamental parameters. Another relevant scale in nuclear and particle physics is the scale $\Lambda_{\rm QCD} \sim 300$ MeV, which is generated by "dimensional transmutation" in QCD; this scale is the typical scale of the masses of light hadrons and also governs the running (i.e. the dependence on the renormalization scale μ) of the strong coupling "constant" $\alpha_s(\mu)$.

When considering weak interactions, the typical scale is set by the W boson mass M_W which at low energies manifests itself in the Fermi coupling constant $G_F \sim 1/M_W^2$ relevant for the four-fermion coupling in the EFT for weak interactions. When studying a weak decay of a bottom hadron, the typical scale is set by the b-quark mass m_b. Due to confinement of QCD, this b quark is bound in a hadron, and the relevant scale for this binding in $\Lambda_{\rm QCD}$.

The elementary interaction for weak processes is expressed in terms of quark currents, however, the observed states are hadrons. Therefore we have to deal with the effects of strong interactions, which are described in QCD. One important feature of QCD is its *asymptotic freedom*, which implies that its running coupling constant $\alpha_s(\mu) \to 0$ as $\mu \to \infty$. In practical terms this means that $\alpha_s(M_W)$ as well as $\alpha_s(m_b)$ is a small parameter which allows us to perform a perturbative expansion.

A hadronic matrix element of a quark current evaluated at the scale $\mu \sim m_b$ still contains perturbatively computable pieces, which can be extracted by switching to an EFT description, which for the cases to be discussed below is the "Heavy Quark Effective Theory" (HQET). By applying HQET, the hadronic matrix elements are expressed as a combination of perturbatively computable coefficients and new, suitably defined matrix elements, which contain the "real" non-perturbative contributions.

For some cases it is convenient to also treat the mass of the charm quark m_c as a perturbative scale, which requires to also describe the c quark in HQET. However, once we arrive at scales $\mu \sim \Lambda_{\rm QCD}$, the strong coupling $\alpha_s(\mu)$ becomes order one or bigger, indicating that perturbation theory becomes useless.

Thus weak decays involve a sequence of vastly different mass scales. Assuming that the Standard Model (SM) itself is an EFT and that we have physics beyond the SM (BSM) at some high scale $\Lambda_{\rm NP}$, we have $\Lambda_{\rm NP} \gg M_W \gg m_b \gg m_c \gg \Lambda_{\rm QCD}$ where - except for the last step - the QCD effects can be treated perturbatively by a tower of suitably constructed EFTs, until finally the non-perturbative QCD effects remain as matrix elements depending solely on the small scale $\Lambda_{\rm QCD}$.

Up to that point this describes the ubiquitous machinery of EFT in general. However, EFT's for heavy quarks are in one aspect quite special. In weak interactions at low scales all effects of e.g. the W boson appear only in the couplings (like the Fermi coupling), and at low scales this is the only remnant of the heavy W boson. However, consider now a bottom quark in QCD. In pure QCD, the bottom number is a conserved quantity, and this statement is independent of the scale. Thus a hadron with one unit of bottom quantum number will at low scales (i.e. below the bottom quark mass) still have the bottom quark inside, however, *this bottom quark will behave like a static source of a color field*. This is in full analogy to the hydrogen atom: Although it is a two particle problem, it is a very good approximation to treat the proton inside the H atom as a static source of a Coulomb field, in which the electron moves; any corrections to this picture will be of order $m_{\rm electron}/m_{\rm Proton}$. The simplest type of such a theory is the already mentioned HQET which describes systems with a single heavy quark, where all light degrees of freedom are "soft", i.e. all components of their momenta are of the order $\Lambda_{\rm QCD}$. In the first part of the lectures we will mainly discuss such systems.

The second part of these lectures is devoted to inclusive processes. Using the Operator Product Expansion (OPE), a standard method in quantum firld theory, we will set up an expansion, called *Heavy Quark Expansion*, which has become the basis of many precision calculations in heavy quark physics.

Heavy quarks can decay weakly into light quarks, and hence there is also the kinematic situation, where light quarks acquire energies (in the rest frame of the decaying heavy quark) which scale with the heavy quark mass. For these situations an EFT has been developed, which is called *Soft Collinear Effective Theory* (SCET). This theory has also many applications in high-energy collider physics and will be covered by a different lecture at this school.

A second class are systems with a heavy quark and a heavy antiquark forming a bound system such as a charmonium, a bottomonium and also a B_c. Such systems require to set up yet a different kind of EFT, which is called "Non-relativistic QCD" (NRQCD). However, do to space and time limitations, this type of theory cannot be covered in these lectures.

The original work on HQET dates back to the papers by Eichten and Hill [1, 2], Grinstein [3] and Georgi [4]. The main impact of HQET are the additional symmetries emerging in the infinite mass limit; this has been noticed first in the papers by Shifman and Voloshin [5] and Isgur and Wise [6, 7]. The OPE-based method for inclusive processes has been first set up in the work by Chay, Georgi and Grinstein [8], by Bigi et al. [9, 10], by Manohar and Wise [11] and also in [12].

There are many reviews on this subject, which are too numerous to be listed here; a subjective selection is [13–16]. Finally, there are in the meantime a few textbooks on this subject [17–20], where the different aspects are elaborated on.

2
Heavy Quark Effective Theory

We start with the simplest heavy quark expansion, which is the heavy quark effective theory for systems with a single heavy quark. We shall first construct its Lagrangian by integrating out heavy degrees of freedom. The remarkable and for phenomenology very relevant features of HQET are the Heavy Quark Symmetries (HQS) which eventually yield constraints on the non-perturbative matrix elements at low scales, which are not evident in full QCD. Since $\alpha_s(m_b)$ is a perturbative scale, we will compute the one-loop matching of full QCD to HQET, which will give us some insight into the anatomy of HQET. Finally we will collect a few results that are used in current phenomenology.

2.1 Construction of the HQET Lagrangian

There are two ways to construct the Lagrangian of HQET. One follows straight the idea of EFT by identifying the heavy degrees of freedom and integrating them out from the functional integral [21]. This approach is quite instructive, since it can be explicitly performed at tree level and also at one loop; it also leads to closed form for the HQET Lagrangian, at least at tree level.

A second approach follows usual non-relativistic reduction of the Dirac equation [22], leading finally to a recursive construction of the terms of higher order. This approach has the disadvantage that it does not explicitly involve the typical steps of the construction of an EFT. We will not discuss this alternative approach here.

The two approaches seem to have different results, since the Lagrangians derived in the two cases look different. However, it has been shown that the two approaches are related by a field redefinition, and that the results for physical quantities are the same in both cases.

We will first consider the derivation of the HQET Lagrangian from the usual machinery of EFT, following [21]. The starting point is the Lagrangian of QCD with a single heavy quark Q written as

$$\mathcal{L}_{\mathrm{QCD}} = \bar{Q}(i\not{D} - m_Q)Q + \mathcal{L}_{\mathrm{light}} \tag{2.1}$$

where m_Q is the mass of the heavy quark, $D_\mu = \partial_\mu + igA_\mu$ is the usual QCD covariant derivative including the interaction with the gluon A_μ, and $\mathcal{L}_{\mathrm{light}}$ is the Lagrangian for the light quarks and gluons[1].

[1] $\mathcal{L}_{\mathrm{light}}$ also contains all terms relevant for the gauge fixing and possibly ghost fields needed for the quantization of QCD.

To obtain the Green functions of the corresponding quantum field theory one may gather them in a generating functional, which is expressed as a functional integral over the field variables. Thus we write

$$Z(\eta, \bar{\eta}, \lambda) = \int [dQ][d\bar{Q}][d\phi_\lambda] \exp\left\{ i \int d^4x \, \mathcal{L}_{\rm QCD} + i \int d^4x \, (\bar{\eta} Q + \bar{Q} \eta + \phi_\lambda \lambda) \right\} , \quad (2.2)$$

where $\phi_\lambda = q,\ A^a_\mu$ denotes the light degrees of freedom (light quarks q and gluons A_μ). Functional differentiation with respect to the source terms $\eta, \bar{\eta}$ and λ and subsequently setting the sources to zero yields the Green functions of QCD[2], e.g.

$$\frac{\delta}{\delta \eta(x)} \frac{\delta}{\delta \bar{\eta}(y)} Z(\eta, \bar{\eta}, \lambda)|_{\eta=0, \bar{\eta}=0, \lambda=0} = \langle 0 | T[Q(y) \bar{Q}(x)] | 0 \rangle \quad (2.3)$$

In order to derive the HQET Lagrangian we consider a system with a single heavy quark which is bound in a heavy hadron. This hadron has a mass m_H and moves with a certain momentum p_H. In case the hadron contains only a single heavy quark, its mass will scale with the heavy quark mass, likewise its momentum will scale with the heavy quark mass. To this end, it is convenient to define a four velocity

$$v = \frac{p_H}{m_H}, \quad v^2 = 1, \quad v_0 > 0 \ . \quad (2.4)$$

which is independent of the heavy quark mass. This vector defines a specific frame, e.g. $v = (1, 0, 0, 0)$ is the rest frame of the heavy hadron.

Eventually we want to consider the heavy quark inside the heavy hadron; since most of the hadron mass is given by the quark mass, the heavy quark moves with almost the same velocity as the heavy hadron. Thus the momentum of the heavy quark my be written as $p_Q = m_Q v + k$ where k is a small "residual" momentum satisfying $k \ll m_Q$.

To implement this idea on the technical side, we use this "external" velocity vector v to decompose the heavy-quark field Q into an "upper" (or "large") component ϕ and a "lower" (or "small") component χ

$$\phi_v = \frac{1}{2}(1 + \not{v}) Q \equiv P_+ Q, \ \ \not{v} \phi_v = \phi, \quad (2.5)$$

$$\chi_v = \frac{1}{2}(1 - \not{v}) Q \equiv P_- Q \ \ \not{v} \chi_v = -\chi, \quad (2.6)$$

and to define a decomposition of the covariant derivative into a "time" and a "spatial" ($\perp$) part

$$D_\mu = v_\mu (v \cdot D) + D^\perp_\mu \, , \quad D^\perp_\mu = (g_{\mu\nu} - v_\mu v_\nu) D^\nu, \quad \left\{ \not{D}^\perp \, , \not{v} \right\} = 2(v \cdot D^\perp) = 0. \quad (2.7)$$

In terms of these new fields (2.5,2.6) and using (2.7), the Lagrangian of the heavy quark field (i.e. the first term of (2.1)) takes the form

[2] We note that the funtional integral (2.2) is mathematically ill defined. For our purposes we take the practitioner's point of view and look at (2.2) as a short-hand notation for perturbation theory, which results from expanding in the strong coupling g_s.

$$\mathcal{L}_{\rm heavy} = \bar{\phi}_v(i(v\cdot D) - m_Q)\phi_v - \bar{\chi}_v(i(v\cdot D) + m_Q)\chi_v + \bar{\phi}_v i\not{D}^\perp \chi_v + \bar{\chi}_v i\not{D}^\perp \phi_v \quad (2.8)$$

To proceed further, we now implement the decomposition of the heavy quark momentum into a "large" and a residual piece. This is achieved by multiplying the heavy quark field by a phase

$$\phi_v = e^{-im_Q(v\cdot x)} h_v\,, \qquad \chi_v = e^{-im_Q(v\cdot x)} H_v\,. \quad (2.9)$$

Note that the momentum of a field is the derivative acting on the field, i.e.

$$p_Q^\mu \sim i\partial^\mu Q(x) \quad \text{hence} \quad i\partial^\mu \phi_v(x) = e^{-im_Q(v\cdot x)}(m_Q v^\mu + i\partial^\mu) h_v(x)$$

which means that the derivative acting on the field h_v reproduces the residual momentum introduced above. This observation provides us with the power counting of HQET: once we have reformulated the theory in terms of h_v, we aim at an expansion in the residual momentum, i.e. in iD_μ/m_Q.

We express the Lagrangian of the heavy quark in term of the fields h_v and H_v and obtain

$$\mathcal{L}_{\rm heavy} = \bar{h}_v i(v\cdot D) h_v - \bar{H}_v(i(v\cdot D) + 2m_Q)H_v + \bar{h}_v i\not{D}^\perp H_v + \bar{H}_v i\not{D}^\perp h_v\,. \quad (2.10)$$

With this form of the Lagrangian we can now easily identify the degrees of freedom. The field h_v does not have a mass term, while the field H_v has acquired a mass term $2m_Q$; the remaining terms are couplings between h_v and H_v. Thus in the sense of EFT the field H_v is the heavy degree of freedom, while the field h_v is light.

To construct the EFT, we have to "integrate out" the heavy degree of freedom, which is H_v. In the language of functional integrals this means that we have to integrate over the field H_v in the generating functional (2.2). It is interesting to note, that in the case at hand this functional integration can be explicitly performed, at least for the tree-level Lagrangian (2.10), since there are only quadratic dependences on the relevant field, hence the functional integral over the field H_v is a Gaussian integral.

In order to integrate over the heavy field H_v, we first split also the source terms in (2.2) according to

$$\int d^4x\,(\bar{\eta}Q + \bar{Q}\eta) = \int d^4x\,(\bar{\rho}_v h_v + \bar{h}_v\rho_v + \bar{R}_v H_v + \bar{H}_v R_v), \quad (2.11)$$

where ρ_v and R_v are now source terms for the upper-component field h_v and the lower-component part H_v, respectively. When studying processes at scales well below the scale $2m_Q$, no Green function involving the heavy field H_v will be relevant, hence we can put the corresponding sources to zero. Performing the Gaussian integral over the field H_v we obtain

$$Z(\rho_v, \bar{\rho}_v, \lambda) = \int [dh_v][d\bar{h}_v][d\lambda]\,\Delta \\ \times \exp\left\{ iS + S_\lambda + i\int d^4x\,(\bar{\rho}_v h_v + \bar{h}_v\rho_v + \phi_\lambda\lambda)\right\}\,, \quad (2.12)$$

where now the action functional for the heavy quark becomes a non-local object

$$S = \int d^4x \left[\bar{h}_v i(v \cdot D) h_v - \bar{h}_v \not{D}^\perp \left(\frac{1}{i(v \cdot D) + 2m_Q - i\epsilon} \right) \not{D}^\perp h_v \right] . \tag{2.13}$$

depending solely on the field h_v and (via the covariant derivatives) on gluon fields.

The quantity Δ is the determinant resulting form the Gaussian integration, which may formally be written as

$$\begin{aligned} \Delta &= \exp\left(\frac{1}{2} \ln\left[i(v \cdot D) + 2m_Q\right] \right) \\ &= \text{const} \exp\left(\frac{1}{2} \ln\left[1 + \frac{1}{i(v \cdot \partial) - 2m_Q + i\epsilon} g_s (v \cdot A) \right] \right) \end{aligned} \tag{2.14}$$

However, unlike in other quantum field theories, this determinant is a constant (i.e. independent of the gluon fields). This can bee seen by either chosing the gauge $v \cdot A = 0$ or by expanding the logarithm which leads to an expression which looks like the fermion bubble diagrams in ordinary QCD; however, here the particles propagate only in forward time-like directions, since the propagator in configuration space is

$$\frac{1}{i(v \cdot \partial) - 2m_Q + i\epsilon} = \delta^3(x^\perp)\theta(v \cdot x) e^{i2m(v \cdot x)} . \tag{2.15}$$

Hence a closed loop always yields a zero result.

In general, integrating our degrees of freedom yields non-local action functionals such as (2.13). However, if the degree of freedom that has been integrated out is heavy, it is in general possible to expand the result in inverse powers of the mass of the heavy scale. In our case this is quite evident, since we have $(v \cdot D) \ll 2m_Q$ because $(v \cdot D)$ is related to the residual momentum of the heavy quark. Consequently we may expand to get

$$\frac{1}{i(v \cdot D) + 2m_Q - i\epsilon} = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(\frac{-i(v \cdot D)}{2m_Q} \right)^n \tag{2.16}$$

which expresses the non-local distribution on the left-hand side as a series of local distributions.

Truncating at some order N yields a local action functional, and hence we get as the Lagranian

$$\mathcal{L}_{1/\mathrm{m_Q}-\mathrm{Expansion}} = \bar{h}_v i(v \cdot D) h_v - \frac{1}{2m_Q} \bar{h}_v \not{D}^\perp \sum_{n=0}^{N} \left(\frac{-i(v \cdot D)}{2m_Q} \right)^n \not{D}^\perp h_v \tag{2.17}$$

This expression is the expansion of the QCD Lagrangian up to the order $1/m_Q^{N+1}$. The leading term

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v i(v \cdot D) h_v \tag{2.18}$$

is the Lagrangian for a static heavy quark moving with the four velocity v, i.e the Lagrangian of Heavy Quark Effective Theory (HQET).

By itself this Lagrangian is not very useful; by choosing an axial gauge $v \cdot A = 0$ the coupling to the gluons can even be made to vanish. However, this Lagrangain becomes useful as soon as additional interactions are implemented which are "hard", meaning that these interactions change the velocity of the heavy quark. In most applications we shall consider these are typically electroweak interactions which inject a large momentum transfer into the system.

To illustrate this in some more detail, let us consider the semileptonic decay $B \to D\ell\bar{\nu}$. The relevant hadronic matrix element is

$$\langle B(p)|\bar{b}\gamma_\mu c|D(p')\rangle$$

which may be obtained from inserting the weak transition current into the generating functional (2.2)

$$\begin{aligned} Z^{(b\to c)}(\eta_b, \bar{\eta}_b, \eta_c, \bar{\eta}_c, \lambda) = \int [db][d\bar{b}][dc][d\bar{c}][d\phi_\lambda]\ \bar{b}(0)\gamma_\mu c(0) \qquad (2.19) \\ \times \exp\left\{ i\int d^4x\, \mathcal{L}_{\rm QCD} + i\int d^4x\, (\bar{\eta}_b b + \bar{b}\eta_b + \bar{\eta}_c c + \bar{c}\eta_c + \phi_\lambda\lambda) \right\}\,, \end{aligned}$$

corresponding to an insertion of the weak $b \to c$ current into the QCD Green functions.

At scales below m_c we may use the static limit for both the b and the c quark, however, the two mesons have different velocities $v = p/M_b$ and $v' = p'/M_D$, so we need to introduce two static quarks b_v and c_v with different velocities. Going through the same steps as before, now for two heavy quarks with different velocities, we get

$$\mathcal{L}_{\rm HQET}^{b\to c} = \bar{b}_v i(v\cdot D)b_v + \bar{c}_{v'} i(v'\cdot D)c_{v'} \qquad (2.20)$$

Although this looks like a Lagrangian with two heavy quarks, the weak current ensures that for $x_0 \le 0$ we have only the bottom quark (moving with velocity v) which at $x_0 = 0$ decays into a charm quark, moving with velocity v'.

This kind of approximation is well known since almost one century. It has been used already in the context of the infrared problem of QED, which for soft photons becomes "Heavy Electron Effective Theory" [23, 24]. Furthermore, once there are two velocities v and v' in the game, there is no possibility to trivialize the theory by a choice of gauge.

Once one considers operator insertions in the Greens functions as in (2.19) one also needs to re-write the fields appearing in the current, which amounts to re-express the full QCD field by the static field h_v. We get

$$\begin{aligned} Q(x) &= e^{-im_Q vx}\,[h_v + H_v] = e^{-im_Q vx}\left[1 + \left(\frac{1}{2m + ivD}\right) i\not{D}_\perp\right] h_v \\ &= e^{-im_Q vx}\left[1 + \frac{1}{2m_Q}\not{D}_\perp + \left(\frac{1}{2m_Q}\right)^2 (-ivD)\not{D}_\perp + \cdots\right] h_v\,. \qquad (2.21) \end{aligned}$$

Note that this expression is inserted in the functional integral (2.19), so we intergrate out the heavy field(s) H_v, resulting in the replacement of H_v in the second step of (2.21).

The Hamiltonian which can be derived from (2.17) has the unusual property that it contains “time derivatives” (i.e. terms involving $(iv\partial)Q_v$). However, these can be removed by field redefinitions, resulting in an Hamiltonian without time derivatives. This Hamiltonian can also be constructed from the start by performing a transformation (a so-called Foldy-Wouthuysen transformation), which decouples the “large” and the “small” components of the spinor Q. This yields a Lagrangian which looks different from the one derived above, starting at $1/m_Q^3$; likewise, the expansion of the field in terms if the static field also looks different.

For physical matrix elements both approaches eventually yield the same answer. To see how this works, we consider a matrix element with a heavy-to-light current of the form $\bar{q}\Gamma Q$ with a heavy meson in the initial state $|M(v)\rangle$ and some final state $|A\rangle$. Computing to order $1/m_Q$ we get

$$\begin{aligned}\langle A|\bar{q}\Gamma Q|M(v)\rangle &= \langle A|\bar{q}\Gamma h_v|H(v)\rangle + \frac{1}{2m_Q}\langle A|\bar{q}\Gamma P_- i\not{D} h_v|H(v)\rangle \\ &-i\int d^4x \langle A|T\{L_1(x)\bar{q}\Gamma h_v\}|H(v)\rangle + \mathcal{O}(1/m^2)\ , \qquad (2.22)\end{aligned}$$

where L_1 are the $1/m$ corrections to the Lagrangian as given in (2.17). In addition, $|M(v)\rangle$ is the state of the heavy meson in full QCD, including all of its mass dependence, while $|H(v)\rangle$ is the corresponding state in the infinite-mass limit.

A contribution to L_1 with a time derivative will become – upon insertion into the T product – a local operator, which in turn means that it could as well be absorbed into the first term by a field redefinition. Using the Hamiltonian without time derivative (such as the one derived from the Foldy-Wouthuysen transformation) will not have any local contributions in the second term, while the closed expression (2.17) and (2.21) will generate such terms, which in the other approach will be contained in the first term. In this way the result for the the physical matrix element will be the same.

2.2 Symmetries of HQET

Probably the most important property of HQET for phenomenology are the Heavy Quark Symmetries (HQS) [5–7]. These appear in the infinite-mass limit and are not present in full QCD. These symmetries have very simple physical origins and are already manifest in the Lagrangians derived in the last section. In addition, there is another symmetry which we shall briefly discuss. It is related to the fact, that the construction of HQET requires to introduce a four velocity vector, which is not present in full QCD. Thus a change of this velocity vector by an amount of the order $1/m_Q$ may not change the physics. This so-called “reparametrization invariance” [25–28] has interesting consequences, since it relates different order in the $1/m_Q$ expansion.

2.2.1 Flavour Symmetry

The QCD Lagrangian is known to have flavour symmetries in the case where quarks become mass-degenerate: The approximate degeneracy of the up and the down quark leads to the isospin symmetry, in case all quarks are assumed to be massless, QCD has a chiral symmetry, of which the flavour $SU(3)$ symmetry is manifest. The underlying reason is that the interaction of the quarks with the gluons does not depend on the mass, it depends only on the color charge of the quarks which is defined by putting all quarks into the fundamental representation of color $SU(3)$.

This still remains true in the infinite mass limit. Once a heavy quark becomes a static source of color, its flavour becomes irrelevant. To make this explicit, we consider the $b \to c$ HQET Hamiltonian (2.20) for the case of two equal velocities

$$\mathcal{L}_{\text{HQET}}^{b,c} = \bar{b}_v i(v \cdot D) b_v + \bar{c}_v i(v \cdot D) c_v = (\bar{b}_v \, , \, \bar{c}_v) \begin{pmatrix} i(v \cdot D) & 0 \\ 0 & i(v \cdot D) \end{pmatrix} \begin{pmatrix} b_v \\ c_v \end{pmatrix} \qquad (2.23)$$

which as a manifest $SU(2)$ symmetry: for any unitary 2×2 matrix U we define the transformation

$$\begin{pmatrix} b_v \\ c_v \end{pmatrix}' = U \begin{pmatrix} b_v \\ c_v \end{pmatrix} ,$$

under which the Lagrangian (2.23) remains invariant. Note that this symmetry relates only heavy quarks moving with the same velocity v.

As a practical application, consider a semileptonic decay of a B meson into a D meson. Assuming both b and c to be heavy, we may look into the point of maximal momentum transfer to the leptons, which is $q^2_{\text{max}} = (m_B - m_D)^2 \approx (m_b - m_c)^2$. Looking at this decay in the rest frame of the B meson (which is also the rest frame of the b quark as $m_b \to \infty$), the final state D meson (as well as the c quark as $m_c \to \infty$) remains at rest at this kinematic point, while the two leptons carry away the energy difference $m_B - m_D \approx m_b - m_c$ in a back-to-back momentum configuration. As a consequence of heavy flavour symmetry, the light degrees of freedom (the light quark(s) and gluons forming the meson) cannot be affected by this transition (at this special kinematic point), which means that their state did not change! We will return to this example when discussing weak transition form factors.

2.2.2 Spin Symmetry

The second HQS is the so-called heavy quark spin symmetry. It originates from the fact that in gauge theories like QED and QCD the interaction of the spin of a particle is always of the form $\vec{\sigma} \cdot \vec{B}$, where $\vec{B}$ is the corresponding (chromo)magnetic field. However, this is a dimension-five operator, and its coupling constant is $g/(2m_Q)$, which is the QCD analogue of the Bohr magneton of the particle. As a consequence, the spin of a particle decouples in QCD and hence the rotations of the particle's spin become a symmetry.

To make this explicit, we look at the HQET Lagrangian and decompose the heavy quark field into the two spin components. This is achieved by introducing a spin vector s with $s \cdot v = 0$ and $s^2 = -1$ such that we can define the projections

$$h_v^{\pm s} = \frac{1}{2}(1 \pm \gamma_5 \not{s}) h_v \qquad h_v = h_v^{+s} + h_v^{-s} \, . \qquad (2.24)$$

In terms of these projections we have

$$\mathcal{L} = \bar{h}_v^{+s}(ivD)h_v^{+s} + \bar{h}_v^{-s}(ivD)h_v^{-s} = (\bar{h}_v^{+s} \, , \, \bar{h}_v^{-s}) \begin{pmatrix} i(v \cdot D) & 0 \\ 0 & i(v \cdot D) \end{pmatrix} \begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix} \quad (2.25)$$

Then, similarly as before we have an $SU(2)$ symmetry: for any unitary 2×2 matrix U we define the transformation

$$\begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix}' = U \begin{pmatrix} h_v^{+s} \\ h_v^{-s} \end{pmatrix} ,$$

under which the HQET Lagrangian remains invariant. Note that this symmetry relates again only heavy quarks moving with the same velocity v.

2.2.3 Consequences of Heavy Quark Symmetries

These symmetries have a few interesting consequences which are important to make HQET a useful tool, since they constrain the non-perturbative matrix elements of HQET.

The spin symmetry of the heavy quark has the consequence that all the heavy-hadron states moving with the velocity v fall into spin-symmetry doublets as $m_Q \to \infty$. In Hilbert space, this symmetry is generated by operators $S_v(\epsilon)$ as

$$[h_v, S_v(\epsilon)] = i \not{\epsilon} \not{v} \gamma_5 h_v \ , \tag{2.26}$$

where ϵ, with $\epsilon^2 = -1$, is the rotation axis. The simplest spin-symmetry doublet in the mesonic case consists of the pseudoscalar meson $H(v)$ and the corresponding vector meson $H^*(v, \epsilon)$, since a spin rotation yields

$$\exp\left(iS_v(\epsilon)\frac{\pi}{2}\right)|H(v)\rangle = (-i)|H^*(v,\epsilon)\rangle \ , \tag{2.27}$$

where we have chosen an arbitrary phase to be $(-i)$.

Thus the pseudoscalar ground state meson forms a spin-symmetry doublet with the vector ground state meson; assuming that the bottom is heavy we have the doublets

$$\begin{array}{ccc} |(b\bar{u})_{J=0}\rangle = |B^-\rangle & \longleftrightarrow & |(b\bar{u})_{J=1}\rangle = |B^{*-}\rangle \\ |(b\bar{d})_{J=0}\rangle = |\overline{B}^0\rangle & \longleftrightarrow & |(b\bar{d})_{J=1}\rangle = |\overline{B}^{*0}\rangle \\ |(b\bar{s})_{J=0}\rangle = |\overline{B}_s\rangle & \longleftrightarrow & |(b\bar{s})_{J=1}\rangle = |\overline{B}_s^*\rangle \end{array} \tag{2.28}$$

which become degenerate in the infinite-mass limit.

For baryons, the situation is more complicated, since the two light quarks can have either spin 0 or spin 1. The doublets with u and d quarks are

$$\left|[(ud)_0 Q]_{1/2}\right\rangle = |\Lambda_Q\rangle \qquad |\Lambda_Q \Uparrow\rangle \longleftrightarrow |\Lambda_Q \Downarrow\rangle \tag{2.29}$$

$$\left|[(uu)_1 Q]_{1/2}\right\rangle, \left|[(ud)_1 Q]_{1/2}\right\rangle, \left|[(dd)_1 Q]_{1/2}\right\rangle = |\Sigma_Q\rangle \tag{2.30}$$

$$\left|[(uu)_1 Q]_{3/2}\right\rangle, \left|[(ud)_1 Q]_{3/2}\right\rangle, \left|[(dd)_1 Q]_{3/2}\right\rangle = \left|\Sigma_Q^*\right\rangle \qquad |\Sigma_Q\rangle \longleftrightarrow \left|\Sigma_Q^*\right\rangle \tag{2.31}$$

and similar relation for the strange baryons Ξ_b and Ω_b. Note that for the Λ_b baryon, the two spin directions of the Λ_b are the spin-symmetry doublet, since the light degrees of freedom are in a spinless state and thus the spin of the baryon is the heavy-quark spin, at least to leading order in $1/m_b$.

To leading order, the mass of a heavy Q hadron is the mass of the quark m_Q. However, we may expand the hadron mass in terms of the quark mass, which reads for the mesonic ground states

$$m_H = m_Q + \bar{\Lambda} - \frac{\lambda_1 - 3\lambda_2}{2m_Q} + ... \tag{2.32}$$

$$m_{H^*} = m_Q + \bar{\Lambda} - \frac{\lambda_1 + \lambda_2}{2m_Q} + ... \tag{2.33}$$

where we have introduced new parameters $\bar{\Lambda}$, λ_1 and λ_2. $\bar{\Lambda}$ is the binding-energy parameter for the heavy hadron

$$\bar{\Lambda} = \frac{\langle 0|q\ i\overleftarrow{vD}\ \gamma_5 h_v|\tilde{H}(v)\rangle}{\langle 0|q\gamma_5 h_v|\tilde{H}(v)\rangle}\ , \tag{2.34}$$

while λ_1 and λ_2 are defined by the HQET matrix elements

$$2m_H\lambda_1 = \langle\tilde{H}(v)|\bar{h}_v(iD^\perp)^2 h_v|\tilde{H}(v)\rangle \tag{2.35}$$

$$2m_H\lambda_2 = \langle\tilde{H}(v)|\bar{h}_v(iD^\perp_\mu)(iD^\perp_\nu)(i\sigma^{\mu\nu})h_v|\tilde{H}(v)\rangle \tag{2.36}$$

where $|\tilde{H}(v)\rangle$ is the pseudoscalar Q meson ground state in the infinite-mass limit.

These parameters have a simple physical interpretation: λ_1 is the kinetic energy induced by the residual motion of the heavy quark, λ_2 corresponds to the interaction of the chromomagnetic moment of the heavy quark induced by the interaction with the chromomagnetic field $\vec{\sigma}\cdot\vec{B}$ produced by the light degrees of freedom. This implies in particular that (taking the b and the c quark to be heavy)

$$m^2_{B^*} - m^2_B = m^2_{D^*} - m^2_D = 4\lambda_2 + \mathcal{O}(1/m_Q) \tag{2.37}$$

from which we get $\lambda_2 \approx 0.12$ GeV^2 which is indeed of the order of Λ^2_{QCD}. Similar relations can be written for the Q baryons [31,32]; in particular, the chromomagnetic parameter λ_2 vanishes for Λ_Q baryons, since the light degrees are in a spin-0 state and hence cannot induce a chromomagnetic field.

HQS also constrain hadronic matrix elements. In order to extract the corresponding relations, it is useful to write down a representation for the spins in the ground state mesons. Introducing a spinor $v(v,\pm)$ with spin direction $\pm$ for the light anti-quarks and $u(v,\pm)$ for the heavy quark, we may couple the spins to get the total spin of the meson

$$|(b\bar{u})_{J=0}(v)\rangle \to \frac{1}{\sqrt{2}}\left[u_\alpha(v,+)\bar{v}_\beta(v,-) - u_\alpha(v,-)\bar{v}_\beta(v,+)\right] \propto \left(\gamma_5\frac{\not{v}-1}{2}\right)_{\alpha\beta}$$

$$|(b\bar{u})_{J=1,M=0}(v)\rangle \to \frac{1}{\sqrt{2}}\left[u_\alpha(v,+)\bar{v}_\beta(v,-) + u_\alpha(v,-)\bar{v}_\beta(v,+)\right] \propto \left(\not{\epsilon}_{\mathrm{long}}\frac{\not{v}-1}{2}\right)_{\alpha\beta}$$

where α and β are spinor indices. Including the proper normalization of the states, we define the representation matrices for these states

$$H(v) = \frac{1}{2}\sqrt{m_H}\gamma_5(\not{v} - 1) \qquad \text{for the pseudoscalar meson,} \tag{2.38}$$

$$H^*(v,\epsilon) = \frac{1}{2}\sqrt{m_H}\not{\epsilon}(\not{v} - 1) \qquad \text{for the vector meson,} \tag{2.39}$$

where the two indices of the matrices correspond to the indices of the heavy quark and the light anti-quark, respectively, and ϵ is the polarization vector of the vector meson.

We may now use these representation matrices to exploit the consequences of spin symmetry in a very simple fashion. We look at a transition current of the form $\bar{h}_{v'}\Gamma h_v$ induced e.g. by a weak transition (such as a $b \to c$ semileptonic process). Spin symmetry implies that the spin of the heavy quark in the current is the same as the one of the quark inside the meson, which means that the heavy-quark index of the representation matrix has to hook directly to the Dirac matrix Γ in the current. Thus for a $0^- \to 0^-$ transition we have

$$\langle M(v')|\bar{h}_{v'}\Gamma h_v|M(v)\rangle = \mathrm{Tr}\left[\overline{H}(v')\Gamma H(v)\mathcal{M}(v,v')\right] \tag{2.40}$$

where the two light-quark indices of $\overline{H}(v')\Gamma H(v)$ will be contracted with a Dirac-matrix valued function $\mathcal{M}(v,v')$ of v and v' which describes the dynamics of the light quarks in the transition. This matrix can be decomposed into the basis of the sixteen Dirac matrices, thus we can write (note that due to parity conservation in strong interactions there are no contributions with γ_5 and $\gamma_\mu\gamma_5$)

$$\mathcal{M}(v,v') = \mathbf{1}\,\xi_1(v\cdot v') + \not{v}\,\xi_2(v\cdot v') + \not{v}'\,\xi_3(v\cdot v') + \not{v}\not{v}'\,\xi_3(v\cdot v') \tag{2.41}$$

with scalar functions ξ_i. Inserting this into (2.40) we see that for any Γ this collapses into

$$\langle M(v')|\bar{h}_{v'}\Gamma h_v|M(v)\rangle = \mathrm{Tr}\left[\overline{H}(v')\Gamma H(v)\right]\,\xi(v\cdot v') \tag{2.42}$$

with $\xi(v\cdot v') = \xi_1(v\cdot v') + \xi_2(v\cdot v') - \xi_3(v\cdot v') - \xi_4(v\cdot v')$

Likewise we can discuss the transitions between the transitions $0^- \to 1^-$ and $1^- \to 1^-$ between ground state mesons. Spin symmetry tells us that the function $\mathcal{M}(v,v')$ for the light degrees of freedom is the same in all cases, and hence

> *Any transition within the ground-state spin flavour multiplet $\mathcal{H}(v)$ to the ground-state multiplet $\mathcal{H}(v')$, where $\mathcal{H}(v)$ denotes either $H(v)$ or $H^*(v,\epsilon)$ is described by a single nonperturbative function $\xi(v\cdot v')$.*

The function ξ is called the Isgur Wise (IW) function, and relation (2.42) is one of the "Wigner-Eckart Theorems" of Spin symmetry and can be written as

$$\langle \mathcal{H}(v')|\bar{h}_{v'}\Gamma h_v|\mathcal{H}(v)\rangle = \xi(v\cdot v')\ \mathrm{Tr}\ \left\{\overline{\mathcal{H}}(v')\Gamma\mathcal{H}(v)\right\}\ , \tag{2.43}$$

This relation has remarkable consequences. Assuming that both b and c are heavy, we find that the six form factors describing the semileptonic transitions $B \to D$ and $B \to D^*$ in the infinite-mass limit for both b and c quark reduce to a single one, the

IW function. Furthermore, the current $\bar{b}\gamma_\mu b$ is a conserved current in pure QCD, which translates into a normalization statement for the IW function

$$\xi(v \cdot v' = 1) = 1 \tag{2.44}$$

where the physical argument for this normalization has been given in the last paragraph of sec. 2.2.1. Note that the point $v \cdot v' = 1$ corresponds exactly to the point $q^2_{\max}$ of maximal recoil to the leptons discussed in sec. 2.2.1.

For phenomenological applications these symmetries are very useful, however, only once the corrections to the symmetry limit can be somehow handled. There are two sources of corrections, which are on the one hand the radiative corrections through hard gluons, on the other hand the ones induced by subleading terms in the $1/m_Q$ expansion.

We will discuss the latter and consider the $1/m_Q$ corrections to the normalization statement (2.44) which originated from the conservation of the heavy quark current, which in turn is related to HQS. In this case we can apply a general theorem originally derived by Ademollo and Gatto [29] to the case of HQS. The theorem, derived by Luke in [30] in the context of HQS, states that

In the presence of explicit symmetry breaking, the matrix elements of the currents that generate the symmetry are normalized up to terms which are second-order in the symmetry-breaking interaction.

For the case of HQS, the argument can be outlined in a simple way, taking as an expample the $b \to c$ case. The relevant symmetry is the heavy-flavor symmetry between b and c in the case $m_{b,c} \to \infty$. This symmetry is an $SU(2)$ symmetry and is generated by three operators $Q_\pm$ and Q_3, where[3]

$$\begin{aligned}
&Q_+ = \int d^3x\, \bar{b}_v(x)\gamma_0 c_v(x)\ , \quad Q_- = \int d^3x\, \bar{c}_v(x)\gamma_0 b_v(x)\ , \\
&Q_3 = \int d^3x\, (\bar{b}_v(x)\gamma_0 b_v(x) - \bar{c}_v(x)\gamma_0 c_v(x))\ , \\
&[Q_+, Q_-] = Q_3\ , \qquad [Q_+, Q_3] = -2Q_+\ , \qquad (Q_+)^\dagger = Q_-\ .
\end{aligned} \tag{2.45}$$

Let us denote the ground-state flavour symmetry multiplet by $|B\rangle$ and $|D\rangle$. The operators then act in the following way:

$$\begin{aligned}
&Q_3|B\rangle = |B\rangle\ , \qquad Q_3|D\rangle = -|D\rangle\ , \\
&Q_+|D\rangle = |B\rangle\ , \qquad Q_-|B\rangle = |D\rangle\ , \\
&Q_+|B\rangle = Q_-|D\rangle = 0\ .
\end{aligned} \tag{2.46}$$

The Hamiltonian of this system has a $1/m_Q$ expansion which is decomposed into a symmetric and a symmetry breaking part

[3]We set $x_0 = 0$ in the following relations, such that we have to compute equal-time (anti)commutators. In case the symmetry is exact, the operators do not depend on x_0 anyway, but once the symmetry is broken, there will be a "small" dependence on x_0.

$$\begin{aligned} H &= H_0^{(b)} + H_0^{(c)} + \frac{1}{2m_b} H_1^{(b)} + \frac{1}{2m_c} H_1^{(c)} + \cdots \\ &= H_0^{(b)} + H_0^{(c)} + \frac{1}{2}\left(\frac{1}{2m_b} + \frac{1}{2m_c}\right)(H_1^{(b)} + H_1^{(c)}) \\ &\qquad\qquad + \frac{1}{2}\left(\frac{1}{2m_b} - \frac{1}{2m_c}\right)(H_1^{(b)} - H_1^{(c)}) + \cdots \\ &= H_{symm} + H_{break} \ . \end{aligned} \tag{2.47}$$

Note that the symmetry breaking term does not commute any more with $Q_\pm$ but it still commutes with Q_3 (which only means that we still can distinguish B and D). Thus to order $1/m_Q$ we still have common eigenstates of H and Q_3, which we shall denote by $|\tilde{B}\rangle$ and $|\tilde{D}\rangle$. Sandwiching the commutation relation, we obtain

$$\begin{aligned} 1 &= \langle \tilde{B}|Q_3|\tilde{B}\rangle = \langle \tilde{B}|[Q_+, Q_-]|\tilde{B}\rangle \\ &= \sum_n \left[\langle \tilde{B}|Q_+|\tilde{n}\rangle\langle \tilde{n}|Q_-|\tilde{B}\rangle - \langle \tilde{B}|Q_-|\tilde{n}\rangle\langle \tilde{n}|Q_+|\tilde{B}\rangle\right] \\ &= \sum_n \left[|\langle \tilde{B}|Q_+|\tilde{n}\rangle|^2 - |\langle \tilde{B}|Q_-|\tilde{n}\rangle|^2\right] \ , \end{aligned} \tag{2.48}$$

where the $|\tilde{n}\rangle$ form a complete set of states of the Hamiltonian $H_{symm} + H_{break}$. The matrix elements may be written as

$$\langle \tilde{B}|Q_\pm|\tilde{n}\rangle = \frac{1}{E_B - E_n}\langle \tilde{B}|[H_{break}, Q_\pm]|\tilde{n}\rangle \ , \tag{2.49}$$

where E_B and E_n are the energies of the states $|\tilde{B}\rangle$ and $|\tilde{n}\rangle$, respectively. In the case $|\tilde{n}\rangle = |\tilde{D}\rangle$ the matrix element on the left-hand side will be of order unity, since both the numerator and the energy difference in the denominator are of the order of the symmetry breaking. For all other states, the energy difference in the denominator is non-vanishing in the symmetry limit, and hence this difference is of order unity; thus the matrix element for these states will be of the order of the symmetry breaking. From this we conclude that

$$\langle \tilde{B}|Q_+|\tilde{D}\rangle = 1 + \mathcal{O}\left[\left(\frac{1}{2m_b} - \frac{1}{2m_c}\right)^2\right] . \tag{2.50}$$

For simplicity we have used states normalized to unity.

In order to relate this to the form factor normalization, we observe that the generators are obtained from integrating over the time-components of the current; the general expression for the matrix elements reads ($q = p - p'$)

$$\langle B(p)|\bar{b}\gamma_\mu c|D(p)\rangle = \frac{1}{\sqrt{4v_0v_0'}}\left((v_\mu + v_\mu')f_+(q^2) + (v_\mu - v_\mu')f_-(q^2)\right) \ . \tag{2.51}$$

Switching to HQET for b and c, taking the time component and integrating over $\vec{x}$ yields

$$\int d^3\vec{x}\,\langle B(p)|\bar{b}_v(x)\gamma_0 c_v(x)|D(p)\rangle = (2\pi)^3\delta^3(\vec{p}-\vec{p}')f_+(q^2_{\rm max}) \tag{2.52}$$

where the δ function appears because we are using momentum eigenstates, thus it corresponds to the above normalization to unity. Furthermore, $q^2_{\rm max} = (m_B - m_D)^2$ is the maximal momentum transfer in the $B \to D$ transition, corresponding to the point $v = v'$. Comparing to (2.50) we find

$$f_+(q^2_{\rm max}) = 1 + \mathcal{O}\left[\left(\frac{1}{2m_b} - \frac{1}{2m_c}\right)^2\right] \tag{2.53}$$

Note that the statement on the corrections only holds for the form factors which are normalized due to the symmetry; we also have $f_-(q^2) = 0$ from HQS, however, including corrections this means

$$f_-(q^2_{\rm max}) = \mathcal{O}\left[\frac{1}{2m_b} - \frac{1}{2m_c}\right] \tag{2.54}$$

2.2.4 Reparametrization Invariance

Finally, there is another symmetry in HQET called Reparametrization Invariance (RPI). It originates from the fact that our starting point was full QCD which is a Lorentz invariant theory. Clearly, when introducing the velocity vector v we explicitly break Lorentz invariance by fixing a time like direction which, to some extent is arbitrary and could be also varied sightly. To this end, a HQET constructed with v and a HQET constructed with $v' = v + \delta v$ should give the same physical results [25–28].

In order to study the consequences of this simple fact, we write down the variation $\delta_{\rm RPI}$ of the relevant quantities under a small change in the velocity

$$\begin{aligned}
&v \to v + \delta v\ , \qquad (v+\delta v)^2 = 1 \text{ and thus } v\cdot\delta v = 0\ ,\\
&h_v \to h_v + \frac{\delta\!\!\!/ v}{2}\left(1 + P_-\frac{1}{2m_Q + ivD}i\not{D}\right)h_v\ ,\\
&iD \to iD - m_Q\,\delta v\ .
\end{aligned} \tag{2.55}$$

In particular the last relation, which originates from the splitting of the heavy-quark momentum, leads to the observation that the transformation (2.55) relates different orders in the $1/m_Q$ expansion.

This can be easily illustrated using the Lagrangian as an example. We start from the expression (2.13) for the action of the heavy quark after integrating out the the small-component field H_v. This (non-local) expression is invariant under (2.55). Expanding (2.13) in local operators according to (2.17) shows that (2.55) actually relates terms of subsequent orders such that

$$\delta_{\rm RPI}\mathcal{L}_{1/{\rm m_Q}-{\rm Expansion}} = \mathcal{O}(1/m_Q^{N+2}) \tag{2.56}$$

since (2.17) includes all terms up to and including terms of order $1/m_Q^{N+1}$. Looking at the leading term we find

$$\delta_{\rm RPI}\bar{h}_v(ivD)h_v = \bar{h}_v(i\,\delta v\,D)h_v$$

which is exactly cancelled by the variation of the first subleading term

$$\delta_{\rm RPI}\frac{1}{2m_Q}\bar{h}_v(i\not{D}^\perp)^2h_v = -\bar{h}_v(i\,\delta v\,D)h_v$$

and as a consequence we have[4]

$$\delta_{\rm RPI}\left(\bar{h}_v(ivD)h_v + \frac{1}{2m_Q}\bar{h}_v(iD^\perp)^2h_v\right) = \mathcal{O}(1/m_Q^2)$$

These relations are all tree level relations; however, RPI has to hold also including QCD corrections in HQET, which means the the relations derived from RPI should hold to all order in α_s. For the Lagangain this means that one may derive relations between the renormalization constants of the operators appearing in (2.17) which are true to any oder in α_s. In particular it means for renormalization constants of the first few terms

$$\begin{aligned} &Z_h\bar{h}_v(ivD)h_v + (Z_hc_1)\frac{1}{2m_Q}\bar{h}_v(iD^\perp)^2h_v \\ &\quad = Z_h\left(\bar{h}_v(ivD)h_v + c_1\frac{1}{2m_Q}\bar{h}_v(iD^\perp)^2h_v\right) \end{aligned} \tag{2.57}$$

where Z_h is the renormalization constant of the the static heavy-quark field. Thus RPI fixes the renormalization constant of the kinetic energy term to be $c_1 \equiv 1$.

2.3 HQET at one loop

Up to now, all discussions refer to the tree-level expressions. We have set up an expansion in $\Lambda_{\rm QCD}/m_Q$ which is, however, only one of the small parameters we can expand in. To become a useful tool, also the perturbative QCD corrections have to be taken into account.

The strong coupling constant taken at the scale $\mu \geq 1$ GeV $\alpha_s(m_Q)$ constitutes another small parameter which may serve as an expansion parameter. In particular, the heavy quark-mass scale $\mu = m_Q$ is large enough to warrant a perturbative expansion. This has the advantage, that many contributions can be computed perturbatively, in particular the matching between HQET and full QCD. In the following we discuss the underlying technology and study the one loop diagrams.

The reader who is interested in the technical aspects of perturbative calculations in HQET will find all details in the textbook by Grozin [17], which includes the relevant master integrals even up to two loops.

2.3.1 The Feynman rules of HQET

I assume that the reader is to some extend familiar with the Feynman rules of QCD, including the discussion of gauge fixing, so I will not repeat here the standard technology of calculations within QCD.

[4]The antisymmetric combination $(i\sigma^{\mu\nu})(iD_\mu^\perp)(iD_\nu^\perp)$ is reparametrization invariant.

However, to compute within HQET, we need to set up the Feynman rules of HQET. These are derived from the HQET Lagrangian (2.18)

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v i(v \cdot D) h_v = \bar{h}_v i(v \cdot \partial) h_v + ig\bar{h}_v i(v \cdot A) h_v \qquad (2.58)$$

The propagator can be read off from the first term, while the heavy quark-gluon coupling is encoded in the second term.

The recipe to obtain the propagator from the first term is to invert the distribution appearing between the two fields according to

$$(v \cdot \partial) P(x) = \delta^4(x) \,. \qquad (2.59)$$

Fourier transforming this relation yields

$$P(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{v \cdot k + i\epsilon} e^{-ikx}$$

where we have already fixed the boundary conditions by adding a small imaginary part $i\epsilon$, ensuring that particles propagate into forward time direction. The interpretation of this propagator becomes evident by performing the k integration in the rest frame $v = (1, 0, 0, 0)$; we get

$$P(x) = \theta(x_0)\delta^3(\vec{x}) \qquad (2.60)$$

which is the propagator of a static quark sitting at the origin. In order to insert this into a general Feynman diagram we still have to multiply this by the projector P_+ defined in (2.5).

The second term in (2.58) yields the coupling of a static quark to the gluon field. The resulting Feynman rule has the same form as the usual one for the quark-gluon coupling, with the matrix γ_μ replaced by v_μ, which reflects the heavy quark spin symmetry,

Fig. 2.1 shows the two resulting additional Feynman rules; here k denotes the residual momentum of the heavy quark moving with velocity v.

2.3.2 One loop diagrams 1: Quark Self Energy

We are now ready to compute Feynman diagrams. As in full QCD there is a set of divergent diagrams, and the handling of these divergencies requires renormalization. We shall discuss this here for a few examples at the one-loop level.

We start with a sample calculation of the self energy; fig. 2.2 (a) is the self energy in full QCD, while fig. 2.2 (b) shows the corresponding diagram in HQET. The expression in full QCD (fig. 2.2 (a)) is well known and reads

$$\Sigma_{\mathrm{QCD}}(p) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^Dl}{(2\pi)^D} \frac{1}{(l^2 + i\epsilon)} \frac{\gamma_\mu(\not{p} + \not{l} + m_Q)\gamma^\mu}{(p+l)^2 - m_Q^2 + i\epsilon} \,. \qquad (2.61)$$

Making use of the Feynman rules we get the expression corresponding to diagram (b)

$$\Sigma(v \cdot k) = -ig^2 T^a T^a \mu^{4-D} \int \frac{d^Dl}{(2\pi)^D} \frac{1}{(l^2 + i\epsilon)(v \cdot k + v \cdot l + i\epsilon)} P_+ \,, \qquad (2.62)$$

where we have anticipated a divergence in $D = 4$ and regularize the diagrams by dimensional regularization. As usual, the factor μ^{4-D} is introduced to keep the dimension of Σ fixed as D varies.

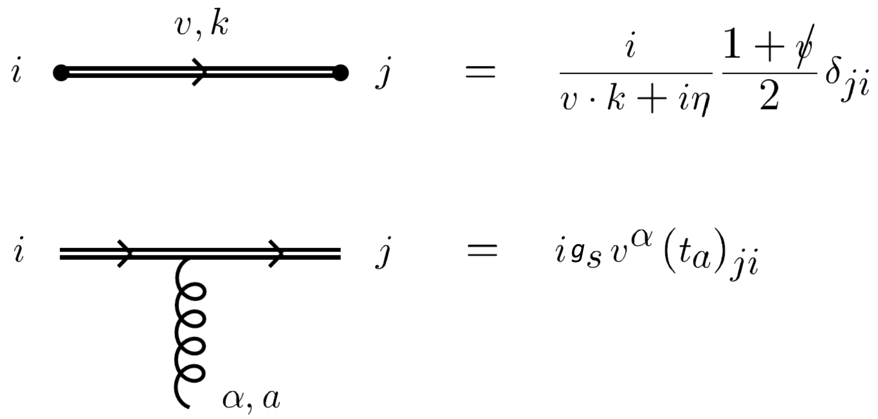


Fig. 2.1 Feynman rules of HQET. All other elements are the same as in full QCD. i and j are color indices, k is the residual momentum of the heavy quark moving with the velocity v and $g_S = g$ is the strong coupling constant.

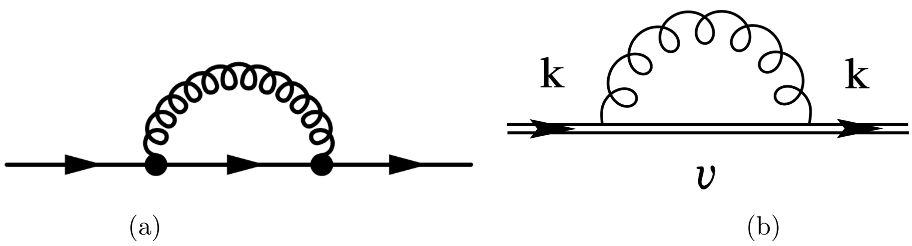


Fig. 2.2 One-loop self energy diagram of a light and a heavy quark

.

In order to evaluate (2.62), we quote a useful relation which we shall use to combine denominators of propagators

$$\frac{1}{A^n B^m} = 2^m \frac{\Gamma(m+n)}{\Gamma(n)\Gamma(m)} \int_0^\infty d\lambda \frac{\lambda^{m-1}}{(A + 2\lambda B)^{m+n}} \tag{2.63}$$

where this relation also holds for non-integer m and n. Using this we can combine the denominators in (2.62) into

$$\Sigma(v \cdot k) = -2ig^2 \mu^{4-D} C_F \int_0^\infty d\lambda \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + 2\lambda[v \cdot k + v \cdot l] + i\epsilon)^2} \tag{2.64}$$

where we inserted $T^a T^a = C_F \mathbf{I}$ where $C_F = 4/3$. In order to apply the one-loop master formula of dimensional regularization

$$\int \frac{d^Dl}{(2\pi)^D} \frac{(l^2)^\alpha}{(l^2 - M^2)^\beta} = (-1)^{\alpha+\beta} \frac{i}{2^D \pi^{D/2}} (M^2)^{\alpha-\beta+D/2} \frac{\Gamma(\alpha + D/2)\Gamma(\beta - \alpha - D/2)}{\Gamma(D/2)\Gamma(\beta)} \tag{2.65}$$

we need to shift the integration variable $l \to l - \lambda v$ which removes the term linear in l in the denominator, leaving us with

$$\Sigma(v \cdot k) = -ig^2 \mu^{2\varepsilon} \frac{8}{3} \int_0^\infty d\lambda \int \frac{d^Dl}{(2\pi)^D} \frac{1}{(l^2 - \lambda^2 + \lambda v \cdot k + i\epsilon)^2}\,, \tag{2.66}$$

where we have defined $D = 4-2\varepsilon$. Performing the integration over the loop momentum with the help of (2.65) we find

$$\Sigma(v \cdot k) = C_F \frac{\alpha_s}{2\pi} \Gamma(\varepsilon) \int_0^\infty d\lambda \left(\frac{4\pi\mu^2}{\lambda^2 - 2\lambda v \cdot k} \right)^\varepsilon \tag{2.67}$$

For the renormalization, we are interested in the divergence as $D \to 4$ (or $\varepsilon \to 0$), which manifests itself as a simple pole

$$\Sigma(v \cdot k) = C_F \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} (v \cdot k) + \text{finite Terms}\,. \tag{2.68}$$

Renormalization proceeds in the usal way. We insert the self energy into the heavy-quark propagator and get

$$\begin{aligned} S^{(1)}_{\text{HQET}}(v \cdot k) &= \frac{i}{(v \cdot k)} + \frac{i}{(v \cdot k)} (-i\Sigma(v \cdot k)) \frac{i}{(v \cdot k)} + \cdots \\ &= \left(1 + C_F \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} \right) \frac{i}{(v \cdot k)} + \cdots \end{aligned} \tag{2.69}$$

from which we can read off the wave function renormalization constant of HQET (in the $\overline{MS}$ scheme)

$$Z_{\text{HQET}} = 1 + C_F \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} \tag{2.70}$$

This can be compared to the result in full QCD. A similar calculation yields for the wave function renormalization of the quark field in full QCD

$$Z_Q = 1 - C_F \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \tag{2.71}$$

We note that the two renormalization constants are different, which is not a surprise, since the UV behaviour of the two theories is different. In fact, the divergencies in HQET are related to logarithmic mass dependencies of full QCD, which become divergent as $m_Q \to \infty$. For our case this can be made explicit by looking at the result in full QCD obtained form (2.61), which reads (for $p^2 < m^2$)

$$\Sigma(p) = \not{p} \left[-\frac{\alpha_s}{4\pi} C_F \frac{1}{\bar{\varepsilon}} - \frac{\alpha_s}{4\pi} C_F \frac{(m_Q^2 + p^2)(m_Q^2 - p^2)}{(p^2)^2} \ln \left(\frac{m^2 - p^2}{m^2} \right) \right] + \cdots \tag{2.72}$$

where we have used

$$\frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon} - \gamma + \ln 4\pi$$

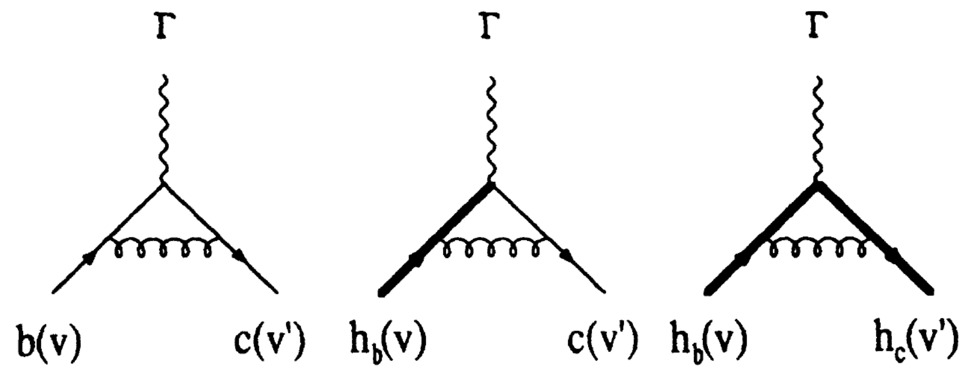
, and the ellipses denote non-logarithmic terms and contributions proportional to m_Q. Inserting $p = mv + k$ and keeping the leading terms only yields

$$\Sigma(v) = \not{v}\left[-\frac{\alpha_s}{4\pi}m_Q C_F \frac{1}{\bar{\varepsilon}} + \frac{\alpha_s}{2\pi}C_F(v\cdot k)\ln\left(\frac{m_Q}{v\cdot k}\right)\right] + \cdots . \tag{2.73}$$

The first term is taken care of by the renormalization of full QCD, i.e. the pole term defines the field renormalization of the full quark field. However, as can be seen in (2.73) the finite term in (2.72) develops in the heavy quark limit a logarithmic divergence, the prefactor of which defines the renormalization constant of the static heavy quark field.

2.3.3 One loop diagrams 2: The $b \to c$ Current

As the next example we will discuss the $b \to c$ vector current and consider the QCD radiative corrections at one loop. Starting at a high scale above the b quark mass, we compute the one-loop diagrams shown in diagram (a) of fig. 2.3, together with the corresponding diagrams with self-energy insertions in the external legs. Adding the three contributions and including the proper renormalization, the one-loop result is UV finite, in other words, the current $\bar{b}\gamma_\mu c$ does not have an anomalous dimension, This actually is true at all orders, since this current is conserved in the limit of vanishing masses.



(a) (b) (c)

Fig. 2.3 Feynman diagrams for the $b \to c$ vertex corrections in full QCD (a), in the theory with a static b quark (b) and in a theory with static b and c quarks (c). Thick lines denote quarks in HQET.

Since the anomalous dimension of this current vanishes, we will not encounter any large logarithms of the form $(\alpha_s/\pi)\ln(M_W^2/m_b^2)$ when we run down to the bottom mass scale. At $\mu = m_b$ we have to match the vector current $V_\mu^{(b\to c)}$ of full QCD to operators in a theory where we use HQET for the b quark. We have schematically

$$V_\alpha^{(b\to c)} = \sum_i C_i^{(0)}(\mu) J_{i,\alpha}^{(b\to c)} + \frac{1}{2m_b} \sum_k C_k^{(1)}(\mu) O_{k,\alpha}^{(b\to c)} + \cdots \tag{2.74}$$

where $J_{i,\mu}^{(b\to c)}$ and $O_{k,\mu}^{(b\to c)}$ are local operators and $C_i^{(0)}$ and $C_k^{(1)}(\mu)$ are (computable) Wilson coefficients, and the ellipses denote even higher orders in the $1/m_b$ expansion. In our example we will consider only the leading term which is expressed in terms of two operators

$$J_{1,\mu}^{(b\to c)} = \bar{c}\gamma_\mu h_v \tag{2.75}$$

$$J_{2,\mu}^{(b\to c)} = \bar{c} h_v\, v_\mu \tag{2.76}$$

The relations (2.74) are operator relations, and in order to compute the matching we may use any states we prefer. For the case at hand we want to compute the perturbative corrections, and thus it is convenient to use on shell states for the b and c quarks. Furthermore, since we are interested in scales well above the charm-quark mass, we compute with a massless charm quark.

Computing the one-loop diagrams in full QCD shown in diagram (a) of fig. 2.3 (together with the diagrams with self-energy insertions in the external legs) and expanding in the result in $1/m_b$ yields the result

$$\langle V_\mu^{(b\to c)} \rangle = \left(1 + \frac{\alpha_s}{2\pi}\left[\ln\frac{m_b^2}{\lambda^2} - \frac{11}{6}\right]\right)\gamma_\mu + \frac{2\alpha_s}{3\pi} v_\mu\,. \tag{2.77}$$

As stated above, the result is UV finite, but we had to introduce an infrared regulator λ which is e.g. a small gluon mass. This is due to the fact, that we are using on-shell "free" quark states in the calculation; if we could compute the matrix element with hadronic states, these IR singularities would be absent.

The next step in the matching procedure is to compute the corresponding diagrams in an HQET where the b quark is replaced by a static quark. Computing the one-loop contribution shown in diagram (b) of fig. 2.3 (together with the diagrams with self-energy insertions in the external legs) and performing the proper renormalization of the heavy and light quark fields we obtain (using again a small gluon mass to regulate the infrared divergence of the amplitudes)

$$\langle J_{1,\mu}^{(b\to c)} \rangle = \left(1 + \frac{\alpha_s}{2\pi}\left[\frac{1}{\bar{\varepsilon}} + \ln\frac{\mu^2}{\lambda^2} - \frac{5}{6}\right]\right)\gamma_\mu\,. \tag{2.78}$$

Note that this result is UV divergent which is to be expected, since we changed the high-energy behaviour of the theory by switching to a static b quark. Consequently we need to renormalize the current operator, in the $\overline{MS}$ scheme this just amounts to removing the $1/\bar{\varepsilon}$ pole.

We can now read off the coefficients in (2.74) by taking the corresponding matrix elements of (2.74), we obtain

$$C_1^{(0)}(\mu) = 1 + \frac{\alpha_s}{2\pi}\left[\ln\frac{m_b^2}{\mu^2} - \frac{8}{3}\right] \tag{2.79}$$

$$C_2^{(0)}(\mu) = \frac{2\alpha_s}{3\pi} \tag{2.80}$$

Note that the IR regulator λ has dropped out which is a general feature of a matching calculation. It is due to the fact that the IR behaviour in the full and the effective theory have to be the same, once the effective theory is properly constructed.

The UV divergence in the effective theory is related to the mass dependence in the full theory, which can be seen by comparing (2.77) to (2.78), since the prefactor of the $\ln m_b^2$ term in (2.77) is the same as the coefficient in front of the $1/\bar{\varepsilon}$ pole in (2.78). This fact allows us to make use of renormalization-group (RG) methods in order to re-sum logarithms of the mass m_b. However, these logarithms will be of the form $\ln(m_b^2/m_c^2)$ once we scale down to the charm-quark mass m_c, and the RG methods will allow us perform a resummation of terms of order $(\alpha_s/\pi)^n \ln^n(m_b^2/m_c^2)$, which makes sense as soon as the log is so large that it overwhelms the α_s supression. We shall assume this as we go on, although the constant term $-3/8$ in (2.79) is numerically comparable to the term with the logarithm.

In order to obtain the RG equation, we note that the left hand side of (2.74) is independent of the scale μ . The μ dependence on the right-hand side originates from the fact that we decided to shift the contributions of scales between m_b and $\mu \leq m_b$ into the Wilson coefficient, while the pieces from scales below μ are still contained in the matrix element of the operator $J_{1,\mu}^{(b\to c)}$. This observation leads to

$$0 = \mu\frac{d}{d\mu}\langle V_\alpha^{(b\to c)}\rangle = \left(\mu\frac{d}{d\mu}C_1^{(0)}(\mu)\right)\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu + C_1^{(0)}(\mu)\left(\mu\frac{d}{d\mu}\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu\right) \tag{2.81}$$

The logarithmic derivative acting on the matrix element of the current is the anomalous dimension, which is computed from the divergence occurring in (2.78). In our one-loop case the anomalous dimension is given by

$$\left(\mu\frac{d}{d\mu}\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu\right) = -\gamma_{h_b\to c}\,\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu = \frac{\alpha_s}{\pi}\,\langle J_{1,\alpha}^{(b\to c)}\rangle_\mu \tag{2.82}$$

This translates into an evolution equation for the coeffiient $C_1^{(0)}(\mu)$

$$\left(\mu\frac{d}{d\mu} - \gamma_{h_b\to c}\right)C_1^{(0)}(\mu) = 0\,. \tag{2.83}$$

The coefficient $C_1^{(0)}(\mu)$ is evaluated as a power series in α_s, and hence the μ dependence has actually two sources: Aside form the explicit dependence (see (2.79)) there is also the μ dependence of α_s. To make this explicit, we write

$$C_1^{(0)}(\mu) = C_1^{(0)}(\alpha_s,\mu)\,,$$

and write the total derivative as

$$\mu \frac{d}{d\mu} C_1^{(0)}(\mu) = \left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) C_1^{(0)}(\alpha_s, \mu)$$

where we have introduced the QCD β function as

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s(\mu)) = -2\alpha_s(\mu) \frac{\alpha_s(\mu)}{4\pi} \left(11 - \frac{2}{3} n_f \right) + \cdots , \tag{2.84}$$

where n_f is the number of active flavours. Inserting all this yields

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \gamma_{h_b \to c} \right) C_1^{(0)}(\alpha_s, \mu) = 0 . \tag{2.85}$$

The general solution of this equation is given by

$$C_1^{(0)}(\alpha_s(\mu), \mu) = \exp \left(- \int\limits_{\alpha_s(\mu)}^{\alpha_s(m_b)} \frac{\gamma_{h_b \to c}(a)}{\beta(a)} da \right) C_1^{(0)}(\alpha_s(m_b), m_b) , \tag{2.86}$$

which gives the coefficient $C_1^{(0)}(\alpha_s(\mu), \mu)$ in terms of the initial value $C_1^{(0)}(\alpha_s(m_b), m_b)$ obtained form the matching calculation.

In order to obtain the leading log result, we insert the one-loop results for $\gamma_{h_b \to c}$ and β and use the tree-level value $C_1^{(0)}(\alpha_s(m_b), m_b) = 1$ for the matching coefficient, which yields finally

$$C_1^{(0)}(\alpha_s(\mu), \mu) = \left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{-6/25} . \tag{2.87}$$

Expanding this result in $\alpha_s(\mu)$ using the one-loop result for the running coupling reproduces the logarithmic term in (2.74).

Scaling further down we eventually arrive at the charm-quark mass m_c. Assuming that we can also treat the charm quark as a heavy quark, we may again replace the charm quark by a static quark. This allows us to scale further down to scales below m_c, however, at some point we arrive at $\mu = \Lambda_{\text{QCD}}$ where we cannot compute perturbatively any more.

We shall again look at the vector current, however, since we now compute in HQET only, due to spin symmetry the results will also hold for other currents. At one-loop, we need to compute diagram (c) of fig. 2.3 and match it to the result obtained in the theory where only the b quark is taken to be static.

The diagram (c) of fig. 2.3 contains also an UV divergence, which is related to the logarithmic m_c dependence of diargam (b), if we had included the charm mass in the calculation. Thus we need to include a renormalization of the heavy-to-heavy current, which due to this renormalization has an anomalous dimension, for which we find at one loop

$$\gamma_{h_b \to h_c}(v \cdot v') = \frac{4\alpha_s}{3\pi} [(v \cdot v') r(v \cdot v') - 1] \tag{2.88}$$

with

$$r(x) = \frac{1}{\sqrt{x^2-1}} \ln\left(x + \sqrt{x^2-1}\right) .$$

This result is remarkable, since usually anomalous dimensions do not depend on kinematic variables. However, the velocities in HQET are external variables and thus this is not a problem. We also note that at $v = v'$ the anomalous dimension vanishes, which is necessary, since this current is a generator of HQS at this kinematic point and thus cannot have an anomalous dimension.

The running below m_c is governed by the RGE

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \gamma_{h_b \to h_c}(v \cdot v')\right) \tilde{C}^{(0)}(\alpha_s, \mu) = 0 \,, \tag{2.89}$$

where the number of active flavours is now 3. In the effective theory with both b and c as heavy quarks the matrix element of the current is (up to trivial factors) the Isgur-Wise function, and thus we can write our result as a renormalization of this function

$$\xi(v \cdot v') = \zeta(v \cdot v', m_b, m_c, \mu) \xi_0(v \cdot v', \mu) \tag{2.90}$$

where $\xi_0(v \cdot v', \mu)$ is the "bare" Isgur-Wise function and

$$\zeta(v \cdot v', m_b, m_c, \mu) = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-6/25} \left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{(8/27)[(v \cdot v') r(v \cdot v') - 1]} \tag{2.91}$$

where the first factor originates from the running from m_b to m_c, while the second one comes from the running from m_c to some small scale μ. This result has been derived first in [33].

In full QCD, the amplitude for a $b \to c$ transition via the $\bar{c}\gamma_\mu b$ current can be analytically continued to values of $q^2 \geq (m_B + m_D)^2$ which correspond to a creation of a B and a D meson by the current. In terms of the velocities this is the region where $v \cdot v' \leq -1$, in which case the anomalous dimension (2.88) picks up an imaginary part [34]. At the first look this is puzzling, however, it is related to the coulombic phases which appear once the two particles are both in the final state and can rescatter through soft gluons.

2.4 Applications to Phenomenology

Finally we discuss a few phenomenological results obtained from HQET. The most prominent result is the fact that due to HQS all transitions between ground-state heavy mesons mediated by a bilinear quark current are given in terms of a single form factor, the Isgur Wise function introduced in (2.43). Assuming that both b and c quarks are heavy, we can consider the decays $B \to D\ell\bar{\nu}$ and $B \to D^*\ell\bar{\nu}$ for which the hadronic matrix element of the current $\bar{c}\gamma_\mu(1-\gamma_5)b$ is exactly of the form of (2.43).

For heavy quarks it is convenient to use the four velocities of the hadrons v and v' as kinematic variables, which are at leading order the same as the velocities of the heavy quarks. The general parametrization of the matrix elements requires a total of six form factor which can be defined as

$$\langle D(v')|\bar{c}\gamma_\mu b|B(v)\rangle = \sqrt{m_B m_D}\left[\xi_+(y)(v_\mu + v'_\mu) + \xi_-(y)(v_\mu - v'_\mu)\right] \ , \tag{2.92}$$

$$\langle D^*(v',\epsilon)|\bar{c}\gamma_\mu b|B(v)\rangle = i\sqrt{m_B m_{D^*}}\xi_V(y)\varepsilon_{\mu\alpha\beta\rho}\epsilon^{*\alpha}v'^\beta v^\rho \ , \tag{2.93}$$

$$\begin{aligned}\langle D^*(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B(v)\rangle = \sqrt{m_B m_{D^*}}&\left[\xi_{A1}(y)(vv'+1)\epsilon^*_\mu - \xi_{A2}(y)(\epsilon^* v)v_\mu\right.\\ &\left.-\xi_{A2}(y)(\epsilon^* v)v'_\mu\right] \ ,\end{aligned} \tag{2.94}$$

where ϵ is the polarization of the charmed vector meson and $y = v \cdot v'$. Applying now (2.43) to (2.92 - 2.94) we find five relations among the form factors ξ_i

$$\xi_i(y) = \xi(y) \quad \text{for } i = +, V, A1, A3 \ , \qquad \xi_i(y) = 0 \quad \text{for } i = -, A2 \ . \tag{2.95}$$

which eventually reduces the number of independent form factors to only one.

In addition, we may make use of Lukes Theorem derived in section 2.2.3 which yields a statement about the size of the corrections; one finds

$$\begin{aligned}\xi_i(1) &= 1 + \mathcal{O}\left(\left[\frac{1}{2m_c} - \frac{1}{2m_b}\right]^2\right) \qquad \text{for } i = +, V, A1, A3 \ ,\\ \xi_i(1) &= \mathcal{O}\left(\frac{1}{2m_c} - \frac{1}{2m_b}\right) \qquad \text{for } i = -, A2 \ .\end{aligned} \tag{2.96}$$

This has interesting phenomenological applications. Computing the rates for the exclusive decays $B \to D\ell\bar{\nu}$ and $B \to D^*\ell\bar{\nu}$ in terms of the form factors ξ_i we get

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D\ell\nu_\ell) &= \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B + m_D)^2\left(m_D\sqrt{y^2-1}\right)^3\\ &\times\left|\xi_+(y) - \frac{m_B - m_D}{m_B + m_D}\xi_-(y)\right|^2\end{aligned} \tag{2.97}$$

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D^*\ell\nu_\ell) &= \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B - m_{D^*})^2 m_{D^*}^2\left(m_{D^*}\sqrt{y^2-1}\right)\\ &\times(y+1)^2|\xi_{A1}(y)|^2\sum_{i=0,\pm}|H_i(y)|^2\end{aligned} \tag{2.98}$$

with the squared helicity amplitudes

$$|H_\pm(y)|^2 = \frac{m_B^2 - m_{D^*}^2 - 2y m_B m_{D^*}}{(m_B - m_{D^*})^2}\left[1 \mp \sqrt{\frac{y-1}{y+1}}R_1(y)\right]^2 \ , \tag{2.99}$$

$$|H_0(y)|^2 = \left(1 + \frac{m_B(y-1)}{m_B - m_{D^*}}\left[1 - R_2(y)\right]\right)^2 \ . \tag{2.100}$$

Here we have defined the form factor ratios

$$R_1(y) = \frac{\xi_V(y)}{\xi_{A1}(y)} \ , \quad R_2(y) = \frac{\xi_{A3}(y) + \frac{m_B}{m_{D^*}}\xi_{A2}(y)}{\xi_{A1}(y)} \ . \tag{2.101}$$

These expression collaps in the limit $m_b, m_c \to \infty$ into

$$\frac{d\Gamma}{dy}(B \to D\ell\nu_\ell) \to \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B + m_D)^2 \left(m_D\sqrt{y^2-1}\right)^3 |\xi(y)|^2 \ , \tag{2.102}$$

$$\begin{aligned}\frac{d\Gamma}{dy}(B \to D^*\ell\nu_\ell) \to & \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B - m_{D^*})^2 m_{D^*}^2 \left(m_{D^*}\sqrt{y^2-1}\right)(y+1)^2 \\ & \times \left[1 + \frac{4y}{y+1}\frac{m_B^2 - m_{D^*}^2 - 2ym_Bm_{D^*}}{(m_B - m_{D^*})^2}\right] |\xi(y)|^2 \ . \end{aligned} \tag{2.103}$$

The impact of these relations is that the absolute normalization of the form factor is given by HQS, and hence a model independent extraction of the CKM matrix element V_{cb} becomes possible by extrapolating the measured differential rates to the kinematic point $y = 1$.

For the decay $B \to D^*\ell\bar{\nu}$ we find

$$\lim_{y\to 1}\frac{1}{\sqrt{y^2-1}}\frac{d\Gamma}{dy}(B \to D^*\ell\nu_\ell) = \frac{G_F^2}{4\pi^3}(m_B - m_{D^*})^2 m_{D^*}^3 |V_{cb}|^2|\xi_{A1}(1)|^2 \ . \tag{2.104}$$

where the form factor $\xi_{A1}(1) = \xi(1) = 1$ is normalized by HQS. In fact, ξ_{A1} is also protected against linear corrections in $1/m_c$ due to (2.96), and hence one expects a determination of V_{cb} from (2.104) with an uncertainty of about ten percent.

One can also use the process $B \to D\ell\bar{\nu}$ for a determination of V_{cb}, however, there is an additional factor of y^2-1 which makes the extrapolation more difficult, furthermore, due to the presence of $\xi_-(1) \sim 1/m_c$ we expect this to be not as precise as for $B \to D^*\ell\bar{\nu}$.

The state of the art is by now far more advanced. First of all, QCD corrections have been computed or both the vector and the axial vector current [35]

$$\langle c(v)|\bar{c}\gamma^\mu b|b(v)\rangle = 1 + \frac{2\alpha_s}{3\pi}\left[\frac{3m_b^2 + 2m_cm_b + 3m_c^2}{2(m_b^2 - m_c^2)}\ln\left(\frac{m_b}{m_c}\right) - 2\right] \tag{2.105}$$

$$\langle c(v)|\bar{c}\gamma^\mu\gamma_5 b|b(v)\rangle = 1 - \frac{\alpha_s}{\pi}\left[\frac{m_b + m_c}{m_b - m_c}\ln\left(\frac{m_c}{m_b}\right) + \frac{8}{3}\right] \ . \tag{2.106}$$

Numerically (including also the known α_s^2 corrections) one finds [36]:

$$\langle c(v)|\bar{c}\gamma^\mu b|b(v)\rangle = \eta_V = 1.022 \pm 0.004 \tag{2.107}$$

$$\langle c(v)|\bar{c}\gamma^\mu\gamma_5 b|b(v)\rangle = \eta_A = 0.960 \pm 0.007 \tag{2.108}$$

Furthermore, QED corrections have been compute as well and amount to an enhancement of the rates by a factor $\eta_{\mathrm{ew}} = 1.007$. In addition the recoil corrections have been estimated by using QCD sum rules which indicate a furhter decrease of the matrix element of the axial current by another 10%.

More recently, lattice calculations of the form factors have become available at the non-recoil point as well as for $y \neq 1$, even for finite values of the quark masses. All this yields a quite consistent picture giving us a quite reliable value for V_{cb}, a recent analysis [37] yields

$$|V_{cb}| = \left(41.9^{+2.0}_{-1.9}\right) \times 10^{-3} \, . \tag{2.109}$$

3
Heavy Quark Expansion

In inclusive processes one often make use of the so-called Operator Product Expansion (OPE) which is a standard tool in quantum field theory. In fact the OPE lies at the heart of the EFT approach, since it is actually this tool which allows us to separate scales.

The most prominent example is deep inelastic scattering $e + p \to e' + X$ (DIS) which is an inclusive process governed by a large scale set by the momentum transfer Q^2 of the electron. Clearly the amplitudes will contain pieces related to this large scale Q^2 which we expect to be computable in perturbation theory, since $\alpha_s(Q^2)$ is small. The nonperturbative parts are eventually the parton distributions of the quarks inside the proton, which are determined by the binding effects of the quarks inside the proton. The expansion which is set up in this case in powers of $\Lambda^2_{\mathrm{QCD}}/Q^2$ which is very small, such that usually only the leading term is considered.

In the case at hand we shall proceed along the same lines as in DIS. The expansion, the Heavy Quark Expansion (HQE) will in this case be in powers of $\Lambda_{\mathrm{QCD}}/m_Q$; however, in our case we will also take subleading terms into account.

3.1 Inclusive Decays

Inclusive decays are all processes where a summation over final states is performed. In case we sum over all possible decay channels, we obtain the "most inclusive" quantity, which is the total decay width of a particle.

However, there are cases where we can single out certain final states. In particular, in weak decays we may also produce leptons, and we may want to sum over the final-state hadrons

$$\Gamma(B \to X\ell\bar{\nu}) = \sum_f \Gamma(B \to f\ell\bar{\nu})$$

Likewise, we may also consider radiative processes, where photons are emitted.

In both cases we are interested in kinematic distributions such as the energy spectrum of the charged lepton or the photon or the invariant mass spectrum of the leptons

$$\frac{d\Gamma(B \to X\ell\bar{\nu})}{dE_\ell} = \sum_f \frac{d\Gamma(B \to f\ell\bar{\nu})}{dE_\ell}$$

In the chapter we will discuss, how to set up an expansion in inverse powers of the b quark mass for inclusive processes.

3.2 Operator Product Expansion (OPE)

We start with the total decay rate of a heavy hadron $H(p_H)$. Assuming that H is a ground-state hadron, it can only decay by a weak decay, which is mediatied by an effective hamiltonian density $\mathcal{H}_{eff}(x)$. To leading order in the weak interaction we obtain - up to trivial factors - for the total rate[1]

$$\Gamma \propto \sum_X (2\pi)^4 \delta^4(p_H - p_X) |\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle|^2 \,, \tag{3.1}$$

where X is the final state with momentum p_X, and we sum over all final states taking into account four-momentum conservation. We use the relation

$$\mathcal{H}_{eff}(x) = e^{-i\hat{P}x} \mathcal{H}_{eff}(0) e^{i\hat{P}x} \,,$$

where $\hat{P}_\mu$ is the (four) momentum operator, and write

$$\begin{aligned}
&\sum_X (2\pi)^4 \delta^4(p_H - p_X) |\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle|^2 \\
&= \sum_X \int d^4y \, \exp\left(i(p_H - p_X)y\right) \langle H(p_H)|\mathcal{H}_{eff}(0)|X\rangle\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
&= \sum_X \int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)|X\rangle\langle X|\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
&= \int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)|H(p_H)\rangle \,,
\end{aligned} \tag{3.2}$$

where in the final step we made use of the fact, that

$$\sum_X |X\rangle \langle X| = \mathbf{1}$$

is the unit operator, since we sum over all states in the Hilbert space.

Finally we may use the optical theorem to relate the matrix element of the product of the Hamiltonian to the time-ordered product

$$\begin{aligned}
&\int d^4y \, \langle H(p_H)|\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)|H(p_H)\rangle \\
&\qquad = 2 \, \mathrm{Im} \int d^4y \, \langle H(p_H)|T\{\mathcal{H}_{eff}(y)\mathcal{H}_{eff}(0)\}|H(p_H)\rangle \,.
\end{aligned} \tag{3.3}$$

This relation is the starting point of all further considerations. The matrix element in (3.3) still contains the heavy quark mass m_Q and our goal is to set up an expansion

[1] In the following we often drop the argument of field operators or of other space time dependent operators $\mathcal{O}(x)$, we define $\mathcal{O} \equiv \mathcal{O}(0)$. Likewise we write $\partial_\mu \mathcal{O} \equiv (\partial_\mu \mathcal{O}(x))|_{x=0}$.

in inverse powers of this mass. Since $\mathcal{H}_{eff}$ induces a decay of the heavy quark, we expect it to be of the form

$$\mathcal{H}_{eff} = \bar{Q}R + \text{h.c.}\,, \tag{3.4}$$

where R consists of light(er) quarks and possibly gluons. In order to make the dependence on the heavy mass explicit, we use (2.9) and write[2]

$$Q(x) = \exp(-im_Q(v\cdot x))Q_v(x)\,, \quad v = \frac{p_H}{m_H} \tag{3.5}$$

corresponding to the splitting of the heavy-quark momentum into the large part $m_Q v$ and a residual part related to the derivative acting on Q_v. Note that we do not use here the static field introduced above, rather $Q_v(x)$ is still the field of full QCD, up to the above phase redefinition.

With this phase redefinition we get

$$\begin{aligned}\int d^4y\, \langle H(p_H)|T\{\mathcal{H}_{eff}(y)\mathcal{H}^\dagger_{eff}(0)\}|H(p_H)\rangle \\ = \int d^4y\, \exp(im_Q(v\cdot y))\langle H(p_H)|T\{\tilde{\mathcal{H}}_{eff}(y)\tilde{\mathcal{H}}^\dagger_{eff}(0)\}|H(p_H)\rangle\end{aligned} \tag{3.6}$$

where $\tilde{\mathcal{H}}_{eff}$ is obtained from $\mathcal{H}_{eff}$ by the replacement (3.5)

$$\tilde{\mathcal{H}}_{eff} = \bar{Q}_v R + \text{h.c.}\,.$$

Expression (3.6) is the starting point of an Operator Product Expansion (OPE), which is a standard method in quantum field theory (for a textbook presentation see e.g. [38]). Without going into details, the main relation is

$$\int d^4y\, e^{-iqx}\, T[O_1(x)O_2(0)] = \sum_n C_n(q)\mathcal{O}_n(0)\,, \tag{3.7}$$

where O_1 and O_2 are renormalized local operators and $\mathcal{O}_n$ are renormalized local operators which can be ordered by increasing dimension, and $C_n(q)$ are coefficients depending on the momentum transfer q. Note that each term on the right hand side must have the same dimension, so the increasing dimension of the operators will be compensated by inverse powers of q. Thus for sufficiently large momentum transfer q one may truncate the series on the right hand side, and one obtains an approximation scheme in terms of powers of $1/q$.

Applying the OPE in the context of QCD, one may make use of the fact that at large q QCD becomes perturbative. This means in particular that we may compute the coefficients in QCD perturbation theory, while the matrix elements of the operators contain the non-perturbative information. This scheme is at the heart of all applications of QCD-based EFT's and has been used in many different contexts such as weak interactions and in DIS.

[2]We note that $Q_v(0) = Q(0)$; however, once a derivative is acting on the field Q_v it corresponds to the residual momentum $i\partial_\mu Q_v(0) \sim k_\mu$.

Inclusive differential rates can be computed for processes with leptons and/or photons in the final state. These rates are inclusive with respect to the final-state hadrons, but we may consider the kinematic distributions of the final state photons and leptons. To be explicit, let us study a semileptonic transition based on the quark decay $Q \to q + \ell + \bar{\nu}$. The effective Hamiltonian can be written as

$$\mathcal{H}_{eff} = \frac{4G_F V_{\mathrm{CKM}}}{\sqrt{2}} J_\mu L^\mu \tag{3.8}$$

where $J_\mu = \bar{Q}_L \gamma_\mu q_L$ is the left-handed hadronic current and $L_\mu = \bar{\ell}_L \gamma_\mu \nu_L$ is the leptonic current and G_F is the Fermi-coupling constant. Inserting this into (3.2), we get

$$\begin{aligned} &8G_F^2 |V_{\mathrm{CKM}}|^2 \sum_{X,\ell\bar{\nu}} (2\pi)^4 \delta^4(p_H - p_X - k - k') |\langle X \ell \bar{\nu} | J_\mu L^\mu | H(p_H) \rangle|^2 \qquad (3.9) \\ &= 8G_F^2 |V_{\mathrm{CKM}}|^2 \sum_X \widetilde{dk}\widetilde{dk}' (2\pi)^4 \delta^4(p_H - p_X - k - k') |\langle X | J_\mu | H(p_H) \rangle \, \langle \ell(k) \bar{\nu}(k') | L^\mu | 0 \rangle|^2 \,, \end{aligned}$$

where $\widetilde{dk}$ and $\widetilde{dk}'$ denote the phase-space integrations over the leptons. Since the leptons do not have any strong interaction, we can decompose this expression into an hadronic and a leptonic part. We get

$$\begin{aligned} &\sum_X \widetilde{dk}\widetilde{dk}' (2\pi)^4 \delta^4(p_H - p_X - k - k') |\langle X | J_\mu | H(p_H) \rangle \, \langle \ell(k) \bar{\nu}(k') | L^\mu | 0 \rangle|^2 \qquad (3.10) \\ &= \int \frac{d^4 q}{(2\pi)^4} \sum_X (2\pi)^4 \delta^4(p_H - p_X - q) \langle H(p_H) | J_\alpha^\dagger | X \rangle \langle X | J_\beta | H(p_H) \rangle \\ &\qquad \times \int \widetilde{dk}\widetilde{dk}' (2\pi)^4 \delta^4(q - k - k') \langle 0 | L^{\alpha\dagger} | \ell(k) \bar{\nu}(k') \rangle \langle \ell(k) \bar{\nu}(k') | L^\beta | 0 \rangle \,. \end{aligned}$$

The leptonic part can be evaluated separately and is taken usually to lowest order in perturbation theory; the hadronic part is encoded in the hadronic tensor, which can be decomposed into scalar functions W_i, $i = 1, .., 5$

$$\begin{aligned} W^{\alpha\beta}(q) &= \sum_X (2\pi)^4 \delta^4(p_H - p_X - q) \langle H(p_H) | J^{\alpha\dagger} | X \rangle \langle X | J^\beta | H(p_H) \rangle \qquad (3.11) \\ &= -g^{\alpha\beta} W_1 + v^\alpha v^\beta W_2 - i\epsilon^{\alpha\beta\mu\nu} v_\mu q_\nu W_3 + q^\alpha q^\beta W_4 + (v^\alpha q^\beta + v^\beta q^\alpha) W_5 \,, \end{aligned}$$

where we introduced $p_H = m_H v$. These scalar functions depend on the two invariants q^2 and $v \cdot q$; in terms of these we get e.g. for the triply differential rate ($E_\ell = v \cdot k$, $E_\nu = v \cdot k'$)

$$\frac{d\Gamma}{dq^2 \, dE_\ell \, dE_\nu} = \frac{G_F^2 |V_{\mathrm{CKM}}|^2}{2\pi^3} \left[W_1 q^2 + W_2 \left(2E_\ell E_\nu - \frac{1}{2} q^2 \right) + W_3 q^2 (E_\ell - E_\nu) \right] \tag{3.12}$$

where the phase space is restricted by $4E_\ell E_\nu - q^2 \geq 0$.

With the hadronic tensor we can go through the same steps (3.1, ... ,3.3), but we have to insert the phase factor $\exp(-iqy)$ into the y integration:

$$\int d^4y \exp(-iqy) \langle H(p_H)|J^\dagger_\mu(y)J_\nu(0)|H(p_H)\rangle$$
$$= 2\,\mathrm{Im}\int d^4y \exp(-iqy)\langle H(p_H)|T\{J^\dagger_\mu(y)J_\nu(0)\}|H(p_H)\rangle \quad (3.13)$$

Performing the replacement (3.5) we end up with

$$\int d^4y \exp(-iqy)\langle H(p_H)|T\{J^\dagger_\mu(y)J_\nu(0)\}|H(p_H)\rangle = \quad (3.14)$$
$$\int d^4y \exp(iy(m_Q v - q))\langle H(p_H)|T\{\widetilde{J}^\dagger_\mu(y)\widetilde{J}_\nu(0)\}|H(p_H)\rangle\,.$$

The time-ordered product of the two hadronic currents has the same decomposition (3.11) as the hadronic tensor with scalar functions T_i, $i = 1, ..., 5$. These functions have an analytic structure as depicted in fig. 3.1: for a fixed value of q^2 we have $p_H - q = p_X$ where p_X is the momentum of the final hadronic state, thus $m_H^2 + q^2 - 2m_H(v\cdot q) = m_X^2$. Thus the maximal value of $v \cdot q$ is given by

$$(v\cdot q)_{\max} = \frac{1}{2m_H}(m_H^2 + q^2 - m^2_{X\,\min})$$

where $m_{X\,\min}$ is the mass of the lightest hadronic state with the correct quantum numbers. Thus for the states with a q quark in the final state, the T_i exhibit a cut

$$-\infty \le (v\cdot q) \le \frac{1}{2m_H}(m_H^2 + q^2 - m^2_{X\,\min}) \quad (3.15)$$

However, there can also be intermediate states with two Q quarks and a q antiquark, which yield a branch cut

$$\frac{1}{2m_H}(m_H^2 + q^2 - m^2_{X(QQ\bar{c})\,\min}) \le (v\cdot q) \le \infty \quad (3.16)$$

where $m_{X(QQ\bar{c})\,\min}$ denotes the mass of the lightest state with the quark content $QQ\bar{c}$. The relevant W_i are given by the discontinuity T_i of the left hand cut according to (3.13).

To compute a doubly differential rate one needs to integrate over one of the variables, which in the present case is e.g. the neutrino energy. Using (3.13) this integration can be replaced by the contour integration depicted in Fig. 3.1. Note that there is a gap between the two cuts such that the contour does not get close to the singularity, which indicates that a perturbative calculation is possible for sufficiently "smeared" quantities.

Before closing the general set-up I need to point out some subtleties. The proof that an OPE exists can strictly only be performed in the deep euclidean region, i.e. for $q^2 \to -\infty$ in (3.7). However, in all applications in heavy quark physics we are

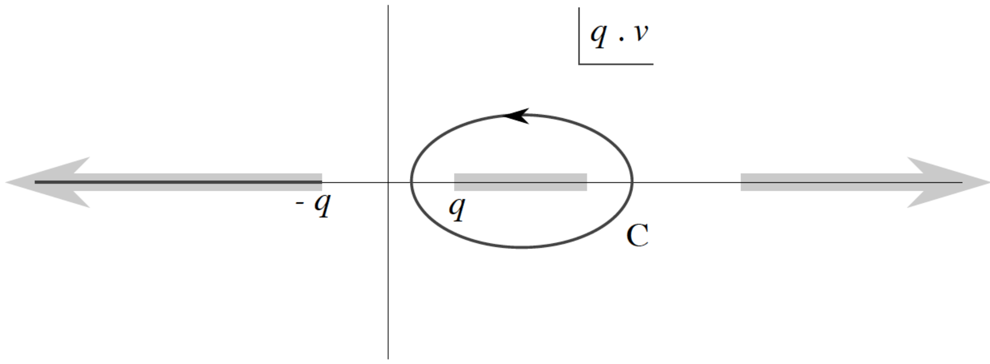


Fig. 3.1 Sketch of the analytic structure of the T_i in the $v \cdot q$ plane for fixed q^2 (Figure taken from [11]).

actually in the minkowskian region; the momentum in (3.6) is $m_Q v$ which is time-like, as well as the momentum $m_Q v - q$ for the differential rate. This innocent looking point of analytically continuing from the euclidean to the minkowskian region is, however, quite subtle; strictly speaking the OPE in the minkowskian region is not proven.

Annother, to some extent related issue is the issue of duality, i.e. the question, to what extend partonic results are “dual” to the real hadronic results. Originally this question has been raised in the context of $e^+e^- \to$ hadrons: How can the differential cross sections obtained from the calculation of $e^+e^- \to$ partons (quarks and gluons) be related to $e^+e^- \to$ hadrons? It has been argued in [39] that such a comparison becomes possible once as suitable “smearing” (i.e. a convolution with smooth weight functions) has been performed.

In the context of heavy-quark physics this has been made more precise in [40], where the question of duality has been connected to the HQE. As we shall see below, the leading term of the HQE for inclusive decays is the parton model, i.e. the decay of a “free” quark. Form a naive notion of quark-hadron duality one would assume this to be a reasonable approxmation, even for suitably “smeared” differential quantities. A more quantitative definition of duality as in [40] links this to the HQE, which means on the one hand the convergence of the expansion itself as well as the absence (or at least smallness) of non-analytic terms in the expansion parameters.

3.3 Tree level Results

To be specific, we start out by constructing the OPE for an inclusive semileptonic $b \to c$ decay. The starting point is the expression (3.14) in the form (now we have $J_\mu = \bar{c}\gamma_\mu(1-\gamma_5)b$)

$$\int d^4y \, \exp(-iqy) \, T\{J_\mu^\dagger(y) J_\nu(0)\} = \int d^4y \, \exp(iy(m_Q v - q)) \, T\{\widetilde{J}_\mu^\dagger(y)\widetilde{J}_\nu(0)\} \quad (3.17)$$

for which we want to perform an OPE according to (3.7)

$$\int d^4y \exp(iy(m_Q v - q))\, T\{\widetilde{J}^\dagger_\mu(y)\widetilde{J}_\nu(0)\} = \sum_n C^{(n)}_{\mu\nu}\mathcal{O}_n\,. \tag{3.18}$$

The key point of the OPE is that it is an operator relation, which means that we can take any matrix element of this relation to compute the coefficients $C^{(n)}_{\mu\nu}$. Hence the simplest way to proceed (having discussed above that we may compute the coefficients in perturbation theory) is to take a matrix element with free quark and gluon states.

We start with a free b quark with momentum $p_b = m_b v + k$ corresponding to the splitting of the quarks momentum into a large part $m_b v$ and a residual part k. The leading tree-level diagram is simply given by the propagator of the free charm quark, leaving us with

$$R^{(0)}_{\mu\nu} = \bar{u}(p_b)\gamma_\mu(1-\gamma_5)\left[\frac{1}{\not{Q}+\not{k}-m_c}\right]\gamma_\nu(1-\gamma_5)u(p_b) \tag{3.19}$$

where we introduce $Q = m_b v - q$. At tree level, the construction of the OPE proceeds by expanding in the residual momentum k which is assumed to be small in all its components. For the propagator we get

$$\frac{1}{\not{Q}+\not{k}-m_c} = \frac{1}{\not{Q}-m_c} - \frac{1}{\not{Q}-m_c}\not{k}\frac{1}{\not{Q}-m_c} + \frac{1}{\not{Q}-m_c}\not{k}\frac{1}{\not{Q}-m_c}\not{k}\frac{1}{\not{Q}-m_c} + \cdots \tag{3.20}$$

Starting with the leading term, we get

$$\begin{aligned} R^{(0,0)}_{\mu\nu} &= \frac{1}{Q^2-m_c^2}\bar{u}(p_b)\gamma_\mu(1-\gamma_5)(\not{Q}+m_c)\gamma_\nu(1-\gamma_5)u(p) \\ &= \frac{2}{Q^2-m_c^2}\bar{u}(p)\gamma_\mu\not{Q}\gamma_\nu(1-\gamma_5)u(p_b) \end{aligned} \tag{3.21}$$

Before we continue, a subtlety should be mentioned. We did not expand the spinors $u(p_b)$ in powers of k, which looks a bit inconsistent on first sight. The matching procedure we employ is to compare the matrix element between fixed states of the right-hand and the left-hand side of (3.18). Consequently, we also would need to compute the matrix element between free quark states with the same momentum $p_b = m_b v + k$ on both sides of the equation. Thus the same spinors $u(p_b)$ will appear on both sides, and thus the expansion of the spinors would cancel. Therefore we can as well drop the expansion of the spinors in both the left and the right hand side.

Integrating over the leptonic phase space, neglecting the lepton mass, one finds that we have to contract this expression with the tensor

$$L^{\mu\nu} = q^2 g^{\mu\nu} - q^\mu q^\nu \tag{3.22}$$

For illustrative reasons we only discuss the first term (even without the factor q^2, so we contract only with the metric tensor) and leave it as an exercise to do the full calculation. We thus get

$$
\begin{aligned}
R^{(0,0)} &= \frac{2}{Q^2 - m_c^2} \bar{u}(p_b) \gamma_\mu \not{Q} \gamma^\mu (1-\gamma_5) u(p_b) \\
&= \frac{-4}{Q^2 - m_c^2} \bar{u}(p_b) \not{Q} (1-\gamma_5) u(p_b)
\end{aligned} \tag{3.23}
$$

At leading order (with this particular contraction of the indices) we see that the leading order expression for the OPE is

$$
\int d^4 y \, \exp(iyQ) \, T\{\widetilde{J}_\mu^\dagger(y) \widetilde{J}^\mu(0)\} = \frac{-4}{Q^2 - m_c^2} \, (b_v(0) \not{Q} (1-\gamma_5) b_v(0)) + \mathcal{O}(k) \tag{3.24}
$$

This is the simplest example of a matching calculation, i.e. the comparison of the right-hand side of (3.18) to the expansion of the left-hand side. The leading terms turn out to be a dimension-three operators; as we shall see, the higher-order terms involve higher-dimensional operators.

The next step is to take the matrix elements with the real B meson states. In our mini-example we need to discuss the matrix element of a dimension-three operator

$$
\langle B(v) | \bar{b}_v(0) \gamma_\lambda (1-\gamma_5) b_v(0) | B(v) \rangle = \langle B(v) | \bar{b}(0) \gamma_\lambda (1-\gamma_5) b(0) | B(v) \rangle = 2 m_B v_\lambda \tag{3.25}
$$

which does not contain any unknown parameter, since the vector current $\bar{b}_v(0)\gamma_\lambda b_v(0)$ is a conserved current, hence its forward matrix elements between B meson states is normalized, while the corresponding matrix element with the axial current vanishes.

In fact, in other applications different dimension-three matrix elements can appear, which differ from the case at hand only by the Dirac matrix between the heavy quark operators Q_v,

$$
\bar{Q}_v \Gamma Q_v = \bar{Q} \Gamma Q \quad : \quad \text{General dimension three operator} \tag{3.26}
$$

where Γ is an arbitrary Dirac matrix. However, taking a forward matrix element between the pseudoscalar ground state meson, only $\Gamma = 1$ and $\Gamma = \gamma_\mu$ are non-vanishing.

As pointed out above, the vector current of the heavy quark Q is conserved, with the consequence that it does not induce an unknown hadronic matrix element:

$$
\langle H(p_H) | \bar{Q}_v \gamma_\lambda Q_v | H(p_H) \rangle = \langle H(p_H) | \bar{Q} \gamma_\lambda Q | H(p_H) \rangle = 2 p_{H\lambda} \tag{3.27}
$$

The matrix element of $\bar{Q}_v Q_v = \bar{Q} Q$ can also be related to the vector current by the equations of motion

$$
\not{v} Q_v = Q_v - \frac{i\not{D}}{m_Q} Q_v \tag{3.28}
$$

$$
(ivD) Q_v = -\frac{1}{2m_Q} (i\not{D})(i\not{D}) Q_v \tag{3.29}
$$

Using (3.28) we get

$$
\begin{aligned}
\langle H(p_H) | \bar{Q}_v Q_v | H(p_H) \rangle &= \langle H(p_H) | \bar{Q}_v \not{v} Q_v | H(p_H) \rangle + \frac{1}{m_Q} \langle H(p_H) | \bar{Q}_v (i\not{D}) Q_v | H(p_H) \rangle \\
&= 2 m_H + \frac{1}{m_Q} \langle H(p_H) | \bar{Q}_v (i\not{D}) Q_v | H(p_H) \rangle
\end{aligned} \tag{3.30}
$$

which means that to leading order in the HQE no unknown hadronic parameter is induced in general. Furthermore, it is easy to see that (3.30) has no contribution of order $1/m_Q$: Starting form the equation of motions (3.28) we get the relation

$$\bar{Q}_v \gamma_\alpha Q_v = \bar{Q}_v \gamma_\alpha \not{v} Q_v + \frac{1}{m_Q} \bar{Q}_v \gamma_\alpha (i\not{D}) Q_v \,. \tag{3.31}$$

Taking the conjugate of (3.28) and multiply form the right with Q_v we get

$$\bar{Q}_v \gamma_\alpha Q_v = \bar{Q}_v \not{v} \gamma_\alpha Q_v + \frac{1}{m_Q} \bar{Q}_v (i\not{D}) \gamma_\alpha Q_v + \text{total derivative} \tag{3.32}$$

where we do not need to take into account the total derivative, since it will not contribute to the forward matrix elements. Averaging (3.31) and (3.32) and taking the forward matrix element ($\langle ... \rangle = \langle H(p_H)|...|H(p_H)\rangle$) yields

$$\langle \bar{Q}_v \gamma_\alpha Q_v \rangle = v_\alpha \, \langle \bar{Q}_v Q_v \rangle + \frac{1}{m_Q} \, \langle \bar{Q}_v (iD_\alpha) Q_v \rangle \tag{3.33}$$

Contracting this with v^α yields the relation

$$\langle \bar{Q}_v \not{v} Q_v \rangle = \langle \bar{Q}_v Q_v \rangle + \frac{1}{m_Q} \, \langle \bar{Q}_v (ivD) Q_v \rangle \tag{3.34}$$

Comparing this to (3.30) we find that

$$\langle \bar{Q}_v (i\not{D}) Q_v \rangle = -\langle \bar{Q}_v (ivD) Q_v \rangle = \frac{1}{2m_Q} \langle \bar{Q}_v (i\not{D})(i\not{D}) Q_v \rangle \,, \tag{3.35}$$

where we have used (3.29). Inserting this into (3.30) we finally get

$$\langle \bar{Q}_v Q_v \rangle = 2m_H + \frac{1}{2m_Q^2} \langle \bar{Q}_v (i\not{D})(i\not{D}) Q_v \rangle \tag{3.36}$$

which proofs that the corrections are $\mathcal{O}(1/m_Q^2)$.

In order to obtain the total rate, we have to take the imaginary part of (3.24). To this end, we re-install the $i\epsilon$ prescription into the propagator and use the relation

$$2 \, \mathrm{Im} \, \frac{1}{x + i\epsilon} = (2\pi) \delta(x) \tag{3.37}$$

from which we finally obtain

$$\Gamma \sim 2 \, \mathrm{Im} \int d^4y \, \exp(iyQ) \, \langle B(v)|T\{\widetilde{J}^\dagger_\mu(y) \widetilde{J}^\mu(0)\}|B(v)\rangle \tag{3.38}$$

$$= -8(2\pi)\delta(Q^2 - m_c^2) m_B (v \cdot Q) + \cdots \tag{3.39}$$

This result is in fact a general statement: in combination with (3.30,3.36) we get

The leading term in the HQE is the partonic result, i.e. the decay of a "free" heavy quark.

Next we look at the first term in the k expansion

$$\begin{aligned} R^{(0,1)} &= 2\left(\frac{1}{Q^2-m_c^2}\right)^2 \bar{u}(p_b)\gamma_\mu \not{Q}\not{k}\not{Q}\gamma^\mu(1-\gamma_5)u(p_b) \\ &= -4\left(\frac{1}{Q^2-m_c^2}\right)^2 \bar{u}(p_b)[Q^2\not{k}-2(Q\cdot k)\not{Q}](1-\gamma_5)u(p_b) \end{aligned} \tag{3.40}$$

Comparing this to the OPE (3.18) we find ($k_\mu \to iD_\mu$)

$$\int d^4y\, \exp(iyQ)\, T\{\widetilde{J}_\mu^\dagger(y)\widetilde{J}^\mu(0)\} = \text{leading term} \tag{3.41}$$
$$+2\left(\frac{1}{Q^2-m_c^2}\right)^2 \left[Q^2(b_v(0)(i\not{D})(1-\gamma_5)b_v(0)) - 2Q^\mu Q^\nu(b_v(0)(iD_\mu)\gamma_\nu(1-\gamma_5)b_v(0))\right]$$

Again we have to take the forward matrix element of this expression. We use the equations of motion (3.28) and (3.29) and get for the general case (the contribution with γ_5 vanish due to parity)

$$\langle H(p_H)|\bar{Q}_v(iD_\mu)\Gamma Q_v|H(p_H)\rangle = \mathcal{O}\left(\frac{1}{m_Q}\right) \tag{3.42}$$

where the explicit relation for the first term in (3.41) is given by (3.35). This shows that these contributions are actually at least of order $1/m_Q^2$. This is in fact a general statement

$$\boxed{\text{There are no contributions of order } 1/m_Q \text{ in the HQE .}}$$

This does not mean that the first term in the k expansion vanishes, rather the dimension-four matrix elements are $1/m_Q$ suppressed.

In order to obtain the corresponding contribution to the rate, we have to take the imaginary part by re-inserting the $i\epsilon$ prescription into the propagator. Using

$$2\,\mathrm{Im}\left(\frac{1}{x+i\epsilon}\right)^2 = -2\,\mathrm{Im}\,\frac{d}{dx}\left(\frac{1}{x+i\epsilon}\right) = -(2\pi)\delta'(x) \tag{3.43}$$

we obtain a contribution to the differential rate proportional to the derivative of the "on-shell" δ function, which, however is of order $1/m_b^2$.

In order to obtain the full $1/m^2$ contributions, one needs to expand $R_{\mu\nu}$ to second order in k_μ, which yields for our toy example

$$R^{(0,2)} = 2\left(\frac{1}{Q^2-m_c^2}\right)^3 \bar{u}(p_b)\gamma_\mu\not{Q}\not{k}\not{Q}\not{k}\not{Q}\gamma^\mu(1-\gamma_5)u(p_b) \tag{3.44}$$

which eventually matches on operators with two derivatives:

$$\bar{Q}_v\Gamma(iD_\mu)(iD_\nu)Q_v \;:\quad \text{General dimension five operator.} \tag{3.45}$$

However, at that order an obvious problem arises: While $k_\mu k_\nu$ is clearly a symmetric tensor, the product of two covariant derivatives contains an antisymmetric part, since the covariant derivatives do not commute, their commutator is the field-strength tensor

$$\bar{Q}_v \Gamma [(iD_\mu),(iD_\nu)] Q_v = -i g_s \bar{Q}_v \Gamma G_{\mu\nu} Q_v \tag{3.46}$$

Obviously the expansion in k cannot give us this antisymmetric piece. However, the antisymmetric part is related to the field strength, i.e. to the emission of a gluon. In order to pin this down we thus have to compute a matrix element of (3.18) between a quark state and a state with a quark and a gluon. For the left hand side of (3.6) the leading order result is

$$S^{(0)}_{\mu\nu} = \bar{u}(p_b)\gamma_\mu(1-\gamma_5)\left[\frac{1}{\not{Q}+\not{k}-m_c}\right] T^a \not{\epsilon}(q) \left[\frac{1}{\not{Q}+\not{k}+\not{q}-m_c}\right]\gamma_\nu(1-\gamma_5)u(p_b) \tag{3.47}$$

where q is the momentum of the gluon with color a and polarization ϵ. Note that also the momentum of the gluon is soft, so we have to perform a combined expansion in k and q. Furthermore, the gluon appears as part of the covariant derivative, so also the polarization ϵ counts as one power in the $1/m_b$ expansion; this means that in order to arrive at the second order, we have to expand (3.47) only to first order in k and q.

To obtain the coefficient of the antisymmetric combination (3.46) we thus have to find the coefficient in front of the combination

$$G_{\alpha\beta} \longleftrightarrow q_\alpha \epsilon_\beta - q_\beta \epsilon_\alpha \,.$$

This concludes the sketch of the practical aspects of the matching procedure to obtain the coefficients in (3.18). The procedure remains the same even once α_s corrections are included, which means that the expansions in k and gluon momenta and polarization has to be performed for the expression including α_s corrections. By comparing the two sides of (3.6) one thus obtains the perturbative expansion of the coefficients.

Finally it is worthwhile to point out, that the tree level expressions for the case of semileptonic decays can be obtained systematically [41], since the ordering of the covariant derivatives can be traced by using for the charm propagator an external field propagator of the form

$$\left[\frac{1}{\not{Q}+i\not{D}-m_c}\right] .$$

Expanding this under the assumption that the components of iD do not commute yields formally the same expression as (3.20)

$$\frac{1}{\not{Q}+i\not{D}-m_c} = \frac{1}{\not{Q}-m_c} - \frac{1}{\not{Q}-m_c} i\not{D} \frac{1}{\not{Q}-m_c} + \frac{1}{\not{Q}-m_c} i\not{D} \frac{1}{\not{Q}-m_c} i\not{D} \frac{1}{\not{Q}-m_c} + \cdots \tag{3.48}$$

but now the ordering of the covariant derivatives in the correct one, i.e. for the second order term we have

$$\frac{1}{\not{Q}-m_c} i\not{D} \frac{1}{\not{Q}-m_c} i\not{D} \frac{1}{\not{Q}-m_c} = (iD_\mu)(iD_\nu)\left[\frac{1}{\not{Q}-m_c}\gamma^\mu \frac{1}{\not{Q}-m_c}\gamma^\nu \frac{1}{\not{Q}-m_c}\right]$$

where the term in the bracket has the correct antisymmetric piece in the indices μ and ν. However, this unfortunately only works at tree level.

3.4 HQE parameters

The non-perturbative input in the HQE is given in terms of the hadronic matrix elements of operators, which have generically the form[3]

$$\langle H(p_H)|\bar{Q}_v(iD_{\mu_1})(iD_{\mu_2})\cdots(iD_{\mu_n})\Gamma Q_v|H(p_H)\rangle\,,$$

where Γ is some Dirac matrix. Note that these operators have dimension $n+3$ and are defined in full QCD, which implies that they still depend on the mass. In principle one can perform an expansion in $1/m_Q$ and the leading term will be just

$$\langle\tilde{H}(v)|\bar{h}_v(iD_{\mu_1})(iD_{\mu_2})\cdots(iD_{\mu_n})\Gamma h_v|\tilde{H}(v)\rangle\,,$$

with the static field h_v and the meson state $|\tilde{H}(v)\rangle$ in the infinite mass limit.

We have already discussed the dimension-three operators and have shown that there is no unknown matrix element at dimension three, since all matrix elements can be related to the conserved Q-quark vector current, up to terms of order $1/m_Q^2$. We also saw already that all the matrix elements of dimension-four operators are suppressed by one power of $1/m_Q$, and thus the first nontrivial contribution appears at dimension five, i.e. for $n=2$.

Before going into the technicalities a historic remark is in order. The idea that the decay of a ground-state hadron with a single heavy quark can be approximately described by the decay of the "free" quark inside the hadron is quite old [42]. However, the HQE proves this to be the leading term of a systematic expansion, where the leading non-perturbative corrections turn out to be of the order $\Lambda_{\mathrm{QCD}}^2/m_Q^2$. In the early days of the HQE this was seen as an embarrassment: As a consequence the lifetimes of all ground state hadrons of a specific heavy flavour should be identical to leading order, the corrections should be of order $\Lambda_{\mathrm{QCD}}^2/m_Q^2$, and, as we shall see below, the lifetime differences should even be of the order $\Lambda_{\mathrm{QCD}}^3/m_Q^3$. Before the precise measurement of bottom-hadron lifetimes, the lifetimes of charmed hadrons were available; with $m_D \approx m_c \sim 1.8$ GeV and $\Lambda_{\mathrm{QCD}} \sim 0.3$ GeV we naively expect lifetime differences to be below one percent. However, the lifetimes of ground-state charmed hadrons vary by a factor of five, which is hard to explain as a $\Lambda_{\mathrm{QCD}}^3/m_Q^3$ effect. Nevertheless, in the meantime we have a qualitative understanding how such large lifetime differences can emerge.

At each order in the $1/m_Q$ expansion we need to identify, how many independent parameters actually appear. At dimension five the two independent parameters can be defined as

$$-2m_H\hat{\mu}_\pi^2 = \langle H(p_H)|\bar{Q}_v(iD)^2 Q_v|H(p_H)\rangle \tag{3.49}$$

$$-2m_H\hat{\mu}_G^2 = \langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\nu)(i\sigma^{\mu\nu})Q_v|H(p_H)\rangle \tag{3.50}$$

and any general matrix element can be related to these two through the "trace formula"

[3] In fact, this does not cover all possible operators; there can also be operators with light quarks, which need to be discussed separately.

$$\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\nu)\Gamma Q_v|H(p_H)\rangle = -2m_H \frac{\hat{\mu}_\pi^2}{6}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}\Gamma\right)(g_{\mu\nu} - v_\mu v_\nu)$$
$$-2m_H \frac{\hat{\mu}_G^2}{12}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}(-i\sigma_{\mu\nu})\frac{1+\not{v}}{2}\Gamma\right) + \mathcal{O}\left(\frac{1}{m_Q}\right) \tag{3.51}$$

From this we get our third statement on the HQE

The first subleading corrections in the HQE are given by $\hat{\mu}_\pi$ and $\hat{\mu}_G$.

The parameter $\hat{\mu}_\pi^2$ is called the kinetic energy parameter, since it is related to the term $\vec{p}^2/(2m_Q)$ appearing in the Schrödinger equation, the parameter $\hat{\mu}_G$ is called the chromomagnetic moment, since it describes the coupling $\vec{\sigma}\cdot\vec{B}$ of the heavy-quark spin to the chromomagnetic field $\vec{B}$.

The values of these parameters have to be taken from either experimental data or calculations in Lattice QCD or in a model. As an example, $\hat{\mu}_G$ can be obtained from hadron spectroscopy. Looking at the expansions of heavy hadron masses in inverse powers of the quark mass (2.32,2.33), we infer that we may use (2.37) to fix the value of the chomomagnetic moment, while the kinetic energy parameter cannot be obtained from spectroscopy.

Before we continue to higher orders, we point out a few details. In many applications it turns out to be useful to split the covariant derivative into a "time derivative" and a "spatial" part according to (2.7). To this end, one may as well use the definitions

$$-2m_H\mu_\pi^2 = \langle H(p_H)|\bar{Q}_v(iD^\perp)^2 Q_v|H(p_H)\rangle \tag{3.52}$$
$$-2m_H\mu_G^2 = \langle H(p_H)|\bar{Q}_v(iD_\mu^\perp)(iD_\nu^\perp)(i\sigma^{\mu\nu})Q_v|H(p_H)\rangle \tag{3.53}$$

and we have $\mu_\pi = \hat{\mu}_\pi + \mathcal{O}(1/m_Q^2)$ and $\mu_G = \hat{\mu}_G + \mathcal{O}(1/m_Q)$. Furthermore, all these parameters still depend on the heavy quark mass; expanding also this mass dependence yields mass independent parameters λ_1 and λ_2 defined in HQET by

$$2m_H\lambda_1 = \langle \tilde{H}(v)|\bar{h}_v(iD^\perp)^2 h_v|\tilde{H}(v)\rangle \tag{3.54}$$
$$2m_H\lambda_2 = \langle \tilde{H}(v)|\bar{h}_v(iD_\mu^\perp)(iD_\nu^\perp)(i\sigma^{\mu\nu})h_v|\tilde{H}(v)\rangle \tag{3.55}$$

The advantage of expanding any mass dependence and to define "static" quantities λ_1 and λ_2 is that these will be the same for *any* heavy quark, thus one might compare inclusive bottom with inclusive charm decays. The advantage of using μ_π and μ_G, or $\hat{\mu}_\pi$ and $\hat{\mu}_G$ becomes clear only when going to higher orders: Starting at $1/m_Q$ one also need to take into account the expansion of the state, since we have $|H(p_H)\rangle = |\tilde{H}(v)\rangle + \mathcal{O}(1/m_Q)$, which in general leads to non-local matrix elements involving the subleading terms in the Lagrangian (2.17).

At dimension six we will have three derivatives

$$\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\alpha)(iD_\nu)Q_v|H(p_H)\rangle$$

together with four-quark operators. At tree level, we can express all dimension six matrix elements in terms of two parameters which are given by

$$2m_H\hat{\rho}_D^3 = \langle H(p_H)|\bar{Q}_v(iD_\mu)(ivD)(iD^\mu)Q_v|H(p_H)\rangle \tag{3.56}$$
$$2m_H\hat{\rho}_{LS}^3 = \langle H(p_H)|\bar{Q}_v(iD_\mu)(ivD)(iD_\nu)(-i\sigma^{\mu\nu})Q_v|H(p_H)\rangle \tag{3.57}$$

where we have again used the covariant definitions. For these parameters, the same remarks apply as for $\hat{\mu}_\pi$ vs μ_π vs. λ_1 etc., which will have differences appearing as terms of subleading order in the $1/m_Q$ expansion.

In a similar fashion as for the terms of dimension five we can write a trace formula, which reads in this case

$$\begin{aligned}\langle H(p_H)|\bar{Q}_v(iD_\mu)(iD_\alpha)(iD_\nu)\Gamma Q_v|H(p_H)\rangle &= 2m_H\,\frac{\hat{\rho}_D^3}{6}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}\Gamma\right)(g_{\mu\nu}-v_\mu v_\nu)v_\alpha \\ &+2m_H\,\frac{\hat{\rho}_{LS}^3}{12}\,\mathrm{Tr}\left(\frac{1+\not{v}}{2}(-i\sigma_{\mu\nu})\frac{1+\not{v}}{2}\Gamma\right)v_\alpha+\mathcal{O}\left(\frac{1}{m_Q}\right).\end{aligned} \tag{3.58}$$

Note that a consistent calculation of higher order terms requires to also take into account the subleading terms in the trace formula (3.51).

One may continue in the same fashion to higher orders [43], however, the number of independent parameters will grow strongly as one proceeds to orders higher than $1/m_Q^3$. At order $1/m_Q^4$ there is a total of 11 independent parameters, at $1/m_Q^5$ there are already 25 new parameters. While the four parameters up to $1/m_Q^3$ can be extracted from the data, the large number of parameters appearing at even higher orders have to be modeled or may one day be taken from lattice calculations.

3.5 QCD Corrections

The HQE has the potential to compute total and specific differential rates with extremely high precision. However, as pointed out above, the leading term is always the decay of the heavy quark inside the heavy hadron, where the result is the same as if we were discussing a "free" quark. If we ignore for the moment the mass of the final state particles, the decay width will be

$$d\Gamma \sim G_F^2|V_{\mathrm{CKM}}|^2 m_Q^5\,, \tag{3.59}$$

which induces an enormously strong dependence on the heavy quark mass. In the early days of the HQE this was considered to be problem, since the heavy-quark mass is not a straightforward observable. Unlike for an electron, this mass cannot be just measured as a pole in the propagator, since there are no asymptotic states of outgoing quarks. The quark mass is thus just a parameter in the QCD Lagrangian and, in fact, depends on the scheme one chooses to define it.

It seems that any ambiguity or uncertainty related to the heavy quark mass enters into the predictions of HQET enhanced by a factor of five, however, as we shall discuss below, this problem can be controlled and is related to a suitable choice of a scheme in which the mass is actually defined.

3.5.1 Why do we need a mass scheme?

When computing Feynman diagrams we insert a quark mass into the quark propagators. This mass is defined by the location of the pole of the propagator, which is the usual definition of what is called the pole mass m_Q^{Pole}. When constructing HQET we redefine the heavy quark momentum by $p_Q = m_Q v + k$, using some mass definition, which we choose to be also the pole mass. However, due to

$$m_H = m_Q + \bar{\Lambda} + \mathcal{O}(1/m_Q)$$

we may compensate any redefinition of the mass by a corresponding shift in the parameter $\bar{\Lambda}$.

The mass renomalization is related to the quark propagator. Including the (one particle irreducible) self energy contributions $\Sigma(p)$, the renormalized quark propagator becomes

$$S(p) = \frac{-iZ_2^{\mathrm{OS}}}{\not{p} - m_0 + \Sigma(p, m_Q^{\mathrm{Pole}})} \longrightarrow \frac{-i}{\not{p} - m_Q^{\mathrm{Pole}}} \quad \text{as } p^2 \to (m_Q^{\mathrm{Pole}})^2 \tag{3.60}$$

where the pole mass is related to the bare mass by a formal (perturbative) series

$$m_0 = Z_m^{\mathrm{OS}} m_Q^{\mathrm{Pole}} = \left(1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\mathrm{Pole}} \tag{3.61}$$

The coefficients c_n are divergent and need to be regularized. The standard way to regularize QCD is dimensional regularization (DimReg) where the loop integrals over momenta are computed in $D = 4 - 2\epsilon$ space-time dimensions. At one loop one obtains

$$c_1 = -C_F \left(\left[\frac{1}{\epsilon} + \gamma_E - 4\pi\right] \frac{3}{4} + 1 + \frac{3}{4} \ln \frac{\mu^2}{(m_Q^{\mathrm{Pole}})^2} + \mathcal{O}(\epsilon) \right)$$

where in the case of the pole mass the scale μ is fixed by the on-shell condition (3.60) and $C_F = 4/3$ is the value of the $SU(3)$ Casimir operator in the fundamental representation.

Alternatively one may use another definition of the quark mass, such as the the $\overline{\mathrm{MS}}$ definition, for which we have a relation similar to (3.61)

$$m_0 = Z_m^{\overline{\mathrm{MS}}} m_Q^{\overline{\mathrm{MS}}} = \left(1 + \sum_{n=1}^{\infty} b_n \left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\overline{\mathrm{MS}}} \tag{3.62}$$

where the $\overline{\mathrm{MS}}$ scheme is defined by removing only the $1/\epsilon + \gamma_E - 4\pi$ term, which means at one loop order

$$b_1 = -C_F \left[\frac{1}{\epsilon} + \gamma_E - 4\pi\right] \frac{3}{4} \tag{3.63}$$

Note that the $\overline{\mathrm{MS}}$ mass depends on the scale μ and is a running parameter.

The key point relevant for our discussion is that different mass definitions can be related by perturbation theory with finite coefficients. We have

$$m_Q^{\mathrm{Pole}} = z^{\mathrm{Pole} \to \overline{\mathrm{MS}}}\, m_Q^{\overline{\mathrm{MS}}} = \frac{Z_m^{\overline{\mathrm{MS}}}}{Z_m^{\mathrm{OS}}} m_Q^{\overline{\mathrm{MS}}} \tag{3.64}$$

and

$$z^{\text{Pole}\to\overline{\text{MS}}} = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{\alpha_s}{\pi}\right)^n \quad \text{and} \quad a_1 = -C_F \left(\frac{3}{4} \ln \frac{\mu^2}{(m_Q^{\text{Pole}})^2} + 1\right) \tag{3.65}$$

Consider now a rate of the form (3.59) and assume that we have fixed the mass scheme to be e.g. the pole mass. Computing radiative corrections to (3.59) takes the schematic form

$$d\Gamma \sim G_F^2 |V_{\text{CKM}}|^2 (m_Q^{\text{Pole}})^5 \left(1 + \frac{\alpha_s}{\pi} r_1 + \left(\frac{\alpha_s}{\pi}\right)^2 r_2 + \cdots\right) \tag{3.66}$$

with (after proper renormalization) finite coefficients r_i. Switching now to another mass definition such as e.g. the $\overline{\text{MS}}$ scheme, we find

$$\begin{aligned} d\Gamma &\sim G_F^2 |V_{\text{CKM}}|^2 (m_Q^{\overline{\text{MS}}})^5 (z^{\text{Pole}\to\overline{\text{MS}}})^5 \left(1 + \frac{\alpha_s}{\pi} r_1 + \left(\frac{\alpha_s}{\pi}\right)^2 r_2 + \cdots\right) \\ &= G_F^2 |V_{\text{CKM}}|^2 (m_Q^{\overline{\text{MS}}})^5 \left(1 + \frac{\alpha_s}{\pi}(r_1 + 5a_1) + \cdots\right) \end{aligned} \tag{3.67}$$

Thus we conclude that the choice of a mass scheme determines the size of the radiative corrections. In other words, with a clever choice of the mass definition one can absorb radiative corrections into the definition of the mass. Clearly such a mass definition must also allow us to obtain the numerical value for the mass from independent data as precisely as possible, since the dependence on the fifth power is still present.

It turns out that the pole mass is a particularly bad choice for a mass scheme, since the coefficients r_1 are large and do not seem to converge well [44–46]. Related to the bad convergence is another problem with the pole mass, since it has an intrinsic uncertainty of the order of Λ_{QCD} due to an infrared renormalon, which we discuss below. Better definitions are so-called short distance masses (e.g. the $\overline{\text{MS}}$ mass) which do not have this problem and can thus be determined in principle with arbitrary precision. For most of these short-distance masses the QCD corrections converge much better; there are even mass definitions especially designed for the HQE.

3.5.2 Short Distance Masses

We use again the pole mass as a starting point. In terms of the $\overline{\text{MS}}$ mass we have

$$m_Q^{\text{Pole}} = z^{\text{Pole}\to\overline{\text{MS}}}\, m_Q^{\overline{\text{MS}}} = \left(1 + \sum_{n=1}^{\infty} a_n \left(\frac{\alpha_s}{\pi}\right)^n\right) m_Q^{\overline{\text{MS}}} \tag{3.68}$$

The main point of the following discussion is the fact that this perturbative relation is not converging, rather it is an asymptotic series. This is due to factorially growing contributions in the coefficients $a_n \sim n!$. In fact, one can show that the asymptotic behavior of the perturbative series is [44]

$$z^{\text{Pole}\to\overline{\text{MS}}} = 1 + \frac{C_F e^{5/6}}{\pi} \frac{\mu}{m_Q^{\overline{\text{MS}}}} \alpha_s \sum_n (-2\hat{\beta}_0 \alpha_s)^n\, n! \tag{3.69}$$

where

$$\hat{\beta}_0 = \frac{1}{4\pi}\left(11 - \frac{2n_f}{3}\right)$$

is the leading term of the $\hat{\beta}$ function of QCD and n_f is the number of active flavors.

In order to consider the consequences of this observation, we study the Borel transform of the perturbative series, defined by

$$z(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^{n+1} \quad \longrightarrow \quad B[z](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{3.70}$$

If both series for z and $B[z]$ were convergent, one could define the reverse operation by

$$z(\alpha) = \int_0^{\infty} dt \, \exp\left(-\frac{t}{\alpha}\right) B[z](t) \tag{3.71}$$

which indeed has the same series expansion as the original z. However, the terms shown in (3.69) lead to poles on the positive real axis in the Borel transform, which are called renormalons. For the case at hand, the leading term originates from a singularity at $t = 1/2$ and hence the integral in (3.71) cannot be computed without a prescription of how to avoid this pole. This leads to an ambiguity which can be expressed by shifting the singularity in the complex t plane by a small amount ϵ either upwards or downwards, hence we use

$$\frac{1}{t - 1/2 + i\epsilon} - \frac{1}{t - 1/2 - i\epsilon} = 2\pi\delta(t - 1/2)$$

leaving us with an ambiguity of the form

$$\Delta z(\alpha) \propto \frac{\mu}{m_Q^{\overline{\mathrm{MS}}}} \exp\left(-\frac{1}{2\alpha(\mu)}\right) \sim \frac{\Lambda_{\mathrm{QCD}}}{m_Q^{\overline{\mathrm{MS}}}} \tag{3.72}$$

where we have inserted the running coupling of QCD in terms of Λ_{QCD}.

Although these arguments can still be made more stringent, we have at least seen the essence of the reasoning which leads to the conclusion that the pole mass has an intrinsic uncertainty of the order of Λ_{QCD}, related to infrared contributions, which can be related to the coulombic self interactions of a heavy quark.[4]

To this end, it means that the pole mass cannot be used for precise predictions. In particular, inserting the pole mass into (3.66) yields large QCD corrections which are mainly due to this particular choice of the mass. In other words, a more clever choice of the mass definition can minimize the size of the QCD corrections and lead to a much better convergence.

Another problem induced by this becomes apparent once power corrections are included. Given an intrinsic uncertainty of the order Λ_{QCD} in the mass in (3.66)

[4]We note that the mass $m_Q^{\overline{\mathrm{MS}}}$ does not suffer from this problem. This can be seen from its relation to the bare mass, where only the ultraviolet $1/\epsilon$ poles are removed.

renders the power corrections, which are by themselves $\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/m_Q^2)$, completely meaningless.

Thus it is obvious that we have to switch to a "short-distance" mass such as the $\overline{\mathrm{MS}}$ mass. This mass depends on the scale μ which is usually taken to be $\mu \geq m_Q^{\overline{\mathrm{MS}}}$; below this scale one should switch to HQET, and thus it becomes clear that one should use mass definitions which are "designed" to go to scales as low as 1 GeV. There are two mass schemes which are frequently used in the context of the HQE which are the kinetic mass scheme and the $1S$ mass scheme.

3.5.3 Kinetic Mass Scheme

As we discussed above, the pole mass contains a renormalon ambiguity of the order of Λ_{QCD}. In order to avoid this problem, we look at the expansion of the heavy hadron mass of the pseudoscalar ground state meson (2.32) we have a physical quantity (the hadron mass) on the left hand side, which cannot suffer from such an ambiguity. However, on the right hand side, we have not yet specified, what mass definition is used. If we use the pole mass, we find that this ambiguity has to cancel between m_Q^{Pole} and the binding-energy parameter $\bar{\Lambda}$ defined in (2.34).

Clearly $\bar{\Lambda}$ as well as the parameters in the HQE of the heavy hadron mass are non-perturbative. However, one may write down a QCD sum rule, called a "Small-Velocity" sum rule [47], which allows us to estimate these parameters. This sum rules also allows us to compute the perturbative contribution to these parameters in a hard cut-off scheme. As we pointed out in the discussion of the HQE for the heavy quark mass, all ambiguities in the heavy quark mass have to cancel against the ambiguities in the HQE parameters $\bar{\Lambda}$ and μ_π^2, such that the meson mass is a well defined physical quantity. Writing down an expression similar as for the hadron mass up to the kinetic energy term and inserting the perturbative expressions for $\bar{\Lambda}$ and μ_π^2 yields the definition of the quark mass in the "kinetic" scheme.

$$m_Q^{\mathrm{kin}}(\mu) = m_Q^{\mathrm{Pole}} - [\bar{\Lambda}(\mu)]_{\mathrm{pert}} - \frac{1}{2m_Q^{\mathrm{kin}}(\mu)}[\mu_\pi^2(\mu)]_{\mathrm{pert}}\,, \tag{3.73}$$

which is a short distance mass like the $\overline{\mathrm{MS}}$ mass, since the renormalon ambiguities in the pole mass cancel against the ones in $\bar{\Lambda}$ and μ_π^2. Here, the leading order expression for $\bar{\Lambda}$ and μ_π^2 read [47]

$$[\bar{\Lambda}(\mu)]_{\mathrm{pert}} = \frac{16}{9}\frac{\alpha_s(\mu)}{\pi}\mu + \mathcal{O}(\alpha_s^2) \tag{3.74}$$

$$[\mu_\pi^2(\mu)]_{\mathrm{pert}} = \frac{4}{3}\frac{\alpha_s(\mu)}{\pi}\mu^2 + \mathcal{O}(\alpha_s^2)\,, \tag{3.75}$$

where μ is the hard cut-off.

The kinetic mass is a short distance mass and can be extracted e.g. from the thresholds in $e^+e^- \to$ hadrons with a very high precision; currently the uncertainty is about 50 MeV [48].

3.5.4 1S Mass Scheme

As has been discussed in [44, 45] the infrared contributions of the pole mass can be attributed to the color Coulomb field of the heavy quark. In a nonrelativistic picture, this Coulomb field is the main contribution to the binding of a system with heavy quarks, and hence it is suggestive to try to find a definition of the b quark mass in terms of a bottomonium state. If one chooses the ground state of the bottomonium, one may compute this in terms of the mass parameter in the Lagrangian in a non-relativistic picture, i.e. in NRQCD (see the last part of the lectures).

To this end, we consider the mass of the lowest lying $J^{PC} = 1^{--}$, ${}^3S_1$ bottomonium state, called Υ, for which precise measurements exist. The (perturbative) relation between the Υ mass and the pole mass reads schematically

$$m_\Upsilon = 2m_b^{\text{pole}} \left(1 - \frac{(\alpha_s C_F)^2}{8} \left\{1 + \frac{\alpha_s\beta_0}{\pi}(\ell+1) + \left(\frac{\alpha_s\beta_0}{\pi}\right)^2 \left(\frac{1}{2}\ell^2 + \ell + 1\right) + \cdots + \right.\right.$$
$$\left.\left. + \left(\frac{\alpha_s\beta_0}{\pi}\right)^n \left(\frac{1}{n!}\ell^n + \frac{1}{(n-1)!}\ell^{n-1} + \ldots + \ell + 1\right) + \cdots \right\}\right) \tag{3.76}$$

where

$$\ell = \ln\left[\frac{\mu}{m_b\alpha_s C_F}\right], \quad \beta_0 = 4\pi\hat{\beta}_0 = 11 - \frac{2n_f}{3} \quad \text{and} \quad C_F = \frac{4}{3}$$

We note that there is a mismatch concerning the orders in α_s, since the binding energy of the nonrelativistic binding energy is α_s^2 and hence the series expansion is in powers of $\{\alpha_s^2, \alpha_s^3\beta_0, ..., \alpha_s^{n+2}\beta_0^n, ...\}$. This is in contrast to the usual perturbative expansion (e.g. the relation of the $\overline{\text{MS}}$ mass to the pole mass) which is in terms of $\{\alpha_s, \alpha_s^2\beta_0, ..., \alpha_s^{n+1}\beta_0^n, ...\}$. It has been noted in [49, 50] that this mismatch disappears in high orders, since we have for large n

$$\left(\frac{1}{n!}\ell^n + \frac{1}{(n-1)!}\ell^{n-1} + \ldots + \ell + 1\right) \approx \exp(\ell) = \frac{\mu}{m_b\alpha_s C_F} \tag{3.77}$$

which fixes the mismatch in the series expansion. In fact, this exponentiation also ensures the cancellation of the renormalons of the pole mass [49].

Thus at low orders one has to compare different orders in α_s which can be made explicit by introducing a counting parameter $\epsilon = 1$ such that

$$m_\Upsilon = 2m_b^{\text{pole}} \left(1 - \frac{(\alpha_s C_F)^2}{8} \left\{\epsilon + \frac{\alpha_s\beta_0}{\pi}(\ell+1)\epsilon^2 + \left(\frac{\alpha_s\beta_0}{\pi}\right)^2 \left(\frac{1}{2}\ell^2 + \ell + 1\right)\epsilon^3 + \right.\right.$$
$$\left.\left. + \cdots + \left(\frac{\alpha_s\beta_0}{\pi}\right)^n \left(\frac{1}{n!}\ell^n + \frac{1}{(n-1)!}\ell^{n-1} + \ldots + \ell + 1\right)\epsilon^{n+1} + \cdots \right\}\right) \tag{3.78}$$

while for the decay rate for a B decay we write for the leading order (schematically)

$$\Gamma = \frac{G_F^2|V_{\text{CKM}}|^2}{192\pi^3}(m_b^{\text{Pole}})^5 \left[1 + c_1\frac{\alpha_s}{\pi}\epsilon + c_2\frac{\alpha_s^2}{\pi^2}\beta_0\epsilon^2 + \ldots + c_n\frac{\alpha_s^n}{\pi^n}\beta_0^{n-1}\epsilon^n + \ldots\right] \tag{3.79}$$

Replacing the pole mass in this relation by the $\Upsilon(1S)$ mass (3.78) and combining the corresponding orders in ϵ yields

$$\Gamma = \frac{G_F^2 |V_{\rm CKM}|^2}{192\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 \left[1 + \hat{c}_1 \epsilon + \hat{c}_2 \epsilon^2 + ... + \hat{c}_n \epsilon^n + ...\right] \tag{3.80}$$

which exhibits a quick convergence. In combination with a precise measurement of m_Υ one can get precise predictions for semileptonic decays as wel as for moments of spectra.

3.6 End-Point Regions

When studying the spectra of photons and leptons within the HQE one finds in some regions of phase space a pathological behaviour which prevents us to interpret the spectra point by point. These regions are related to endpoints of the spectra where the HQE breaks down. As an example, let us consider the endpoint of the electron spectrum in semileptonic B decays. The maximal lepton energy is given by

$$E_{\rm max} = \frac{m_B^2 - m^2}{2m_B}$$

where m is the mass of the lightest final state that can be produced. Close to this energy the possible final states are very few, in the extreme case only the single state with mass m contributes. Clearly one cannot expect an inclusive calculation to be correct here, in other words, the HQE breaks down in this region.

Neglecting the mass of the final-state quark (which we expect to be a good approximation for the $b \to u$ case) already the partonic result behaves pathologically in the endpoint region, since it is a θ function. In fact, one finds for the charged lepton energy spectrum up to $1/m_b^2$

$$\begin{aligned}\frac{d\Gamma}{dy} = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3} \Bigg[&\theta(2E - m_b) y \left\{(3-2y)y - \frac{5y^2}{3}\frac{\mu_\pi^2}{m_b^2} + \frac{y}{3}(6+5y)\frac{\mu_G^2}{m_b^2}\right\} \\ &+ \frac{\mu_\pi^2 - 11\mu_G^2}{6m_b^2}\delta(1-y) + \frac{\mu_\pi^2}{6m_b^2}\delta'(1-y)\Bigg]\end{aligned} \tag{3.81}$$

with $y = 2E/m_b$. Nevertheless, the integrated inclusive rate exists and can be compute in a $1/m_b$ expansion as shown above, for the case at hand we get

$$\Gamma = \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3} \left[1 - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2}\right] . \tag{3.82}$$

In addition, one can show by the same steps as for the total rate also moments of the spectra can be computed in the HQE.

Obviously the spectrum cannot be interpreted point by point, in particular close to the endpoint, since the true expansion parameter is $\Lambda_{\rm QCD}/(m_b - 2E)$, which becomes large close to the endpoint. Very close to the endpoint we have a region which is dominated by single states or resonances, where a description in terms of (a sum over a few) exclusive states is appropriate, and this region is defined by $0 \leq (m_b - 2E)^2 \leq \Lambda^2_{\mathrm{QCD}}$.

This particularly means that such a fine "resolution" of the spectrum in the end-point region is impossible within the HQE. However, if we look at the structure of the terms of the HQE, we see that the "most singular" term (i.e. the term with the highest derivative of the δ-function) is the last term in (3.81); in fact, proceeding to $1/m_b^3$ exhibits a term with $\delta''(1-x)$ etc. These terms can be summed by a technique analogous to to what is done in Deep Inelastic Scattering (DIS), leading to nonperturbative functions instead of nonperturbative parameters. These techniques have been set up in [51–53] and put the model suggested in [54] on a firm theoretical basis.

In order to illustrate this technique, we will (instead of $B \to X_u \ell \bar{\nu}$) consider $B \to X_s \gamma$. The leading contribution to this process is mediated by the operator

$$O_7 = \frac{e^2}{16\pi^2} m_b \, \bar{s}_L \sigma^{\mu\nu} b_R \, F_{\mu\nu} \qquad H_{\mathrm{eff}} = \frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* C_7(\mu) O_7(\mu) \,. \tag{3.83}$$

Computing the inclusive rate for $B \to X_s \gamma$ using only this operator yields up to order $1/m_b^2$

$$\Gamma(B \to X_s \gamma) = \frac{\alpha G_F^2 m_b^5}{16\pi^4} |V_{tb} V_{ts}^*|^2 |C_7(m_b)|^2 \left[1 - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2} \right] \,. \tag{3.84}$$

However, one may also compute the photon spectrum for this decay, which at tree level and to leading order is a δ function, fixing the photon energy to the value $E_\gamma = m_b/2$ determined by the two-particle kinematics of the partonic process. This behaviour persists also for the tree-level expressions at higher orders in the HQE, leading to derivatives of δ functions.

Up to terms of order $1/m_b^2$ one finds for the spectrum

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{\alpha G_F^2 m_b^5}{32\pi^4} |V_{tb} V_{ts}^*|^2 |C_7(m_b)|^2 \qquad (3.85) \\ &\times \left(\delta(1-y) - \frac{\mu_\pi^2 + 3\mu_G^2}{2m_b^2} \delta(1-y) + \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \delta'(1-y) + \frac{\mu_\pi^2}{6m_b^2} \delta''(1-y) \right) . \end{aligned}$$

Gluon emission will eventually lead to a nontrivial spectrum, however, a perturbative calculation is possible only in the region where the gluon and the final-state strange quark have a sizable invariant mass to warrant a perturbative treatment. Close to the endpoint we face the same situation as in $B \to X_u \ell \bar{\nu}$: The spectrum computed for the HQE cannot be interpreted point by point.

However, instead of studying the spectrum point by point, one may take moments of the spectrum. In fact, one may interpret the result (3.85) in terms of an expansion in singular functions, i.e. a moment expansion of the form [55]

$$\frac{1}{\Gamma} \frac{d\Gamma}{dy} = \sum_{n=0}^{\infty} \frac{M_n}{n!} \delta^{(n)}(1-y) \,, \tag{3.86}$$

where $\delta^{(n)}$ denotes the n^{th} derivative of the δ function, and the moments are defined as

$$M_n = \int_0^\infty (y-1)^n \left(\frac{1}{\Gamma} \frac{d\Gamma}{dy} \right) . \tag{3.87}$$

From the structure of the HQE we infer that the moments M_n have a $1/m_b$ expansion, the leading term of which is of order $1/m_b^n$. For the case of $B \to X_s \gamma$ we get

$$\begin{aligned} M_1 &= \mathcal{O}(1/m_b^2) = \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2} \\ M_2 &= \frac{\mu_\pi^2}{6m_b^2} + \mathcal{O}(1/m_b^3) \\ M_3 &= -\frac{\rho_D^3}{18m_b^3} + \mathcal{O}(1/m_b^4) . \end{aligned} \tag{3.88}$$

From this structure it is evident that a re-summation scheme would be desirable in which the leading contribution to each moment is re-summed. In order to set this up we take a look at the tree-level calculation of $B \to X_s \gamma$. Taking the time ordered product of two effective Hamiltonians from (3.83) and using the external-field propagator as in (3.20) (in this case of the massless s quark) we have

$$\frac{1}{\not{Q} + i\not{D}} = \frac{\not{Q} + i\not{D}}{Q^2 + 2(Q \cdot iD) + (i\not{D})^2} \tag{3.89}$$

where $Q = m_b v - q$, and q is the photon momentum. In the case where Q^2 is large compared to the terms with the covariant derivatives, one obtains the usual power counting and we may perform the expansion as in (3.20) with $m_c \to 0$. However, we have $Q^2 = m_b^2(1-y)$ and thus this quantity is not large compared to the other terms in the denominator, in which case we cannot expand as in (3.20). Instead we are in the kinematic region, where Q^2 is small and $v \cdot Q$ is of the order m_b.

The region we are interested in is the one where Q^2 and $(Q \cdot iD)$ are of the same order, which is $m_b \Lambda_{\rm QCD}$. Note that this is not the resonance region, where - as discussed above - Q^2 is actually of order $\Lambda_{\rm QCD}^2$. Thus in the endpoint region $m_b(1-y) \sim \Lambda_{\rm QCD}$ we can re-sum the leading contributions to the moments by approximating

$$\frac{1}{\not{Q} + i\not{D}} = \frac{\not{Q}}{Q^2 + 2(Q \cdot iD)} + \cdots \tag{3.90}$$

Since Q is (almost) a light-like vector, it is convenient to introduce light cone vectors n and $\bar{n}$ according to

$$n^2 = 0 = \bar{n}^2 \quad n \cdot \bar{n} = 2 \quad v = \frac{n + \bar{n}}{2} \quad \text{and} \quad Q \cdot iD \approx (v \cdot Q)(n \cdot iD) . \tag{3.91}$$

Introducing the shape function (or light-come distribution function) f according to

$$2M_B f(\omega) = \langle B(v) | \bar{b}_v \delta(\omega + i(n \cdot D)) | B(v) \rangle$$

allows us now to write this as a convolution, so we get (e.g. for $\Gamma = 1$)

$$\langle B(v)|\bar{b}_v \frac{\not{Q}}{Q^2 + 2(Q \cdot iD)} b_v |B(v)\rangle = \int d\omega\, f(\omega)\, \frac{v \cdot Q}{Q^2 + 2\omega(v \cdot Q)}\,. \tag{3.92}$$

The shape function is a nonperturbative function, where the moments of f are given in terms of HQE parameters

$$f(\omega) = \delta(\omega) + \frac{\mu_\pi^2}{6}\delta''(\omega) - \frac{\rho_D^3}{18}\delta'''(\omega) + \cdots \tag{3.93}$$

Using this function and using (3.90) one obtains for the spectrum of $B \to X_s\gamma$

$$\begin{aligned} \frac{d\Gamma}{dy} &= \frac{\alpha G_F^2 m_b^6}{32\pi^4}|V_{tb}V_{ts}^*|^2|C_7(m_b)|^2 f(m_b(y-1)) \qquad (3.94)\\ &= \frac{\alpha G_F^2 m_b^5}{32\pi^4}|V_{tb}V_{ts}^*|^2|C_7(m_b)|^2 \left(\delta(1-y) + \frac{\mu_\pi^2}{6m_b^2}\delta''(1-y) - \frac{\rho_D^3}{18m_b^3}\delta'''(1-y) + \cdots\right) \end{aligned}$$

showing that the shape function indeed re-sums the most singular terms, i.e. the terms with the highest derivatives of δ functions.

The shape function f plays the same role as the parton distributions of DIS. It is genuinely non-perturbative, however, it is also universal. For the case of B decays, this means that this shape function appears in the end point regions $m_b(1-y) \sim \Lambda_{\rm QCD}$ of any inclusive heavy-to-light transition. In other words, it also appears in the description of the end-point region of $B \to X_u \ell\bar{\nu}$. This leads to a relation between this decay and $B \to X_s\gamma$ which is exploited in phenomenological analyses.

The shape function has a few interesting properties. First of all, we note that the first moment vanishes, i.e., the term with the first derivative of the δ function is absent. This is a consequence of the equations of motion. The second moment is non-vanishing; since this moment is taken with respect to the partonic end-point (defined by the quark mass), this means that the shape function has to extend beyond the partonic endpoint $y = 1$ corresponding to the photon energy $E_\gamma = m_b/2$ and $\omega = 0$. The shape function is non-vanishing for $-\infty \le \omega \le \bar{\Lambda}$, where the region $0 \le \omega \le \bar{\Lambda}$ is entirely non-perturbative. The parameter $\bar{\Lambda}$ is exactly the same as the one appearing in the expansion of the heavy hadron masses (2.32,2.33), since the true phase space (ignoring the masses of the final-state hadrons) has a maximal photon energy $E_\gamma^{\rm max} = m_B/2$. Thus the shape function ensures the correct phase-space boundary, which is given in terms of the B meson mass.

In fact, the vanishing of the first moment corresponds to a definition of the quark mass. A measurement of the photon spectrum of $B \to X_s\gamma$ yields directly the shape function, and the reference point for which the first moment vanishes yields a measurement of $\bar{\Lambda}$ and hence a definition of the quark mass.

All further discussion, including the way to include radiative corrections, requires more heavy machinery. Since the end-point region in heavy-to-light decays is related to (in the restframe of the B meson) energetic light degrees of freedom, the proper tool is in this case "Soft Collinear Effective Theory" which is beyond the scope of these lectures, and I refer the reader to the lectures of Thomas Becher at the same school.

4
Summary

In these lectures the physical foundations of heavy quark methods have been outlined, with a focus on technical issues. There are many aspects of heavy quark theory that could not be covered:

- Heavy quark expansions (HQET as well as HQE) have an enormous impact on particle physics phenomenology. Unitil the development of these methods, the hadronic matrix elements had to be modelled, introducing an uncontrollable systematic uncertainty into the theoretical predictions. Heavy quark methods have not made models fully obsolete; however, the use of a model is often necessary only for an estimate of subleading terms, for which one cannot make use of a QCD lattice calculation. To this end, many constraints on physics beyond the Standard Model coming from heavy flavour physics could be made much more stringent on the basis of heavy quark methods.
- There are still other types of heavy quark methods, which could not be covered by the lectures. As an example, in an exclusive non-leptonic decay like $B \to \pi\pi$, the two pions have (in the rest frame of the B meson) energies of the order of the B meson mass, thus there are light quarks and gluons which have large momenta. Since the settings we have discussed in these lectures are such that the light degrees of feedom have "small" momenta, it becomes clear that a different effective theory needs to be used in such cases. The relevant theory for this case is "Soft Collinear Effective Theory" (SCET) which has been invented in the context of B decays, but has many applications also in collider physics. There has been an extra lecture on SCET at this school.
- Another class of heavy quark systems are hadrons with two heavy quarks, such as the quarkonia $b\bar{b}$ and $c\bar{c}$ as well as the $B_c = (b\bar{c})$ or "doubly heavy baryons" containing two or even three heavy quarks. Also in this case one has to set up a slightly modified effective theory, since the static approximation turns out to be insufficient. Rather one needs to include the kinetic energy $\vec{p}^2/(2m)$ into the leading Lagrangian. This leads to non-relativistic QCD (NRQCD) which corresponds to the leading term of a systematic expansion in the relative velocity v between the heavy constituents. The structure of NRQCD turns out to be also more complicated than HQET, since the dynamics of binding generates mass scales, which are the inverse Bohr radius $m_Q v$ and the binding energy $m_Q v^2$. For small v this generates a hierarchy of mass scales, and - depending on the sizes of the dynamically generated mass scales relative to $\Lambda_{\rm QCD}$ - requires to set up different effective field theories.

Overall, heavy quark methods have put the flavour physics of heavy hadrons onto a solid basis, allowing us to perform in many cases precision calculations including the control over uncertainties. Starting from the original idea encoded in HQET, more and more applications have been discovered and elaborated; it seems that these ideas have even still a potential which is not yet fully explored.

References

[1] E. Eichten and B. R. Hill, Phys. Lett. B **234**, 511 (1990). doi:10.1016/0370-2693(90)92049-O
[2] E. Eichten and B. R. Hill, Phys. Lett. B **243**, 427 (1990). doi:10.1016/0370-2693(90)91408-4
[3] B. Grinstein, Nucl. Phys. B **339**, 253 (1990). doi:10.1016/0550-3213(90)90349-I
[4] H. Georgi, Phys. Lett. B **240**, 447 (1990). doi:10.1016/0370-2693(90)91128-X
[5] M. A. Shifman and M. B. Voloshin, Sov. J. Nucl. Phys. **47**, 511 (1988) [Yad. Fiz. **47**, 801 (1988)].
[6] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989). doi:10.1016/0370-2693(89)90566-2
[7] N. Isgur and M. B. Wise, Phys. Lett. B **237**, 527 (1990). doi:10.1016/0370-2693(90)91219-2
[8] J. Chay, H. Georgi and B. Grinstein, Phys. Lett. B **247**, 399 (1990). doi:10.1016/0370-2693(90)90916-T
[9] I. I. Y. Bigi, N. G. Uraltsev and A. I. Vainshtein, Phys. Lett. B **293**, 430 (1992) Erratum: [Phys. Lett. B **297**, 477 (1992)] doi:10.1016/0370-2693(92)90908-M, 10.1016/0370-2693(92)91287-J [hep-ph/9207214].
[10] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Phys. Rev. Lett. **71**, 496 (1993) doi:10.1103/PhysRevLett.71.496 [hep-ph/9304225].
[11] A. V. Manohar and M. B. Wise, Phys. Rev. D **49**, 1310 (1994) doi:10.1103/PhysRevD.49.1310 [hep-ph/9308246].
[12] T. Mannel, Nucl. Phys. B **413**, 396 (1994) doi:10.1016/0550-3213(94)90625-4 [hep-ph/9308262].
[13] M. Neubert, Phys. Rept. **245**, 259 (1994) doi:10.1016/0370-1573(94)90091-4 [hep-ph/9306320].
[14] I. I. Y. Bigi, M. A. Shifman and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. **47**, 591 (1997) doi:10.1146/annurev.nucl.47.1.591 [hep-ph/9703290].
[15] B. Grinstein, In *Boulder 1994, Proceedings, CP violation and the limits of the standard model* 307-375, and Calif. U. San Diego - UCSD-PTH-94-24 (94/11,rec.Dec.) 74 p [hep-ph/9411275].
[16] T. Mannel, Rept. Prog. Phys. **60**, 1113 (1997). doi:10.1088/0034-4885/60/10/003
[17] A. G. Grozin, *Heavy quark effective theory*, Springer Tracts Mod. Phys. **201**, 1 (2004). doi:10.1007/b79301
[18] T. Mannel, *Effective Field Theories in Flavour Physics*, Springer Tracts Mod. Phys. **203**, 1 (2004). doi:10.1007/b79301
[19] A. V. Manohar and M. B. Wise, *Heavy quark physics*, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10**, 1 (2000).
[20] A. A. Petrov and A. E. Blechman, *Effective Field Theories*, World Scientific 2016

[21] T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. B **368**, 204 (1992). doi:10.1016/0550-3213(92)90204-O

[22] S. Balk, J. G. Korner and D. Pirjol, Nucl. Phys. B **428**, 499 (1994) doi:10.1016/0550-3213(94)90211-9 [hep-ph/9307230].

[23] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937). doi:10.1103/PhysRev.52.54

[24] D. R. Yennie, S. C. Frautschi and H. Suura, Annals Phys. **13**, 379 (1961). doi:10.1016/0003-4916(61)90151-8

[25] M. J. Dugan, M. Golden and B. Grinstein, Phys. Lett. B **282**, 142 (1992). doi:10.1016/0370-2693(92)90493-N

[26] M. E. Luke and A. V. Manohar, Phys. Lett. B **286**, 348 (1992) doi:10.1016/0370-2693(92)91786-9 [hep-ph/9205228].

[27] Y. Q. Chen, Phys. Lett. B **317**, 421 (1993). doi:10.1016/0370-2693(93)91018-I

[28] J. Heinonen, R. J. Hill and M. P. Solon, Phys. Rev. D **86**, 094020 (2012) doi:10.1103/PhysRevD.86.094020 [arXiv:1208.0601 [hep-ph]].

[29] M. Ademollo and R. Gatto, Phys. Rev. Lett. **13**, 264 (1964). doi:10.1103/PhysRevLett.13.264

[30] M. E. Luke, Phys. Lett. B **252**, 447 (1990). doi:10.1016/0370-2693(90)90568-Q

[31] A. F. Falk, Nucl. Phys. B **378**, 79 (1992). doi:10.1016/0550-3213(92)90004-U

[32] M. Neubert and C. T. Sachrajda, Nucl. Phys. B **483**, 339 (1997) doi:10.1016/S0550-3213(96)00559-7 [hep-ph/9603202].

[33] A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. B **343**, 1 (1990). doi:10.1016/0550-3213(90)90591-Z

[34] W. Kilian, T. Mannel and T. Ohl, Phys. Lett. B **304**, 311 (1993) doi:10.1016/0370-2693(93)90301-W [hep-ph/9303224].

[35] M. Neubert, Phys. Lett. B **264**, 455 (1991). doi:10.1016/0370-2693(91)90377-3

[36] K. Melnikov, Phys. Lett. B **666**, 336 (2008) doi:10.1016/j.physletb.2008.07.089 [arXiv:0803.0951 [hep-ph]].

[37] B. Grinstein and A. Kobach, Phys. Lett. B **771**, 359 (2017) doi:10.1016/j.physletb.2017.05.078 [arXiv:1703.08170 [hep-ph]].

[38] J. C. Collins, *Renormalization : An Introduction to Renormalization, The Renormalization Group, and the Operator Product Expansion*

[39] E. C. Poggio, H. R. Quinn and S. Weinberg, Phys. Rev. D **13**, 1958 (1976). doi:10.1103/PhysRevD.13.1958

[40] M. A. Shifman, doi:10.1142/9789812810458-0032 hep-ph/0009131.

[41] B. M. Dassinger, T. Mannel and S. Turczyk, JHEP **0703**, 087 (2007) doi:10.1088/1126-6708/2007/03/087 [hep-ph/0611168].

[42] B. Guberina, S. Nussinov, R. D. Peccei and R. Ruckl, Phys. Lett. **89B**, 111 (1979). doi:10.1016/0370-2693(79)90086-8

[43] T. Mannel, S. Turczyk and N. Uraltsev, JHEP **1011**, 109 (2010) doi:10.1007/JHEP11(2010)109 [arXiv:1009.4622 [hep-ph]].

[44] M. Beneke and V. M. Braun, Nucl. Phys. B **426**, 301 (1994) doi:10.1016/0550-3213(94)90314-X [hep-ph/9402364].

[45] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Phys. Rev. D **50**, 2234 (1994) doi:10.1103/PhysRevD.50.2234 [hep-ph/9402360].

[46] M. Neubert and C. T. Sachrajda, Nucl. Phys. B **438**, 235 (1995) doi:10.1016/0550-3213(95)00032-N [hep-ph/9407394].

[47] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Phys. Rev. D **52**, 196 (1995) doi:10.1103/PhysRevD.52.196 [hep-ph/9405410].
[48] M. Antonelli *et al.*, Phys. Rept. **494**, 197 (2010) doi:10.1016/j.physrep.2010.05.003 [arXiv:0907.5386 [hep-ph]].
[49] A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. Lett. **82**, 277 (1999) doi:10.1103/PhysRevLett.82.277 [hep-ph/9809423].
[50] A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. D **59**, 074017 (1999) doi:10.1103/PhysRevD.59.074017 [hep-ph/9811239].
[51] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Int. J. Mod. Phys. A **9**, 2467 (1994) doi:10.1142/S0217751X94000996 [hep-ph/9312359].
[52] M. Neubert, Phys. Rev. D **49**, 3392 (1994) doi:10.1103/PhysRevD.49.3392 [hep-ph/9311325].
[53] T. Mannel and M. Neubert, Phys. Rev. D **50**, 2037 (1994) doi:10.1103/PhysRevD.50.2037 [hep-ph/9402288].
[54] G. Altarelli, N. Cabibbo, G. Corbo, L. Maiani and G. Martinelli, Nucl. Phys. B **208**, 365 (1982). doi:10.1016/0550-3213(82)90226-7
[55] M. Neubert, Phys. Rev. D **49**, 4623 (1994) doi:10.1103/PhysRevD.49.4623 [hep-ph/9312311].