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Gravitational Waves - A new window to the Universe

THEORY OF THE GRAVITATIONAL WAVES

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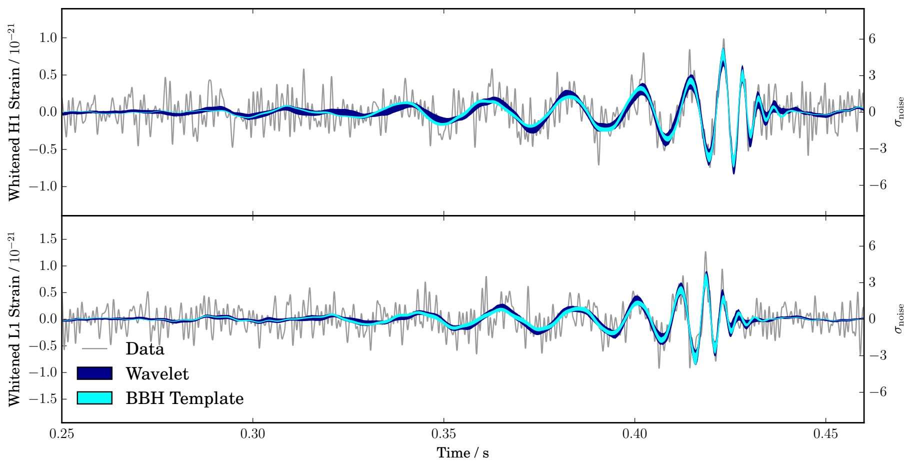
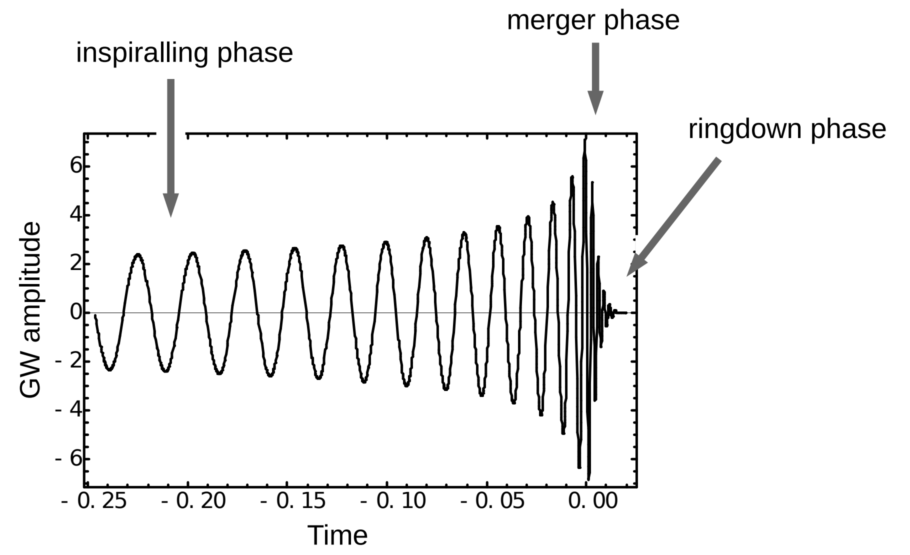
6 Juillet 2021

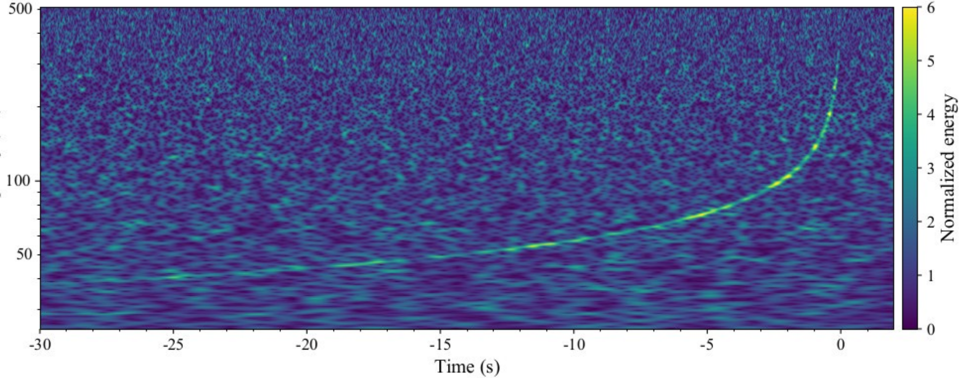
Binary black-hole event GW150914 [LIGO/Virgo 2016]



The gravitational chirp of binary black holes



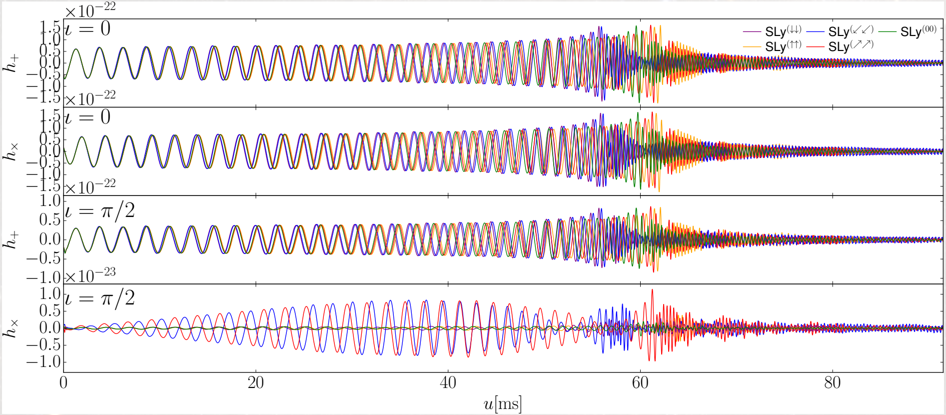
Binary neutron star event GW170817 [LIGO/Virgo 2017]



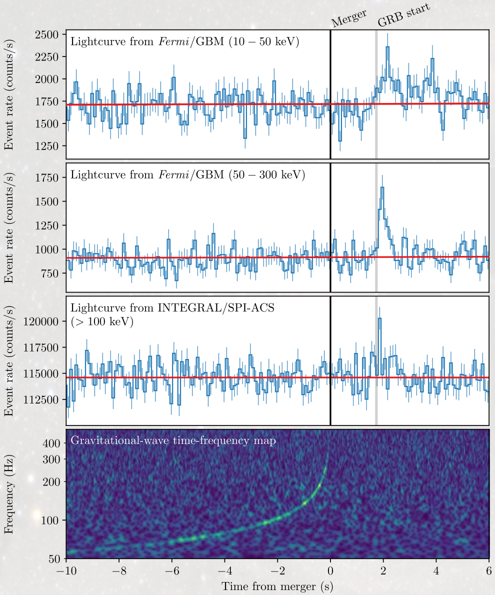
- The signal is observed during $\sim 100\,\mathrm{s}$ and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\,M_{\odot}$
- The distance is measured from the gravitational signal as $R = 40$ Mpc

Post-merger waveform of neutron star binaries

[Dietrich, Bernuzzi, Bruegmann, Ujevic & Tichy 2018]

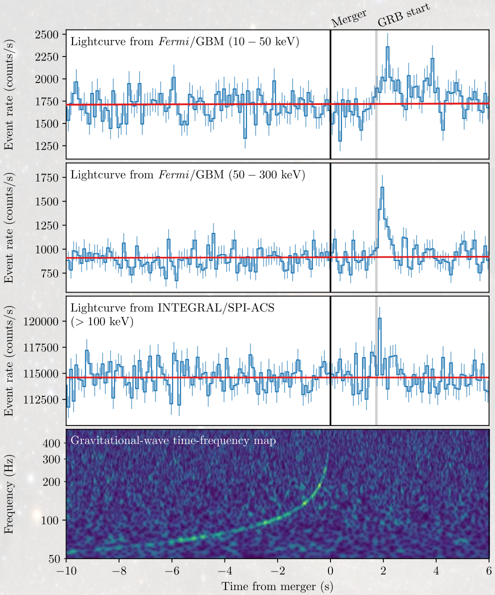


The advent of multi-messenger astronomy



- The gamma-ray burst has been detected 1.7 second after the instant of merger
- This is the closest gamma-ray burst whose distance is known and is probably seen off-axis with respect to the relativistic jet

Speed of gravitational waves versus speed of light



- The observed time delay between GW170817 and GRB170817A gives a strong constraint

$$\boxed{|c_\mathrm{g} - c_\mathrm{em}| \lesssim 10^{-15} c}$$

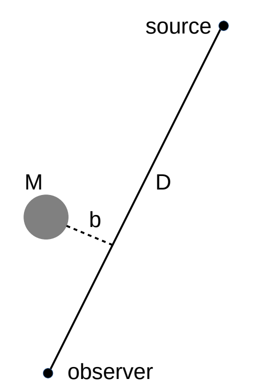
- This eliminated a series of alternative theories

[Bettoni *et al.*, Creminelli & Vernizzi 2017]

Test of the strong equivalence principle [Desai & Kahya 2016]

- The test involves the cumulative Shapiro time delay due to the gravitational potential of the dark matter distribution
- The violation of the equivalence principle is quantified by a PPN like parameter γ_a depending on the type of radiation $a = \text{GW}, \text{EM}$. For a spherical mass distribution

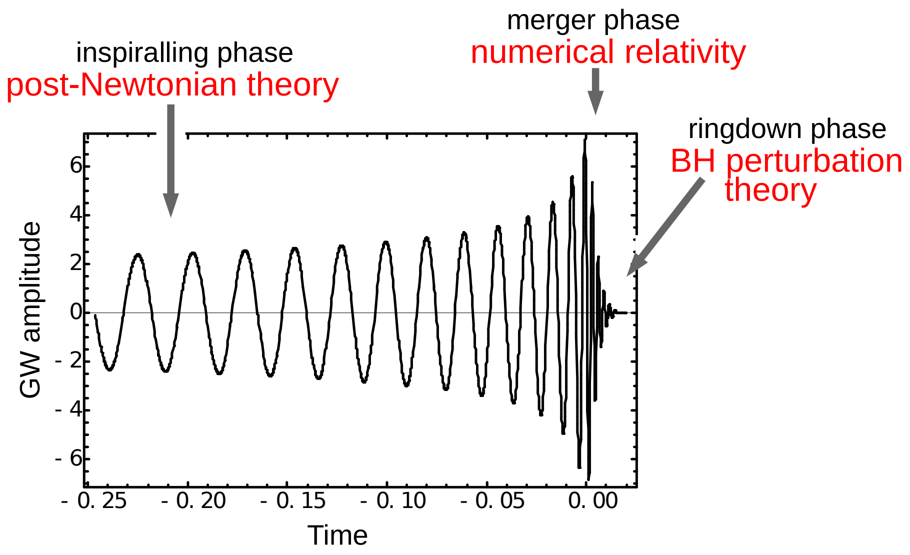
$$\Delta t^a_{\text{Shapiro}} = (1 + \gamma_a) \frac{GM}{c^3} \ln \left(\frac{D}{b} \right)$$



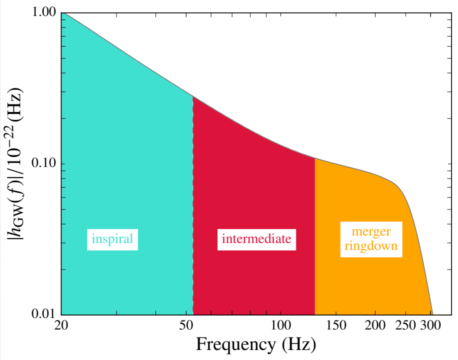
- The main contributions come from the galaxy NGC4993 and our own Galaxy with mass $M_{\text{MW}} = 5.6\,10^{11}\,M_{\odot}$
- Assuming an isothermal density profile for dark matter this yields about 400 days delay in GR
- The observed difference in arrival time $\Delta t = 1.7\,\text{s}$ yields

$$\boxed{|\gamma_{\text{GW}} - \gamma_{\text{EM}}| \lesssim 10^{-7}}$$

Methods to compute gravitational waves



The inspiral-merger-ringdown models

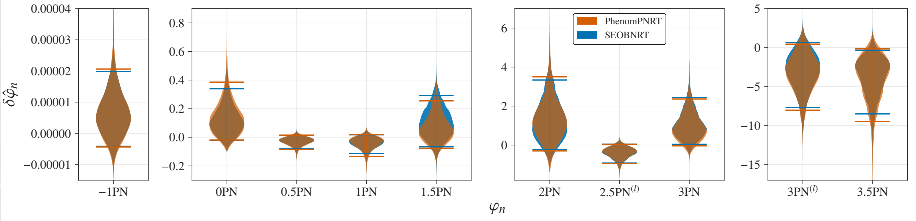


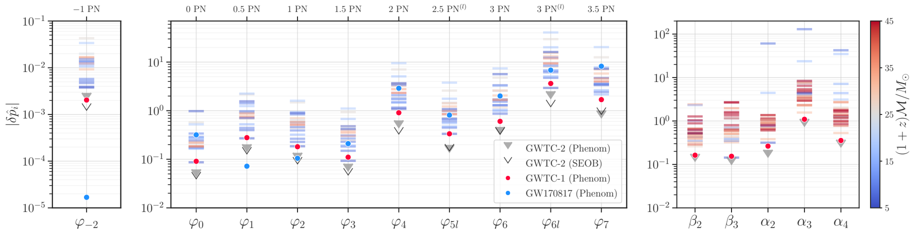
These models interpolate between the different phases play a crucial role

- The effective-one-body (EOB) approach [Buonanno & Damour 1999]
- The inspiral-merger-ringdown (IMR) [Ajith *et al.* 2008]

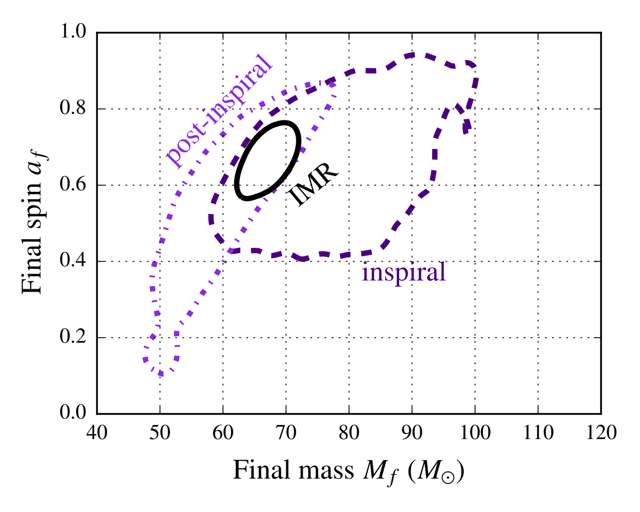
$$\Big\{\underbrace{\text{PN parameters};}_{\text{inspiral}}\;\underbrace{\beta_2,\beta_3}_{\text{intermediate}}\;;\;\underbrace{\alpha_2,\alpha_3,\alpha_4}_{\text{merger-ringdown}}\Big\}$$

Measurement of PN parameters [LIGO/Virgo]





Inspiral-Merger-Ringdown consistency test [LIGO/Virgo]



Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1945]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu}\dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu}\dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

$$\boxed{h_{ij}^{\text{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Radiation reaction and balance equations

1. Conserved Newtonian energy in the source

$$E = \int \mathrm{d}^3\mathbf{x}\,\rho\left[\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right]$$

2. Eulerian equations of motion in the source

$$\rho\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\partial_i P + \rho\partial_i U - \overbrace{\frac{2G}{5c^5}\rho\, x^j\,\frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}}^{\boldsymbol{F}^{\text{reac}}}$$

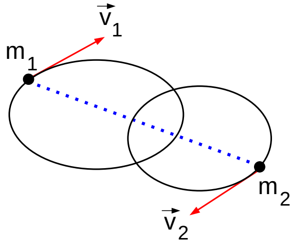
3. Energy loss is due to the work of the radiation reaction force

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\boldsymbol{v}\cdot\boldsymbol{F}^{\text{reac}} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \text{total time derivative}}$$

4. Obtain the balance equation after averaging over one period

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}^{\text{GW}}\rangle \quad\Longrightarrow\quad \phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\left\{\begin{array}{l} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{array}\right.$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

Averaged energy and angular momentum balance equations

$$\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = -\langle\mathcal{F}^{\text{GW}}\rangle \qquad \left\langle\frac{\mathrm{d}J_i}{\mathrm{d}t}\right\rangle = -\langle\mathcal{G}_i^{\text{GW}}\rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\boxed{\begin{aligned} \left\langle\frac{\mathrm{d}P}{\mathrm{d}t}\right\rangle &= -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi GM}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \\ \left\langle\frac{\mathrm{d}e}{\mathrm{d}t}\right\rangle &= -\frac{608\pi}{15c^5}\nu\frac{e}{P}\left(\frac{2\pi GM}{P}\right)^{5/3}\frac{1+\frac{121}{304}e^2}{(1-e^2)^{5/2}} \end{aligned}}$$

Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. Compact binaries are circularized when they enter the detector's bandwidth

$$\boxed{E = -\frac{Mc^2}{2}\nu\, x \qquad \mathcal{F}^{\mathsf{GW}} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5}$$

   where $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ denotes a small PN parameter defined with ω

2. Equating $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$ gives a differential equation for x

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}x^5 \quad \Longleftrightarrow \quad \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}}$$

3. This permits to solve for the orbital phase

$$\phi = \int \omega\, \mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

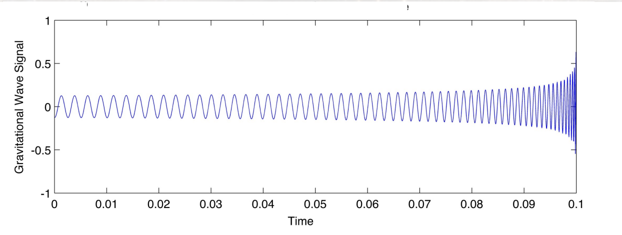
Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. The amplitude and phase evolution follow an adiabatic chirp in time

$$\boxed{\begin{aligned} a(t) &= \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4} \\ \phi(t) &= \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8} \end{aligned}}$$

2. The amplitude and orbital frequency diverge at the instant of coalescence t_c and the merger phase is to be described numerically



The quadrupole formula works for GW150914

- The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi} \left[\frac{256}{5} \frac{G\mathcal{M}^{5/3}}{c^5} (t_c - t) \right]^{-3/8}$$

- Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96} \frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f} \right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

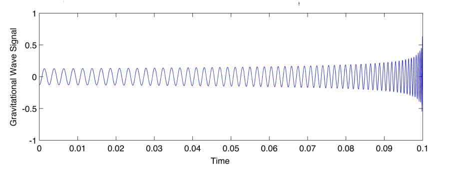
- The GW amplitude is predicted to be[1]

$$h_\text{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left(\frac{100\,\text{Mpc}}{R} \right) \left(\frac{100\,\text{Hz}}{f_\text{merger}} \right)^{-1/6} \sim 1.6 \times 10^{-21}$$

- The distance $R = 400\,\text{Mpc}$ is measured from the signal itself [Schutz 1986]

[1] $h_\text{eff} \sim h\sqrt{N}$ where $N \sim \omega^2/\dot{\omega}$ is the number of cycles around frequency ω

PN parameters in the orbital phase evolution



- The PN parameters come from a mixture of conservative and dissipative effects through the energy balance equation

$$\frac{\mathrm{d}\overbrace{E}^{\text{conservative energy}}}{\mathrm{d}t} = - \underbrace{\mathcal{F}^{\text{GW}}}_{\text{dissipative energy flux}}$$

- The orbital phase $\phi = \int \omega\, \mathrm{d}t$ is obtained as a function of $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ and the symmetric mass ratio $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$

$$\boxed{\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \Big(\varphi_{p\text{PN}}(\nu) + \varphi_{p\text{PN}}^{(l)}(\nu)\, \log x \Big)\, x^p + \mathcal{O}[(\log x)^2]}$$

The known 3.5PN parameters [Blanchet 2014 Living Review in Relativity]

They are computed with the Multipolar-post-Minkowskian-PN formalism

$$\varphi_{\text{0PN}} = 1 \qquad \Longleftarrow \quad \text{Einstein quadrupole formula}$$

$$\varphi_{\text{1PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{\text{1.5PN}} = -10\pi$$

$$\varphi_{\text{2PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(l)}_{\text{2.5PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

$$\begin{aligned}\varphi_{\text{3PN}} &= \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_{\text{E}} - \frac{3424}{21}\ln 2 \\ &\quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\end{aligned}$$

$$\varphi^{(l)}_{\text{3PN}} = -\frac{856}{21}$$

$$\varphi_{\text{3.5PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Tail effects in PN parameters

tail terms

$$\varphi_{0\mathrm{PN}} = 1$$

$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = \boxed{-10\pi}$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

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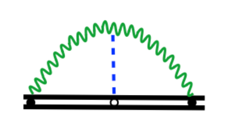
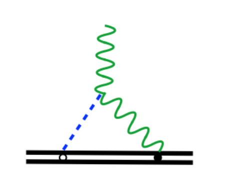
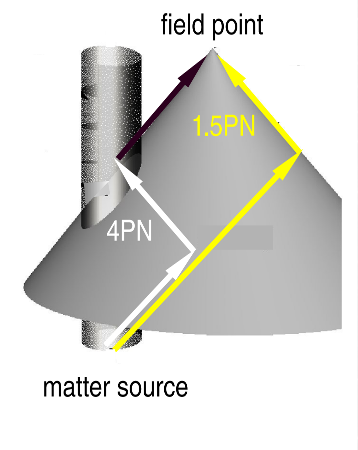
The gravitational wave tail effect [Blanchet & Damour 1988, 1992]







- In the near zone (4PN effect)

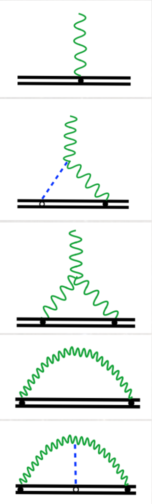
$$\boxed{S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')}$$

- In the far zone (1.5PN effect)

$$\boxed{h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{u} \mathrm{d}u'\, I^{(4)}_{ij}(u') \ln\left(\frac{u-u'}{P}\right)}$$

Diagrammatic expansion in EFT

Effective Field Theory



$\Longleftrightarrow$

Post-Newtonian

- emission from a quadrupole source
- tail effect in radiation field (1.5PN)
- non-linear memory effect (2.5PN)
- radiation reaction (2.5PN)
- tail in radiation reaction (4PN)

The EFT is equivalent to the traditional PN at the level of tree diagrams

Tail effects in PN parameters

tail terms

tail-of-tail terms

$$\varphi_{0\mathrm{PN}} = 1$$

$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = \boxed{-10\pi}$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(l)}_{2.5\mathrm{PN}} = \boxed{\left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi}$$

$$\varphi_{3\mathrm{PN}} = \frac{12348611926451}{18776862720} \boxed{- \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{3424}{21}\ln 2}$$
$$+ \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3$$

$$\varphi^{(l)}_{3\mathrm{PN}} = \boxed{-\frac{856}{21}}$$

$$\varphi_{3.5\mathrm{PN}} = \boxed{\left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi}$$

Toward 4.5PN parameters

- The 4.5PN term is also known and due to the 4.5PN tail-of-tail-of-tail integral for circular orbits [Marchand, Blanchet & Faye 2017; Messina & Nagar 2017]

$$\varphi_{4.5\text{PN}} = \left(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_\text{E} + \frac{3424}{21}\ln 2 + \left[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3 \right)\pi$$

$$\varphi^{(l)}_{4.5\text{PN}} = \frac{856}{21}\pi$$

tail-of-tail-of-tail terms

- However the 4PN term is only known from perturbative BH theory in the test-mass limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\varphi_{4\text{PN}} = \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_\text{E} - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu)$$

$$\varphi^{(l)}_{4\text{PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

The 4.5PN radiative quadrupole moment

$$
\begin{aligned}
U_{ij}(t) = I^{(2)}_{ij}(t) &+ \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}} \\
&+ \frac{G}{c^5}\Bigg\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\Bigg\} \\
&+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}} \\
&+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}} \\
&+ \mathcal{O}\left(\frac{1}{c^{10}}\right)
\end{aligned}
$$

The source type multipole moments

Following the matching between the near zone and the exterior zone

$$\boxed{\begin{aligned} I_L &= \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x}\,\left(\tfrac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \biggl\{ \delta_\ell\,\hat{x}_L\,\overline{\Sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\delta_{\ell+1}\,\hat{x}_{iL}\,\overline{\Sigma}_i^{(1)} \\ &\qquad + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\,\delta_{\ell+2}\,\hat{x}_{ijL}\,\overline{\Sigma}_{ij}^{(2)} \biggr\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right) \\ J_L &= \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x}\,\left(\tfrac{r}{r_0}\right)^B \varepsilon_{ab\langle i_\ell} \int_{-1}^{1} \mathrm{d}z \biggl\{ \delta_\ell \hat{x}_{L-1\rangle a} \Sigma_b \\ &\qquad - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1\rangle ac} \overline{\Sigma}_{bc}^{(1)} \biggr\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right) \end{aligned}}$$

$$\overline{\Sigma} = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \qquad \overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c} \qquad \overline{\Sigma}_{ij} = \overline{\tau}^{ij}$$

where $\overline{\tau}^{\mu\nu}$ represents the PN expansion of the matter + gravitation stress-energy pseudo tensor (*a priori* valid only in the near zone)

The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet 2020]

- Using dimensional regularisation for UV but Hadamard regularization for IR

$$I_{ij} = \mu\, A\, x_{\langle i}x_{j\rangle} + \cdots + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$\boxed{\begin{aligned} A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\ &\quad + \underbrace{\gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \cdots\right)}_{\text{3PN terms}} \\ &\quad + \underbrace{\gamma^4\left(-\frac{1023844001989}{12713500800} + \frac{31886}{2205}\ln\left(\frac{r}{r_0}\right) + \cdots\right)}_{\text{4PN terms}} \\ C &= \overbrace{\frac{48}{7} + \gamma\left(-\frac{4096}{315} - \frac{24512}{945}\nu\right)}^{\text{2.5PN and 3.5PN terms}} \end{aligned}}$$

- This result has to be completed by dimensional regularization for the IR

The 3.5PN gravitational-wave $(\ell, m) = (2,2)$ mode

$$h_+ - \mathrm{i} h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}\, Y_{-2}^{\ell m}(\Theta, \Phi)$$

- The modes can be compared directly with results from numerical relativity
- The dominant mass-type quadrupole mode is

$$\boxed{\begin{aligned} H^{22} &= 1 + x\left(-\tfrac{107}{42} + \tfrac{55}{42}\nu\right) + 2\pi x^{3/2} \\ &+ x^2\left(-\tfrac{2173}{1512} - \tfrac{1069}{216}\nu + \tfrac{2047}{1512}\nu^2\right) + x^{5/2}\left(-\tfrac{107\pi}{21} - 24\,\mathrm{i}\,\nu + \tfrac{34\pi}{21}\nu\right) \\ &+ x^3\Bigg(\frac{27027409}{646800} - \frac{856}{105}\,\gamma_\mathsf{E} + \frac{428\,\pi}{105}\mathrm{i} + \frac{2\pi^2}{3} \\ &\qquad + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261}{2772}\nu^2 + \frac{114635}{99792}\nu^3 - \frac{428}{105}\ln(16x)\Bigg) \\ &+ x^{7/2}\left(-\tfrac{2173\pi}{756} + \left(-\tfrac{2495\pi}{378} + \tfrac{14333}{162}\mathrm{i}\right)\nu + \left(\tfrac{40\pi}{27} - \tfrac{4066}{945}\mathrm{i}\right)\nu^2\right) \end{aligned}}$$

- The $(\ell, m) = (2,2)$ mode at 4PN order is in progress

The 3PN current type quadrupole moment

[Henry, Faye & Blanchet 2021]

- We need dimensional regularisation for the UV but Hadamard regularization is sufficient for the IR
- To apply dimensional regularization we define the decomposition of a tensor into irreducible pieces in d dimensions (where we do not have the usual ε_{ijk} to define the current moment)
- The mass moment I_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $J_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

$$I_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline\end{array}$$

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & \multicolumn{3}{c}{} \\ \cline{1-2}\end{array}$$

- The tensor $K_{ij|L}$ is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

The 3PN current type quadrupole moment

[Henry, Faye & Blanchet 2021]

- After dimensional regularization and renormalization

$$J_{ij} = -\mu\Delta\left[A\,L^{\langle i}x^{j\rangle} + \cdots\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

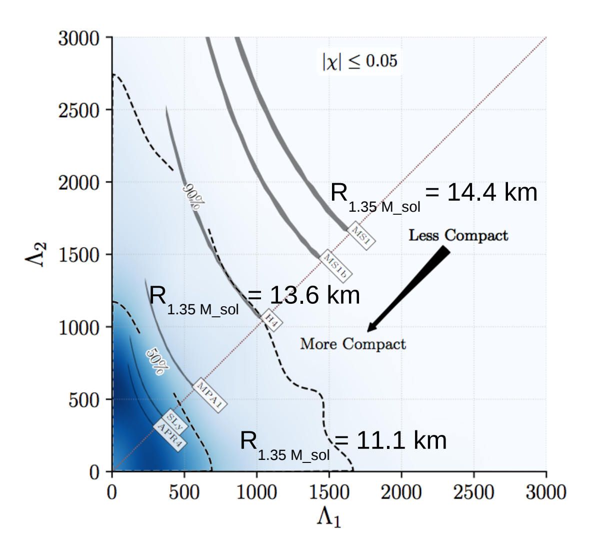
$$A = 1 + \gamma\big(\tfrac{67}{28} - \tfrac{2}{7}\nu\big) + \gamma^2\left(\tfrac{13}{9} - \tfrac{4651}{252}\nu - \tfrac{\nu^2}{168}\right) + \cdots$$

- The corresponding $(\ell, m) = (2,1)$ mode at 3.5PN order reads

$$\boxed{\begin{aligned} H^{21} = \frac{\mathrm{i}}{3}\,\Delta\Big[& x^{1/2} + x^{3/2}\big(-\tfrac{17}{28} + \tfrac{5\nu}{7}\big) + x^2\big(\pi + \mathrm{i}\big[-\tfrac{1}{2} - 2\ln 2\big]\big) \\ & + x^{5/2}\big(-\tfrac{43}{126} - \tfrac{509\nu}{126} + \tfrac{79\nu^2}{168}\big) \\ & + x^3\big(\pi\big[-\tfrac{17}{28} + \tfrac{3\nu}{14}\big] + \mathrm{i}\big[\tfrac{17}{56} + \nu\big(-\tfrac{353}{28} - \tfrac{3}{7}\ln 2\big) + \tfrac{17}{14}\ln 2\big]\big) \\ & + x^{7/2}\Big(\tfrac{15223771}{1455300} + \tfrac{\pi^2}{6} - \tfrac{214}{105}\gamma_\mathsf{E} - \tfrac{107}{105}\ln(4x) - \ln 2 - 2(\ln 2)^2 \\ & \quad + \nu\big[-\tfrac{102119}{2376} + \tfrac{205}{128}\pi^2\big] - \tfrac{4211}{8316}\nu^2 + \tfrac{2263}{8316}\nu^3 + \mathrm{i}\pi\big[\tfrac{109}{210} - 2\ln 2\big]\Big)\Big] \end{aligned}}$$

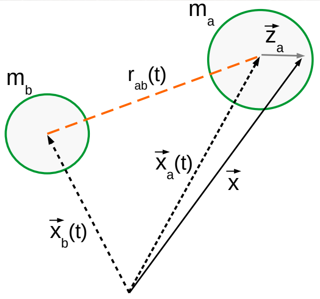
Constraining the neutron star equation of state

[LIGO/Virgo 2017]



$$\Lambda_a = \frac{2}{3}\underbrace{k_a}_{\text{Love number}}\left(\frac{c^2 R_a}{G m_a}\right)^5$$

Equations of motion of N extended bodies



$$m_a = \int_{\mathcal{V}_a} \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x},t) \qquad \boldsymbol{x}_a(t) = \frac{1}{m_a}\int_{\mathcal{V}_a} \mathrm{d}^3\mathbf{x}\,\mathbf{x}\,\rho(\mathbf{x},t)$$

$$\mathbf{x} = \boldsymbol{x}_a(t) + \boldsymbol{z}_a(\mathbf{x},t) \qquad Q_a^{ij} = \int_{\mathcal{V}_a} \mathrm{d}^3\boldsymbol{z}_a\,\rho_a(\boldsymbol{z}_a,t)\Big(z_a^i z_a^j - \frac{1}{3}\delta^{ij}\boldsymbol{z}_a^2\Big)$$

$$\boxed{\alpha \sim \frac{|\boldsymbol{z}_a|}{r_{ab}} \ll 1}$$

Equations of motion of N extended bodies

1. The Newtonian equations of motion of extended (spinless) bodies are

$$m_a \frac{\mathrm{d}v_a^i}{\mathrm{d}t} = G \sum_{b \neq a} \left[m_a m_b \frac{\partial}{\partial x_a^i} \left(\frac{1}{r_{ab}} \right) + \overbrace{\frac{1}{2} \left(m_a\, Q_b^{jk} + m_b\, Q_a^{jk} \right) \frac{\partial^3}{\partial x_a^i \partial x_a^j \partial x_a^k} \left(\frac{1}{r_{ab}} \right)}^{\text{effect of the quadrupole moments}} \right]$$

2. The conserved energy of the N-body system is the sum of the internal energies E_a and of the orbital contributions

$$E = \sum_a \left\{ E_a + \frac{1}{2} m_a \boldsymbol{v}_a^2 - \frac{G}{2} \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} - \frac{1}{2} Q_a^{ij}\, \mathcal{G}_a^{ij} \right\}$$

3. The tidal quadrupole moment felt by the body a is

$$\boxed{\mathcal{G}_a^{ij} = \frac{\partial^2 U_a}{\partial x_a^i \partial x_b^j} \quad \text{where} \quad U_a = \sum_{b \neq a} \frac{G m_b}{r_{ab}}}$$

Equations of motion of N extended bodies

1. The coupling of the quadrupole moments with the external tidal field $\mathcal{G}_a^{ij}$ implies a variation of the internal energy given by
$$\frac{\mathrm{d}E_a}{\mathrm{d}t} = \frac{1}{2}\dot{Q}_a^{ij}\,\mathcal{G}_a^{ij}$$
2. Neglecting tidal dissipation we assume that the quadrupole moment is aligned with the tidal field
$$\boxed{Q_a^{ij} = \mu_a\,\mathcal{G}_a^{ij}}$$
where μ_a is a deformability or polarizability coefficient
3. The conserved energy of the system simplifies in this case
$$E = \sum_a \left\{ \frac{1}{2}m_a \boldsymbol{v}_a^2 - \frac{G}{2}\sum_{b\neq a}\frac{m_a m_b}{r_{ab}} - \frac{\mu_a}{4}\,\mathcal{G}_a^{ij}\mathcal{G}_a^{ij} \right\}$$
4. Very importantly the dynamics admits a Lagrangian formulation
$$\boxed{L = \sum_a \left\{ \frac{1}{2}m_a \boldsymbol{v}_a^2 + \frac{G}{2}\sum_{b\neq a}\frac{m_a m_b}{r_{ab}} + \frac{\mu_a}{4}\,\mathcal{G}_a^{ij}\mathcal{G}_a^{ij} \right\}}$$

GW flux of extended two-body systems

1. We compute the GW flux using the quadrupole formula, where the total quadrupole moment of the system is ($x^i = x_1^i - x_2^i$)

$$Q^{ij} = \overbrace{m\nu\left(x^i x^j - \frac{1}{3}\delta^{ij} r^2\right)}^{\text{orbital quadrupole moment}} + Q_1^{ij} + Q_2^{ij}$$

2. For two bodies moving on a circular orbit this yields

$$\boxed{\mathcal{F}^{\mathrm{GW}} = \frac{32G}{5c^5} r^4 \omega^6 m^2 \nu^2 \left[1 + 6\big(m_1^4 \Lambda_1 + m_2^4 \Lambda_2\big) \frac{G^5 m}{r^5 c^{10}}\right]}$$

3. The internal structure is characterized by the dimensionless parameter

$$\Lambda_a = \frac{c^{10}\mu_a}{G^4 m_a^5} = \frac{2}{3} k_a \left(\frac{c^2 R_a}{G m_a}\right)^5$$

Influence of the internal structure on the phase

1. Applying the energy balance equation $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$ we obtain the modification of the phase due to the internal structure as

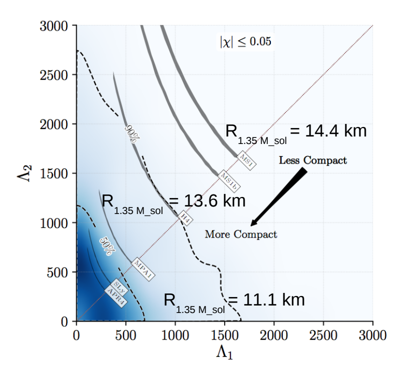
$$\boxed{\phi = \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \overbrace{\frac{39}{8}\tilde{\Lambda}\, x^5}^{\text{5PN effect}}\right]} \qquad x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$$

2. The tidal interaction on two bodies moving on a circular orbit depends on [Flanagan & Hinderer 2008]

$$\tilde{\Lambda} = \frac{16}{13}\left[\frac{(m_1 + 11m_2)m_1^3}{m^4}\Lambda_1 + \frac{(m_2 + 11m_1)m_2^3}{m^4}\Lambda_2\right]$$

3. The effect of the internal structure is formally a very small effect for compact objects comparable to an orbital correction of the order 5PN $\sim 1/c^{10}$

Dominant quadrupole tidal effect in BNS



- Tidal contribution to the GW chirp

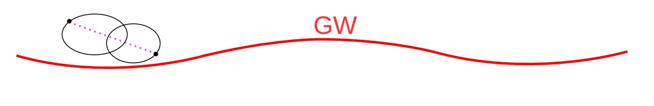
$$\boxed{\begin{aligned} x(t) &= \frac{1}{4}\theta^{-1/4}\left[1 + \frac{39}{8192}\tilde{\Lambda}\,\theta^{-5/4}\right] \\ \phi(t) &= \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \underbrace{\frac{39}{8}\tilde{\Lambda}\,x^5}_{\text{5PN effect}}\right] \end{aligned}}$$

with $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$ and $\theta = \frac{\nu c^3}{5Gm}(t_c - t)$

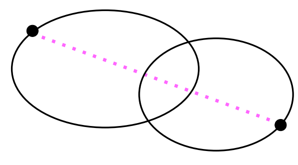
- The polarizability $\tilde{\Lambda}$ depends on the source mass of the NS (for a given EoS) while the point-particle part of the signal depends on the redshifted mass

Effective action for compact binary systems

- Hierarchy of length scales in a compact binary system



$$\Longleftarrow \quad \boxed{a \ll \lambda_{\mathrm{GW}} \sim \frac{ac}{v}}$$



$$\Longleftarrow \quad \boxed{R_1\,,\ R_2 \ll a}$$

- The Newtonian result can be reformulated as an effective matter action

$$\boxed{S_{\mathrm{eff}} = \sum_a \int \mathrm{d}t \Bigg[\overbrace{\frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{1}{2} \sum_{b\neq a} \frac{G m_b}{r_{ab}}}^{\text{point-particle action}} + \overbrace{\underbrace{\frac{\mu_a}{4} \mathcal{G}_a^{ij}\,\mathcal{G}_a^{ij}}_{\text{5PN}}}^{\text{internal structure effect}} \Bigg]}$$

Effective field theory for extended compact objects

[Goldberger & Rothstein 2006; Damour & Nagar 2009]

- Matter action with non-minimal world-line couplings

$$S_{\text{eff}} = \sum_a \int \mathrm{d}\tau_a \left\{ -m_a + \sum_{\ell=2}^{+\infty} \frac{1}{2\ell!} \left[\underbrace{\mu_a^{(\ell)}}_{\substack{\text{mass type}\\ \text{polarizability}}} \left(\mathcal{G}^a_{\hat{L}}\right)^2 + \frac{\ell}{\ell+1} \underbrace{\sigma_a^{(\ell)}}_{\substack{\text{current type}\\ \text{polarizability}}} \left(\mathcal{H}^a_{\hat{L}}\right)^2 \right] + \cdots \right\}$$

- Tidal multipole moments [Thorne & Hartle 1985; Zhang 1986]

$$\mathcal{G}^a_{\hat{L}} = -\left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle \hat{0}}\right]_a$$

$$\mathcal{H}^a_{\hat{L}} = 2\left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C^*_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle \hat{0}}\right]_a$$

where $C_{\hat{i}0\hat{j}0}$ are the components of the Weyl tensor $C_{\mu\nu\rho\sigma}$ projected on a local tetrad and evaluated at the location of the particle using a self-field regularization

High-order PN tidal effects [Henry, Faye & Blanchet 2020abc]

A recent result is the orbital phase (in the stationary phase approximation) at the next-to-next-to-leading order for equal NS binaries on circular orbit

$$\boxed{\begin{aligned}\psi_{\text{tidal}} = -\frac{117}{2}v^5\Bigg\{&\widetilde{\mu}^{(2)} + \overbrace{\left(\frac{3115}{1248}\widetilde{\mu}^{(2)} + \frac{370}{117}\widetilde{\sigma}^{(2)}\right)v^2}^{\text{NLO}}\\ &-\pi\widetilde{\mu}^{(2)}v^3 + \underbrace{\left(\frac{379931975}{44579808}\widetilde{\mu}^{(2)} + \frac{935380}{66339}\widetilde{\sigma}^{(2)} + \frac{500}{351}\widetilde{\mu}^{(3)}\right)v^4}_{\text{NNLO}}\\ &-\pi\left(\frac{2137}{546}\widetilde{\mu}^{(2)} + \frac{592}{117}\widetilde{\sigma}^{(2)}\right)v^5\Bigg\}\end{aligned}}$$

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