Are induced gravitational waves gauge dependent?

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GW Primordial Cosmology
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Non-Linear Theory of Gravitational Instability in the Expanding Universe

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(Received January 5, 1967)

The gravitational instability in the expanding universe is studied in the second-order approximation. This work is an extension of Lifshitz's linearized theory on the basis of general relativity. Basic equations are formulated generally, but their analysis is confined to a special case where pressure effects are negligible and the spatial curvature of the unperturbed model universe is zero. The results show that the second-order density contrast tends to accentuate the increase of the first-order density contrast with time, unless the linear dimension of the perturbation is too great. Moreover it is shown that gravitational wave is induced by deformed density perturbations even if the first-order metric perturbation includes no part of gravitational wave. If time is reversed, our results will be applicable to the problem of the gravitational instability in the contracting universe or in the collapsing star.

Secondary GWs history

- First pointed out by K. Tomita in 1967 [Prog. Theor. Phys. 45, 1747 (1971)]
- Followed by Matarrese, Pantano, Saez in 1993 [*Phys.Rev.Lett*. 72 (1994) 320-323]

Relativistic second-order perturbations of the Einstein-de Sitter Universe

Sabino Matarrese,[1] Silvia Mollerach[2] and Marco Bruni[3]
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[2] *Departamento de Astronomia y Astrofísica, Universidad de Valencia, 46100 Burjassot, Valencia, Spain*
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(July 25, 1997)

VII. CONCLUSIONS

In this paper we considered relativistic perturbations of a collisionless and irrotational fluid up to second order around the Einstein-de Sitter cosmological model. The most important phenomenon of second-order perturbation theory is mode mixing. An interesting consequence of this phenomenon is that primordial density fluctuations act as seeds for second-order gravitational waves. The specific form of these waves is gauge-dependent, as tensor modes are no longer gauge-invariant beyond the linear level. A second interesting effect is the generation of density fluctuations from primordial tensor modes. One can even figure out a scenario in which no scalar perturbations were initially present, but they were later generated, as a second-order effect, by the non-linear evolution of a primordial gravitational-wave background.

The first effect, which is discussed in some detail in Ref. [33], in the synchronous and comoving gauge also contains a term growing like τ^4 and a second one growing like τ^2: the first accounts for the Newtonian tidal induction of the environment on the non-linear evolution of fluid elements, the second is a post-Newtonian tensor mode induced by the growth of the shear field. The remaining parts of this second-order tensor mode (excluding a constant term required by the vanishing initial conditions) oscillate with decaying amplitude inside the horizon and describe true gravitational *waves*. Quite interesting is the fact that these are the only parts of these second-order tensor modes which survive to the transformation leading to the Poisson gauge.

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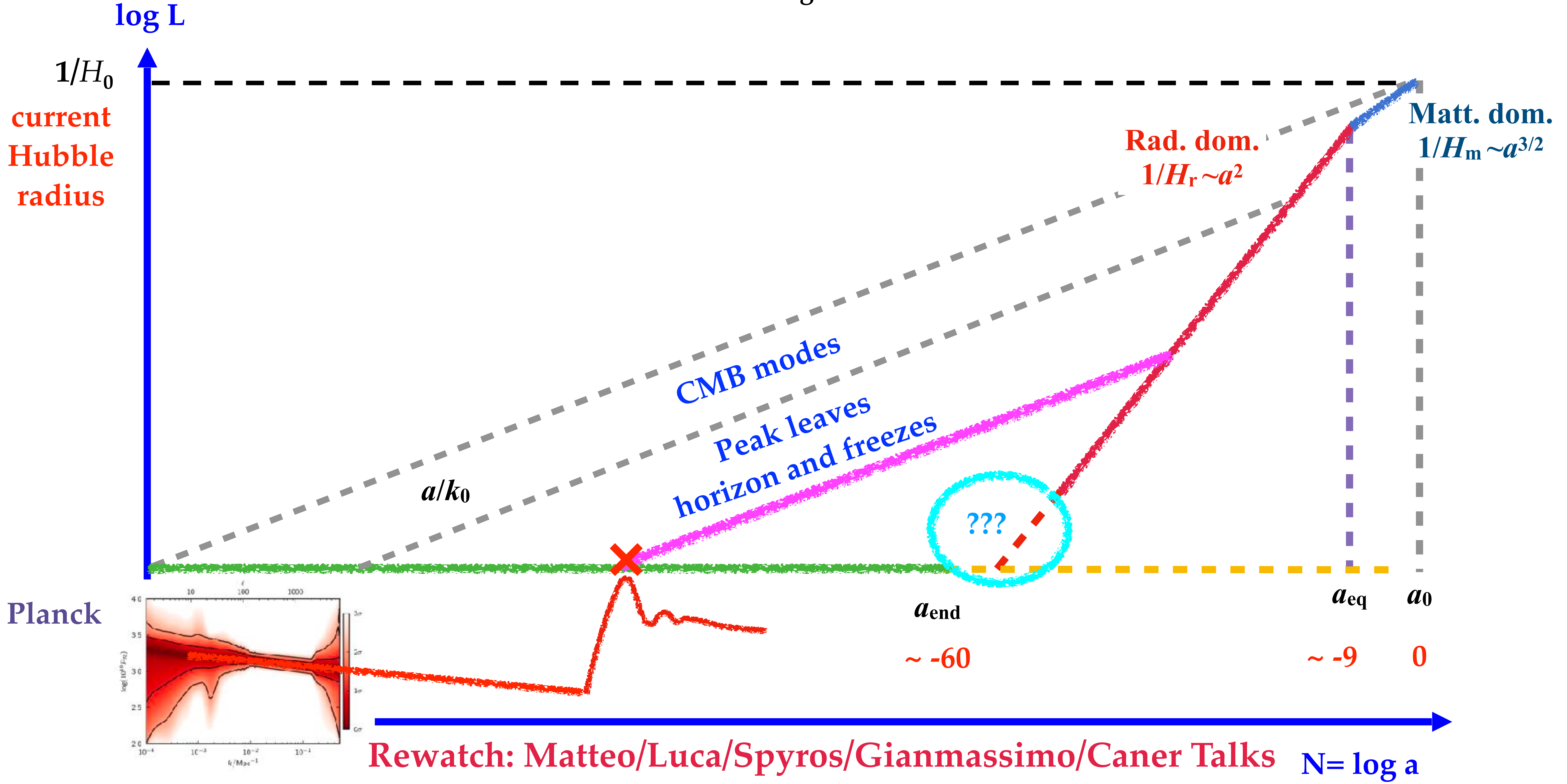
- First pointed out by K. Tomita in 1971 [*Prog. Theor. Phys.* 45, 1747 (1971)]
- Followed by Matarrese, Pantano, Saez in 1993 [*Phys.Rev.Lett.* 72 (1994) 320-323]
- Also Matarrese, Mollerach, Bruni in 1997 [*Phys.Rev.D* 58 (1998) 043504]
- Then Ananda, Clarkson and Wands in 2006 [gr-qc/0612013]
- And Baumann, Ichiki, Steinhardt and Takahashi in 2007 [hep-th/0703290]
- Saito and Yokoyama in 2008: **induced GWs <=> PBHs!** [0812.4339]
- …After the first LIGO detection the publication number keeps growing!
- Hwang, Jeong and Noh in 2017: **induced GWs gauge dependent!** [1704.03500]

[Sorry for missing all the other works… they don't fit here, not even mines!]

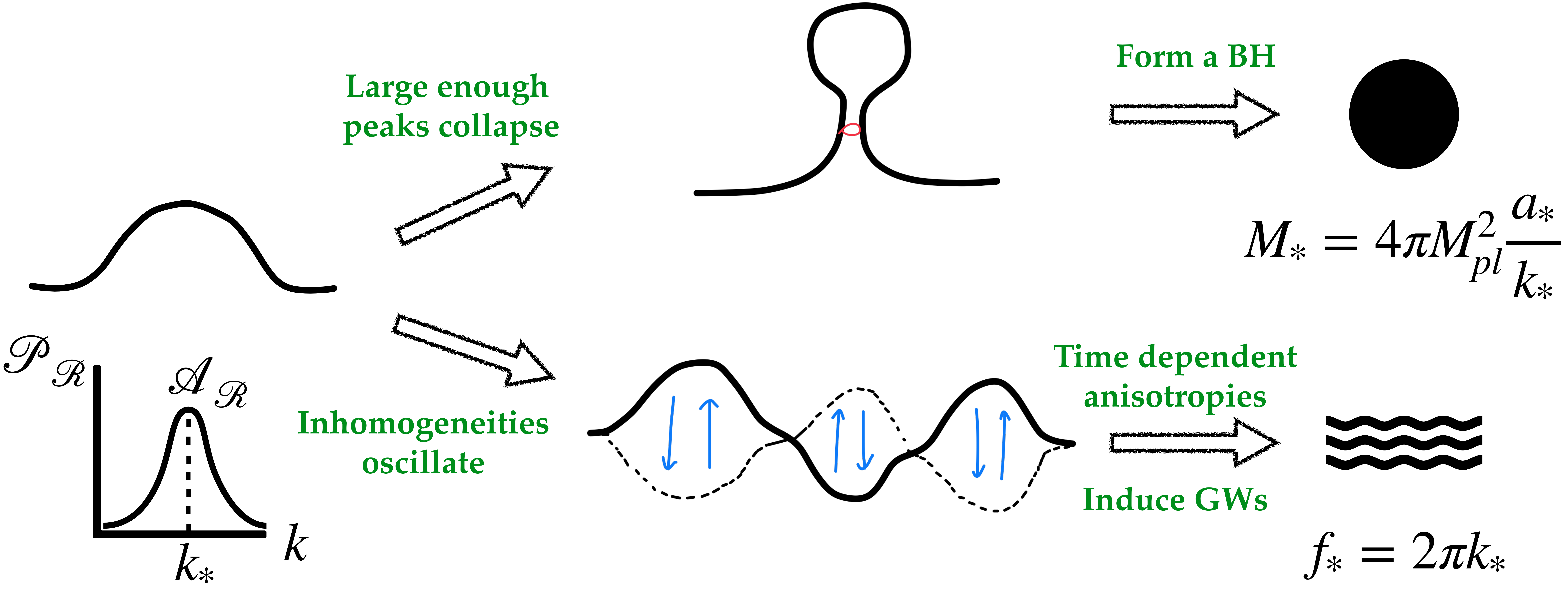
Cosmic spacetime diagram

Generating the seed



Rewatch: Matteo/Luca/Spyros/Gianmassimo/Caner Talks

Large perturbations also produce PBH



LISA band

PBH = CDM : $M_{PBH} \sim 10^{21}$g

Induced GWs with $f \sim 10^{-3}$ Hz

$$f_{GW} \sim 3\text{Hz} \left(\frac{M_{PBH}}{10^{16}\text{g}} \right)^{-1/2}$$

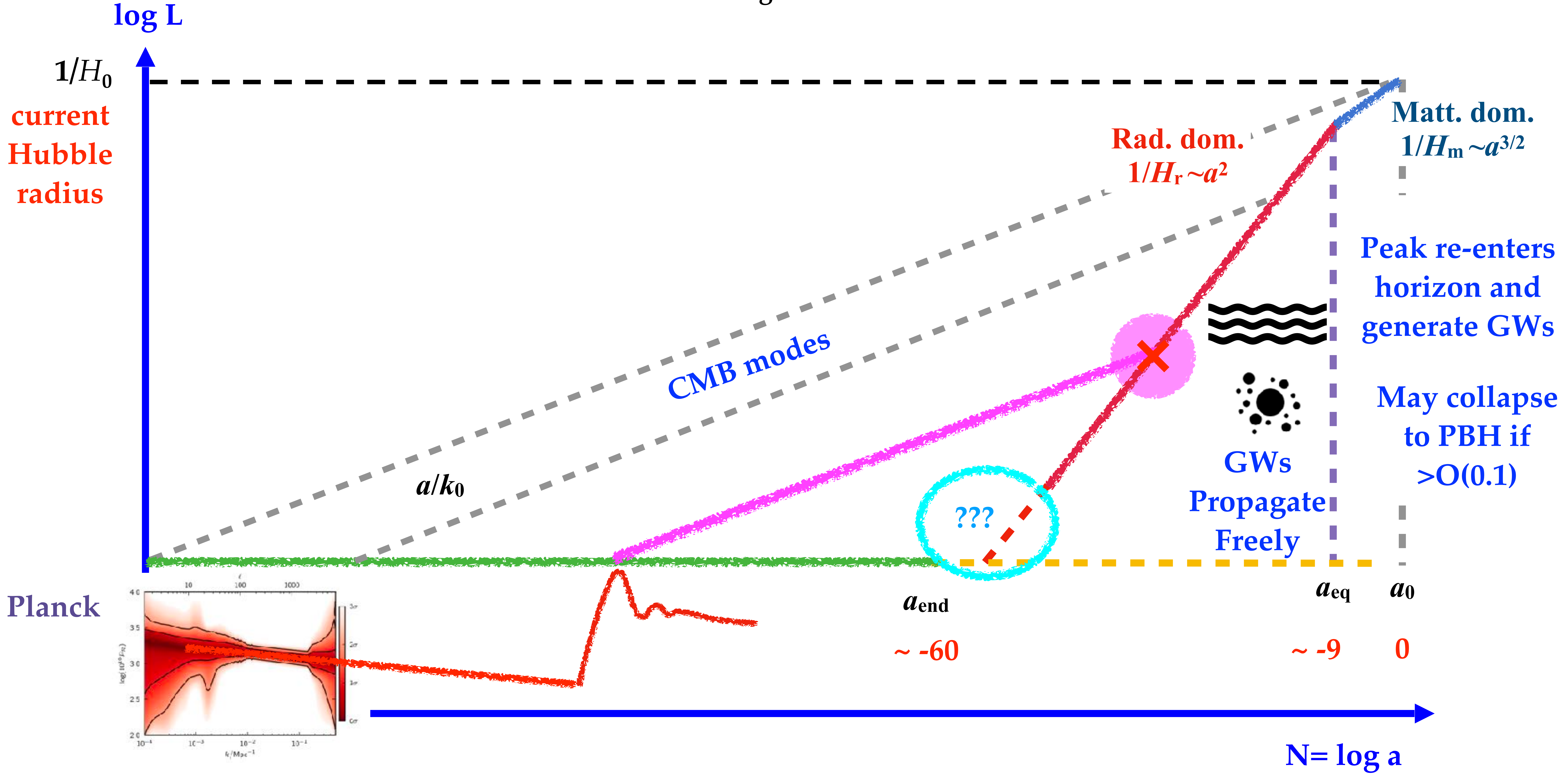
PTA / SKA band

PBH = LIGO BH : $M_{PBH} \sim 10^{34}$ g

Induced GW with $f \sim 10^{-9}$ Hz

Cosmic spacetime diagram

Generating induced GWs



Induced GWs amplitude

After inflation:

1st order: **Free wave propagating** $\Box h_{ij} = 0$

2nd order: **Massless field with source** $\Box h_{ij} \sim \widehat{TT}_{ij}^{ab}(\partial_a \Phi \partial_b \Phi)$

$$\Omega_{\rm GW}(k) = \frac{d\rho_{GW}}{d\ln k} = \frac{k^2}{12\mathscr{H}^2}\mathscr{P}_h(k,\tau) \qquad \Omega_{\rm GW}^{\rm induced} \sim \frac{1}{12}\Omega_{r,0}\mathscr{P}_{\mathscr{R}}^2 \sim 10^{-6}\mathscr{P}_{\mathscr{R}}^2(k \gg k_{\rm CMB})$$

Density ratio of radiation today $\Omega_{r,0} \sim 4\times10^{-5}$

Amplitude of Primordial Fluctuations => Amplitude of induced GWs

Primordial spectrum => Content of universe (w, c_s) => induced GW spectrum

[GD, 1912.05583]

Induced GWs are very interesting!

What can we learn from the primordial universe with IGWs?

1. We can probe the primordial spectrum:

Inomata & Nakama: 1812.00674
Byrnes et al. 2008.03289

$\Omega_{GW}^{induced} \sim 10^{-6}\mathcal{P}_{\mathcal{R}}^2$ $\qquad$ $\mathcal{P}_{\mathcal{R}} \gtrsim 10^{-4}$

2. We can probe the expansion history:

[GD, 1912.05583]
[GD, S.Pi, M.Sasaki, 2005.12314]

GW spectrum sensitive to w

$$\frac{d\Omega_{GW}^{induced}(IR)}{d\log k} \sim 3 - 2\frac{1-3w}{1+3w}$$

3. Might explain the NANOGrav results (and some PBH):

[GD and S.Pi, 2010.03976]

Vaskonen+, Kohri+, Inomata+, De Luca+, Sugiyama+

4.We can constrain epochs of PBH domination:

Papanikolau, Vennin & Langlois 2010.11573
[GD and C.Lin, M.Sasaki, 2012.08151]

Strongly constrain the initial fraction of PBH:

$\beta_{PBH} < 10^{-4} - 10^{-12}$ $\qquad$ $M_{PBH} \sim 1 - 10^9\,\mathrm{g}$

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GW spectr

See Shi Pi's talk!

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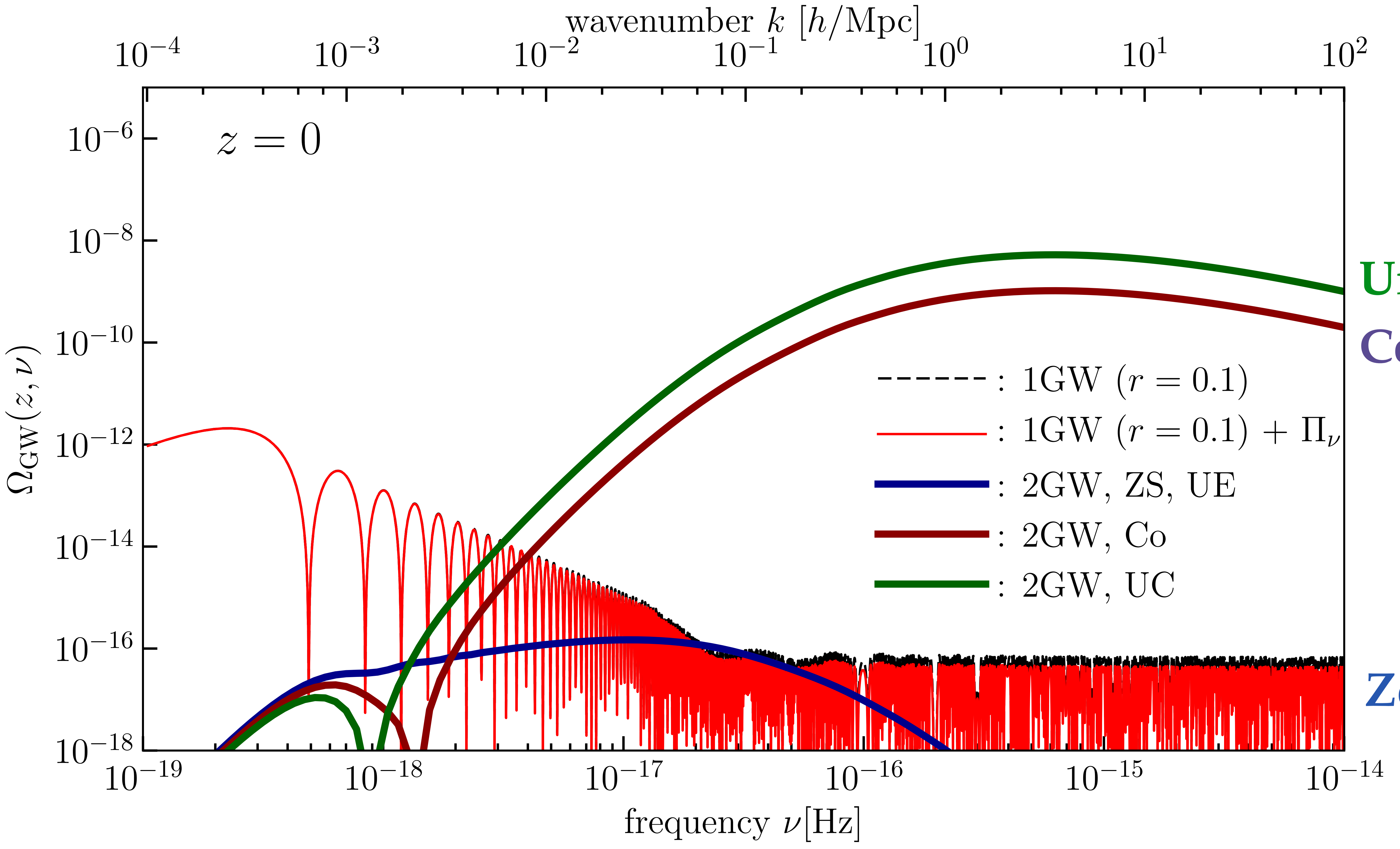
Are induced GWs gauge dependent or not?

[GD and M.Sasaki, 1709.09804 & 2012.14016]

Induced GWs are gauge dependent????

Hwang, Jeong & Noh 1704.03500: Induced GW spectrum (in dust domination) is very much gauge dependent! Also see Gong 1909.12708, Tomikawa & Kobayashi: 1910.01880



Uniform Hubble slicing

Comoving slicing

Zero Shear (Newtonian) slicing

Are the induced GWs computed in the Newtonian gauge reliable/meaningful?

We got used to talk about GWs

It has not been always like this:

In the early stages of General Relativity the existence of GWs was in doubt (by Einstein himself).

Check this article in American Scientist: "The secret history of gravitational waves"

Why is it tricky to get things right?

Because of the equivalence principle

and/or

Choose a very bad coordinate system (e.g. one in which a detector oscillates) and prepare to get confused.

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Way out in Ricci flat spacetimes. Well-defined EMT (Isaacson 1968):

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \quad \Longrightarrow \quad t^{\mathrm{GW}}_{\mu\nu} = \frac{M_{pl}^2}{4}\left\langle \partial_\mu h^{\alpha\beta}\partial_\mu h_{\alpha\beta} - \frac{1}{2}\bar{g}_{\mu\nu}\partial_\sigma h^{\alpha\beta}\partial^\sigma h_{\alpha\beta}\right\rangle$$

Why are induced GWs are gauge dependent?

In cosmology we do the following:

We use the analogy with Ricci flat spacetimes… $\rho_{\rm GW} \sim \left\langle \dot{h}^{ij}\dot{h}_{ij} \right\rangle$

…and do a gauge transformation, e.g.: $\tau \to \tau + T$

$$h_{ij} \to h_{ij} - \widehat{TT}_{ij}^{\;ab}\left[\partial_a T \partial_b T\right]$$

$$\rho_{\rm GW} \sim \left\langle \dot{h}^{ij}\dot{h}_{ij} \right\rangle \longrightarrow \rho_{\rm GW} \sim \left\langle \dot{h}^{ij}\dot{h}_{ij} \right\rangle + \left\langle (\partial_i \dot{T} \partial^i T)^2 \right\rangle$$

Then the energy density is inevitably gauge dependent!

Alternatively: the source term of induced GWs depends on the gauge.

How can we fix $\rho_{\mathrm{GW}} \sim \left\langle \dot{h}^{ij}\dot{h}_{ij} \right\rangle$?

Direction (1):
Gauge invariant formulation

Nakamura (1912.12805)
Chang+(2009.11025)

Direction (2):
What is observable

De Luca+(1911.09689)
Inomata+(1912.00785)
Yuan+(1912.00885)

Direction (3):
GWs well-defined far enough from the source (in a reasonable slicing)

[GD and M.Sasaki, 2012.14016]

Gauge invariant formulation

[GD, M.Sasaki, 1709.09804]
Chang+(2009.11025)

Gauge invariant equations of motion: $\Box h_{ij}^{GI} \sim \widehat{TT}_{ij}^{\ ab}(\partial_a \Phi^{GI} \partial_b \Phi^{GI})$

✗ **But which gauge invariant form goes in** $\rho_{\rm GW} \sim \left\langle \dot{h}^{ij} \dot{h}_{ij} \right\rangle$ **?** **Not known…**

✗ **A choice of gauge invariant variable ~ a choice of gauge!**

Natural in Hamiltonian formalism: [GD and M.Sasaki, 1709.09804]

4-Constraints $\left.\begin{array}{l}\mathscr{H}_N \approx 0 \\ \mathscr{H}_i \approx 0\end{array}\right\}$ **Generate diffeomorphisms** ⟹ **How you solve them fixes gauge invariant variables ~ fixes a gauge (consistently)**

Same question: what is the GI h_{ij} closer to the observable?

What is the observable?

De Luca+(1911.09689)
Inomata+(1912.00785)
Yuan+(1912.00885)

Analogy with Ricci flat: Transverse-traceless gauge

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j \quad \Longrightarrow \quad R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}^{TT}$$

This looks like the synchronous gauge in cosmology!

$$ds^2 = a^2\left[-d\tau^2 + (\delta_{ij} + 2\phi\delta_{ij} + 2\partial_i\partial_j E + h_{ij})dx^i dx^j\right]$$

..more or less

- ✓ **Synchronous & Newtonian gauge same prediction for $\rho_{\rm GW}$!**
- ✗ **Attention to E! It leads to spurious gauge modes.** **Lu et al (2006.03450).**
- ✗ **Argument based on a single gauge. More general principle?**
- ✗ **Only checked for radiation domination. Other cosmological backgrounds?**

GWs far from the source

[GD and M.Sasaki, 2012.14016]

The GW spectrum should be well defined if:

1. **Computed on subhorizon scales once the source is not active**

   => Free GWs

2. **Computed on a gauge which is well-behaved on subhorizon scales**

   => Reasonable coordinates

How to show?

GWs far from the source

[GD and M.Sasaki, 2012.14016]

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How to show?

A. **Assume Newtonian gauge is OK. (e.g. Newtonian limit at short distances)**

B. **Show that any gauge ~ Newtonian gauge => same GW spectrum (inside horizon)**

Well-behaved gauges

We define well-behaved gauges as those similar to Newtonian gauge on small scales:

$$\Phi_G(k \gg \mathscr{H}) = O(\Phi_N(k \gg \mathscr{H}))$$

From a gauge transformation $\tau \to \tau + T \qquad x^i \to x^i + \partial^i L$

$$\mathscr{H} T_G \sim O(\Phi_N) \qquad \Delta L_G \sim O(\Phi_N)$$ **or higher order**

This requirement includes: spatially flat, uniform Hubble & synchronous gauge

This requirement excludes: comoving slicing gauge

Well-behaved gauges

Particular example: perfect fluid w=p/ϱ=constant

Solutions: $\Phi_N \propto \left(\frac{k}{\mathcal{H}}\right)^{-2-\frac{1-3w}{1+3w}}$ $h_{ij}^N \propto \left(\frac{k}{\mathcal{H}}\right)^{-1-\frac{1-3w}{1+3w}}$

Well-behaved gauge $\Longrightarrow$ $T_G \sim \frac{1}{k}\left(\frac{k}{\mathcal{H}}\right)^{-1-\frac{1-3w}{1+3w}}$ **or higher order**

From the gauge transformation of the tensor modes:

$$h_{ij}^G = h_{ij}^N - \widehat{TT}_{ij}^{\;ab}\left[\partial_a T_G \partial_b T_G\right] \Longrightarrow h^G(k \gg \mathcal{H}) = h^N(k \gg \mathcal{H})$$

Includes: spatially flat, uniform Hubble & synchronous gauge

Approximate gauge independence of IGWs

The induced GW spectrum is gauge independent if we focus on:

1. **Sub-horizon scales once the source term is not active**

   i.e. once the scalar source decayed enough on subhorizon scales

2. **Gauges well-behaved on such subhorizon scales**

   i.e. gauges similar to Newtonian gauge on small scales.

   e.g. spatially flat, uniform Hubble & synchronous gauges.

Approximate gauge independence of IGWs

The induced GW spectrum is gauge independent if we focus on:

1. **Sub-horizon scales once the source term is not active**

   i.e. once the scalar source decayed enough on subhorizon scales

   Note: this excludes dust domination! Source terms always active. (See next slide)

2. **Gauges well-behaved on such subhorizon scales**

   i.e. gauges similar to Newtonian gauge on small scales.

   e.g. spatially flat, uniform Hubble & synchronous gauges.

   Note: this excludes comoving slicing gauge! But highly deformed slicing on small scales where fluid velocities oscillate!

   Note: pay extra caution to the synchronous gauge. Gauge modes affect the GW spectrum if not fixed properly. Lu et al (2006.03450) [GD and M.Sasaki, 2012.14016]

The dust dominated universe

What is wrong with dust domination $w=c_s^2=0$?

$$\Phi_N = \text{constant} \longrightarrow \Box h_{ij}^N = \frac{20}{3} \widehat{TT}_{ij}^{ab} \left[\partial_a \Phi_N \partial_b \Phi_N\right]$$

Constant source, always active.

Can they really be called GWs?

Assadullahi & Wands (0901.0989)
Inomata & Terada (1912.00785)

Don't decay as radiation $\rho_{\rm GW} \neq a^{-4}$

Won't be detected by interferometers $\ddot{h}_{ij} = 0$

Can be gauged away $h_{ij}^G = 0$

GD & M.Sasaki (2012.14016)

Dominant contribution to iGW right after reheating

Inomata et al. (1904.12879)

Our gauge independence also applies to dust domination, just after reheating when the source term is not active!

Summary

- PBH + Induced GWs: probe of inflation, the primordial spectrum and the early universe expansion history.
- Distinct signatures of GW spectrum: IR broken power-law, w-dependent slope, resonant peak and cut-off. Constrains PBH domination and might explain NANOGrav results.
- The iGW spectrum is strictly speaking gauge dependent (as is the GW energy density in cosmology)
- The iGW spectrum is gauge independent if we focus: (i) on sub horizon scales and (ii) on well-behaved gauges.

We can rely on the predictions of iGWs
and use it to explore the primordial universe!

≋ The End ≋