Gravitational-Wave Primordial Cosmology Meeting

Oscillations in the stochastic gravitational wave background from small-scale features

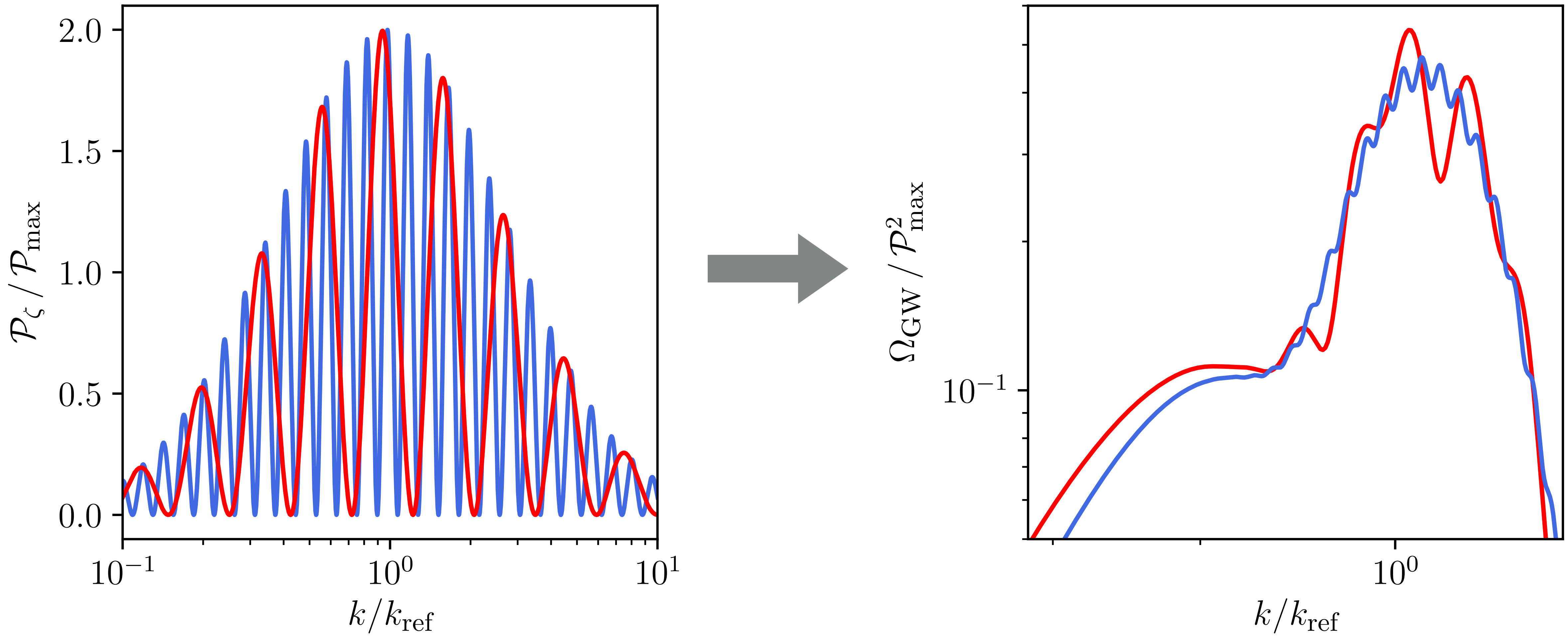
Lukas Witkowski

arXiv:2012.02761 + arXiv:2105.06481

with Jacopo Fumagalli and Sébastien Renaux-Petel

European Research Council
Established by the European Commission

Main message:



Oscillations in $\mathcal{P}_\zeta(k)$ give corresponding modulations in $\Omega_{\mathrm{GW}}(k)$.

<u>This talk</u>: Explain how properties of the oscillations in $\mathcal{P}_\zeta(k)$ determine the characteristics of the modulations in $\Omega_{\mathrm{GW}}(k)$.

$$\mathcal{P}_\zeta(k) \implies \Omega_{\mathrm{GW}}(k)$$

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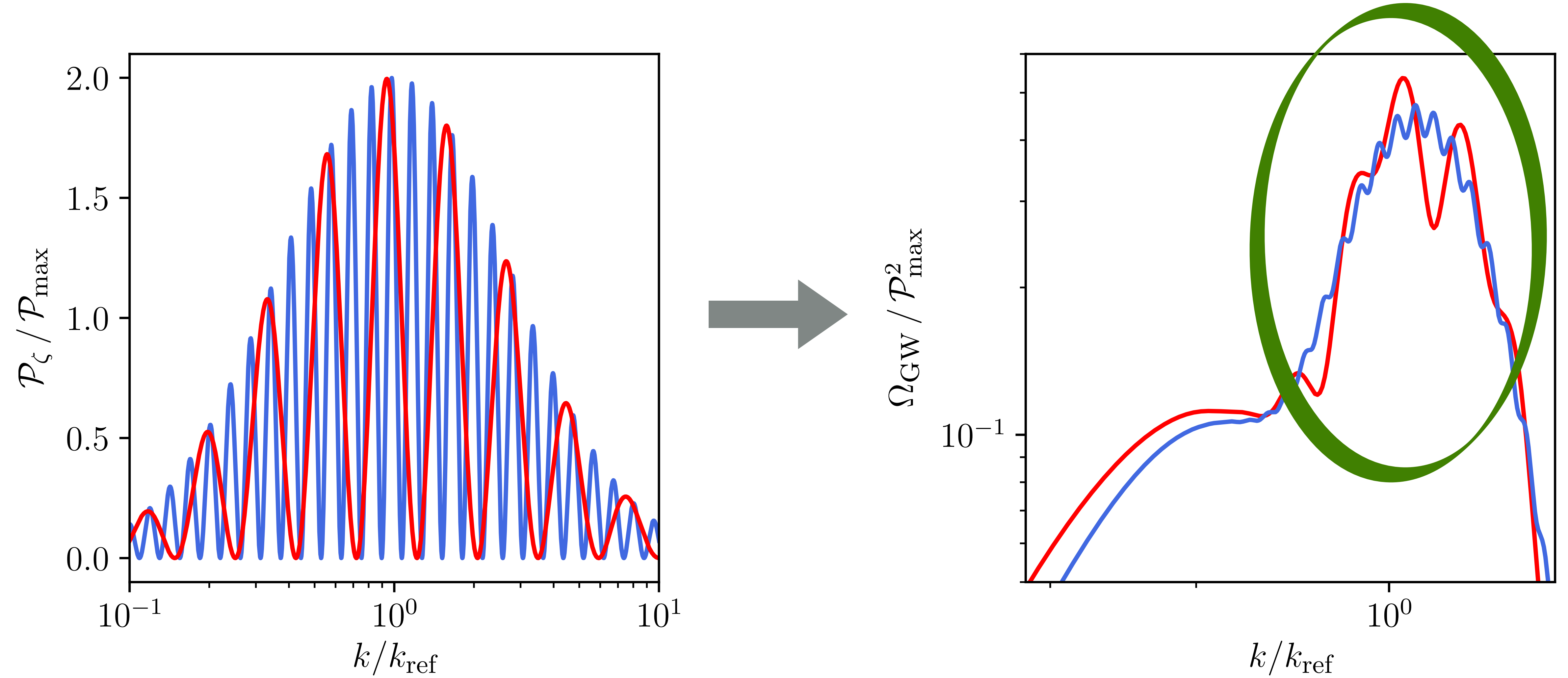

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$$\mathcal{P}_\zeta(k) \quad \Longleftarrow \quad \Omega_{\rm GW}(k)$$

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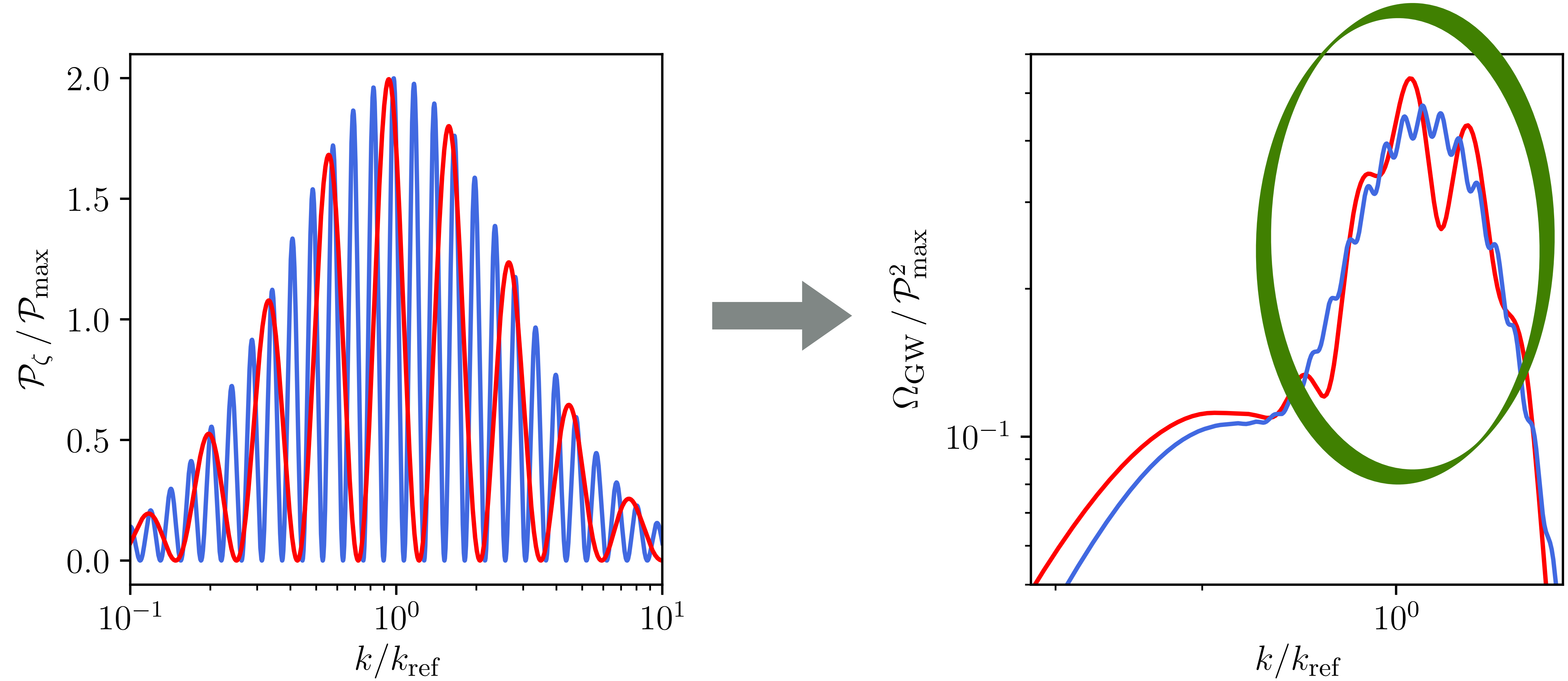


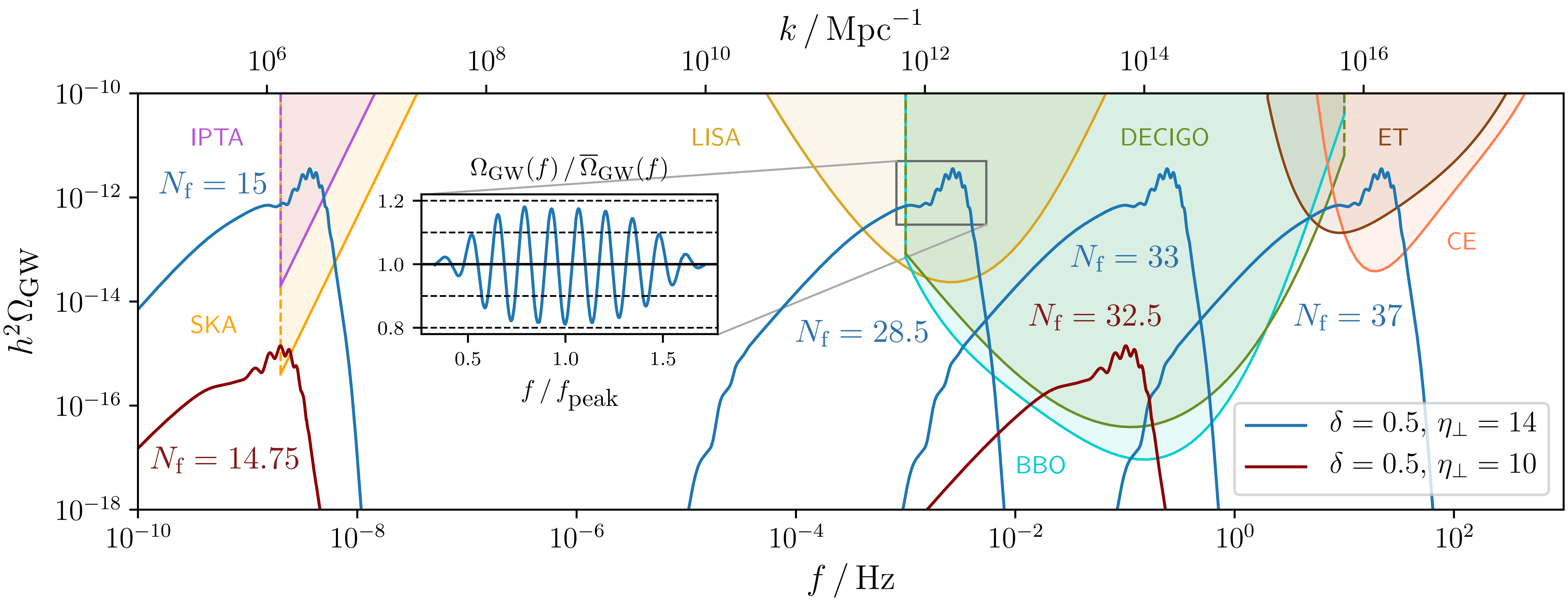
Oscillations in $\mathcal{P}_\zeta(k)$ give corresponding modulations in $\Omega_{\rm GW}(k)$.

<u>This talk</u>: Focus on scalar-induced contributions to the stochastic gravitational wave background (SGWB) sourced in the **post-inflationary era**.

Inflationary-era GWs ⟶ Spyros Sypsas' talk

Main message:

Potentially detectable in the upcoming generation of GW observatories:



Here: examples for an inflation model with a strong sharp turn in the Inflationary trajectory. [Palma et al. 2004.06106] [Fumagalli, Renaux-Petel, **LW** 2012.02761]

$N_{\rm f} =$ time of the sharp turn in numbers of e-folds after horizon exit of CMB modes

Outline

I. Inflation and small-scale features

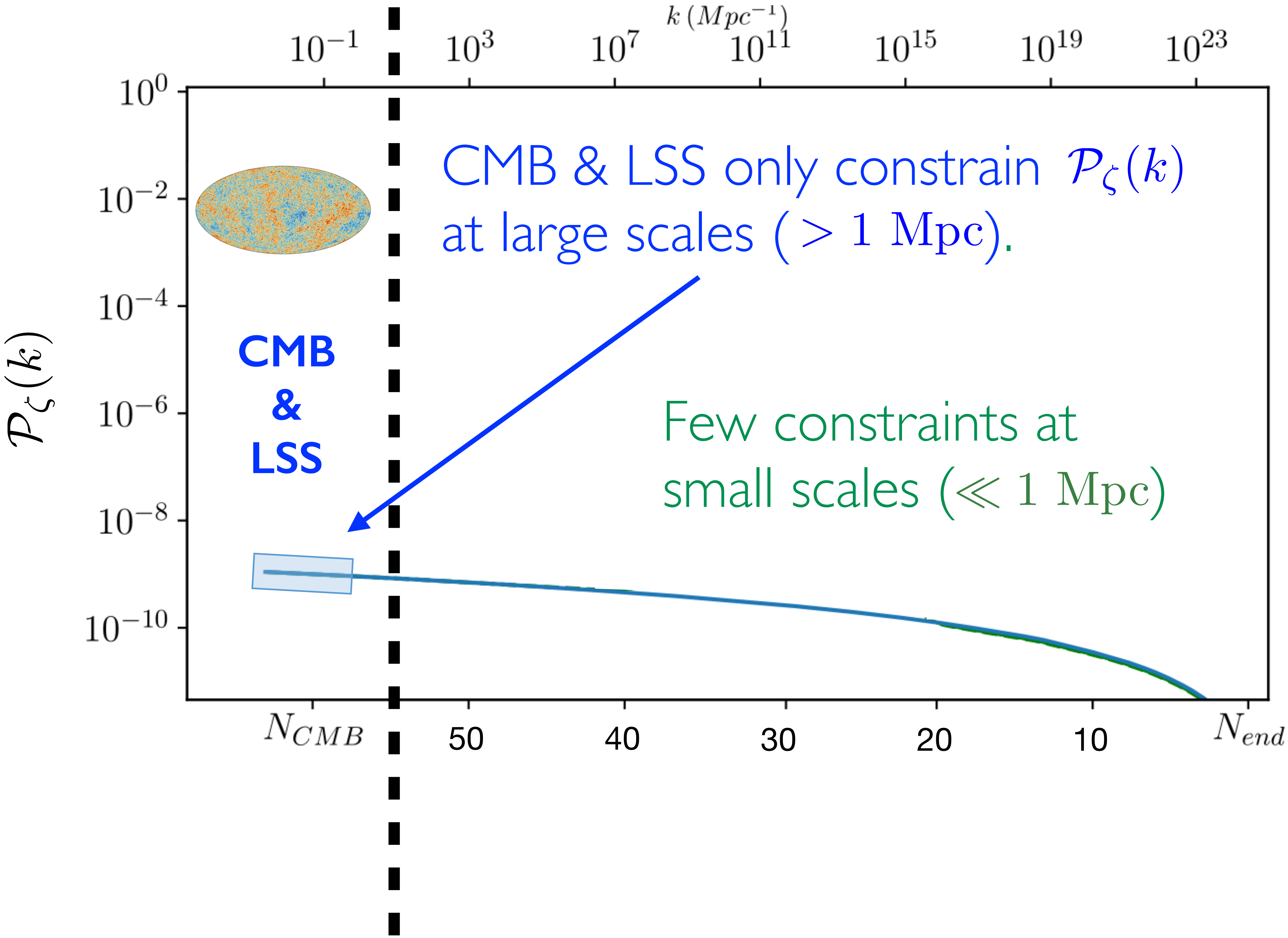
II. Scalar-induced GWs

sharp feature

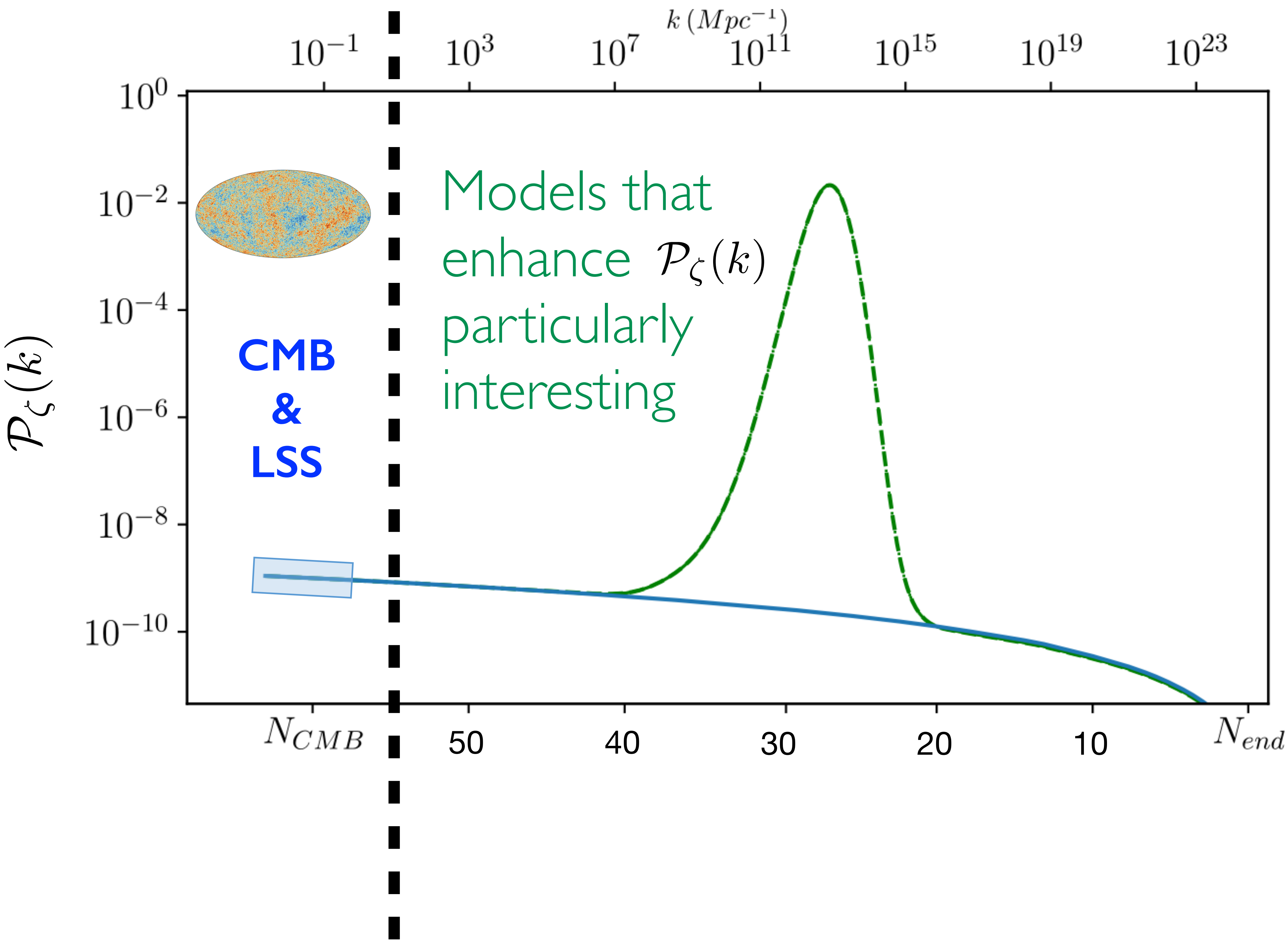
resonant feature

III. Detecting small-scale features with GWs

Inflation and small-scale features



Inflation and small-scale features



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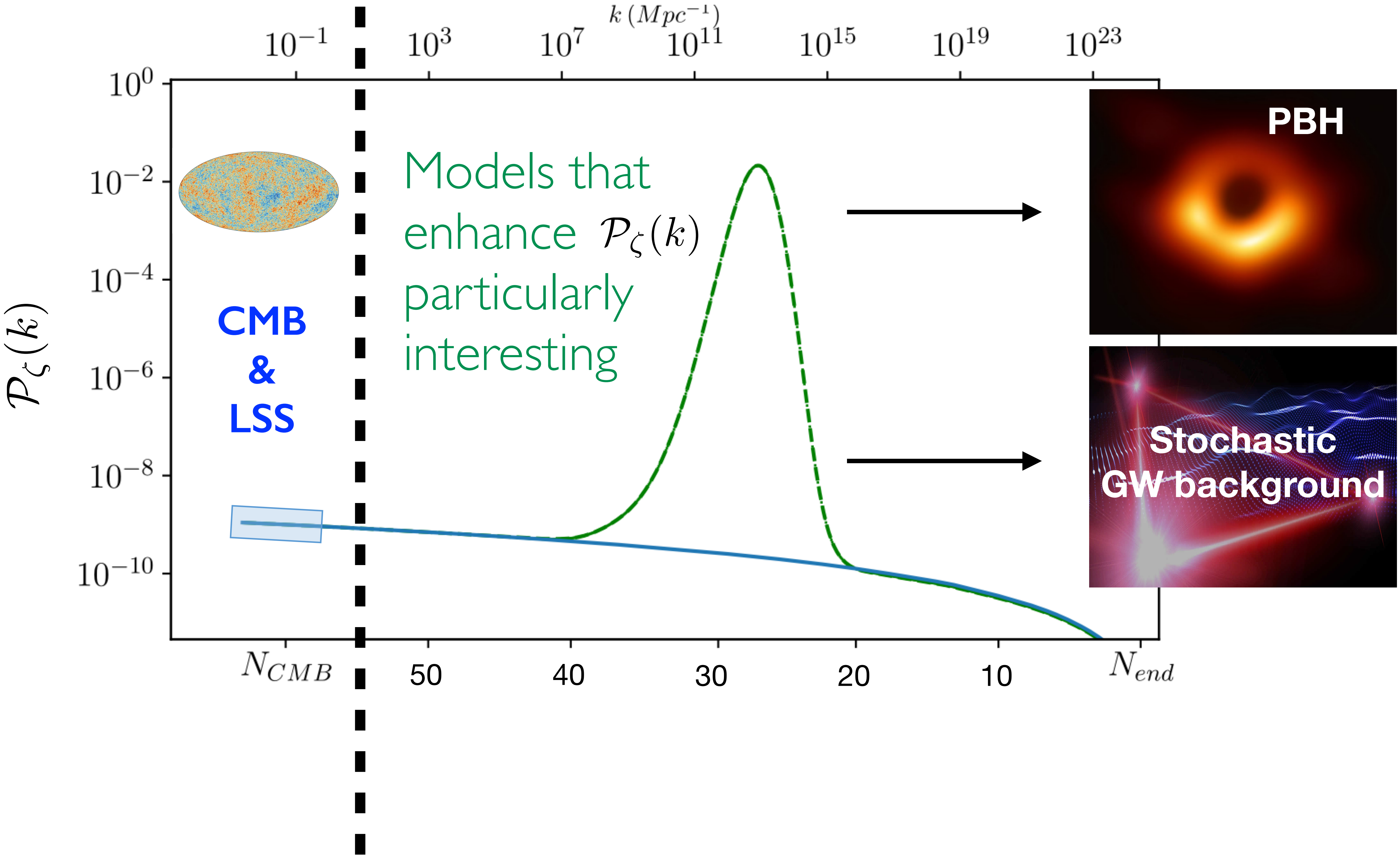

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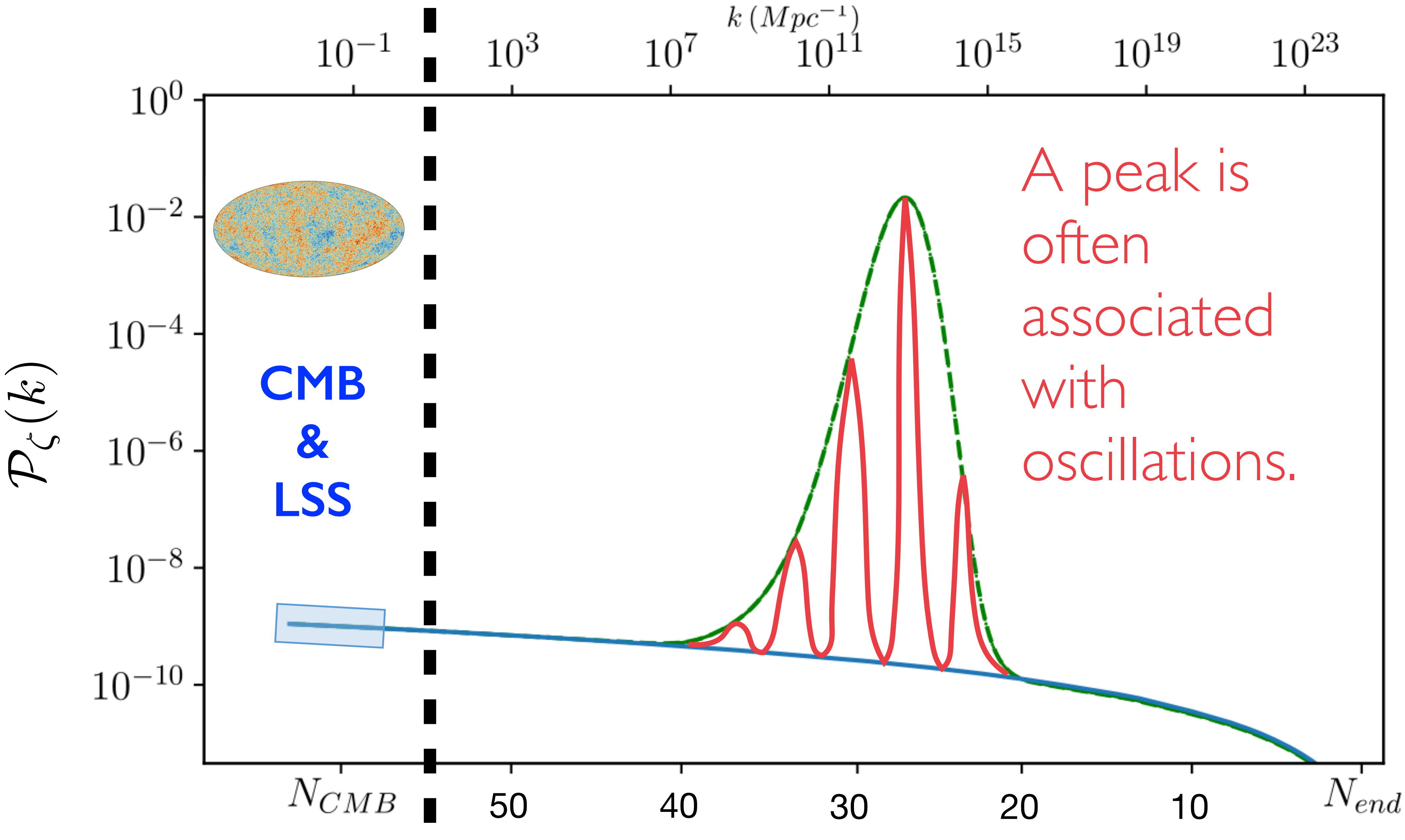

Small-scale primordial features:
departures from single-field slow-roll associated with an (oscillatory) feature in $\mathcal{P}_\zeta(k)$.

Primordial Features

[See e.g. review Slosar et al. 1903.09883]

1.) Sharp feature: caused by a "sharp" transition during inflation (e.g. due to a step in the potential or a sharp turn)

→ k-periodic modulation in $\mathcal{P}_\zeta(k)$.

$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\rm lin}\cos\left(\omega_{\rm lin}k + \phi_{\rm lin}\right)\Big)$$

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$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\rm lin}\cos\left(\omega_{\rm lin}k + \phi_{\rm lin}\right)\Big)$$

Enhancement of $\overline{\mathcal{P}}$ compared to featureless model $\Rightarrow$ $A_{\rm lin} \to 1$.

[e.g. explained in Fumagalli, Renaux-Petel, **LW** 2012.02761]

Primordial Features

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2.) Resonant feature: caused by some components of the background oscillating with a frequency larger than the Hubble scale (as e.g. in monodromy inflation)

→ $\log(k)$-periodic modulation in $\mathcal{P}_\zeta(k)$.

$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\rm log}\cos\left(\omega_{\rm log}\log(k/k_{\rm ref}) + \phi_{\rm log}\right)\Big)$$

Primordial Features

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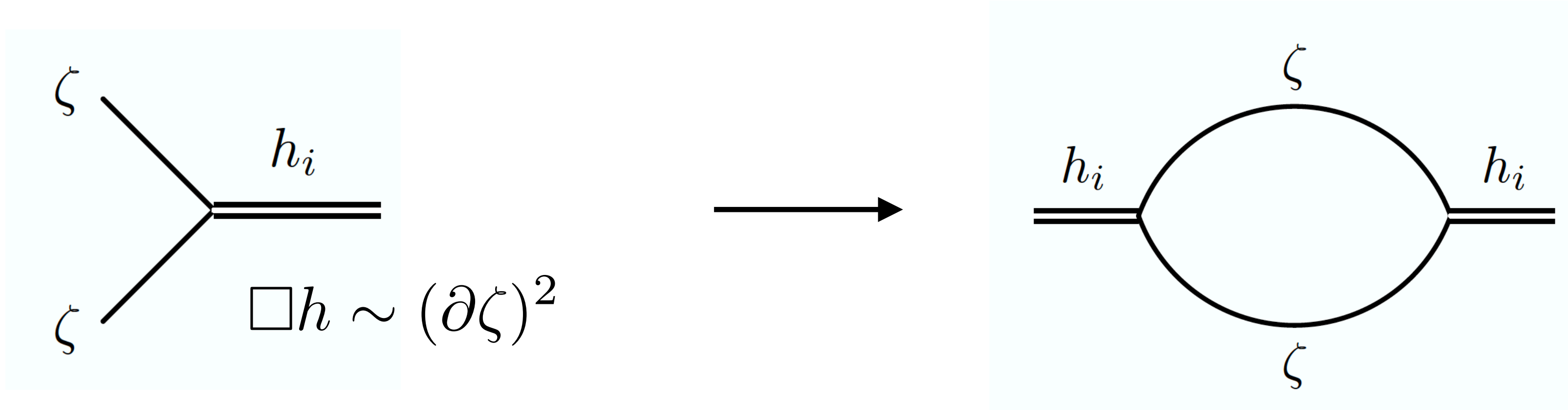
\+ combinations of the two. [Chen, Namjoo 2014; Chen, Namjoo, Wang 2015]

Scalar-induced GWs

Scalar fluctuations source GWs at 2nd order:



[Acquaviva et al. 2002; Mollerach, Harari, Matarrese 2003; Ananda, Clarkson, Wands 2006; Baumann et al. 2007 …]

Energy density per $\log(k)$-interval of post-inflationary GWs:

$$\Omega_{\rm GW}(k) = c_g \Omega_{\rm r,0} \int_0^{\frac{1}{\sqrt{3}}} {\rm d}d \int_{\frac{1}{\sqrt{3}}}^{\infty} {\rm d}s \, \mathcal{T}_{\rm RD}(d,s)\, \mathcal{P}_\zeta\left(\frac{\sqrt{3}k}{2}(s+d)\right) \mathcal{P}_\zeta\left(\frac{\sqrt{3}k}{2}(s-d)\right)$$

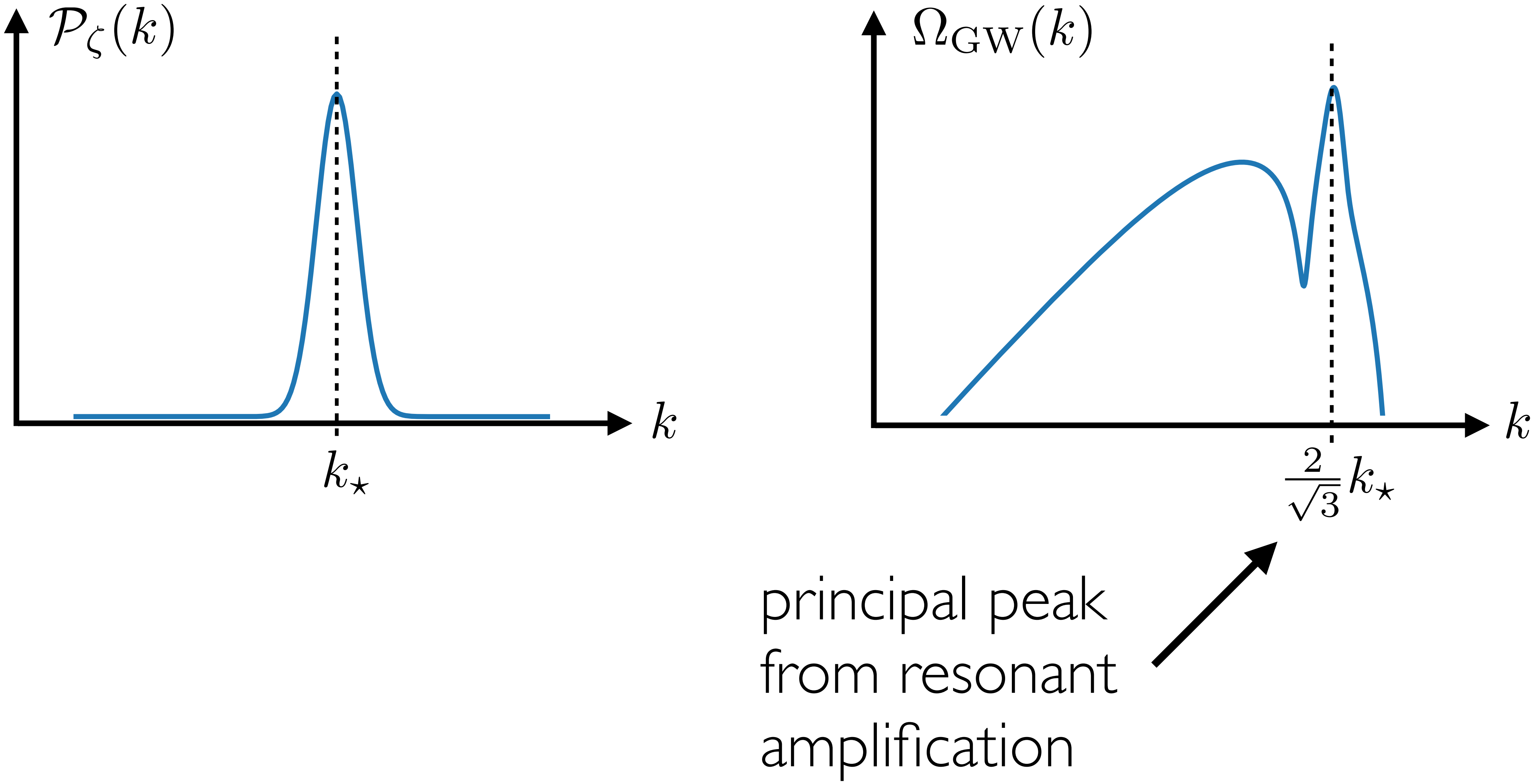
Here consider standard cosmological history with post inflationary-era GWs sourced during epoch of radiation domination.

How does an oscillations in $\mathcal{P}_\zeta(k)$ manifest itself in $\Omega_{\rm GW}(k)$?

Scalar-induced GWs

How does an oscillations in $\mathcal{P}_\zeta(k)$ manifest itself in $\Omega_{\rm GW}(k)$?

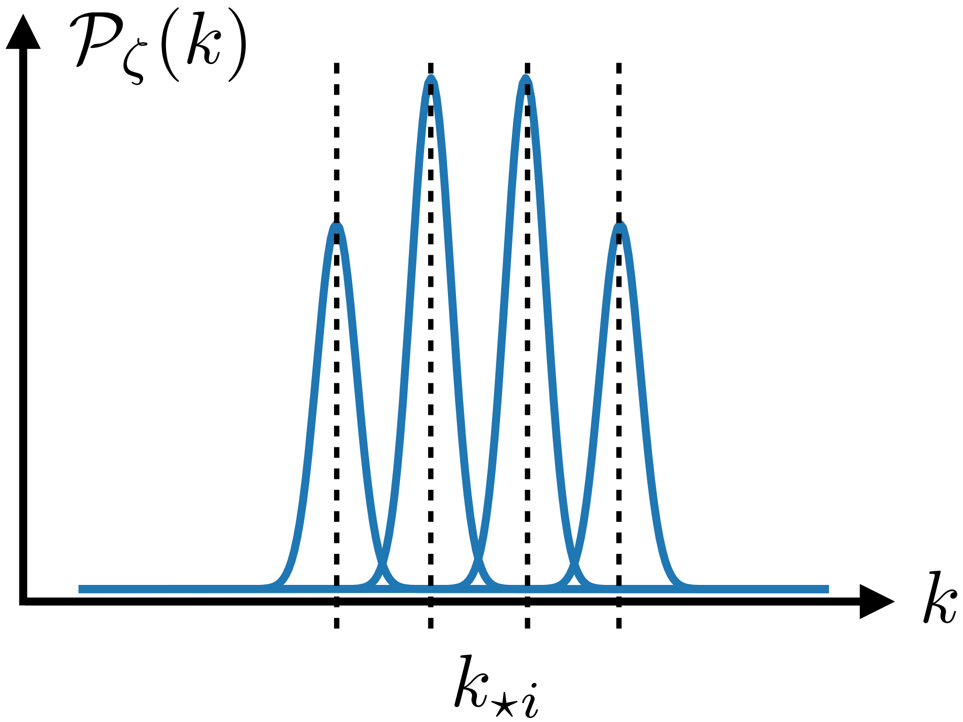
Consider a single narrow peak in $\mathcal{P}_\zeta(k)$:

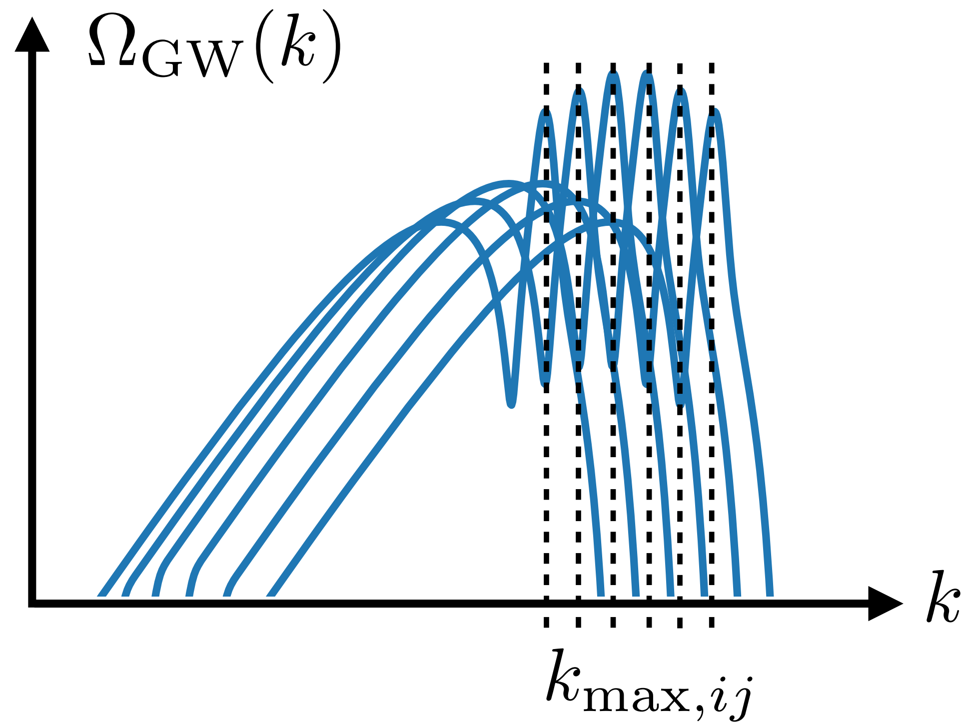


Scalar-induced GWs

Strategy 1: Model the oscillation as a series of peaks:

[Fumagalli, Renaux-Petel, **LW** 2012.02761]





Find a superposition of resonance peaks in $\Omega_{\rm GW}(k)$ at:

$$k_{\max,ij} = \frac{1}{\sqrt{3}}(k_{\star i} + k_{\star j})\,, \quad \text{with} \quad k_{\max,ij} > |k_{\star i} - k_{\star j}|$$ [Cai et al. 1901.10152]

Allows for successful prediction of the location of maxima in $\Omega_{\rm GW}(k)$ due to both a **sharp** and a **resonant feature**. [Fumagalli, Renaux-Petel, **LW** 2012.02761, 2105.06481]

Scalar-induced GWs

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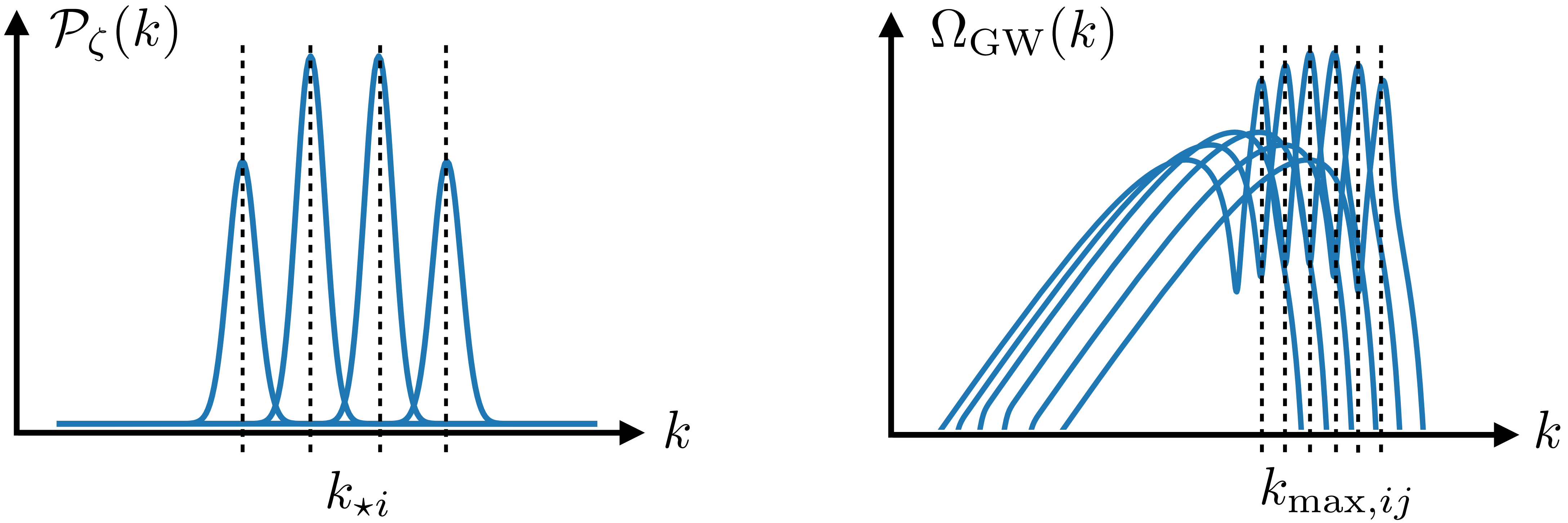

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Sharp feature: $\omega_{\rm lin} \longrightarrow \omega_{\rm lin}^{\rm GW} = \sqrt{3}\,\omega_{\rm lin}$

Scalar-induced GWs

Strategy II: Separate smooth and oscillatory pieces:

[Fumagalli, Renaux-Petel, **LW** 2105.06481]

sharp / resonant feature: $\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + \underline{A}\cos\left(\dots\right)\Big)$

$$\Omega_{\rm GW} \sim \iint \mathcal{P}_\zeta^2 \qquad \Rightarrow \qquad \Omega_{\rm GW}(k) = \Omega_{\rm GW,0}(k) + \underline{A}\,\Omega_{\rm GW,1}(k) + \underline{A^2}\,\Omega_{\rm GW,2}(k)$$

with

$$\Omega_{\rm GW,0}(k) \sim \iint \overline{\mathcal{P}}^2(k)$$

$$\Omega_{\rm GW,1}(k) \sim \iint \overline{\mathcal{P}}^2(k)\cos\left(\dots\right)$$

$$\Omega_{\rm GW,2}(k) \sim \iint \overline{\mathcal{P}}^2(k)\cos^2\left(\dots\right)$$

Scalar-induced GWs

Strategy II: Separate smooth and oscillatory pieces:

[Fumagalli, Renaux-Petel, **LW** 2105.06481]

resonant feature:

$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + \underline{A_{\log}} \cos\left(\underline{\omega_{\log}} \log(k/k_{\rm ref}) + \phi_{\log}\right)\Big)$$

$$\Omega_{\rm GW} \sim \iint \mathcal{P}_\zeta^2 \qquad \Rightarrow \qquad \Omega_{\rm GW}(k) = \Omega_{\rm GW,0}(k) + \underline{A_{\log}}\, \Omega_{\rm GW,1}(k) + \underline{A_{\log}^2}\, \Omega_{\rm GW,2}(k)$$

Particularly powerful for a resonant feature where the oscillatory part can be calculated semi-analytically (for sufficiently broad $\overline{\mathcal{P}}(k)$):

compute numerically once

$$\Omega_{\rm GW,1}(k) = \Omega_{\rm GW,0}(k)\, \mathcal{A}_{\log,1}(\omega_{\log})\ \cos\Big(\underline{\omega_{\log}} \log(k/k_{\rm ref}) + \varphi_{\log,1}(\omega_{\log})\Big)$$

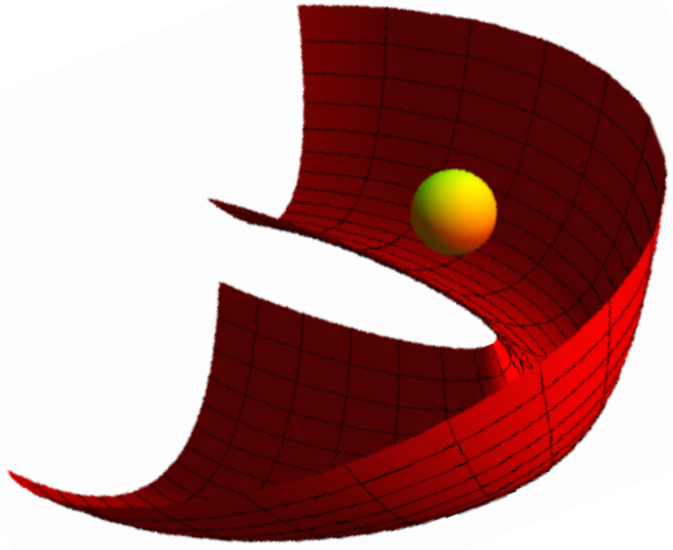
$$\Omega_{\rm GW,2}(k) = \Omega_{\rm GW,0}(k)\, \mathcal{A}_{\log,2}(\omega_{\log})\ \cos\Big(\underline{2\omega_{\log}} \log(k/k_{\rm ref}) + \varphi_{\log,2}(\omega_{\log})\Big)$$

GWs from a sharp feature

Consider explicit realisation of a sharp feature in terms of a sharp turn in the inflationary trajectory in multi-field inflation:

[Palma et al. 2004.06106]
[Fumagalli, Renaux-Petel, **LW** 2012.02761]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}G_{IJ}(\phi^K)\partial_\mu\phi^I\partial^\mu\phi^J - V(\phi^I)\right)$$



The turn is described by three parameters:

δ : duration of turn in e-folds

$\eta_\perp$: dimensionless rate of turning

k_f : scale that leaves horizon during time of turn

⟶ **cf. Matteo Braglia's talk for a different model with sharp & res. features**

GWs from a sharp feature

Consider explicit realisation of a **sharp feature** in terms of a **sharp turn** in the inflationary trajectory in multi-field inflation:

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Amplified part of the power spectrum:

$$\mathcal{P}_\zeta(k) = \mathcal{P}_0 \frac{\eta_\perp^2 k_\mathrm{f}^2}{4(2\eta_\perp k_\mathrm{f} - k)k}\, e^{2\sqrt{(2\eta_\perp k_\mathrm{f} - k)k}\frac{\delta}{k_\mathrm{f}}} \left(1 - \cos\left(\frac{2k}{k_\mathrm{f}} + \arctan\left(\frac{k}{\sqrt{(2\eta_\perp k_\mathrm{f} - k)k}}\right)\right)\right)$$

Envelope with an exponentially enhanced peak

Rapid order one sinusoidal modulations $\omega_\mathrm{lin} \simeq \frac{2}{k_\mathrm{f}}$

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Envelope with an exponentially enhanced peak

Rapid order one sinusoidal modulations $\omega_{\rm lin} \simeq \frac{2}{k_{\rm f}}$

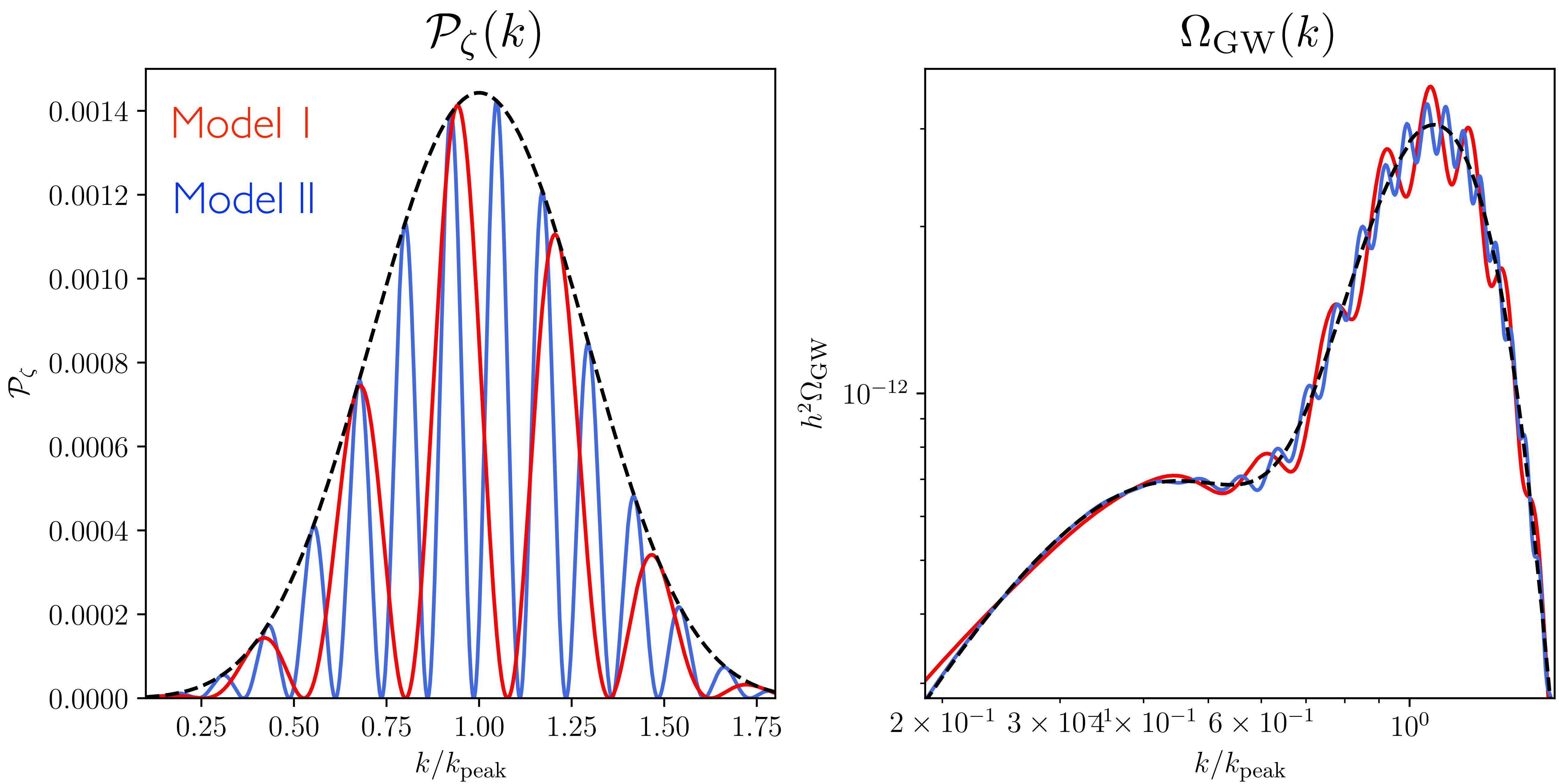
This takes indeed the form of a sharp feature:

$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\rm lin}\cos\left(\omega_{\rm lin} k + \phi_{\rm lin}\right)\Big)$$

GWs from a sharp feature

Consider two example models (I and II) and compute the corresponding (post-inflationary contribution to the) GW spectrum:

[Fumagalli, Renaux-Petel, **LW** 2012.02761]

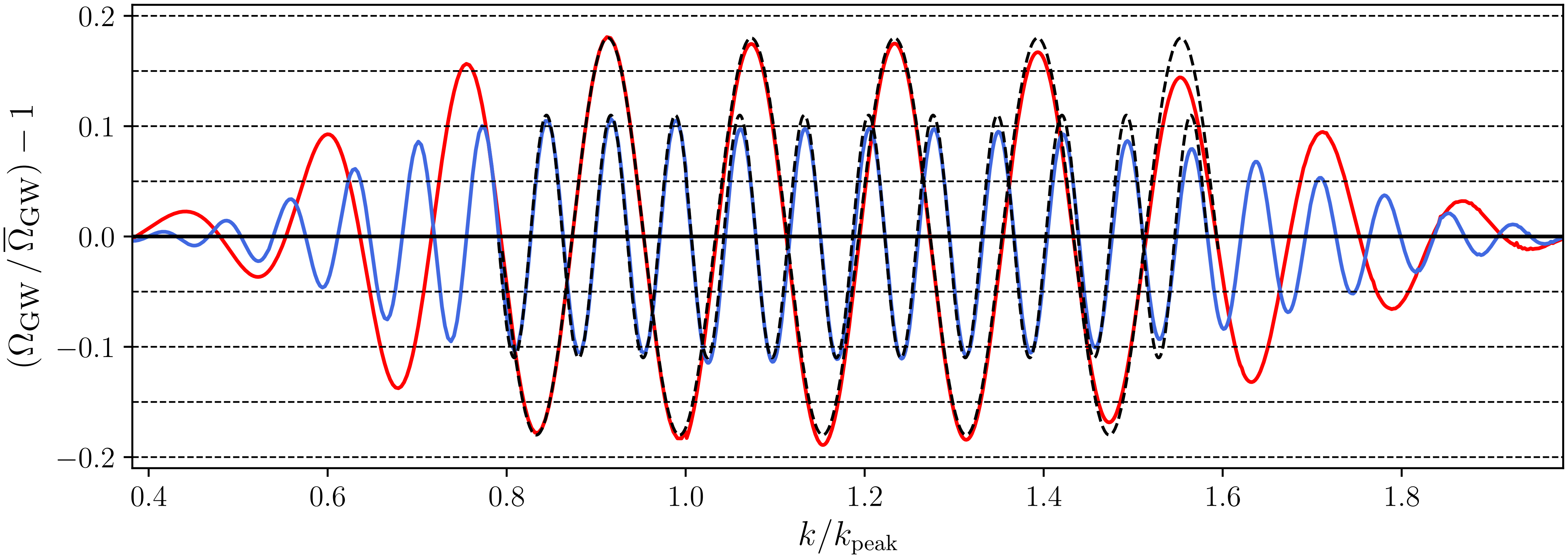


GWs from a sharp feature

Consider two example models (I and II) and compute the corresponding (post-inflationary contribution to the) GW spectrum:

[Fumagalli, Renaux-Petel, **LW** 2012.02761]

$\overline{\Omega}_{\rm GW}(k)$: smoothed GW spectrum



Have sinusoidal oscillations in $\Omega_{\rm GW}(k)$ over the principal peak.

[see also Braglia, Chen, Hazra 2012.05821]

GWs from a sharp feature

$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\rm lin}\cos\left(\omega_{\rm lin}k + \phi_{\rm lin}\right)\Big)$$

[Fumagalli, Renaux-Petel, **LW** 2012.02761]

$$\omega_{\rm lin}^{\rm GW} = \sqrt{3}\,\omega_{\rm lin}$$

$$\Omega_{\rm GW}(k) = \overline{\Omega}_{\rm GW}\Big(1 + \mathcal{A}_{\rm lin}\cos\left(\omega_{\rm lin}^{\rm GW}k + \varphi_{\rm lin}\right)\Big)$$

- Overall shape determined by envelope of power spectrum
- Periodic structure in k ⟶ Periodic structure in k
- Averaging-out effect: at best $\mathcal{A}_{\rm lin} \sim 10\%$ even with $A_{\rm lin} = 1$.

$\mathcal{A}_{\rm lin}$ also decreases as $\omega_{\rm lin}k_{\rm peak}$ is increased for fixed $\overline{\mathcal{P}}(k)$.

GWs from a resonant feature

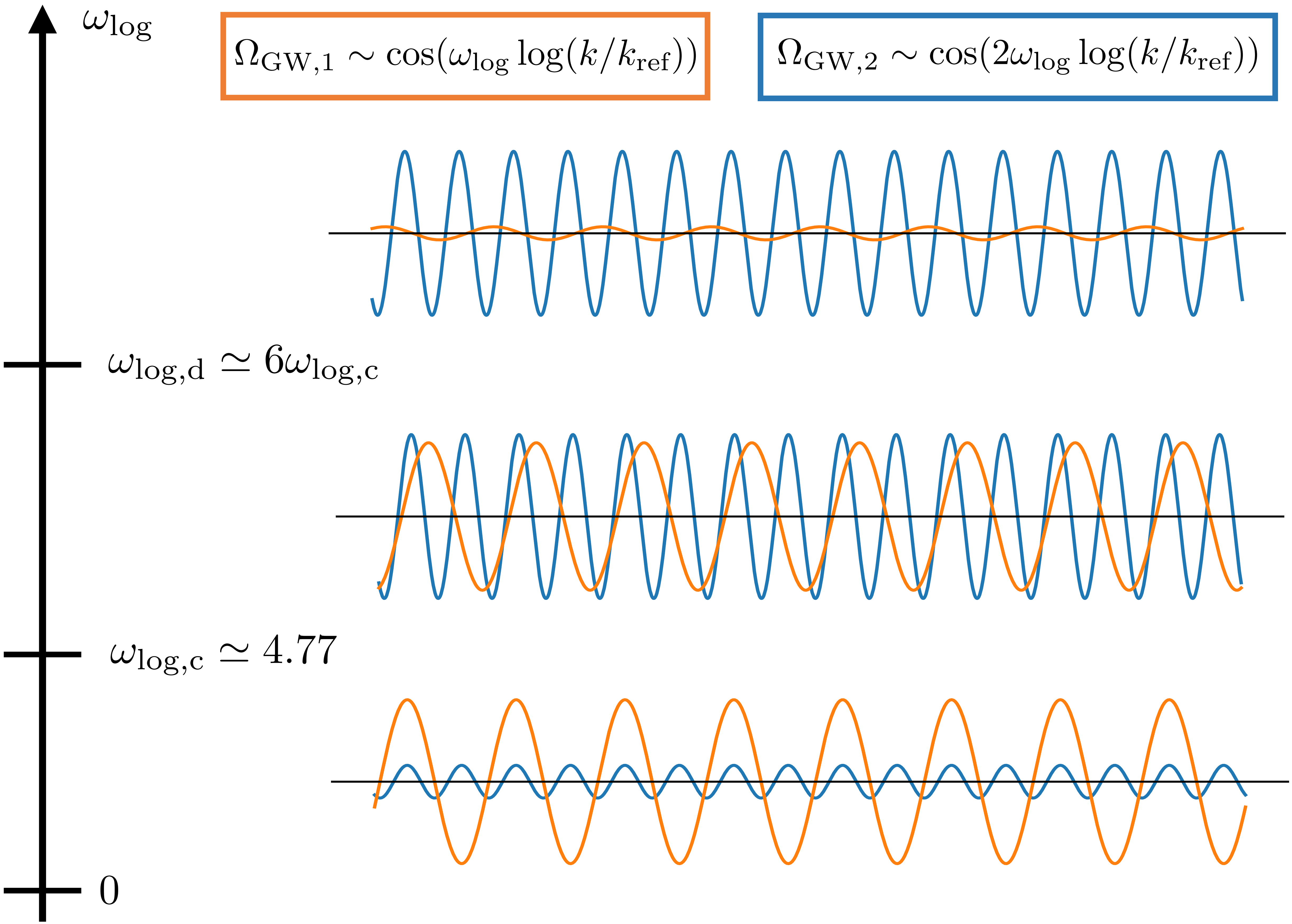
$$\mathcal{P}_\zeta(k) = \overline{\mathcal{P}}(k)\Big(1 + A_{\log}\cos\left(\omega_{\log}\log(k/k_{\rm ref}) + \phi_{\log}\right)\Big)$$

[Fumagalli, Renaux-Petel, **LW** 2012.02761
Fumagalli, Renaux-Petel, **LW** 2105.06481]

$$\Omega_{\rm GW}(k) = \overline{\Omega}_{\rm GW}(k)\Big[1 + \mathcal{A}_{\log,1}\cos\left(\omega_{\log}\log(k/k_{\rm ref}) + \phi_{\log,1}\right) + \mathcal{A}_{\log,2}\cos\left(2\omega_{\log}\log(k/k_{\rm ref}) + \phi_{\log,2}\right)\Big]\,.$$

- (blue) Overall shape determined by envelope of power spectrum
- (red) Find a more complicated periodic structure in log(k)
- (purple) $\mathcal{A}_{\log,1/2}$ decreases as $\omega_{\log}$ is increased, but not equally fast …

GWs from a resonant feature



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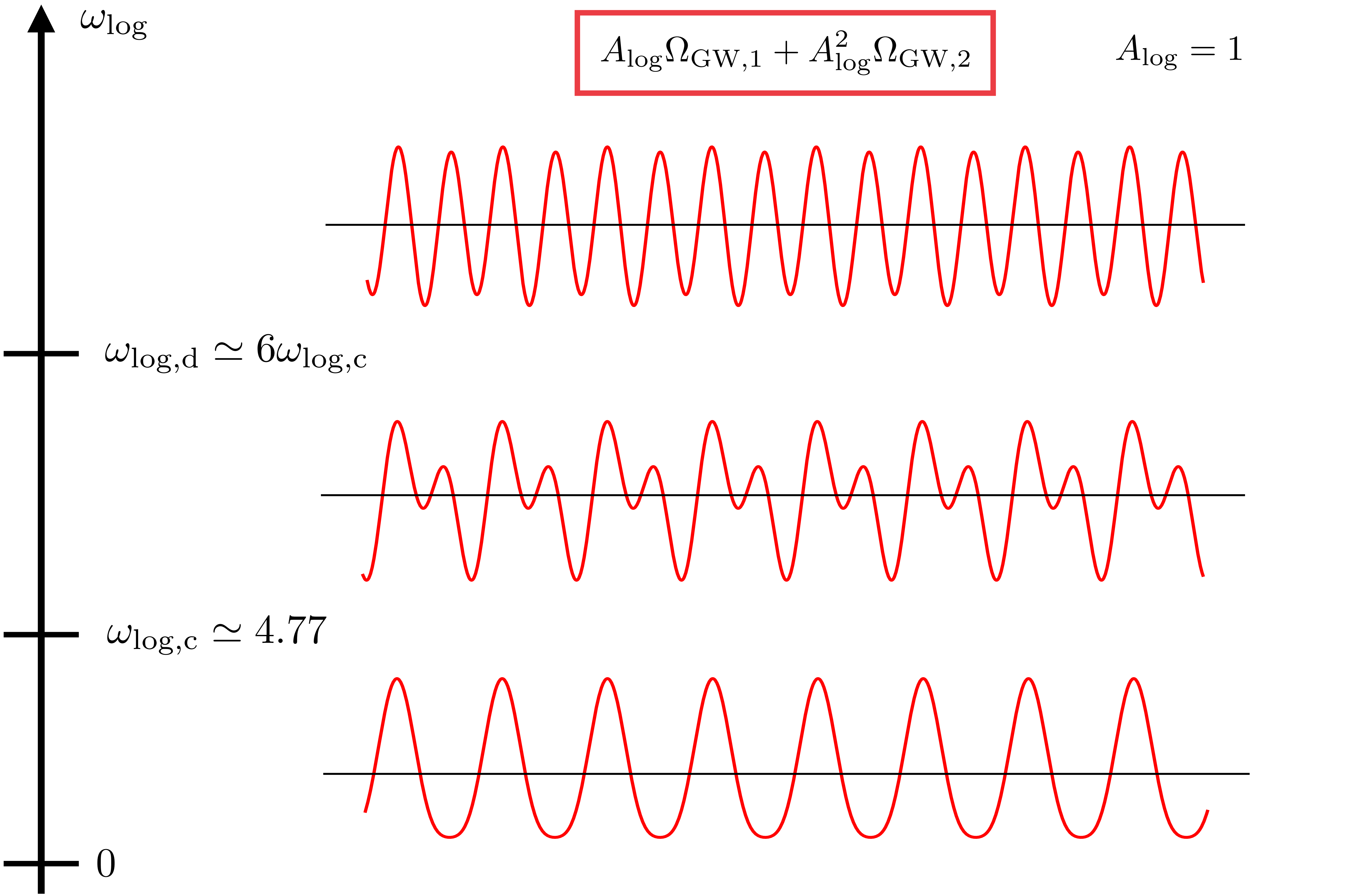
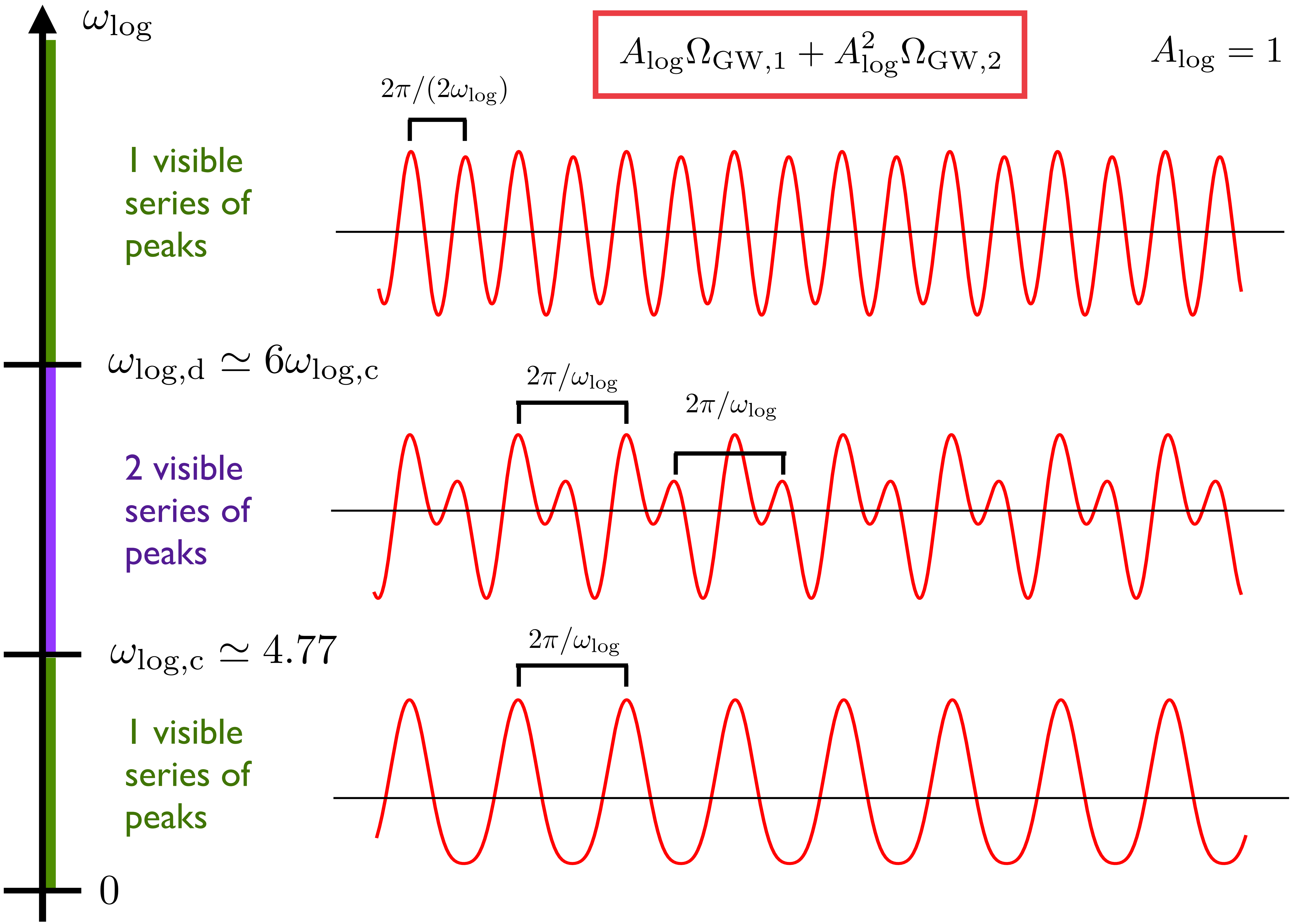


GWs from a resonant feature

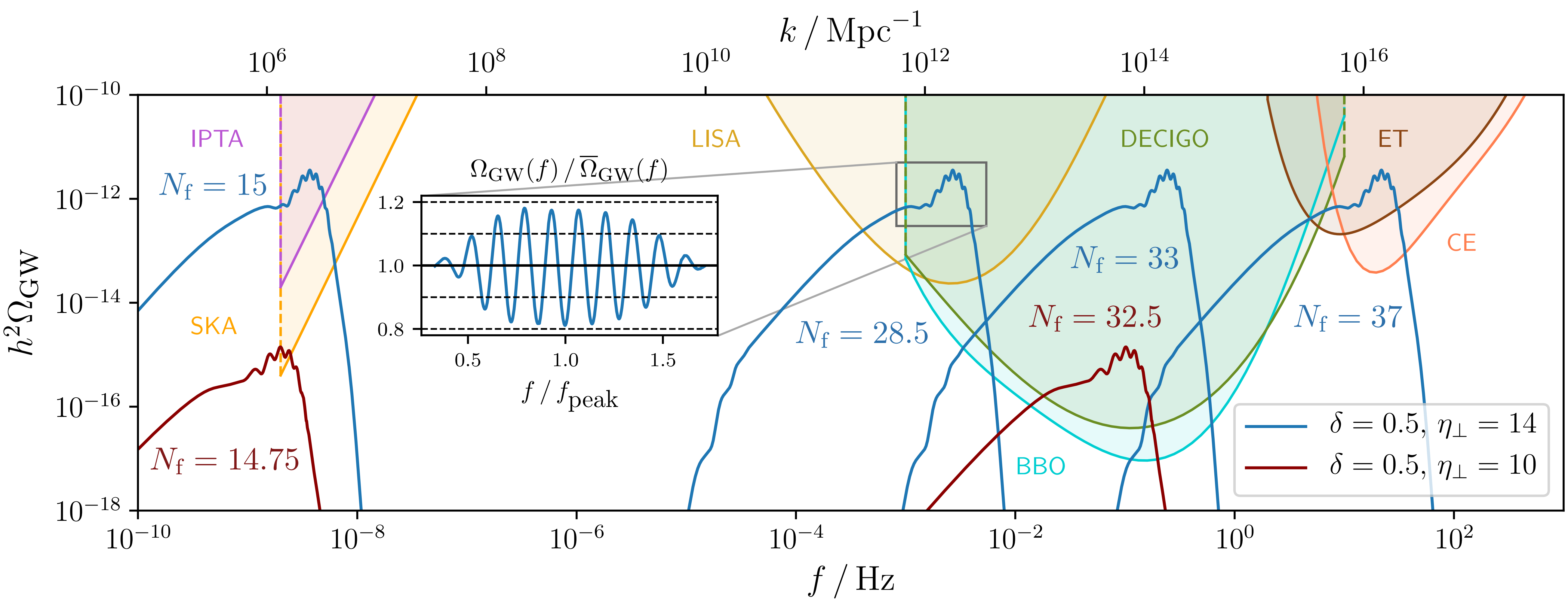
$$A_{\log}\Omega_{\rm GW,1} + A_{\log}^2\Omega_{\rm GW,2}$$

$A_{\log} = 1$



Detecting small-scale features

Depending on the time of the feature / the scale of maximal enhancement of $\Omega_{\rm GW}(k)$ the GW spectrum can peak across the whole frequency range to be probed by upcoming GW experiments:



$N_{\rm f}$ = time of feature (~ max. enhancement) in numbers of e-folds after horizon exit of CMB modes

Detecting small-scale features

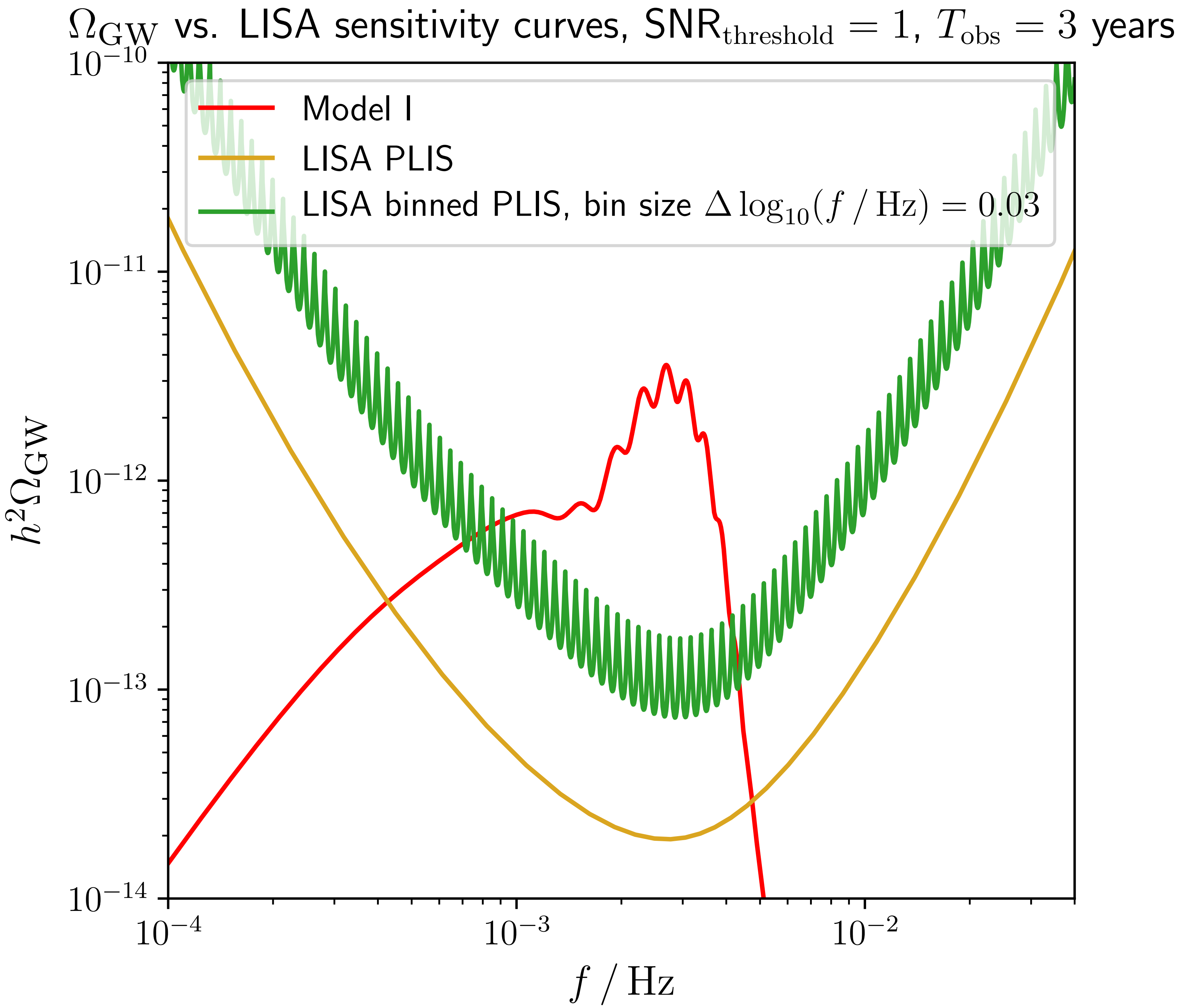
Oscillation detectable?
Can get a first idea by comparing with the power-law-integrated sensitivity (PLIS) curves:
[Thrane, Romano 1310.5300]

LISA PLIS

LISA binned PLIS
[Caprini et al. 1906.09244]

→ **Mauro Pieroni's talk**

Oscillations appear resolvable with LISA if the overall magnitude of $\Omega_{\rm GW}(k)$ is sufficiently high.



Definite answer requires dedicated analysis, to see if 10% modulations can be resolved.

Detecting small-scale features

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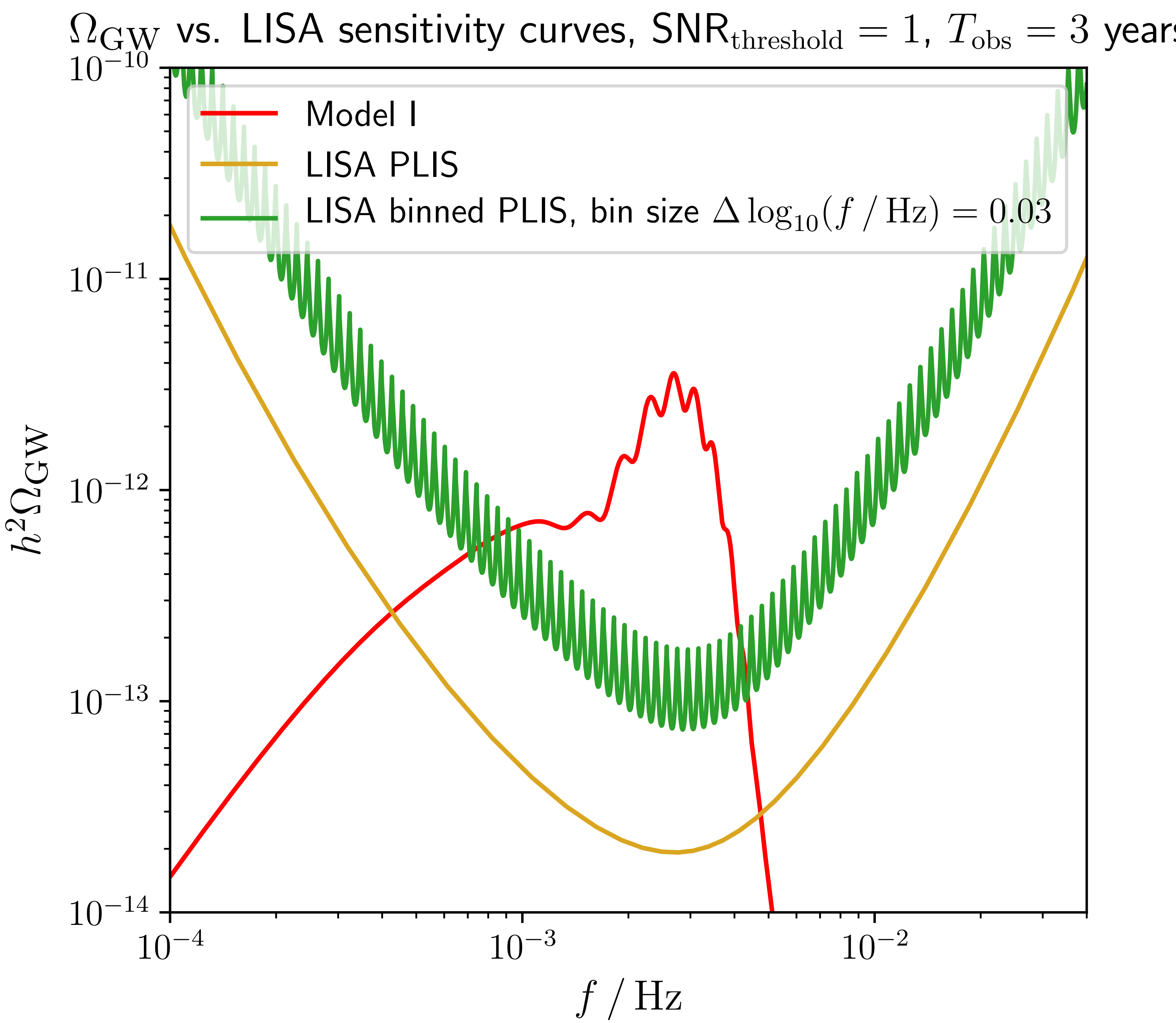
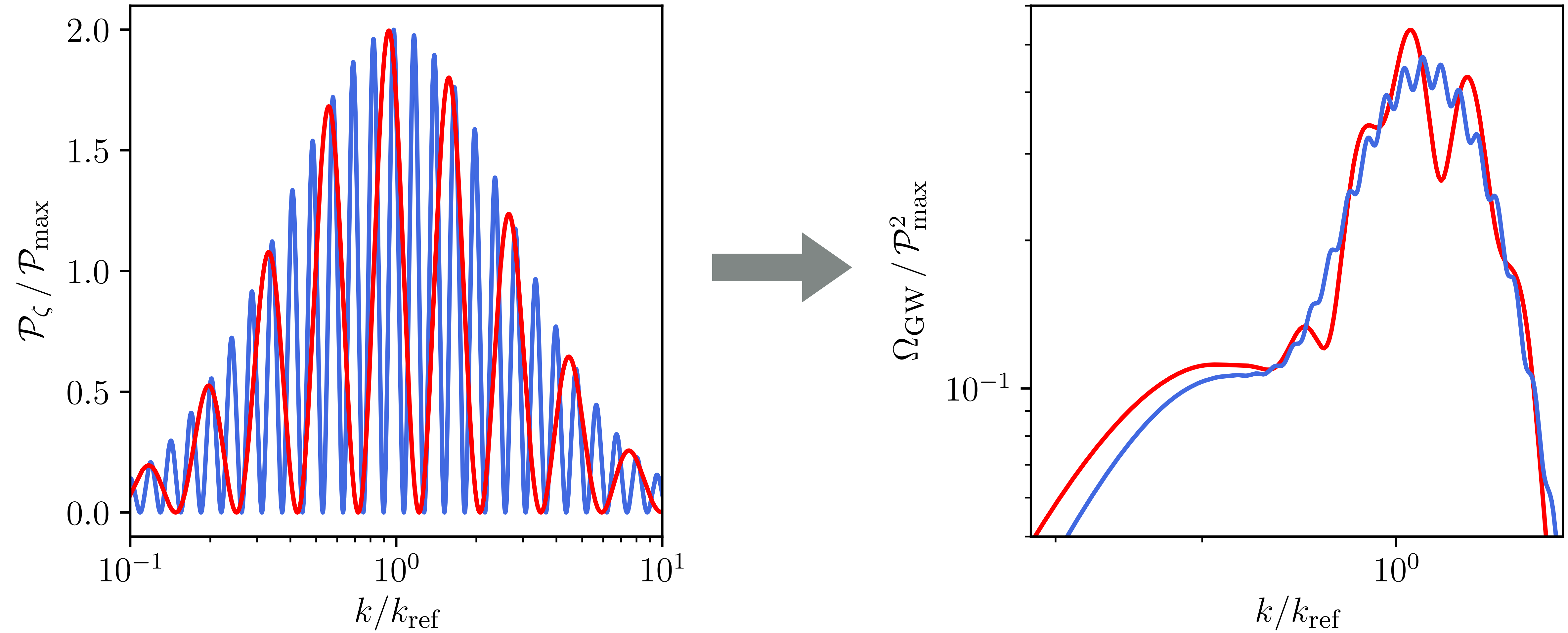


Constraints on enhancements from if excessive **backreaction** and loss of **perturbative control** are to be avoided. [addressed in Fumagalli, Renaux-Petel, **LW** 2012.02761]

Summary



Oscillations in $\mathcal{P}_\zeta(k)$ give corresponding modulations in $\Omega_{\rm GW}(k)$.

Sharp feature: $\omega_{\rm lin} \longrightarrow \omega_{\rm lin}^{\rm GW} = \sqrt{3}\omega_{\rm lin}$

Resonant feature: $\omega_{\rm log} \longrightarrow \omega_{\rm log}^{\rm GW} = \omega_{\rm log},\ 2\omega_{\rm log}$

Summary

Sharp feature:

$$\Omega_{\rm GW}(k) \quad = \quad \overline{\Omega}_{\rm GW}\left(1 + \mathcal{A}_{\rm lin}\cos\left(\omega_{\rm lin}^{\rm GW} k + \varphi_{\rm lin}\right)\right)$$

Resonant feature:

$$\Omega_{\rm GW}(k) \quad = \quad \overline{\Omega}_{\rm GW}(k)\Big[1 + \mathcal{A}_{\log,1}\cos\left(\omega_{\log}\log(k/k_{\rm ref}) + \phi_{\log,1}\right) + \mathcal{A}_{\log,2}\cos\left(2\omega_{\log}\log(k/k_{\rm ref}) + \phi_{\log,2}\right)\Big] \, .$$

- Demonstrated how parameters in the above templates for $\Omega_{\rm GW}$ depend on the parameters in $\mathcal{P}_\zeta$.
- For a resonant feature have precise semi-analytic expressions for $\mathcal{A}_{\log,1/2}$, $\phi_{\log,1/2}$ for suitably broad $\mathcal{P}_\zeta$.
- Above template valid in general (i.e. for both broad & narrow peaks in $\mathcal{P}_\zeta$).

Open questions & further work

- So far only considered scalar-induced GWs produced during post-inflationary era. Also consider inflation-era contribution.

  ⟶ Does this affect the oscillatory part? ⟶ Spyros Sypsas' talk

- To what extent can ~10%-modulations in $\Omega_{\rm GW}(k)$ be reconstructed from GW observatory data?

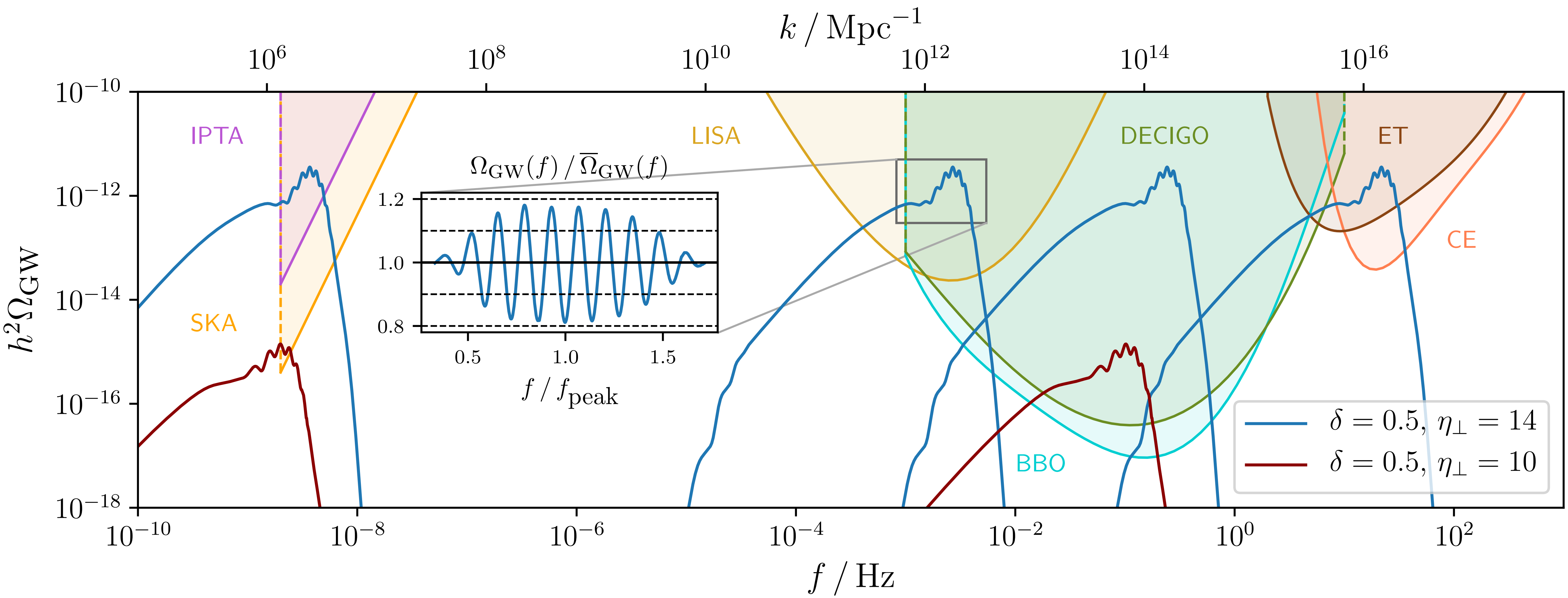
  ⟶ Perform dedicated analysis

- Study explicit inflation models with features:

  ⟶ Can assess constraints on the amplification of scalar fluctuations from backreaction and perturbativity bounds.

- So far examined how the scalar power spectrum affects the GW spectrum:

  ⟶ Invert analysis and study to what extent the scalar power spectrum can be reconstructed from GW data.



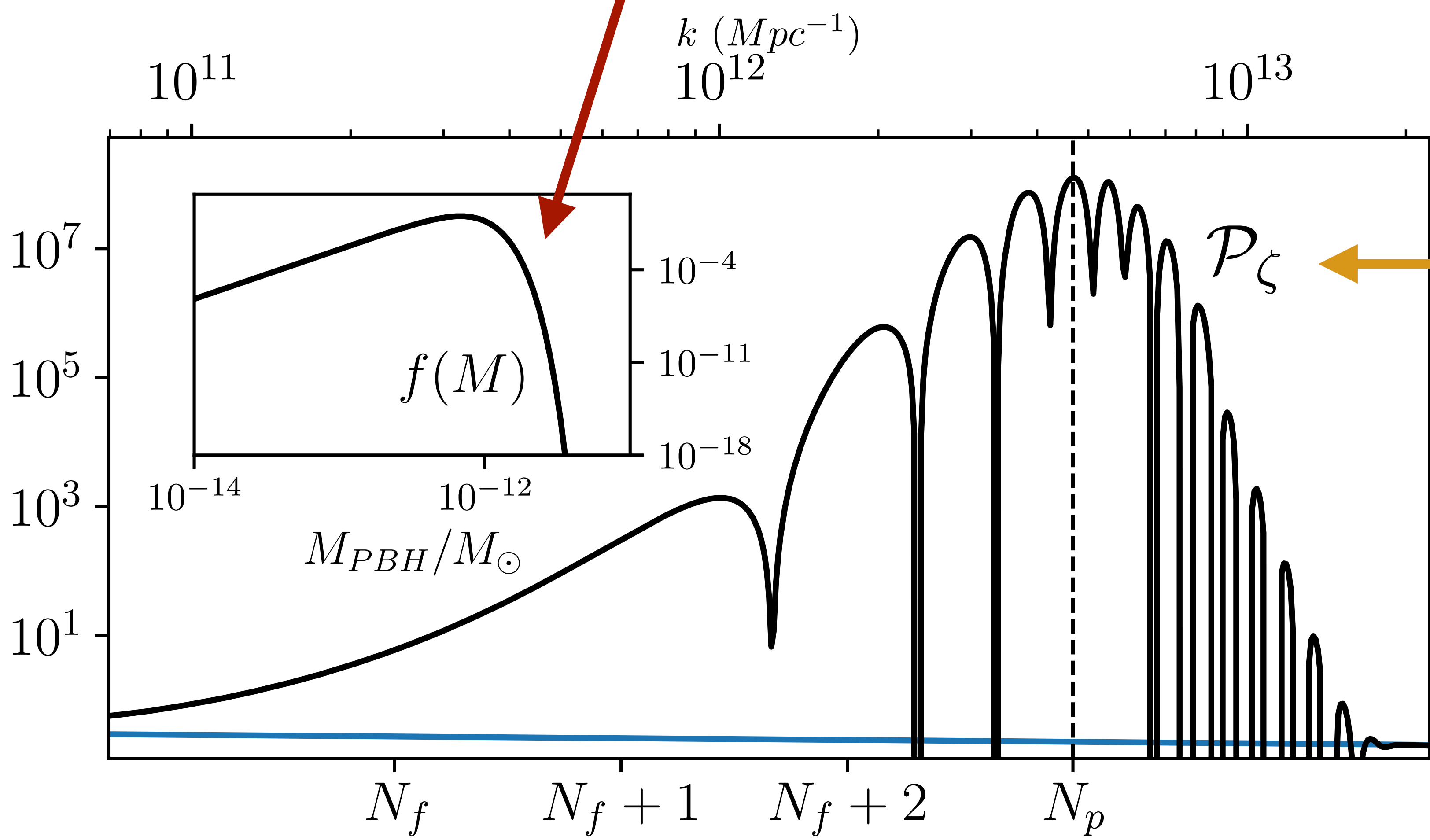
Many thanks for listening!

Extra Slides

PBH abundance

PBH mass function $f(M)$ for a sharp feature model

[Fumagalli, Renaux-Petel, Ronayne, **LW** 2004.08369]



Scalar power spectrum with oscillations

Here: compute $f(M)$ assuming the fluctuations obey Gaussian statistics.

Oscillations washed out in $f(M)$ as a result of smoothing and the integration over the formation time.

Backreaction & perturbative control

Large enhancement of fluctuations can induce **strong backreaction** on background dynamics or lead to **loss of perturbative control**.

May not be fatal for the mechanism, but needs to be taken into account and will certainly affect the phenomenology of results.

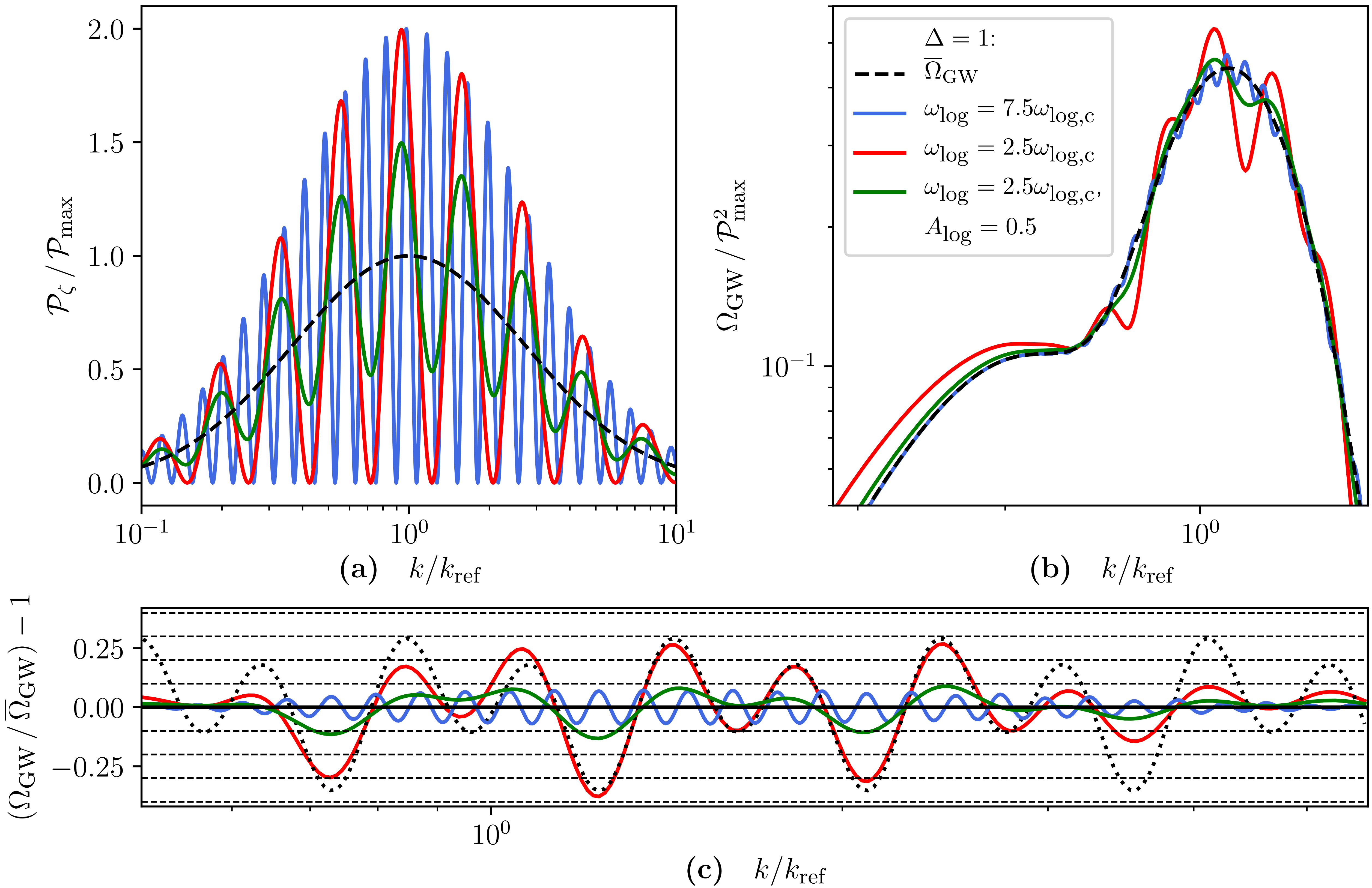
For the sharp turn model:

No excessive backreaction: $$\eta_\perp^4 e^{2\delta\eta_\perp} \lesssim 10^{11}\left(\frac{10^{-9}}{\mathcal{P}_0}\right),$$

Perturbative control: $$\eta_\perp^4 e^{2\delta\eta_\perp} \lesssim 10^{9}\left(\frac{10^{-9}}{\mathcal{P}_0}\right).$$

The perturbativity bound is more stringent, but a more rigorous computation than this estimate is required.

Resonant feature examples



Sharp features

Bunch-Davies

Sharp feature

$$\hat{\zeta}_k(\tau) = \zeta_k^{\rm BD}(\tau)\hat{a}(\boldsymbol{k}) + {\rm h.c.}(-\boldsymbol{k})$$

$$\zeta_k^{\rm BD}(\tau) = \left(\frac{k^3}{2\pi^2}\right)^{-1/2} \mathcal{P}_0^{1/2} e^{-ik\tau}(1 + ik\tau)$$

$$\mathcal{P}_0 = \frac{H^2}{8\pi^2 \epsilon M_{\rm Pl}^2}$$

N

$N_{\rm f} \sim \log(k_{\rm f})$

Sharp features

Bunch-Davies | **Sharp feature** | **Excited state**

$$\hat{\zeta}_k(\tau) = \zeta_k^{\rm BD}(\tau)\hat{a}(\boldsymbol{k}) + {\rm h.c.}(-\boldsymbol{k}) \longrightarrow \hat{\zeta}_k(\tau) = \left[\alpha_k\,\zeta_k^{\rm BD}(\tau) + \beta_k\,\zeta_k^{*\rm BD}(\tau)\right]\hat{a}(\boldsymbol{k}) + {\rm h.c.}(-\boldsymbol{k})$$

$$\zeta_k^{\rm BD}(\tau) = \left(\frac{k^3}{2\pi^2}\right)^{-1/2}\mathcal{P}_0^{1/2}e^{-ik\tau}(1+ik\tau)$$

quantisation: $|\alpha_k|^2 - |\beta_k|^2 = 1$

$$\mathcal{P}_0 = \frac{H^2}{8\pi^2\epsilon M_{\rm Pl}^2}$$

matching: $${\rm ph}\left(\frac{\beta_k}{\alpha_k}\right) \sim e^{2ik/k_{\rm f}}$$

$N_{\rm f} \sim \log(k_{\rm f})$ $\longrightarrow N$

Sharp features

Bunch-Davies — Sharp feature — Excited state

$$\hat{\zeta}_k(\tau) = \zeta_k^{\rm BD}(\tau)\hat{a}(\boldsymbol{k}) + \text{h.c.}(-\boldsymbol{k}) \longrightarrow \hat{\zeta}_k(\tau) = \left[\alpha_k\,\zeta_k^{\rm BD}(\tau) + \beta_k\,\zeta_k^{*\rm BD}(\tau)\right]\hat{a}(\boldsymbol{k}) + \text{h.c.}(-\boldsymbol{k})$$

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$$\mathcal{P}_0 = \frac{H^2}{8\pi^2\epsilon M_{\rm Pl}^2}$$

matching: $\text{ph}\left(\frac{\beta_k}{\alpha_k}\right) \sim e^{2ik/k_{\rm f}}$

N

$N_{\rm f} \sim \log(k_{\rm f})$

$$\mathcal{P}_\zeta(k) \sim \mathcal{P}_0(k)\left(|\alpha_k|^2 + |\beta_k|^2 + 2|\alpha_k||\beta_k|\cos\left(\frac{2k}{k_{\rm f}}\right)\right)$$

Sharp features

Why peaks in $\mathcal{P}_\zeta(k)$ and oscillations go together (for sharp features)

$$\mathcal{P}_\zeta(k) \sim \mathcal{P}_0(k)|\alpha_k|^2\left(1 + \frac{|\beta_k|^2}{|\alpha_k|^2} + 2\frac{|\beta_k|}{|\alpha_k|}\cos\left(\frac{2k}{k_\mathrm{f}}\right)\right) \qquad |\alpha_k|^2 - |\beta_k|^2 = 1$$

Sharp features

Why peaks in $\mathcal{P}_\zeta(k)$ and oscillations go together (for sharp features)

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