Magic properties of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

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LAPTH Annecy

Work in collaboration with

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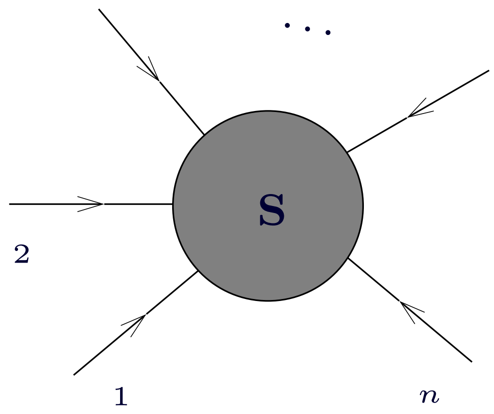
Outline

- ✔ On-shell scattering amplitudes in planar $\mathcal{N} = 4$ SYM
- ✔ MHV Amplitude/Wilson loop duality
- ✔ Dual conformal symmetry – hidden symmetry of planar amplitudes
- ✔ Dual superconformal symmetry for all amplitudes : all-order conjectures
- ✔ Superconformal + Dual superconformal $\Longrightarrow$ Yangian symmetry.

Motivation

We are going to study gluon scattering amplitudes in $\mathcal{N} = 4$ SYM.

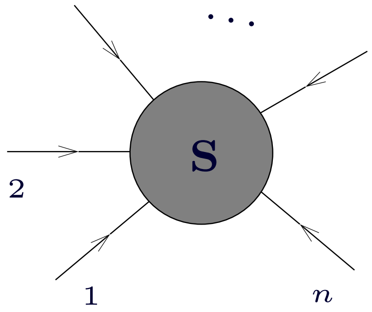
$\mathcal{A}_n =$ 

Why?

- Simpler than QCD amplitudes but they share many of the same properties.
- In planar $\mathcal{N} = 4$ they seem to have a remarkable structure.
- All-order conjectures and a proposal for strong coupling via AdS/CFT.
- New symmetries (integrability) and a new duality - hints at much more to be understood.

Gluon amplitudes

Gluon scattering amplitudes in planar $\mathcal{N}=4$ SYM.

$\mathcal{A}_n =$ 

- ✔ Momenta $(p_i^2 = 0)$, helicities $(h_i = \pm 1)$, colours (a_i).
- ✔ IR divergent.

$$\mathcal{A}_n^{\mathrm{MHV}} = \delta(p_1 + ... + p_n)\,\mathrm{tr}(T^{a_1} ... T^{a_n})\mathcal{A}_{\mathrm{tree}}(p_i, h_i)\mathcal{A}_{\mathrm{loop}}(p_i; \mu, \epsilon)$$

Divergences factorise and exponentiate : [Catani,Collins,Korchemsky,Magnea,Radyushkin,Sterman,...]

$$\ln A_{\mathrm{loop}} = -\frac{1}{4}\sum_{l=1}^{\infty}\lambda^l\left[\frac{\Gamma_{\mathrm{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma_{\mathrm{col}}^{(l)}}{l\epsilon}\right]\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + F_n^{(\mathrm{MHV})} + O(\epsilon)$$

Γ_{cusp} : IR divergences of amplitude related to UV divergences of Wilson loop [Korchemsky,Marchesini,Radyushkin].

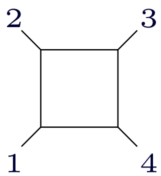
Bern, Dixon and Smirnov proposed an all-order form for $F_n^{(\mathrm{MHV})}$ (not true for $n \geq 6$).

$$F_n^{(\mathrm{BDS})} = \tfrac{1}{2}\Gamma_{\mathrm{cusp}}(\lambda)\mathcal{F}_n + \text{ const .}$$

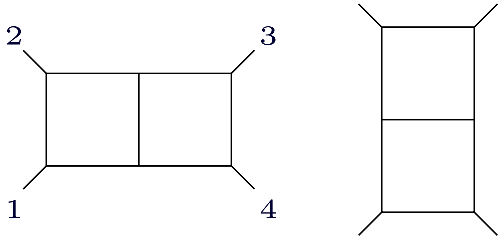
Perturbative MHV amplitudes

Weak coupling corrections to the amplitude can be expressed in terms of scalar integrals:

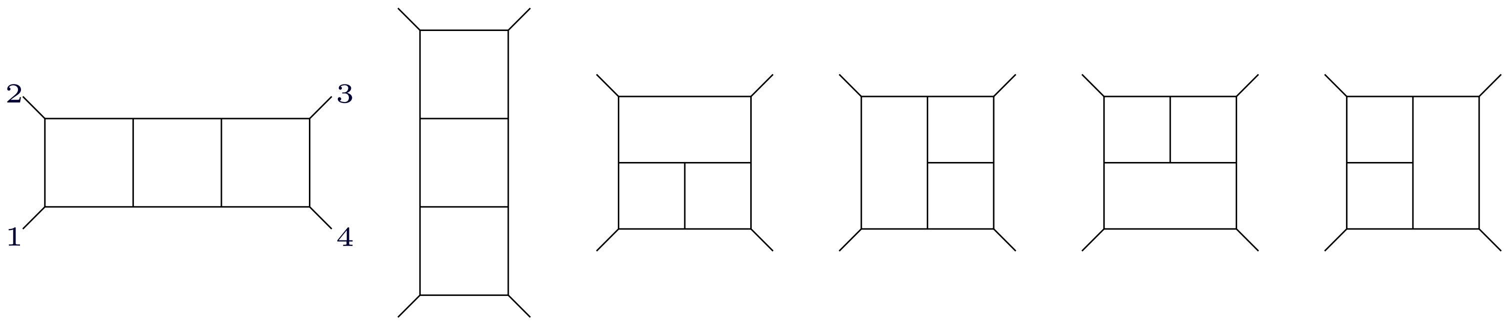
One loop: [Green,Schwarz,Brink]



Two loops: [Bern,Rozowsky,Yan]



Three loops: [Bern,Dixon,Smirnov]



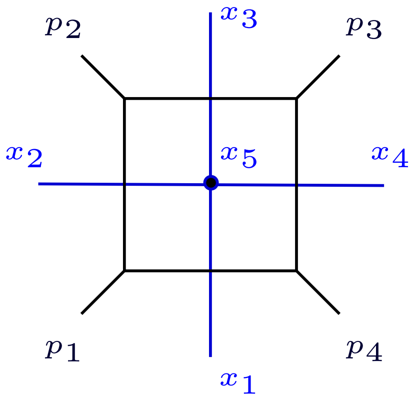
Four loops: scalar integrals of 8 different topologies are identified [Bern,Czakon,Dixon,Kosower,Smirnov]

Hints of a new symmetry

Examine one-loop ‘scalar box’ diagram

- ✔ Change variables to go to a *dual ‘coordinate space’* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^4k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Conformal inversion $x_i^\mu \longrightarrow \frac{x_i^\mu}{x_i^2}$; $x_{ij}^2 \longrightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}$

[Broadhurst],[JMD,Henn,Smirnov,Sokatchev]

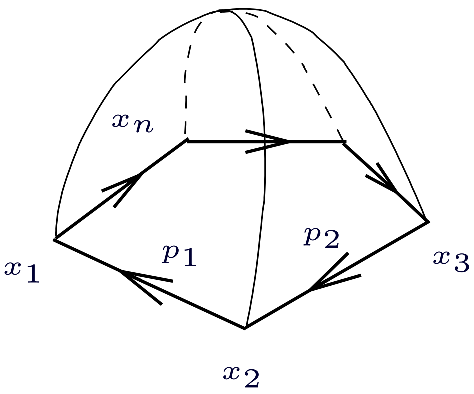
- ✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!
- ✔ The symmetry *is not related* to ordinary conformal symmetry of $\mathcal{N}=4$ SYM (at least not in an obvious way).
- ✔ For *planar* integrals only!
- ✔ IR divergences $\Longrightarrow$ conformal invariance slightly broken.

Amplitudes at strong coupling [Alday, Maldacena]

On-shell scattering amplitude is the area of a classical string world-sheet in AdS$_5$.

Gluons correspond to open strings ending on a brane parallel to the boundary M_4.

T-duality [Kallosh,Tseytlin]: same change of variables $p_i = x_i - x_{i+1}$. Background $\mathrm{AdS}_5 \longleftrightarrow \widetilde{\mathrm{AdS}}_5$.



The closed contour has n cusps with the *dual coordinates* x_i^μ.

Equivalent to the calculation of a Wilson loop in the dual coordinate space at strong coupling.

IR divergences of the amplitude $\longleftrightarrow$ UV divergences of the cusped Wilson loop.

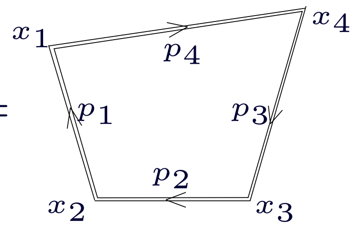
Agrees with the all-loop conjecture for four points, disagrees for a large number of points.

Amplitude/Wilson loop duality

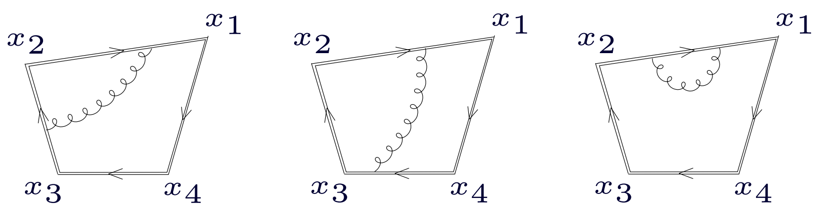
Are Wilson loops are related to gluon amplitudes at weak coupling? - Yes!

$$\ln W(C_n) = -\frac{1}{4}\sum_{l=1}^{\infty}\lambda^l\Big[\frac{\Gamma^{(l)}_{\rm cusp}}{(l\epsilon)^2}+\frac{\Gamma^{(l)}}{l\epsilon}\Big]\big(-\mu^2 x^2_{i-1,i+1}\big)^{l\epsilon} + F_n^{(\rm WL)} + O(\epsilon)$$

$$W(C_4) = \langle {\rm tr}\ {\rm P}\exp ig\oint_{C_4} dx^\mu A_\mu(x)\rangle\,, \qquad C_4 =$$



One loop calculation: [JMD,Korchemsky,Sokatchev]



Agrees with the gluon amplitude! $F_4^{(\rm WL)} = F_4^{(\rm MHV)} +$ const .

Generalised to n-gluon one-loop amplitudes [Brandhuber,Heslop,Travaglini].

Two loop calculations: 4,5,6 points [JMD,Henn,Korchemsky,Sokatchev].

Six point MHV amplitude [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]: $F_6^{(\rm MHV)} = F_6^{(\rm WL)} \neq F_6^{(\rm BDS)}$.

Dual conformal symmetry: MHV loops

Conformal transformations map light-like polygon C_n into another light-like polygon C'_n

If the Wilson loop $W(C)$ were well-defined in four dimensions then we would have

$$W(C) = W(C').$$

However, $W(C_n)$ is UV divergent due to cusps $\Longrightarrow$ breaks conformal invariance.

Anomalous conformal Ward identities for the finite part of the Wilson loop [JMD,Henn,Korchemsky,Sokatchev].

Under special conformal transformations,

$$\mathbb{K}^{\mu} F_n \equiv \sum_{i=1}^{n} \left[2x_i^{\mu}(x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^{\mu}\right] F_n = \frac{1}{2}\Gamma_{\rm cusp}(a) \sum_{i=1}^{n} x_{i,i+1}^{\mu} \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

Unique solution for 4 and 5 points - coincides with BDS.

Invariants start appearing at six points -

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}.$$

Allows deviation from BDS as we have seen.

Dual conformal symmetry at tree-level

So far we ignored the tree-level structures. They are helicity-dependent.

Recall $p_i^2 = 0 \iff p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$. Helicity $h_i = \frac{1}{2}(\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} - \lambda_i^\alpha \partial_{i\alpha})$

MHV Example: $(+\ldots+-_i+\ldots+-_j+\ldots+)$

$$\mathcal{A}_{\rm tree}^{\rm MHV} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \qquad \langle ij\rangle = \lambda_i^\alpha \lambda_{j\alpha}.$$

We need the action of the dual conformal generators on the spinors $\lambda, \tilde\lambda$.

The momenta satisfy two constraints

$$\sum p_i^{\alpha\dot\alpha} = 0 \implies p_i^{\alpha\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha}$$
$$p_i^2 = 0 \implies p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$$

Together these imply the constraints: $x_i - x_{i+1} - \lambda_i\tilde\lambda_i = 0$

Extend dual conformal generators so that they commute with the constraints up to constraints:

$$K_{\alpha\dot\alpha} = \sum_i [x_{i\alpha}{}^{\dot\beta} x_{i\dot\alpha}{}^{\beta} \partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta}]$$

$A_{\rm tree}^{\rm MHV}$ covariant if i and j are neighbours.

$\mathcal{N} = 4$ **Super-amplitudes**

Why are split-helicity gluon amplitudes special? What about other helicities/particles?

$\mathcal{N} = 4$ SYM is special because it is described by PCT self-conjugate supermultiplet:

Chiral representation:

$$\Phi(\eta) = G^+ + \eta^A \Gamma_A + \frac{1}{2}\eta^A\eta^B S_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D + \frac{1}{4!}(\eta)^4 G^-$$

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha A} = \lambda^\alpha \eta^A, \qquad \bar{q}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}.$$

Superamplitudes:

$$\mathcal{A}(\Phi_1 \ldots \Phi_n) = (\eta_1)^4 (\eta_2)^4 \mathcal{A}(G_1^- G_2^- G_3^+ \ldots G_4^+) + \ldots$$

$$p^{\alpha\dot{\alpha}} = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \qquad q^{\alpha A} = \sum_i \lambda_i^\alpha \eta_i^A, \qquad \bar{q}^{\dot{\alpha}}_A = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A}.$$

Symmetries:

$$p\mathcal{A} = q\mathcal{A} = \bar{q}\mathcal{A} = 0 \implies \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \delta^4(p)\delta^8(q)\mathcal{P}(\lambda, \tilde{\lambda}, \eta), \qquad \bar{q}\mathcal{P} = 0.$$

$$\mathcal{P} = \mathcal{P}^{\mathrm{MHV}} + \mathcal{P}^{\mathrm{NMHV}} + \ldots + \mathcal{P}^{\overline{\mathrm{MHV}}}.$$

Conventional superconformal symmetry

Since $\mathcal{N}=4$ SYM is a superconformal theory so we expect an action of the superconformal algebra on amplitudes [Witten].

$$p^{\dot\alpha\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}, \qquad k_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha}\partial_{i\dot\alpha},$$

$$\overline{m}_{\dot\alpha\dot\beta} = \sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, \qquad m_{\alpha\beta} = \sum_i \lambda_{i(\alpha}\partial_{i\beta)},$$

$$d = \sum_i [\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + 1], \qquad r^A{}_B = \sum_i [-\eta_i^A\partial_{iB} + \tfrac{1}{4}\eta_i^C\partial_{iC}],$$

$$q^{\alpha A} = \sum_i \lambda_i^{\alpha}\eta_i^A, \qquad \bar q^{\dot\alpha}_A = \sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA},$$

$$s_{\alpha A} = \sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar s^A_{\dot\alpha} = \sum_i \eta_i^A\partial_{i\dot\alpha}.$$

$$c = \sum_i [1 + \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda^{\dot\alpha}\partial_{i\dot\alpha} - \tfrac{1}{2}\eta^A\partial_{iA}]$$

$$\partial_{i\alpha} = \frac{\partial}{\partial\lambda_i^{\alpha}}, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}, \qquad \partial_{iA} = \frac{\partial}{\partial\eta_i^A}.$$

Some operators are zeroth order, some first order and some second order.

Dual superconformal symmetry [JMD,Henn,Korchemsky,Sokatchev]

Momentum conservation $\delta^4(p)$ suggests the introduction of the dual x_i:

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} - \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} = 0.$$

Supersymmetry $\delta^8(q)$ suggests the introduction of chiral dual θ_i:

$$\theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} - \lambda_i^{\alpha}\eta_i^{A} = 0.$$

Now we can extend dual conformal symmetry to dual superconformal symmetry by extending the standard chiral representation so that all generators commute with the constraints up to constraints.

All generators of dual superconformal symmetry

$$P_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha\dot\alpha},$$

$$Q_{\alpha A} = \sum_i \partial_{i\alpha A},$$

$$\overline{Q}^A_{\dot\alpha} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot\alpha} + \eta_i^A \partial_{i\dot\alpha}],$$

$$D = \sum_i [-x_i^{\dot\alpha\alpha}\partial_{i\alpha\dot\alpha} - \tfrac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} - \tfrac{1}{2}\lambda_i^\alpha \partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}],$$

$$C = \sum_i [-\tfrac{1}{2}\lambda_i^\alpha\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + \tfrac{1}{2}\eta_i^A\partial_{iA}] = \sum_i h_i,$$

$$S_\alpha^A = \sum_i [-\theta_{i\alpha}^B\theta_i^{\beta A}\partial_{i\beta B} + x_{i\alpha}{}^{\dot\beta}\theta_i^{\beta A}\partial_{\beta\dot\beta} + \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\partial_{i\dot\beta} - \theta_{i+1\,\alpha}^B\eta_i^A\partial_{iB}],$$

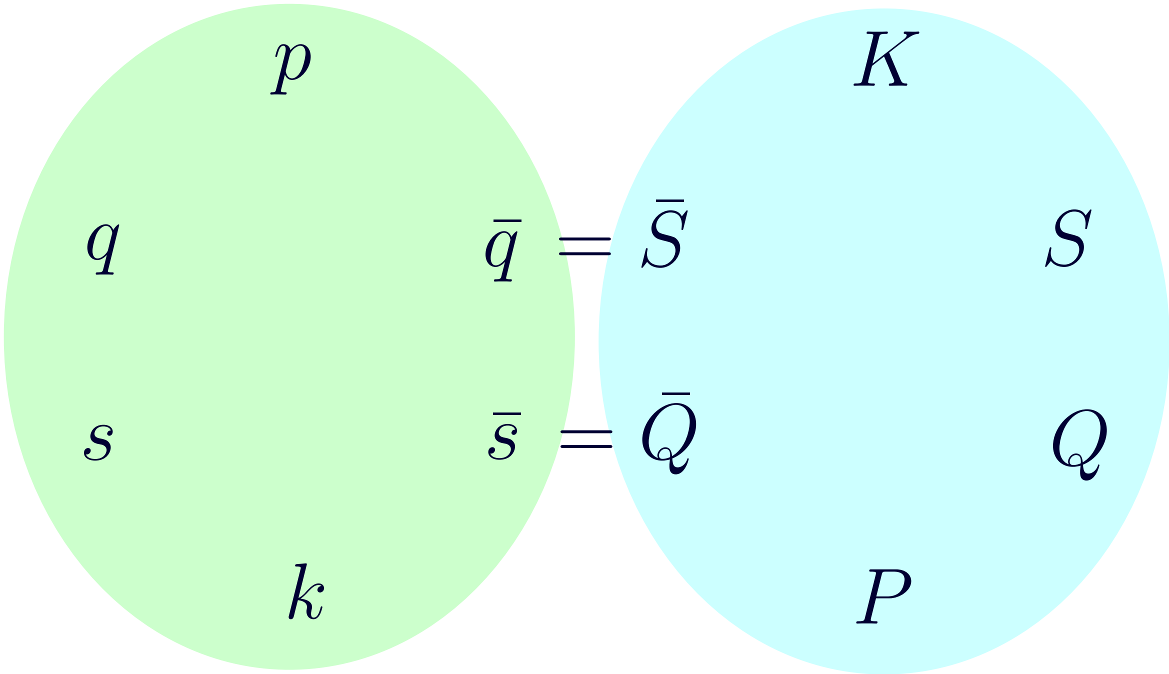
$$\overline{S}_{\dot\alpha A} = \sum_i [x_{i\dot\alpha}{}^{\beta}\partial_{i\beta A} + \tilde\lambda_{i\dot\alpha}\partial_{iA}],$$

$$K_{\alpha\dot\alpha} = \sum_i [x_{i\alpha}{}^{\dot\beta}x_{i\dot\alpha}{}^{\beta}\partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\theta_{i\alpha}^B\partial_{i\beta B} + x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta_{i+1\,\alpha}^B\partial_{iB}].$$

$$\partial_{i\alpha\dot\alpha} = \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \qquad \partial_{i\alpha A} = \frac{\partial}{\partial\theta_i^{\alpha A}}, \qquad \partial_{i\alpha} = \frac{\partial}{\partial\lambda_i^\alpha}, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}, \qquad \partial_{iA} = \frac{\partial}{\partial\eta_i^A}.$$

Conventional and dual superconformal symmetries

The generators of conventional and dual superconformal symmetry are not all independent:



[Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]: Combining bosonic T-duality with a fermionic one shows that dual superconformal symmetry arises naturally from the string theory viewpoint (at least at large N).

How amplitudes behave under the symmetries

At tree level: we expect invariance under conventional superconformal algebra.

Observe covariance under the dual superconformal algebra:

$$\mathcal{A}_{\rm tree} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\big(1 + \tfrac{1}{n}\sum_{r,s,t} R_{r;s,t} + \ldots\big)$$

Here $R_{r;s,t}$ is a dual superconformal invariant [Bern,Dixon,Kosower],[JMD,Henn,Korchemsky,Sokatchev]:

$$R_{r;s,t} = \frac{\langle s\ s-1\rangle\langle t\ t-1\rangle\delta^4(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x_{st}^2\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}$$

Dual conformal covariance of trees confirmed using supersymmetric BCFW recursion relations [Brandhuber,Heslop,Travaglini] .

All tree-level amplitudes solved explicitly [JMD,Henn].

Beyond tree level

Conventional superconformal symmetry $s, \bar{s} = \bar{Q}, k$ broken by infrared divergences. We do not know how to control the breaking.

Would expect dual superconformal symmetry $S, \bar{S} = \bar{q}, K$ also to be broken.

Also for MHV we have seen that K is broken, but in a controlled way (anomalous Ward identity).

We claim that the breaking of K is universal for all amplitudes:

$$\mathcal{A} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\Big(\mathcal{P}^{\mathrm{MHV}} + \mathcal{P}^{\mathrm{NMHV}} + \ldots \mathcal{P}^{\overline{\mathrm{MHV}}}\Big)$$

Factor out $\mathcal{P}^{\mathrm{MHV}}$ (which contains all IR divergences):

$$\mathcal{A} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\mathcal{P}^{\mathrm{MHV}}\mathcal{R}.$$

Claim: After setting $\epsilon \to 0$, $\mathcal{R}$ is dual conformally invariant.

$$\mathcal{R} = 1 + \mathcal{R}^{\mathrm{NMHV}} + \ldots \mathcal{R}^{\overline{\mathrm{MHV}}}.$$

Verified at one loop [JMD,Henn,Korchemsky,Sokatchev],[Brandhuber,Heslop,Travaglini].

Commuting the two algebras

What algebraic structure combines both superconformal and dual superconformal algebras?

[JMD,Henn,Plefka]

First we must reformulate dual superconformal symmetry as an invariance.

Subtract the weight terms:

$$\tilde{K}^{\alpha\dot{\alpha}} = K^{\alpha\dot{\alpha}} + \textstyle\sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}} \qquad \text{and} \qquad \tilde{S}_\alpha^A = S_\alpha^A + \textstyle\sum_{i=1}^{n} \theta_{i\alpha}^A$$

So that: $\tilde{K}\mathcal{A} = 0$ and $\tilde{S}\mathcal{A} = 0$.

We want to remove all x and θ dependence.

Use $P_{\alpha\dot{\alpha}}$ and $Q_{\alpha A}$ to set $x_1 = 0$ and $\theta_1 = 0$. Eliminate all other x_i and θ_i in favour of $\lambda_i, \tilde{\lambda}_i, \eta_i$.

$$S'^{A}_{\alpha} = -\sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} \lambda_j^\gamma \eta_j^A \lambda_{i\alpha} \frac{\partial}{\partial \lambda_i^\gamma} + \sum_{j=1}^{i} \lambda_{j\alpha} \tilde{\lambda}_j^{\dot{\beta}} \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} - \sum_{j=1}^{i} \lambda_{j\alpha} \eta_j^B \eta_i^A \frac{\partial}{\partial \eta_i^B} + \sum_{j=1}^{i-1} \lambda_{j\alpha} \eta_j^A \right]$$

Now on the same footing as the ordinary superconformal generators.

Yangians

Consider a Lie (super)algebra g: $[J_a, J_b] = f_{ab}{}^c J_c$

Can introduce some 'level one' generators

$$[J_a, J_b^{(1)}] = f_{ab}{}^c J_c^{(1)}$$

The Jacobi identity can be 'quantised' (Drinfel'd):

$$[J_a^{(1)}, [J_b^{(1)}, J_c]] + \mathrm{cyc}(a,b,c) = h f_{ar}{}^l f_{bs}{}^m f_{ct}{}^n f^{rst} \{J_l, J_m, J_n\}$$

Then J and $J^{(1)}$ generate the Yangian $Y(g)$.

On a chain the generators J can be given by sums of single site generators

$$J_a = \sum_i J_{ia}$$

Then $J_a^{(1)}$ can take the bilocal form [Dolan,Nappi,Witten]

$$J_a^{(1)} = f_a{}^{cb} \sum_{i<j} J_{ib} J_{jc}$$

if the representation $\mathcal{R}$ of J_i satisfies the condition that the adjoint appears only once in $\mathcal{R} \otimes \bar{\mathcal{R}}$.

From dual superconformal symmetry to the Yangian

We want to identify two bilocal Yangian generators $J_a^{(1)}$ with the symmetries K' and S'

Inspecting the dimensions and Lorentz and $su(4)$ labels suggests the identification

$$p^{(1)}_{\alpha\dot{\alpha}} \sim K'_{\alpha\dot{\alpha}}, \qquad\qquad q^{(1)}{}^{A}_{\alpha} \sim S'^{A}_{\alpha}$$

Indeed we can add terms to S' which annihilate the amplitudes on their own

$$\Delta S^A_\alpha = \tfrac{1}{2}\left[-q^A_\gamma m^\gamma_\alpha + q^A_\alpha \tfrac{1}{2} d_\lambda + n q^A_\alpha + p^{\dot\beta}_\alpha \bar{s}^A_{\dot\beta} + q^B_\alpha r^A_B - q^A_\alpha \tfrac{1}{4} d_\eta + q^A_\alpha\right]$$

and we arrive at the bilocal formula

$$q^{(1)A}_\alpha := \sum_{i>j}\left[m^{\gamma}_{i\alpha} q^A_{j\gamma} - \tfrac{1}{2}(d_i + c_i) q^A_{j\alpha} + p^{\dot\beta}_{i\alpha}\bar{s}^A_{j\dot\beta} + q^B_{i\alpha} r^A_{jB} - (i \leftrightarrow j)\right].$$

The remaining generators in the level one multiplet come by acting with level zero generators.

The generator $p^{(1)}$ so obtained coincides with K' after similarly adding terms which annihilate the amplitude.

Cyclicity

The bilocal representation of Yangians is not normally consistent with the cyclicity of a closed chain.

Here this problem is avoided by a remarkable mechanism.

Consider

$$J_a^{(1)} = f_a{}^{cb} \sum_{1 \le i < j \le n} J_{ib} J_{jc}$$

$$\tilde{J}_a^{(1)} = f_a{}^{cb} \sum_{2 \le i < j \le n+1} J_{ib} J_{jc}$$

Then cyclicity implies $J_a^{(1)} - \tilde{J}_a^{(1)}$ should annihilate the amplitude.

One finds the following term which, in general, does not annihilate the amplitude

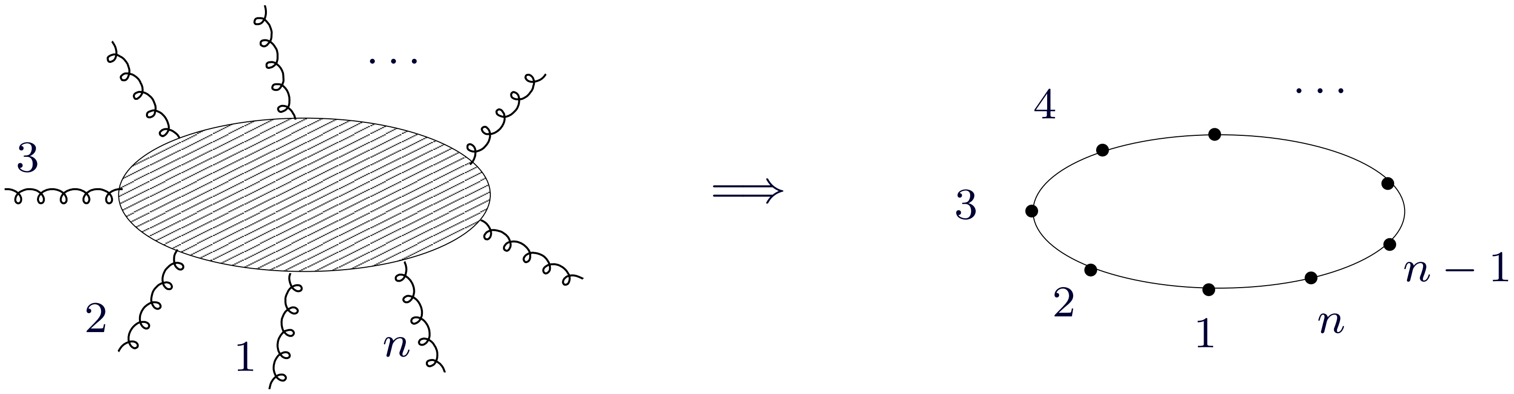
$$f_a{}^{cb} f_{bc}{}^{d} J_{1d}$$

But for certain superalgebras this vanishes identically (those with vanishing Killing form):

$$psl(n|n), osp(2n+2|2n), D(2,1;\alpha), P(n), Q(n)$$

Towards integrability for the S-matrix?

As far as the algebraic representations are concerned, amplitudes are identical to local operators.



Fields in a single trace operator can be written as (e.g.) $\Phi_{AB} = c^\dagger_A c^\dagger_B |0\rangle, \quad \Psi_{\alpha A} = a^\dagger_\alpha c^\dagger_A |0\rangle$.

The free generators get deformed by coupling-dependent corrections [Beisert].

They involve operators which can increase or decrease the number of sites.

Yangian symmetry implies there are extra charges which commute with the spin chain Hamiltonian (anomalous dilatation generator) [Dolan,Nappi,Witten].

Suggests that all the extremely powerful techniques (Bethe Ansatz etc.) applied to the spectral problem should have a version for amplitudes. Can the S-matrix be solved?