QCD at high temperature and density : Renormalization group invariant resummation of perturbative expansion

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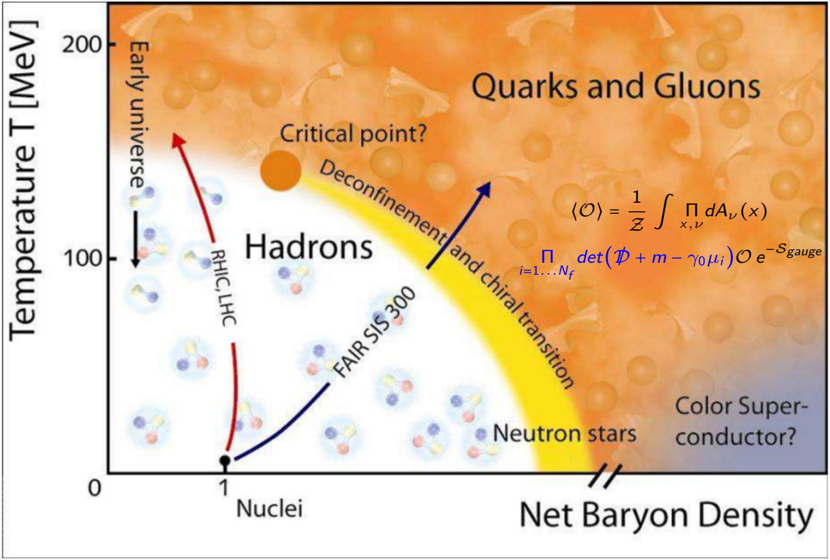
LABORATOIRE CHARLES COULOMB MONTPELLIER

QCD phase diagram



$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\nu} dA_\nu(x) \prod_{i=1\ldots N_f} det\left(\not{D} + m - \gamma_0 \mu_i\right) \mathcal{O}\, e^{-S_{gauge}}$$

Gert Aarts. Journal of Physics:Conference Series, 706:022004, Apr 2016

Quantum Field Theory at finite temperature and density

Partition function of statistical physics : $\mathcal{Z} = Tr\, e^{-\beta(\hat{H}-\mu\hat{N})}$.
Starting from QFT : $\mathcal{Z}_{QFT}$ (Imaginary time formalism)

1. $\int_0^{+\infty} dt \xrightarrow{t=-i\tau} \int_0^{\frac{1}{T}} d\tau \quad , \quad p_0 \to p_0 - i\mu$
2. Periodic/anti-periodic B.Cs for Bosons/Fermions
3. In Fourier Space :

$$\int_0^{\frac{1}{T}} d\tau \to \sum_{n=-\infty}^{\infty} \quad , \quad p_0 \to \omega_n = \begin{cases} 2\pi n T & \textit{Bosons} \\ (2n+1)\pi T - i\mu & \textit{Fermions} \end{cases}$$

4. $$T \sum_{\{\omega_n\}} \int d^3\vec{p} = \sum\!\!\!\!\!\!\int_{\{P\}} \quad , \quad T \sum_{\omega_n} \int d^3\vec{p} = \sum\!\!\!\!\!\!\int_{P}$$

$$\mathcal{Z}_{QFT} = \mathcal{Z}_{free\ gas} + \textit{radiative corrections}$$

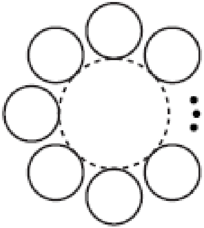
Infrared divergences ($\lambda\phi^4$ model)

$\lambda\phi^4$ shares properties similar to QCD.
If $m = 0$ (gluons) and $n = 0 \Rightarrow \omega_n = 0$ then :

$$\int dp \frac{1}{(p^2)^k},$$

- IR-divergent !

Daisy diagram : Most divergent diagram at order N



IR divergences : Resummation

- Structure of a Taylor Expansion
- Resum a subclass of Daisy diagrams
- $$\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{\lambda_B T^2}{4}\right)^N\left(\frac{d}{dm_B^2}\right)^N\left(\frac{-m_B^3 T}{12\pi}\right)=\frac{-T}{12\pi}\left(m_B^2+\frac{\lambda_B T^2}{4}\right)^{\frac{3}{2}}$$
- Thermal mass generated by the dynamics of the theory
- Expansion in massless theory → IR-divergences
- "Equivalent" resummation : $\mathcal{L} \to \mathcal{L} + m_{thermal}^2\phi^2$
- Other approach of resummation : (OPT/SPT)

Renormalization Group Optimized Perturbation Theory (RGOPT)

- J-L. Kneur, A. Neveu, QCD T=0, '13
- J-L. Kneur, M. Pinto, $\lambda\phi^4$ $T \neq 0$, '16

1. $\mathcal{L}_(\lambda, 0) \rightarrow \mathcal{L}_(\delta\lambda, m(1-\delta)^a)$
2. $m \rightarrow m(1-\delta)^a \quad ; \quad \lambda \rightarrow \delta\ \lambda$
3. Expand in δ at order $\mathcal{O}\left(\lambda^k\right)$ then $\delta \rightarrow\ 1$
4. RG invariance requires : "a" and vacuum substraction terms

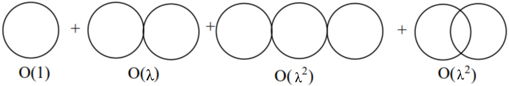
$$\frac{\partial}{\partial m}\mathcal{P}(\lambda, m)\Big|_{m=\bar{m}} = 0 \quad ; \quad \left(M\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \gamma_m m\frac{\partial}{\partial m}\right)\mathcal{P}(\lambda, m) = \ 0$$

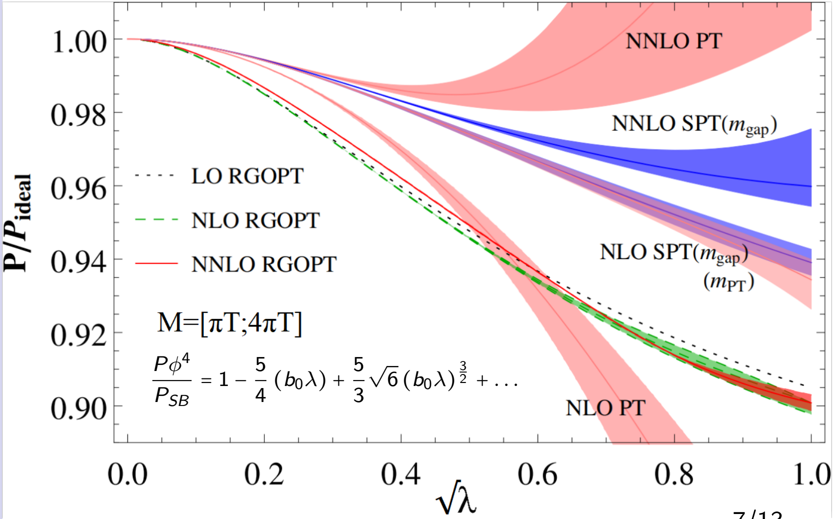
OPT $\oplus$ RG = reduced RG :

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda}\right)\mathcal{P}(\lambda, m, \delta = 1) = 0$$

one-loop Order : RG $\rightarrow$ $a = \frac{\gamma_0}{b_0} = \frac{1}{6}$ & OPT $\rightarrow$ $\bar{m}^2 \sim \mathcal{O}(\lambda T^2)$

$\lambda\phi^4$ *model*





L. Fernandez, J-L. Kneur, to appear soon : 03/21

Mass term for the Gluons : Hard thermal Loops

$m^2 A_\mu A^\mu$: explicitly breaks gauge invariance, (Curci-Ferrari model, cf. Van Egmond's talk)

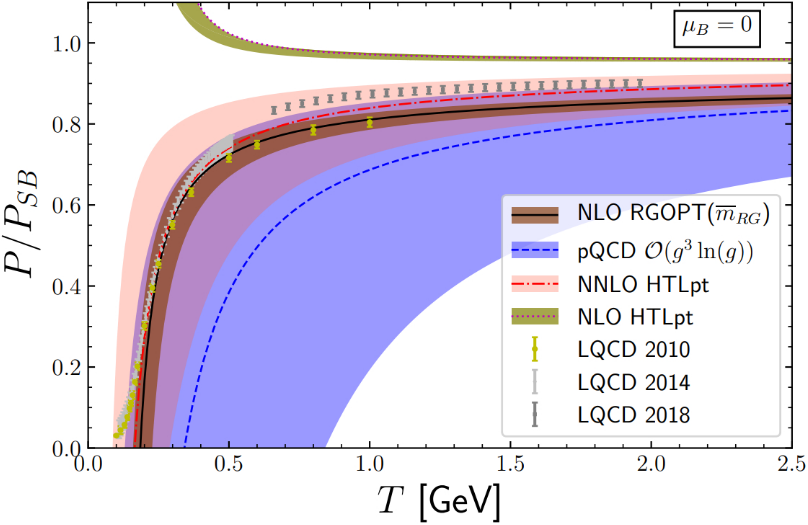
Hard Thermal Loop term (E.Braaten and D.Pisarski (1990)) :

$$-\frac{m^2}{2} Tr\left[G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot \mathcal{D})^2} \right\rangle G^\mu{}_\beta \right] \quad ; \quad y^\mu = (1, \vec{y})$$

Gauge invariant but non-linear and add dressed vertices and propagators !

Systematic high temperature expansion : HTLpt (Andersen et al. 2010-2014) State-of-the-art calculations : NNLO

High Temperature QCD



J-L. Kneur, M.B. Pinto, T.E Restrepo '21

RGOPT LO Cold& Dense

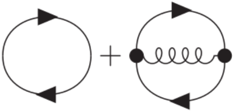
RG invariance needs vacuum substraction :

$$g = 4\pi\alpha_s \quad ; \quad \mathcal{E}_0 = \frac{-m^4}{g}\sum_{n=0}^{k} s_n\, g^n \quad ; \quad LO \rightarrow \ \mathcal{E}_0 = \frac{-m^4\ s_0}{g}$$

(missing in HTL,HDLpt)

$$\mathcal{P}^{PT}_{0,f}(\mu, m) = N_c\frac{m^4}{(4\pi)^2 b_0 g} - N_c\frac{m^4}{8\pi^2}\left(\frac{3}{4} - Log\left(\frac{m}{M}\right)\right)$$
$$+\Theta(\mu^2 - m^2)\frac{N_c}{12\pi^2}\left[\mu\ p_F\left(\mu^2 - \frac{5}{2}m^2\right) + \frac{3}{2}m^4 \ln\left(\frac{\mu + p_F}{m}\right)\right]$$

$$OPT \rightarrow \ \bar{m} \sim \mathcal{O}\left(\sqrt{g}\mu\right) \ \ \& \ \ RG \rightarrow \ a = \frac{\gamma_0}{2b_0}$$

$P^{PT} =$  $+ \mathcal{O}\left(g^2\right)$

Renormalization Scheme Change at NNLO

$$m \to m\left(1 + B3 g^3\right)$$

RSC was not needed for $\lambda\phi^4$
Theoretical constraint on $\bar{m}$:

- Asymptotic Freedom (AF) matching

Expected properties of $\bar{m}$:

- Pressure should not exceed P_{fg}
- $P(M = 4\mu) > P(M = \mu)$ for large enough μ
- Small deviation from $\bar{MS}$ scheme

Prescription :

Fix $B3(g)$ such as to restore reality of the solution

NB : In HTLpt, m_{PT} is used, so no problem of imaginary solutions.
But loose resummation properties. Start-of-the-art : NNLO

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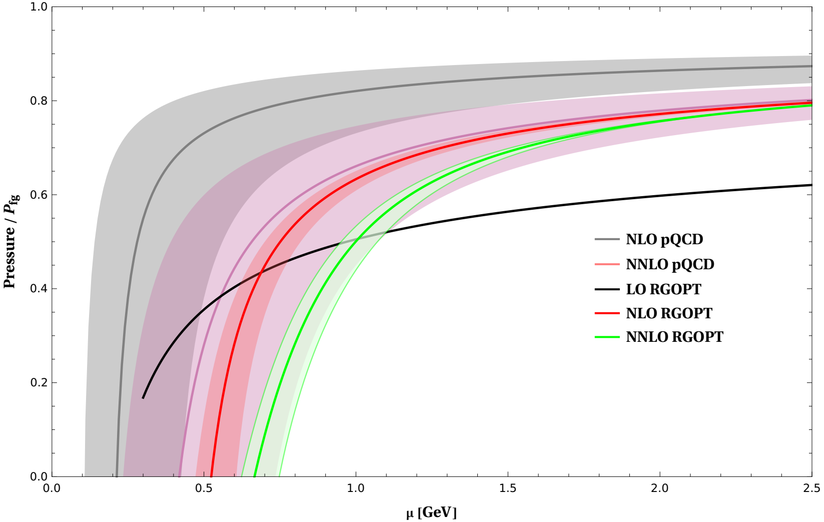
(Preliminary !!) Results



Future of RGOPT

- Equation of State at NNLO for Neutron Star
- Include (HTL) variational mass for gluons
- NLO, and ultimately, NNLO for full QCD at (T,μ)

NLO Cold&Dense QCD

- No real solutions for every μ ($\mathcal{I}m(m) \neq 0$)

One way to recover realness of the solutions : Renormalization Scheme Change (RSC)

$$m \to m\,(1 + B2 g^2)$$

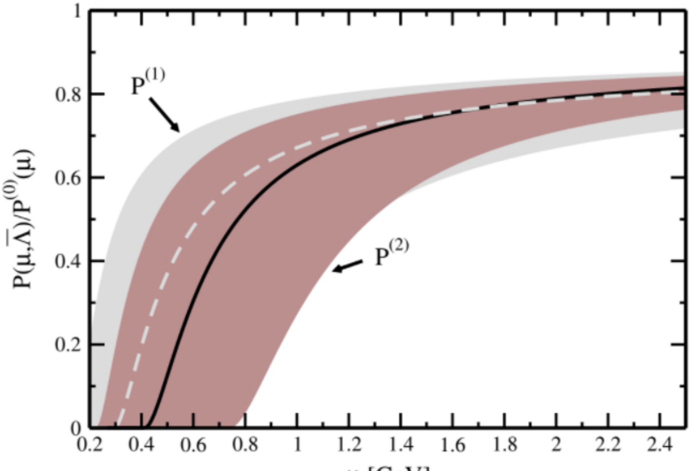
- LO&NLO : $Log\left(\frac{m}{M}\right)$ and $Log\left(\frac{\mu+p_F}{m}\right)$ recombine as $Log\left(\frac{\mu+p_F}{M}\right) = L_\mu$
- RG equation $\longleftrightarrow$ c1(m,g,M) $L_\mu^2 + c2(m,g,M)\ L_\mu + c3(m,g,M)$
- Quadratic equation in L_μ
- Discriminant suggest :

$$B2 = -\frac{7}{81\pi^2 g}$$

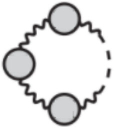
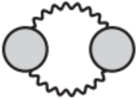
NNLO Cold&Dense QCD

Starting from $\mathcal{P}_2^{PT}(g,\mu,m)$ calculated by :

- A. Kurkela, P. Romatschke, and A. Vuorinen. Cold Quark Matter. *Phys. Rev. D*, 81:105021, 2010

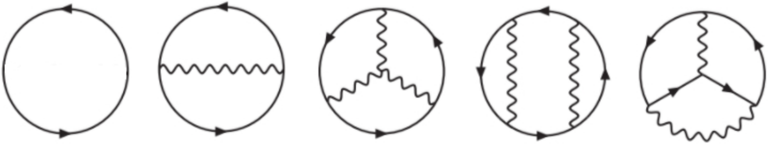


NNLO Cold&Dense QCD

$-\sum_{n=2}^{\infty}$  $-$ 

Model : $N_f = 3$; $m_u = m_d = m_s = 0$; $m_{variationnal}$.
(Investigating $Nf = 2+1$; $m_u = m_d = 0$; m_s ; $m_{v,ud} \neq m_{v,s}$)
Starting from $\mathcal{P}_2^{PT}(g, \mu, m)$ without the Gluons.

1. $\mathcal{P}_{2,f}^{PT}(g, \mu, m, \delta = 1)$ (for one flavor)
2. Reduced RG equation gives a real solution $\bar{m}$ for $\mu \in [1.2, 2.5]$ GeV



Strange quark current mass : $\mathcal{N}_F = 2 + 1$

m_s considered as a perturbation to the (all-order) original massless theory :

$$\mathcal{N}_F\ \mathcal{P}_{2,f}(g,m) \to\ (\mathcal{N}_F - 1)\ \mathcal{P}_{2,f}(g,m) + \mathcal{P}_{2,f}(g,m+m_s)$$

$$M\frac{d}{dM} = M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\, m_s\ \frac{\partial}{\partial m_s}$$

Renormalization Group

Renormalization → Scale dependence : M

$$g \to g(M) \quad ; \quad m \to m(M)$$

$$\alpha_s(M = 1 \ GeV) \simeq 0.42 \to \Lambda_{QCD}$$

Observables are independent of the choice of M

Perturbative RG equation :

$$M\frac{d}{dM}\mathcal{O} = \left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} - \gamma_m m\frac{\partial}{\partial m}\right)\mathcal{O} = 0$$

OPT breaks the RG invariance

$$\begin{aligned}\mathcal{P}^{PT}_{2,f}(\mu,m) =& -N_c\frac{m^4}{8\pi^2}(\frac{3}{4}-L_m)+\Theta(\mu^2-m^2)\frac{N_c}{12\pi^2}\left[\mu\, p_F\left(\mu^2-\frac{5}{2}m^2\right)+\frac{3}{2}m^4\ln\left(\frac{\mu+p_F}{m}\right)\right]\\ &-\frac{d_A g}{4(2\pi)^4}m^4\left(3L_m^2-4L_m+\frac{9}{4}\right)-\Theta(\mu^2-m^2)\frac{d_A g}{4(2\pi)^4}\left\{3\left[m^2\ln\left(\frac{\mu+p_F}{m}\right)-\mu\, p_F\right]^2-2p_F^4\right\}\\ &-\Theta(\mu^2-m^2)\frac{d_A g}{4(2\pi)^4)}m^2(4-6L_m)\left[\mu\, p_F-m^2\ln\left(\frac{\mu+p_F}{m}\right)\right]\\ &+\frac{g^2m^4}{135(4\pi)^6}\left(\alpha_{0,2}+\alpha_{1,2}\,L_m+\alpha_{2,2}\,L_m^2+\alpha_{3,2}\,L_m^3\right)\\ &+\frac{g^2d_A\mu^4}{(4\pi)^4 2\pi^2}\Theta(\mu^2-m^2)\Big\{-\hat{m}^2[(11C_A-2N_f)z+18C_F(2z-\hat{u})](L_m)^2-\frac{1}{3}\Big[C_A\left(22\hat{u}^4-\frac{185}{2}z\hat{m}^2-33z^2\right)\\ &+\frac{9C_F}{2}(16\hat{m}^2\hat{u}(1-\hat{u})-3(7\hat{m}^2-8\hat{u})z-24z^2)-N_f(4\hat{u}^4-13z\hat{m}^2-6z^2)\Big]L_m\\ &+C_A\left(-\frac{11}{3}\ln\frac{\hat{m}}{2}-\frac{71}{9}+G_1(\hat{m})\right)+C_F\left(\frac{17}{4}+G_2(\hat{m})\right)+N_f\left(\frac{2}{3}\ln\frac{\hat{m}}{2}+\frac{11}{9}+G_3(\hat{m})\right)+G_4(\hat{m})\Big\},\end{aligned}\tag{1}$$

$$\begin{aligned}\alpha_{0,2} =& (357315+176\pi^4+960\pi^2(\log 2)^2-960(\log 2)^4-23040\,Li_4(1/2)+12960\,\zeta(3)\\ &+90N_f(-393+224\,\zeta(3)))\\ \alpha_{1,2} =& 180(-3817+286N_f+48\,\zeta(3))\\ \alpha_{2,2} =& -720(-807+26N_f)\\ \alpha_{3,2} =& 2880(-81+2N_f)\end{aligned}$$

$$\hat{m}=\frac{m}{\mu}\ ,\ L_m=\ln\frac{m}{M}\ ,\ \hat{u}=\frac{u}{\mu}=\frac{\sqrt{\mu^2-m^2}}{\mu}\ ,\ z=\hat{u}-\hat{m}^2\ln\frac{1+\hat{u}}{\hat{m}}\ ,\ L=\ln\hat{m}$$

$$\begin{aligned}G_1(\hat{m}) =& 32\pi^4\hat{m}^2(-0.01863+0.02038\hat{m}^2-0.039\hat{m}^2L+0.02581\hat{m}^2L^2-0.03153\hat{m}^2L^3+0.01151\hat{m}^2L^4)\\ G_2(\hat{m}) =& 32\pi^4\hat{m}^2(-0.1998-0.04797L+0.1988\hat{m}^2-0.3569\hat{m}^2L+0.3043\hat{m}^2L^2-0.1611\hat{m}^2L^3+0.09791\hat{m}^2L^4)\\ G_3(\hat{m}) =& 32\pi^4\hat{m}^2(-0.05741-0.02679L-0.002828L^2+0.05716\hat{m}^2-0.08777\hat{m}^2L+0.0666\hat{m}^2L^2\\ &-0.02381\hat{m}^2L^3+0.01384\hat{m}^2L^4)\\ G_4(\hat{m}) =& 32\pi^4\hat{m}^2(0.07823+0.0388L+0.004873L^2-0.07822\hat{m}^2+0.1183\hat{m}^2L-0.08755\hat{m}^2L^2+0.03293\hat{m}^2L^3-0.01644\hat{m}^2L^4).\end{aligned}\tag{2}$$

Sign Problem

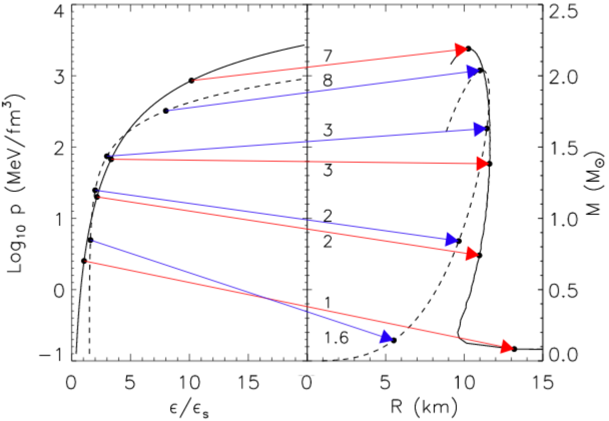
$$\mathcal{Z} = \int \mathcal{D}A_a^\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{C}\mathcal{D}C \; e^{-\int_0^\beta d\tau \int d^3x \, (\mathcal{L}_E - \bar{\psi}\mu\gamma^0\psi)} \tag{3}$$

Expectation value of an operator $\mathcal{O}$:

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\nu} dA_\nu(x) \prod_{i=1\dots N_f} det\left(\not{D} + m - \gamma_0\mu_i\right)\mathcal{O} \, e^{-S_{gauge}}$$

With $\mu_i \neq 0$ the determinant is no longer positive definite which is needed for Monte-Carlo method used for evaluation.

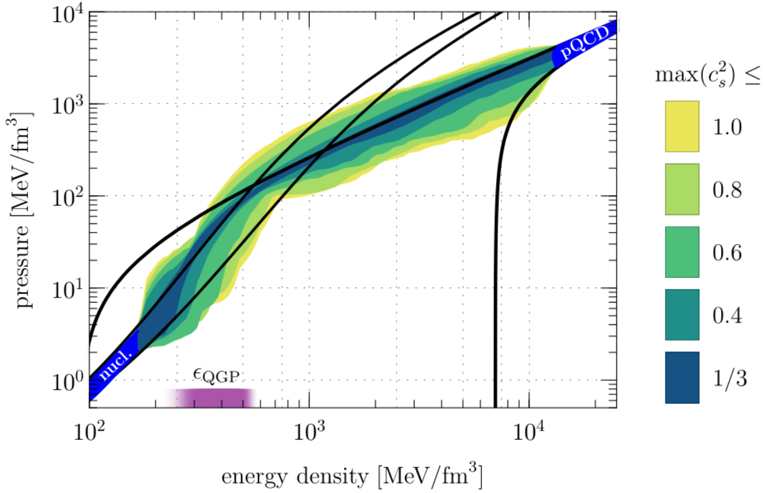
Neutron Star



$M_{MAX} \lesssim 2.16 M_{\odot}$
$M > 0.9\ M_{\odot}$
$\mathcal{C}_s^2 < 1$: Causal limit
$\mathcal{C}_s^2 < 1/3$: Conformal limit

J. M. Lattimer, Annu. Rev. Nucl. Part. Sci. 62, 485 (2012), URL https://doi.org/10.1146/

state-of-the-art of NS Equation of State (EoS)



Tolman Oppenheimer Volkoff equation (TOV)

Assume a static, spherically symmetric perfect fluid.

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = e^{\nu}c^2dt^2 - \left(1 - \frac{2G\ m}{r\ c^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

$$EinsteinEquations \rightarrow \frac{d\nu}{dr} = -\left(1 - \frac{2G\ m}{r\ c^2}\right)\frac{dP}{dr}$$

$$\begin{cases} \dfrac{dp}{dr} = -\dfrac{G\epsilon(r)\mathcal{M}(r)}{c^2r^2}\left[1 + \dfrac{p(r)}{\epsilon(r)}\right]\left[1 + \dfrac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}\right]\left[1 - \dfrac{2G\mathcal{M}(r)}{c^2 r}\right]^{-1} \\ \dfrac{d\mathcal{M}}{dr} = 4\pi r^2\rho(r) = \dfrac{4\pi r^2\epsilon(r)}{c^2} \\ \mathcal{M}(r) = 4\pi\displaystyle\int_0^r dr' r'^2\rho(r') = 4\pi\int_0^r dr' r'^2\frac{\epsilon(r')}{c^2} \end{cases}$$

RGOPT LO : Mass VS Radius