Coordinate Bethe Ansatz in application to Hubbard-like integrable models

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work with L.Frappat and E.Ragoucy

Plan of the talk

On the origin of the Hubbard model

- *Model named after John Hubbard describes electronic correlations in narrow energy band*

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  - only 1 band nearest-neighbour electrons interact
  - small range of Coulomb interaction = forbid double occupancy

On the origin of the Hubbard model

- Second-quantized Hamiltonian in the basis of Wannier states is

$$H_{Hub} = -t \sum_{<i,j>} \sum_{a=\uparrow,\downarrow} c^{\dagger}_{i,a} c_{j,a} + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}$$

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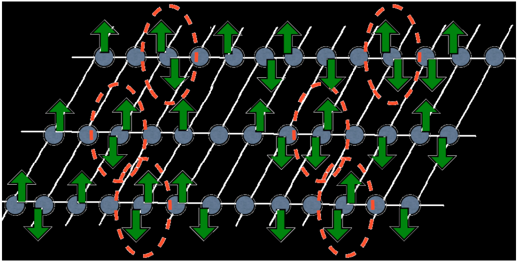
- 

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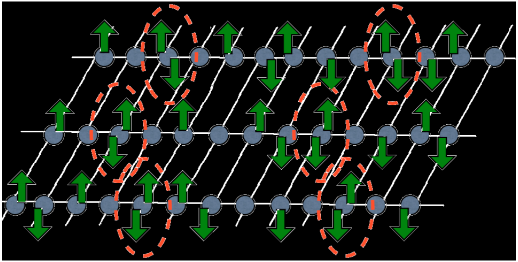
- 
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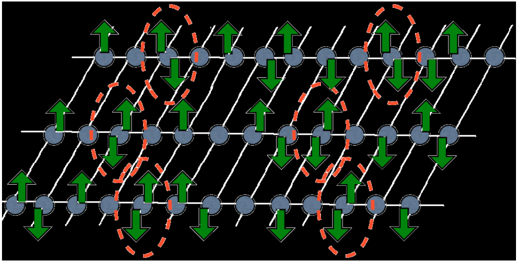
- 
- *Mott metal-insulator transition*
- *relevant to high-T_c super-conductivity*

Integrability and R-matrix formalisme

1D periodic Hubbard model is integrable!

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- `Integrability` $\propto \exists$ independent Q_i such that $[Q_i, H] = 0$ for $i = 1, 2, \ldots$

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  - Charges Q_i are also in $\tau(u)$: $\tau(u) = \sum_{i=0}^{L} u^i Q_i$

Integrability and R-matrix formalisme
One-dimensional Hubbard model

The Hubbard model's R-matrix:

- 16×16 matrix

$$R^{\uparrow\downarrow}_{12}(\lambda_1,\lambda_2) = R^{\uparrow}_{12}(\lambda_{12})\, R^{\downarrow}_{12}(\lambda_{12}) + f(\lambda_1,\lambda_2,U) R^{\uparrow}_{12}(\lambda'_{12})\, C_{\uparrow 1}\, R^{\downarrow}_{12}(\lambda'_{12})\, C_{\downarrow 1}$$

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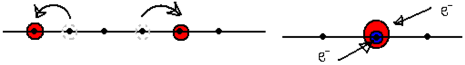
- and f - coupling function $\rightarrow$ Hubbard model constant U

Coordinate Bethe Ansatz I

We diagonalize directly the periodic Hamiltonian:

$$H_{Hub} = \sum_{i=1}^{L} \sum_{a=\uparrow,\downarrow} \left(c^{\dagger}_{i,a} c_{i+1,a} + c^{\dagger}_{i+1,a} c_{i,a} \right) + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}$$



Symmetry: $[H_{Hub}, \sum_{i=1}^{L} n_{i,\sigma}]$ =0 - number of particles is conserved

$N_{\uparrow}, N_{\downarrow}$ are quantum numbers of eigenfunction

- I) we choose pseudo-vacuum:
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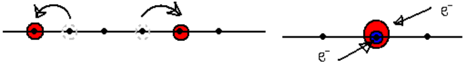

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  $\phi_{N,\vec{\sigma}} = \sum_{\vec{x}\in[1,L]} \Psi_{\vec{\sigma}}(\vec{x}) \prod_{i=1}^{N} c^{\dagger}_{x_i,\sigma_i}\phi_0$ where $\vec{\sigma} = (\sigma_1, ..., \sigma_N)$
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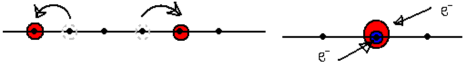

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- $\Psi_{\vec{\sigma}}(\vec{x})$ - ?

Coordinate Bethe Ansatz II

Schrodinger equation

The Hamiltonian applied on N-particle eigenfunction ($H_{Hub}\phi_{N,\vec{\sigma}} = E\phi_{N,\vec{\sigma}}$):

$$\sum_{k=1}^{N}(\Psi_{\vec{\sigma}}(\vec{x}+\vec{e}_k)+\Psi_{\vec{\sigma}}(\vec{x}-\vec{e}_k))+U\sum_{k<l}\delta(x_k=x_l)\delta(\sigma_k\neq\sigma_l)\Psi_{\vec{\sigma}}(\vec{x}) = E\Psi_{\vec{\sigma}}(\vec{x})$$

We have conditions on $\Psi_{\vec{\sigma}}(\vec{x})$:

- periodic boundary conditions: $\Psi_{\vec{\sigma}}(\vec{x}+e_kL) = \Psi_{\vec{\sigma}}(\vec{x})$

One-particle solution:

notation: e_k - vector $(0, ...0, \overset{\overset{k}{\downarrow}}{1}, 0, ...0)$

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- Usual solid state physics result!

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Bethe hypothesis

N-particles solution:

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  - $\Psi^{(Q)}_{\vec{\sigma}}(\vec{x}) = \sum_{P \in \mathfrak{S}_N} (-1)^{sg[PQ]} A_{\vec{\sigma}}(P, QP^{-1}) e^{i \sum_i k_{P(i)} x_{Q(i)}}$,

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- Schrodinger equation + Pauli exclusion principle $\longrightarrow$ "S-matrix"
- for $N = 2$: a) Identical particles or $\sigma_1 = \sigma_2$, only 1 sector: $A_{\vec{\sigma}}(id, id)$ and $A_{\vec{\sigma}}(\Pi_{12}, \Pi_{12})$, b) Non-identical particles, $\sigma_1 \neq \sigma_2$, 2 sectors: $A_{\vec{\sigma}}(id, id)$, $A_{\vec{\sigma}}(\Pi_{12}, \Pi_{12})$, $A_{\vec{\sigma}}(id, \Pi_{12})$ and $A_{\vec{\sigma}}(\Pi_{12}, id)$

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- for $N = 2$: a) Identical particles or $\sigma_1 = \sigma_2$, only 1 sector: $A_{\vec{\sigma}}(id, id)$ and $A_{\vec{\sigma}}(\Pi_{12}, \Pi_{12})$, b) Non-identical particles, $\sigma_1 \neq \sigma_2$, 2 sectors: $A_{\vec{\sigma}}(id, id)$, $A_{\vec{\sigma}}(\Pi_{12}, \Pi_{12})$, $A_{\vec{\sigma}}(id, \Pi_{12})$ and $A_{\vec{\sigma}}(\Pi_{12}, id)$
- Thus, $\begin{pmatrix} A_{\uparrow,\uparrow}(\Pi_{12}P, P^{-1}) \\ \hline A_{\uparrow,\downarrow}(\Pi_{12}P, P^{-1}) \\ A_{\uparrow,\downarrow}(\Pi_{12}P, \Pi_{12}P^{-1}) \\ \hline A_{\downarrow,\downarrow}(\Pi_{12}P, P^{-1}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t_{12} & r_{12} & 0 \\ 0 & r_{12} & t_{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{\uparrow,\uparrow}(P, P^{-1}) \\ \hline A_{\uparrow,\downarrow}(P, P^{-1}) \\ A_{\uparrow,\downarrow}(P, \Pi_{12}P^{-1}) \\ \hline A_{\downarrow,\downarrow}(P, P^{-1}) \end{pmatrix}$,

where $t_{ab} = \frac{2i(\sin k_a - \sin k_b)}{u + 2i(\sin k_a - \sin k_b)}$, $r_{ab} = \frac{-u}{u + 2i(\sin k_a - \sin k_b)}$

Coordinate Bethe Ansatz IV

Bethe equations

N-particles solution: What are k's and $A_{\vec{\sigma}}(P, QP^{-1})$?

- for any N: $\hat{A}(\Pi_{ab}P) = S_{ab}(\sin k_a, \sin k_b)\hat{A}(P)$

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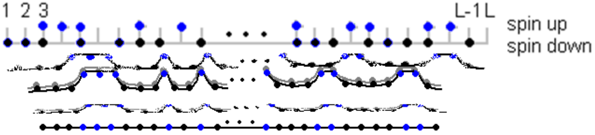
  - $S_{ab}(\sin k_a, \sin k_b)$ acts non-trivially on a and b particles.
- Periodic boundary conditions $\Rightarrow$ "Auxiliary problem"

$$S_{j+1j}...S_{Nj}S_{1j}...S_{j-1j}\hat{A}(id) = e^{ik_jL}\hat{A}(id)$$

Physical chain $\longrightarrow$ "small" chain



Coordinate Bethe Ansatz V

Bethe equations

N-particles solution: What are k's and $A_{\vec{\sigma}}(P, QP^{-1})$ *?*

- Using Coordinate Bethe Ansatz approach to "Auxiliary problem"

Coordinate Bethe Ansatz V

Bethe equations

N-particles solution: What are k's and $A_{\vec{\sigma}}(P, QP^{-1})$?

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  - we choose pseudo-vacuum, for example spin down particles

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- Using Coordinate Bethe Ansatz approach to "Auxiliary problem"
  - we choose pseudo-vacuum, for example spin down particles
  - we find the one-particle solution $f_y(a)$ of new Hamiltonian. a - new Bethe root, y - coordinate on the small chain.

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  - Bethe hypothesis: $\Psi_M^{(1)}(\vec{y}) = \sum_{\pi \in \mathfrak{S}_M} B(P) \prod_{i=1}^{M} f_{y_i}(a_{\pi(i)})$ with $y_1 < ... < y_M$ and $a_1 < ... < a_M$.

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  - $e^{ik_jL} = \prod_{\alpha=1}^M \frac{\sin k_j - a_\alpha + iU/4}{\sin k_j - a_\alpha - iU/4}$, for $j = 1, ..., N$

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Generalized $gl(n) \oplus gl(m)$ 1d Hubbard-like model's

generalized R-matrix and integrability

We introduce the projectors: π and $\bar{\pi}$ such that $\pi + \bar{\pi} = Id$

- Generalized XX spin chain model: $R^{\sigma}_{12}(\lambda)$ depend on choice of π and $\bar{\pi}$

The generalized Hubbard-like model's R-matrix associated with $gl_{\uparrow}(n) \oplus gl_{\downarrow}(m)$:

(G. Feverati, L. Frappat and E. Ragoucy arXiv:0903.0190 [math-ph])

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- with $R^{\uparrow}_{12}(\lambda)$ - generalized $gl(n)$ XX model's R-matrix

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- with $R^{\uparrow}_{12}(\lambda)$ - generalized $gl(n)$ XX model's R-matrix
- with $R^{\downarrow}_{12}(\lambda)$ - generalized $gl(m)$ XX model's R-matrix

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- with $R^{\uparrow}_{12}(\lambda)$ - generalized $gl(n)$ XX model's R-matrix
- with $R^{\downarrow}_{12}(\lambda)$ - generalized $gl(m)$ XX model's R-matrix
- the same coupling function $f(\lambda_1,\lambda_2,U)$

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generalized R-matrix and integrability

The generalized Hubbard-like model's R-matrix :

- satisfies the Yang-Baxter Equation

generalized R-matrix and integrability

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  - $\bar{\pi}$-particles = electron-like particles with spin $\uparrow, \downarrow$ and additional degree of freedom.

generalized R-matrix and integrability

The generalized Hubbard-like model's R-matrix :

- satisfies the Yang-Baxter Equation
- generates Hubbard-like integrable Hamiltonians depending on the choice of C^σ:
  - $\bar{\pi}$-particles = electron-like particles with spin $\uparrow, \downarrow$ and additional degree of freedom.
  - π-particles = heavy particles with spin $\uparrow, \downarrow$ and additional degree of freedom which only interact with $\bar{\pi}$-particles.

Coordinate Bethe Ansatz I

General case: $gl(n) \oplus gl(m)$ 1d Hubbard-like model

The periodic Hamiltonian:

$$H = \sum_{x=1}^{L} \Big((\Sigma P)_{\uparrow\ x,x+1} + (\Sigma P)_{\downarrow\ x,x+1} + u\, C_{\uparrow x}\, C_{\downarrow x} \Big), \qquad \text{mod } L$$

- $n + m - 2$ types particles

Periodic boundary conditions $\rightarrow$ Auxiliary problem

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- $n + m - 2$ types particles
- p types of $\pi \uparrow$-particles and $n - p - 1$ types of $\bar{\pi} \uparrow$-particles

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- $n + m - 2$ types particles
- p types of π $\uparrow$-particles and $n - p - 1$ types of $\bar{\pi}$ $\uparrow$-particles
- q types of π $\downarrow$-particles and $m - q - 1$ types of $\bar{\pi}$ $\downarrow$-particles

Periodic boundary conditions $\rightarrow$ Auxiliary problem

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- $n + m - 2$ types particles
- p types of $\pi \uparrow$-particles and $n - p - 1$ types of $\bar{\pi} \uparrow$-particles
- q types of $\pi \downarrow$-particles and $m - q - 1$ types of $\bar{\pi} \downarrow$-particles

Periodic boundary conditions $\rightarrow$ Auxiliary problem

- $n + m - 3$ types particles

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General case: $gl(n) \oplus gl(m)$ 1d Hubbard-like model

The periodic Hamiltonian:

$$H = \sum_{x=1}^{L} \Big((\Sigma P)_{\uparrow\, x,x+1} + (\Sigma P)_{\downarrow\, x,x+1} + u\, C_{\uparrow x}\, C_{\downarrow x} \Big), \qquad \text{mod } L$$

- $n + m - 2$ types particles
- p types of π $\uparrow$-particles and $n - p - 1$ types of $\bar{\pi}$ $\uparrow$-particles
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Periodic boundary conditions $\rightarrow$ Auxiliary problem

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Periodic boundary conditions $\rightarrow$ Auxiliary problem

- $n + m - 3$ types particles
- p types of π $\uparrow$-particles and $n - p - 1$ types of $\bar{\pi}$ $\uparrow$-particles
- q types of π $\downarrow$-particles and $m - q - 2$ types of $\bar{\pi}$ $\downarrow$-particles
- choice of the new vacuum brakes the symmetry between spin up and down particles

Coordinate Bethe Ansatz II

General case: $gl(n) \oplus gl(m)$ 1d Hubbard-like model

Results:

- $e^{ik_j L} = e^{2\pi i \Phi} \prod_{\alpha=1}^{M} \frac{\sin k_j - a_\alpha + iU/4}{\sin k_j - a_\alpha - iU/4}$, for $j = 1, \ldots, N$

(V.F., L. Frappat and E. Ragoucy `arXiv:0906.4512`)

Coordinate Bethe Ansatz II

General case: $gl(n) \oplus gl(m)$ 1d Hubbard-like model

Results:

- $e^{ik_j L} = e^{2\pi i\Phi} \prod_{\alpha=1}^{M} \frac{\sin k_j - a_\alpha + iU/4}{\sin k_j - a_\alpha - iU/4}$, for $j = 1, \ldots, N$
- $\prod_{j=1}^{N} \frac{\sin k_j - a_\alpha + iU/4}{\sin k_j - a_\alpha - iU/4} = -e^{2\pi i\Psi} \prod_{\beta=1}^{M} \frac{a_\beta - a_\alpha + iU/2}{a_\beta - a_\alpha - iU/2}$, for $\alpha = 1, \ldots, M$

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Outlook

- Application to $N = 4SYM$ theory

Outlook

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- Application to condensed matter physics

Outlook

Thank you