On-off intermittency due to parametric Lévy noise

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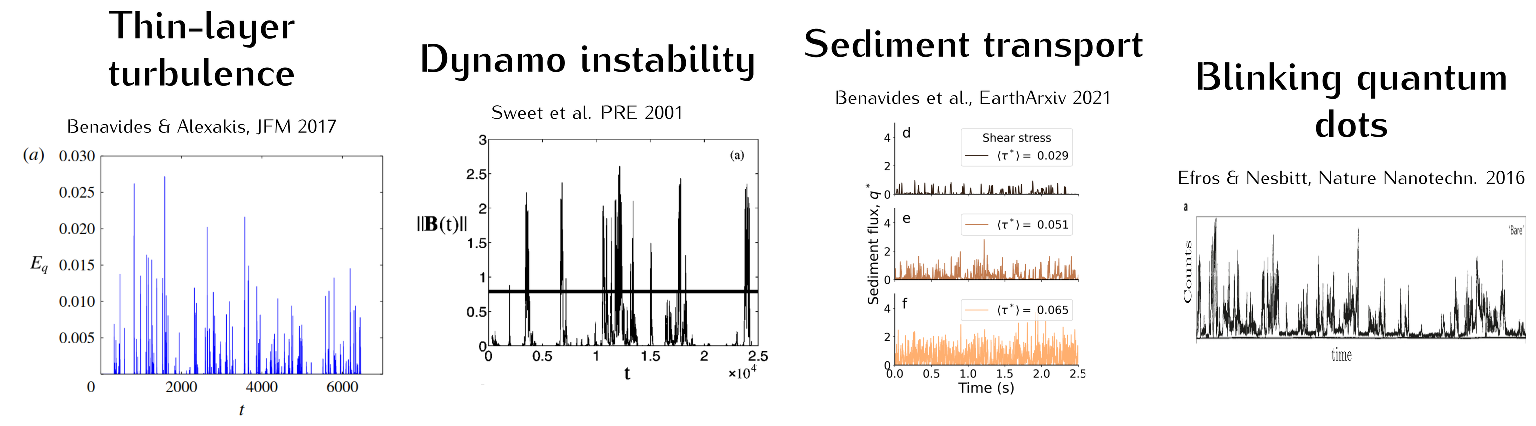
Multiplicative noise and on-off intermittency

- Instabilities arise in many systems at a parameter threshold (e.g. onset of convection, 3D instabilities in quasi-2D flows, dynamo instability, sediment transport, etc.)
- Typically, the system is embedded in an uncontrolled noisy environment.
- The fluctuating properties of the environment affect the control parameters of the instability, which leads to parametric (also known as multiplicative) noise.
- Parametric noise close to an instability threshold causes on-off intermittency, switching aperiodically between a large-amplitude "on" state and a small-amplitude "off" state.



- The noisy supercritical pitchfork bifurcation gives a minimal example of this behaviour

$$\dot{X} = (\mu + f(t))X - \gamma X^3, \tag{1}$$

with mean growth rate μ, nonlinear coefficient $\gamma \geq 0$, and the random noise $f(t)$.

(Generalised) central limit theorem & stable laws

- Typically $f(t)$ in (1) is taken to be Gaussian white noise. This is motivated by the **CLT**:

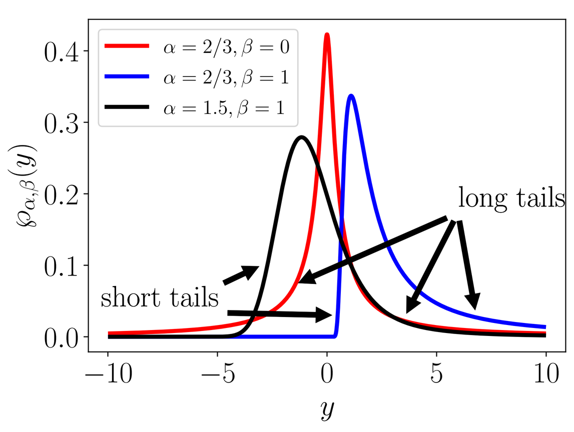
> Given N identically distributed RVs $X_1, \dots, X_N$ with mean 0 and variance σ^2, then $S_N = (X_1 + \dots + X_N)/\sqrt{N} \overset{N\to\infty}{\longrightarrow} S \sim \mathcal{N}(0,\sigma^2)$, if and only if
> 1) The X_i are mutually independent, i.e. $\langle X_i X_j \rangle = 0$ for $i \neq j$ and
> 2) X has finite variance.

- Both assumptions of the CLT may be violated when choosing $f(t)$.
  1) If noise has finite correlation time $\Rightarrow$ spectrum at zero frequency important [?].
  2) If noise has infinite variance $\Rightarrow$ noise from non-equilibrium source (no temperature)
- **Generalised CLT** for 2): the scaled sum of the X_i tends to a stable distribution $\wp_{\alpha,\beta}(x)$,

$$\varphi_{\alpha,\beta}(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} e^{ikx} \wp_{\alpha,\beta}(x)dx = \exp\left\{ -|k|^\alpha \left(1 - i\beta \tan\left(\frac{\alpha\pi}{2}\right)\right)\right\}.$$

- Some simple properties of stable distributions

  → Choosing $\alpha = 2$ gives Gaussian
  → For $\wp_{\alpha,\beta}(x) \geq 0 \Rightarrow 0 < \alpha \leq 2, -1 \leq \beta \leq 1$.
  → For $|\beta| < 1$, $\wp_{\alpha,\beta}(x) \overset{x\to\pm\infty}{\sim} (1 + \beta \mathrm{sign}(x))x^{-1-\alpha}$.
  → This breaks down on one side for $\beta = \pm 1$.
  There, $\wp_{\alpha,\beta}(x) \propto \exp\left(-cst.\ x^{\frac{\alpha}{\alpha-1}}\right)$
  $\Rightarrow$ one-sided distribution for $\beta = \pm 1$, $\alpha < 1$.



The fractional Fokker-Planck equation

- For (1) with α-stable white noise ($f(t)dt = dt^{1/\alpha}F(t)$, $F(t)$ α-stable), the PDF of X obeys

$$\partial_t p_x(x,t) = -\partial_x[(\mu x - \gamma x^3)p_x(x,t)] + \mathcal{D}_x^{\alpha,\beta} p_x(x,t), \tag{2}$$

the (Stratonovich) **fractional Fokker-Planck equation**, where $\mathcal{D}_x^{\alpha,\beta}$ is a linear operator. The variable $\log(X)$ performs a **Lévy flight**.

- For $\alpha = 2$ (Gaussian white noise), $\mathcal{D}_x^{\alpha,\beta} = \partial_x^2$. There for $\mu > 0$, the stationary PDF is

$$p_{st}(x) = N x^{-1+\mu} e^{-\frac{\gamma}{2}x^2} \tag{3}$$

Some important properties
→ Critical transition at $\mu = 0$ (deterministic threshold)
→ Power law divergence at small x with exponent $\to -1$ as $\mu \to 0$, cut-off at large x
→ Anomalous scaling near onset: for all $n > 0$, $\langle X^n \rangle \propto \mu$ as $\mu \to 0$

- **Goal**: extend this result to $\alpha < 2$.
- **Problem**: Can only solve (2) analytically for $\gamma = 0$, the log-stable process,

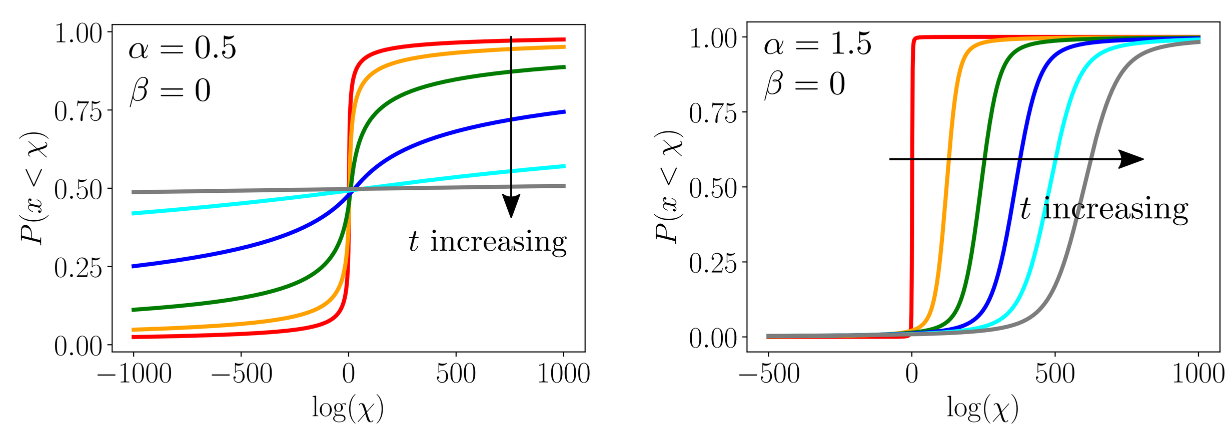
$$p_x(x,t) = \frac{\wp_{\alpha,\beta}\left(\frac{\log(x)-\mu t}{t^{1/\alpha}}\right)}{x t^{1/\alpha}}, \tag{4}$$

which does not converge to a stationary state, since probability escapes to $\pm\infty$.

- For $\gamma > 0$, a stationary state exists and its asymptotics can be computed.

The linear regime ($\gamma = 0$)

The leakage of probability depends on α, β and μ in this case. E.g. for $\mu > 0, \beta = 0$,

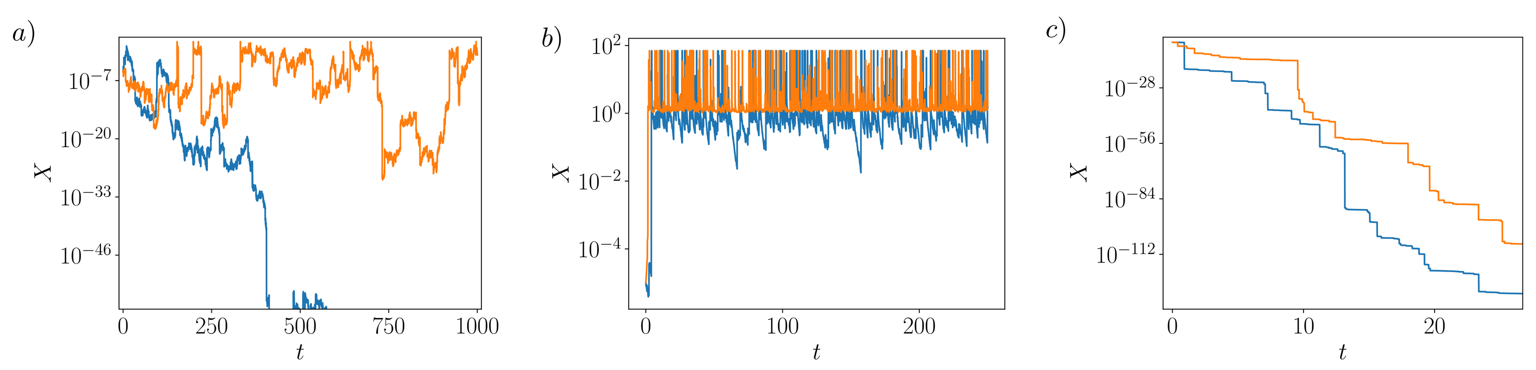


Critical difference: $\alpha > 1$ (mean of noise finite) and $\alpha \leq 1$ (mean of noise infinite)

- For $1 < \alpha < 2$: critical transition at $\mu = 0$ from probability accumulating at $x = 0$ (stable origin) or leaking to $x = \infty$ (unstable origin).
- For $\alpha = 1$, and for $\alpha < 1$, $\beta < 1$, the origin is always stable
- For $\alpha < 1$, $\beta = 1$ (noise positive definite), the origin is always unstable

The nonlinear regime ($\gamma > 0$)

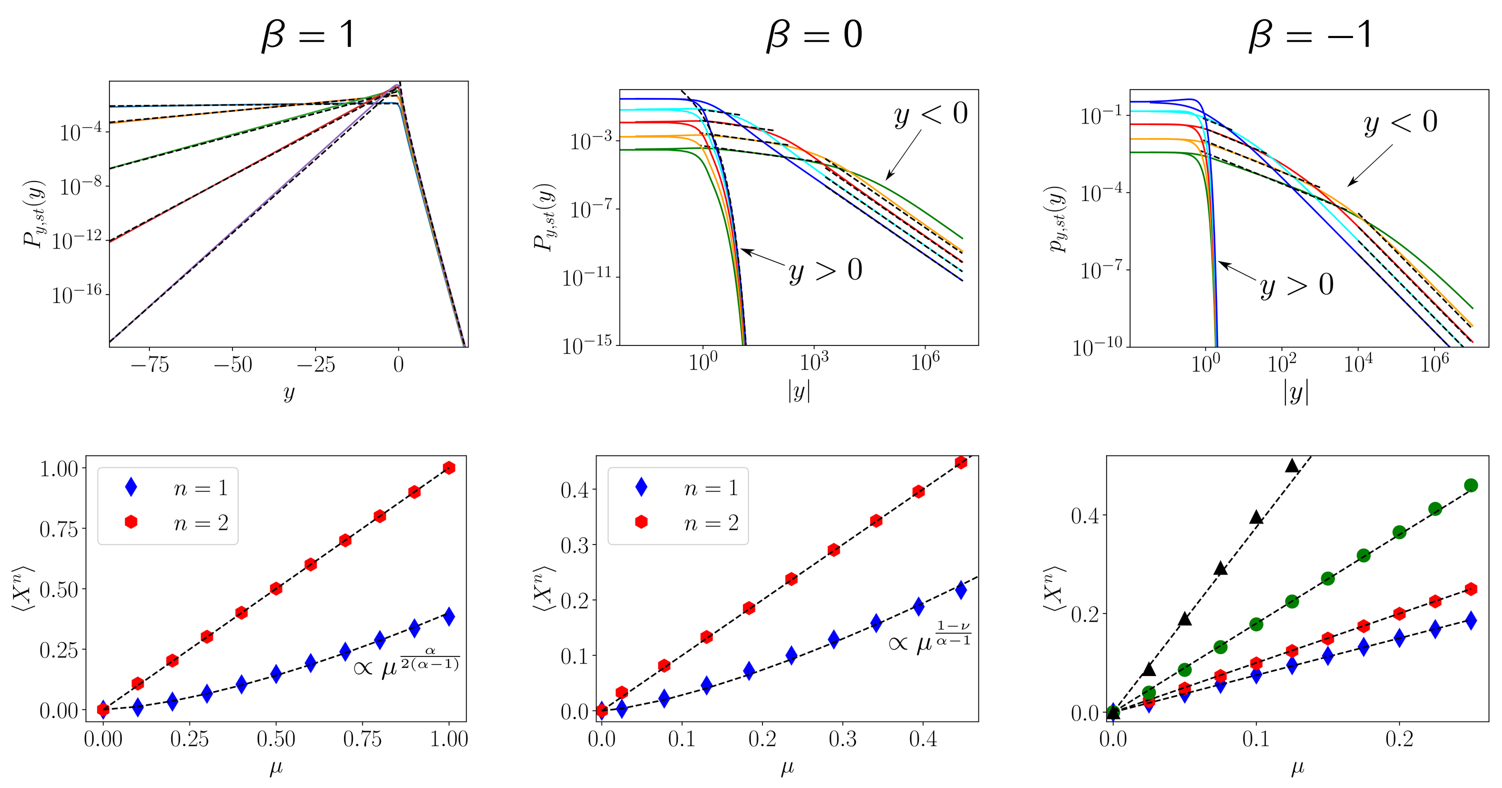
Typical time series a) $\alpha = 1.5, \beta = 0$, b) $\alpha = 0.5$, $\beta = 1.0$, c) $\alpha = 0.5$, $\beta = -1$.



A critical transition only occurs for $\alpha > 1$. Else, the origin is either always stable/unstable. The exact asymptotics of $p_{x,st}$ in steady state are summarised in the table below

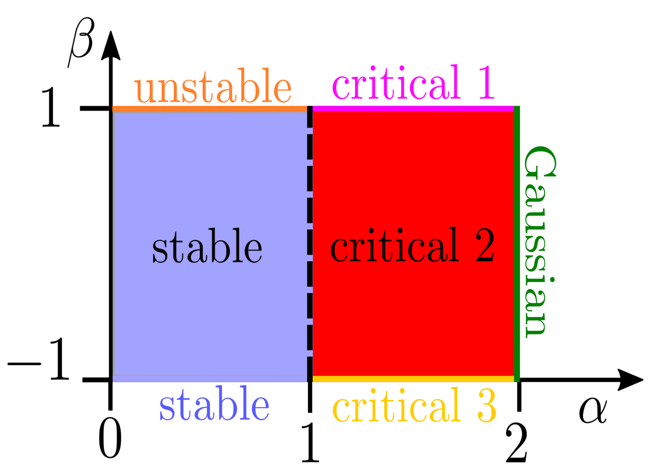
β	$p_{x,st}(x\to 0)$	$p_{x,st}(x\to\infty)$
-1	$C(\mu x)^{-1}\log^{-\alpha}(1/x)$	exponential decay
$(-1,1)$	$C(\mu x)^{-1}\log^{-\alpha}(1/x)$	$C\gamma^{-1}x^{-3}\log^{-\alpha}(x)$
1	$\propto x^{-1+A_\alpha(\mu)}$	$C\gamma^{-1}x^{-3}\log^{-\alpha}(x)$

Numerically integrating (2) confirms asymptotics $\Rightarrow$ predict critical exponents (heuristic)



Conclusion and Outlook

We have shown that instabilities subject to parametric heavy-tailed noise, modeled as Lévy white noise, can display anomalous critical exponents differing from those for Gaussian noise. Our work serves as a first step in the study of instabilities in the presence of multiplicative Levy noise. Many directions can be further pursued. The exponents of the intermediate power laws at $\beta < 1$ remain unknown. Other topics of interest include truncated Lévy noise, combined Lévy-Gaussian noise, finite-velocity Lévy walk, different nonlinearities, higher dimensions and time statistics. Since Lévy statistics are found in many physical systems, one may speculate that the anomalous critical exponents predicted here for instabilities subject to Lévy noise may be observable experimentally.



References

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