PLANCKS2022 French Preliminaries

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February 5, 2022

This PLANCKS2022 French preliminary SOLUTIONS is made of 21 pages (including cover page) and contains the solutions to the set of 4 problems that were presented at the french preliminary competition on the 5th of February 2022.

Part I

The Tautochrone curve

There is a particularly interesting curve, first studied by Christian Huygens in 1673, on which a material point of mass m subject to a uniform gravitational field of acceleration g, and moving with no friction and no initial velocity, can arrive at a destination point in a time Δt which is independent of the initial position. It is called the Tautochrone curve from the ancient Greek 'tauto' which means *the same*, and 'chronos' which means *the time*.

In this problem, we propose to establish the form of this curve in a two dimensional framework.

1 The Brachistochrone curve differential equation

As a first step, we propose to establish the differential equation and the solution for the curve that minimizes the travel time in the set-up above-mentioned; it is called the Brachistochrone curve, from 'brachi' *the least* in ancient Greek.

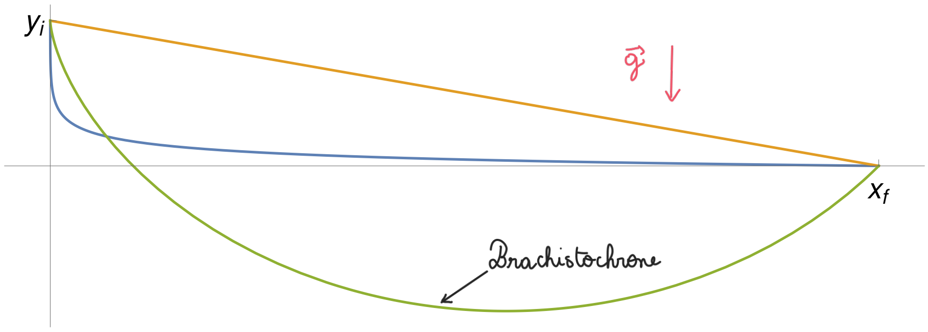

Figure 1: Different curves connecting the initial point of altitude y_i to the destination point x_f. The Brachistochrone curve in green minimizes the time travel.

1.1 [0.75 points] Define the infinitesimal distance noted ds as a function of the derivative $y'(x)$, with y and x the position variables; y corresponds to the height of the material point.

Solution: *If one considers the infinitesimal quantities dx and dy respectively for the variable of the plane x and y, via Pythagoras's theorem, the infinitesimal element of distance is defined as:*

$$ds^2 = dx^2 + dy^2.$$

Moreover, the definition of the derivative $y'(x)$:

$$dy = y'(x)dx,$$

leads to:

$$ds = \sqrt{1 + y'^2(x)}dx.$$

1.2 [1.25 points] Using the conservation of mechanical energy, express the travel time:

$$\Delta t = \int_{x_i}^{x_f} \sqrt{\frac{1 + y'^2(x)}{2g(y_i - y(x))}}\mathrm{d}x, \tag{1}$$

with y_i the initial vertical position of the point, x_i its initial horizontal position and x_f its final final horizontal position.

Solution: *The conservation of mechanical energy E_m:*

$$\begin{aligned} &E_m^i - E_m^f = 0 \\ &\leftrightarrow mgy_i = \frac{1}{2}mv_f^2 + mgy_f \\ &\leftrightarrow v_f = \sqrt{2g(y_i - y_f)}, \end{aligned}$$

with v_i and v_f respectively the initial and final velocity of the material point of mass m subject to the uniform gravitational acceleration g.

By definition, the velocity is:

$$v = \frac{ds}{dt}, \tag{2}$$

such that:

$$\Delta t = \int_0^t dt = \int_0^t \frac{ds}{v} = \int_{x_i}^{xf} \sqrt{\frac{1 + y'^2(x)}{2g(y_i - y(x))}} dx \tag{3}$$

1.3 [0.75 points] Let there be a functional:

$$S(y, y') = \int_{x_i}^{x_f} \mathcal{F}(y(x), y'(x), x) \mathrm{d}x, \tag{4}$$

show that extremalising this functional S, *i.e.* imposing that $\delta S(y, \delta y) = 0$, is equivalent to the Euler-Lagrange formula:

$$\frac{\partial \mathcal{F}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{F}}{\partial y'} = 0, \tag{5}$$

if the variation δy vanishes at x_i and x_f.

Solution: *The variation principle is used:*

$$\delta S = 0 \leftrightarrow \int \delta \mathcal{F}(y, y', x) dx = 0 \leftrightarrow \int \left(\frac{\partial \mathcal{F}}{\partial y} \delta y + \frac{\partial \mathcal{F}}{\partial y'} \delta y' \right) dx = 0$$

Let us note that the term $\frac{\partial \mathcal{F}}{\partial x} \delta x$ is not considered here because the variational principle by definition considers variation of the function y, and not its variable x. It is usual to encounter the Euler-Lagrange equations with the time variable t instead of x, including the term $\frac{\partial \mathcal{F}}{\partial x} \delta x$ would mean considering the time contracting or dilating.

One can write:

$$\frac{d}{dx}\left(\frac{\partial \mathcal{F}}{\partial y'} \delta y\right) = \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} \delta y + \frac{\partial \mathcal{F}}{\partial y'} \delta y',$$

therefore:

$$\int \left[\frac{\partial \mathcal{F}}{\partial y} \delta y + \frac{d}{dx}\left(\frac{\partial \mathcal{F}}{\partial y'} \delta y\right) - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} \delta y \right] dx$$

Considering a vanishing δy on the borders x_i and x_f:

$$\int_{x_i}^{x_f} \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} \delta y dx = \left[\frac{\partial \mathcal{F}}{\partial y'} \delta y \right]_{x_i}^{x_f} = 0$$
$$\leftrightarrow \frac{\partial \mathcal{F}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} = 0$$

1.4 [0.75 points] Assuming that $\mathcal{F}$ does not depend explicitly on x, *i.e.* $\mathcal{F} = \mathcal{F}(y, y')$, show that the Beltrami formula:

$$\mathcal{F} - y'(x)\frac{\partial \mathcal{F}}{\partial y'} = \tilde{C}, \tag{6}$$

with $\tilde{C}$ a constant, implies the Euler-Lagrange formula (5).

Solution: *The constant in the Beltrami formula is equivalent to a null derivative of eq.* (6)*:*

$$\begin{aligned}
\frac{d}{dx}\left(\mathcal{F} - y'(x)\frac{\partial \mathcal{F}}{\partial y'}\right) &= \frac{d\mathcal{F}(y, y')}{dx} - \frac{dy'(x)}{dx}\frac{\partial \mathcal{F}}{\partial y'} - y'(x)\frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} \\
&= \frac{\partial \mathcal{F}}{\partial y} y'(x) + \frac{\partial \mathcal{F}}{\partial y'} y''(x) - y''(x)\frac{\partial \mathcal{F}}{\partial y'} - y'(x)\frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'} \\
&= y'(x)\left(\frac{\partial \mathcal{F}}{\partial y} - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial y'}\right) = 0,
\end{aligned}$$

the Euler-Lagrange formula appears in the last line.

1.5 [1.5 points] Show that extremalising the travel time Δt gives the differential equation for the profile $y(x)$:

$$(y_i - y(x))(1 + y'^2(x)) = C, \tag{7}$$

with C a constant.

Solution: *Let us minimise* Δt *using the Beltrami formula, in which* $\mathcal{F} = \sqrt{\frac{1+y'^2(x)}{2g(y_i-y(x))}}$. *The derivative with regards to* y' *gives:*

$$\frac{\partial}{\partial y'}\left(\sqrt{\frac{1+y'^2(x)}{2g(y_i-y(x))}}\right) = \frac{y'(x)}{\sqrt{2g(y_i-y(x))}}\frac{1}{\sqrt{1+y'^2(x)}}. \tag{8}$$

In the Beltrami formula:

$$\begin{aligned}
&\sqrt{\frac{1+y'^2(x)}{2g(y_i-y)}} - \frac{y'(x)}{\sqrt{2g(y_i-y)}}\frac{y'(x)}{\sqrt{1+y'^2(x)}} = \tilde{C} \\
\leftrightarrow \quad & \frac{1}{\sqrt{2g(y_i-y)}} = \tilde{C}\sqrt{1+y'^2(x)} \\
\leftrightarrow \quad & \tilde{C} = \frac{1}{\sqrt{2g(y_i-y)(1+y'^2(x))}} \quad \leftrightarrow \quad (y_i-y(x))(1+y'^2(x)) = C,
\end{aligned}$$

with C *and* $\tilde{C}$ *some constants.*

2 Parametrized solution

The solution to eq. (7) is best established using the 'angular' variable[1] θ, with the parametrization:

$$y'(x) = \frac{1}{\tan(\theta/2)}. \tag{9}$$

2.1 [1.75 points] Establish that the differential equation eq. (7) has the cycloid solution:

$$\begin{cases} y(\theta) = \frac{k}{2}(C_y - \cos(\theta)) \\ x(\theta) = \frac{k}{2}(\theta - \sin(\theta) + C_x) \end{cases}, \tag{10}$$

with C_x, C_y and k constants.
Help: Some trigonometry formulas are needed, one of which is:

$$\sin(x) = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \tag{11}$$

Solution: *Let us set* $y_i = 0$, *such that the differential equation falls to:*

$$\begin{aligned}
C &= (y_i - y(x))\Big(1 + \frac{1}{\tan^2(\theta/2)}\Big) \\
&= (y_i - y(x))\frac{1}{\tan^2(\theta/2)}\frac{\sin^2(\theta/2)+\cos^2(\theta/2)}{\cos^2(\theta/2)} = \frac{y_i - y(x)}{\sin^2(\theta/2)}.
\end{aligned}$$

We use the trigonometry formula:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}, \tag{12}$$

to write:

$$\begin{aligned}
y_i - y(x) &= \frac{C}{2}(1-\cos(\theta)) \\
y(x) &= -\frac{C}{2}\Big(\frac{2y_i}{C} + 1 - \cos(\theta)\Big),
\end{aligned}$$

[1] The new variable θ has nothing to do with the angular variable of polar coordinates.

We note the constant $C_y = \frac{2y_i}{C} + 1$ *and* $k = -C$ *such that:*

$$y(\theta) = \frac{k}{2}(C_y - \cos(\theta))$$

Now let us care for the variable x*, by definition of the the derivation* $y'(x)$*:*

$$dx = \tan(\theta/2)dy = \tan(\theta/2)\frac{dy}{d\theta}d\theta = \frac{k}{2}\tan(\theta/2)\sin(\theta)d\theta.$$

Using the trigonometry formula (11) *and* (12)*:*

$$x(\theta) = \frac{k}{2}\int (1 - \cos(\theta))d\theta = \frac{k}{2}\Big(\theta - \sin(\theta) + C_x\Big)$$

2.2 [0.5 points] Explain why constants C_y and C_x can respectively be set to 1 and 0?
Solution: *If one sets the following condition:*

$$x(\theta = 0) = y(\theta) = 0,$$

then immediately $C_x = 0$ *and* $C_y = 1$*.*

3 The Tautochrone solution

Finally, let us prove that the cycloid solution to the Brachistochrone curve, is also Tautochrone.

3.1 [1.5 points] What is the differential equation the distance s must follow according to conservation of mechanical energy ?
Help: The parametrisation with the variable θ should be used.
Solution: *Let's recall that* $ds^2 = dx^2 + dy^2$*. With the* θ *parametrisation, one can write:*

$$dx = \frac{k}{2}\Big(1 + \cos(\theta)\Big)d\theta$$
$$dy = \frac{k}{2}\sin(\theta)d\theta$$
$$ds^2 = 2\left(\frac{k}{2}\right)^2(1 + \cos(\theta)).$$

Another trigonometry formula is used

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}, \tag{13}$$

such that the integration is simplified:

$$ds = k\cos(\theta/2)d\theta$$
$$s(\theta) = 2k\big(\sin(\theta/2) + C_s\big),$$

with C_s *easily set to 0. According to the conservation of mechanical energy:*

$$\frac{dE_m}{dt} = \frac{d}{dt}\left(\frac{m}{2}\left(\frac{ds}{dt}\right)^2 + mgy\right) = \frac{d}{dt}\left(\frac{1}{2}\left(\frac{ds}{dt}\right)^2 + g\frac{k}{2}(1 - \cos(\theta))\right) = 0$$

Using the definition of $s(\theta)$*:*

$$1 - \cos(\theta) = 2\sin^2(\theta/2) = \frac{s^2(\theta)}{2k^2}, \tag{14}$$

one can write the differential equation of s*:*

$$\frac{d}{dt}\left(\left(\frac{ds}{dt}\right)^2 + g\frac{s^2}{2k})\right) = \frac{d^2s}{dt^2} + \frac{g}{2k}s = 0 \tag{15}$$

3.2 [0.75 points] Find the solution to this equation, assuming that x and y follow a cycloid.

Solution: *The cycloid is exactly the parametrisation by the variable θ that leads to the values of $x(\theta)$ and $y(\theta)$. Let us find a solution to the differential equation of s that has been established with the cycloid parametrisation. Typical solution for such forms is:*

$$s = A\cos\left(\sqrt{\frac{g}{2k}}t + \phi\right).$$

The initial ($t = 0$) distance is noted s_0 and there is no initial velocity, such that:

$$s(t) = s_0\cos\left(\sqrt{\frac{g}{2k}}t\right).$$

3.3 [0.5 points] Conclude on the Tautochrone nature of the Brachistochrone curve.

Solution: *A Tautochrone curve simply means that whatever the distance s you are on the curve at a starting time $t = 0$, you will arrive at the destination point at the same time. If we note s_0 the original point and $s = 0$ the destination point, solving $s(t) = 0$ implies a travelling time $\Delta t = \frac{\pi\sqrt{k}}{\sqrt{2g}}$. This travelling time is absolutely independent of the initial distance s_0.*

Part II
Fluid mechanics at the scale of stars

1 Introduction

The theory of hydrodynamics is surprising for its wide range of applications from the behaviour of most everyday fluids, water in a river or gas in a ventilation shaft for example, all the way to fluids at astrophysical and cosmological scales, such as the flow of gas around a galaxy or the distribution of dark matter along the cosmic web. Here we will study simple models for stars and how they are believed to form within the context of ideal hydrodynamics. As such we recall the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \tag{16}$$

relating the density, ρ, to the velocity field, $\boldsymbol{v}$. And the Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\phi, \tag{17}$$

in the presence of pressure P and a gravitational potential ϕ, the latter being described by the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho \tag{18}$$

with G Newton's gravitational constant. This problem is structured in two separate and independent parts.

2 Hydrostatic stars of the 19th century

An early model for stellar atmospheres considered the latter to be isotropic spheres of gas in hydrostatic equilibrium. Here we will study the density profile produced by this model.

2.1 [0.5 points] Assuming hydrostatic equilibrium, show that the Euler equation can be rewritten in the form

$$\frac{1}{\rho}\boldsymbol{\nabla} P = -\boldsymbol{\nabla}\phi \tag{19}$$

Solution: *Two assumptions are acceptable to arrive at the result. Either $\boldsymbol{v} = \boldsymbol{0}$, from which finding the result is trivial. Or more formally using the Lagrangian formulation of hydrostatic equilibrium,*

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0,$$

then injecting into the Euler equation gives you the desired result.

2.2 [1 point] Assuming spherical symmetry, rewrite this expression as a relation between P, ρ and r such that:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G \rho \tag{20}$$

We recall the expression for the Laplacian in spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \tag{21}$$

Solution: *This one is a bit of a trap. Indeed if you just take the divergence of Eq. (19) you'll be in for a rough time to find the answer. The simplest way is by explicitly writing the* *gradient. As we have assumed spherical symmetry the only non zero component is along $\hat{\boldsymbol{r}}$. The solution can be found simply by constructing the expression for the Laplacian around it.*

$$
\begin{aligned}
\frac{1}{\rho}\partial_r P &= -\partial_r \phi \\
\frac{r^2}{\rho}\partial_r P &= -r^2\partial_r \phi \\
\partial_r \frac{r^2}{\rho}\partial_r P &= -\partial_r r^2 \partial_r \phi \\
\frac{1}{r^2}\partial_r \frac{r^2}{\rho}\partial_r P &= -\frac{1}{r^2}\partial_r r^2 \partial_r \phi = -\nabla^2 \phi = -4\pi G \rho
\end{aligned}
$$

2.3 [2 points] Assuming a barotropic equation of state for the gas, $P = K\rho^\gamma$ and dimensionless variables, $\rho(r) = \rho_c(D_n(r))^n$ and $r = \lambda_n \xi$ such that $\gamma = 1 + \frac{1}{n}$. Show that Eq. (20) takes the form of the dimensionless Lane-Emden equation,

$$
\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial D_n}{\partial \xi}\right) = -\left(D_n(\xi)\right)^n, \tag{22}
$$

for a certain choice of λ_n.

Solution: *We can rewrite Eq. (20) as:*

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) &= -4\pi G\rho \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial K\rho^\gamma}{\partial r}\right) &= -4\pi G\rho \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}K\gamma\rho^{\gamma-1}\frac{\partial \rho}{\partial r}\right) &= -4\pi G\rho \\
\frac{K\gamma}{r^2}\frac{\partial}{\partial r}\left(r^2\rho^{\gamma-2}\frac{\partial \rho}{\partial r}\right) &= -4\pi G\rho
\end{aligned}
$$

We now insert $\gamma = 1 + \frac{1}{n} = \frac{n+1}{n}$

$$
\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\rho^{\frac{1-n}{n}}\frac{\partial \rho}{\partial r}\right) = -4\pi G\rho
$$

We can then write this equation in terms of the dimensionless quantities.

$$
\rho(r) = \rho_c(D_n(r))^n \textit{ and } r = \lambda_n \xi
$$

This lead to the following

$$
\left(\frac{n+1}{n}\right)\frac{K}{\lambda_n^2\xi^2}\frac{1}{\lambda_n}\frac{\partial}{\partial \xi}\left(\lambda_n^2\xi^2\rho_c^{\frac{1-n}{n}}\left(D_n(\xi)\right)^{1-n}\rho_c n\left(D_n(\xi)\right)^{n-1}\frac{1}{\lambda_n}\frac{\partial D_n}{\partial \xi}\right) = -4\pi G\rho_c\left(D_n\right)^n
$$

Which can be simplified to Eq. (22)

$$
\frac{(n+1)K\rho_c^{\frac{1-n}{n}}}{4\pi G\lambda_n^2} \times \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial D_n}{\partial \xi}\right) = -\left(D_n(\xi)\right)^n
$$

We can choose λ_n such as:

$$
\frac{(n+1)K\rho_c^{\frac{1-n}{n}}}{4\pi G\lambda_n^2} = 1 \iff \lambda_n = \sqrt{\frac{(n+1)K\rho_c^{\frac{1-n}{n}}}{4\pi G}}
$$

And so we obtain

$$
\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial D_n}{\partial \xi}\right) = -\left(D_n(\xi)\right)^n
$$

This is the dimensionless Lane-Emden equation.

2.4 [1.5 points] Derive the solution for $n = 0$ imposing that $D_0(0) = 1$ and $\frac{\partial D_0}{\partial \xi} = 0$ for $\xi = 0$. Do not worry about the $\frac{1}{n}$ as this is simply a mathematical curiosity. Indeed this equation admits only three known analytical solutions for $n = 0$, $n = 1$ and $n = 5$.

Solution: *This equation only has 3 analytical solutions each corresponding to different values of n, these solutions exist for n=0, n=1, n=5. We will derive the solution for n=0.*

$$\begin{aligned}
\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial D_0}{\partial \xi}\right) &= -1 \\
\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial D_0}{\partial \xi}\right) &= -\xi^2 \\
\xi^2 \frac{\partial D_0}{\partial \xi} &= -\frac{1}{3}\xi^3 + C_1 \\
\frac{\partial D_0}{\partial \xi} &= -\frac{1}{3}\xi + \frac{C_1}{\xi^2}
\end{aligned}$$

imposing that $\frac{\partial D_n}{\partial \xi} = 0$ *for* $\xi = 0$ *we deduce that* $C_1 = 0$ *and so we can integrate to obtain the solution.*

$$D_0 = C_2 - \frac{1}{6}\xi^2$$

and now imposing $D_0(0) = 1$ *we deduce* $C_2 = 1$ *we obtain.*

$$D_0(\xi) = 1 - \frac{1}{6}\xi^2$$

This is in fact one of only three analytical solution of the Lane-Emden equation. The other two solutions are for n=1 and n=5.

$$D_1(\xi) = \frac{\sin(\xi)}{\xi} \quad D_5(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

3 Stellar formation

In this second part of the problem we will study a simple model for the formation of stars. To do so we will study the evolution of a small perturbation around the hydrostatic solution. As such we will start by liniarizing Eq. (16) and (17). To do so introduce first order perturbation terms such that,

$$\begin{aligned}
\rho(\boldsymbol{x}, t) &= \rho_0 + \rho_1(\boldsymbol{x}, t) \\
\boldsymbol{v}(\boldsymbol{x}, t) &= \boldsymbol{v}_1(\boldsymbol{x}, t) \\
P(\boldsymbol{x}, t) &= P_1(\boldsymbol{x}, t) \\
\phi(\boldsymbol{x}, t) &= \phi_1(\boldsymbol{x}, t)
\end{aligned}$$

with $\nabla^2 \phi_1 = 4\pi G \rho_1$.

3.1 [1 point] Show that the linearized fluid equations take the form.

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0 \tag{23}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\boldsymbol{\nabla} P_1 - \rho_0 \boldsymbol{\nabla} \phi_1 \tag{24}$$

Solution: *This simply a matter of plugging in and eliminating second order terms and hydrostatic terms*

$$\begin{aligned}
\partial_t(\cancel{\rho_0} + \rho_1) + \boldsymbol{\nabla} \cdot ([\rho_0 + \cancel{\rho_1}]^{2nd}\boldsymbol{v}_1) &= 0 \\
\partial_t \rho_1 + \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v}_1) &= 0 \\
\partial_t \rho_1 + \rho_0 \boldsymbol{\nabla} \cdot (\boldsymbol{v}_1) &= 0
\end{aligned}$$

for the first, and

$$
\begin{aligned}
\partial_t \boldsymbol{v}_1 + \cancelto{2nd}{(\boldsymbol{v}_1 \cdot \boldsymbol{\nabla})\boldsymbol{v}_1} &= -\frac{1}{\rho_0 + \rho 1}\boldsymbol{\nabla} P_1 - \boldsymbol{\nabla}\phi_1 \\
(\rho_0 + \cancelto{2nd}{\rho_1})\partial_t \boldsymbol{v}_1 &= -\boldsymbol{\nabla} P_1 - (\rho_0 + \cancelto{2nd}{\rho_1})\boldsymbol{\nabla}\phi_1 \\
\rho_0 \partial_t \boldsymbol{v}_1 &= -\boldsymbol{\nabla} P_1 - \rho_0 \boldsymbol{\nabla}\phi_1
\end{aligned}
$$

3.2 [2 points] Introducing the sound speed $c_s^2 := \frac{dP}{d\rho}$ such that $P_1 = c_s^2 \rho_1$ show that the previously derived result can be rewritten as.

$$
\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0 \tag{25}
$$

Solution: *Starting from Eq. (24) we can write*

$$
\begin{aligned}
\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} &= -\boldsymbol{\nabla} P_1 - \rho_0 \boldsymbol{\nabla}\phi_1 \\
\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} &= -c_s^2 \boldsymbol{\nabla}\rho_1 - \rho_0 \boldsymbol{\nabla}\phi_1
\end{aligned}
$$

then taking the divergence of this equation we obtain

$$
\rho_0 \boldsymbol{\nabla} \cdot \partial_t \boldsymbol{v}_1 = -c_s^2 \nabla^2 \rho_1 - \rho_0 \nabla^2 \phi_1
$$

Taking the time derivative of Eq. (23)

$$
\partial_t^2 \rho_1 + \rho_0 \partial_t \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0
$$

such that we can isolate

$$
\rho_0 \boldsymbol{\nabla} \cdot \partial_t \boldsymbol{v}_1 = -\partial_t^2 \rho_1
$$

and insert into the previous result to obtain

$$
\partial_t^2 \rho_1 - c_s^2 \nabla^2 \rho_1 - \rho_0 \nabla^2 \phi_1 = 0,
$$

finally inserting the Poisson equation to obtain the final result

$$
\partial_t^2 \rho_1 - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0.
$$

3.3 [1.5 points] Using the Anzatz $\rho_1 = A \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$ recover a dispersion relation of the form

$$
\omega^2 = c_s^2 (k^2 - k_J^2) \tag{26}
$$

and give the expression of k_J^2. What happens if $k^2 < k_J^2$?

Solution: *The Anzatz allows to replace $\partial_t^2 \to -\omega^2$ and $\nabla^2 \to -k^2$. From this obtaining the relation is trivial. As a result one obtains,*

$$
k_J^2 = \frac{4\pi G \rho_0}{c_s^2}.
$$

What can be observed in the case where $k^2 < k_J^2$ is that ω becomes imaginary this in turn means that the perturbation is a linear combination of two modes one that decreases exponentially and one that grows exponentially. With time the growing mode will always superseded the decreasing mode.

3.4 [0.5 points] Defining $\lambda_J = \frac{2\pi}{k_J}$, known as the Jeans length and $M_J = \frac{4\pi}{3}\rho_0 \lambda_J^3$, known as the Jeans Mass. Give a physical interpretation as to the future of a perturbation with mass $M > M_J$ and $M < M_J$.

Solution: *The result can be interpreted as follows if the perturbation is more massive that the Jeans mass it will undergo gravitational collapse. On the other hand if the perturbation is smaller the collapse will be halted by the internal pressure of the gas and the perturbation is dissipated in the form of sound waves.*

Part III

The origin of cosmological structures

1 Introduction

Cosmology is an observational science rather than an experimental one. There is no way we can run the history of the universe several times. In the end, all we can do is observe the universe as it is today and infer its evolution. The astonishing beauty of modern cosmology is that the large scale structures that we see around us – like galaxy clusters – are not randomly distributed but exhibit specific patterns: we say they are correlated. This is all the more remarkable as one can learn a lot from these correlations, and reconstruct the history of the universe, leading to the ΛCDM Standard Model of Cosmology. However, there is still a legitimate question to ask:

Where do these correlations come from in the first place, and how were they generated?

In this problem, we will see that these correlations were generated during an earlier phase of accelerated expansion just after the so-called Big Bang called inflation.

Knowledge of general relativity and quantum field theory is **not** required to solve the following problem.

2 Phase of accelerated expansion

A fundamental hypothesis of modern Cosmology is that the Universe is spatially homogeneous and isotropic on large scales. In simple words, it means that there is no "center of the universe" nor a preferred direction in the sky. Mathematically, it means that the entire universe can be described by a single function of time called the *scale factor*: $a(t)$. One can see the scale factor as the "size of the universe". If the scale factor is an increasing function of time $\dot{a} > 0$ (a dot means derivative with respect to time), then the universe is expanding. The evolution of $a(t)$ is governed by the universe content through the Friedmann equations

$$H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\rho, \qquad \dot{H} + H^2 = -\frac{1}{6M_{\mathrm{pl}}^2}(\rho + 3p), \tag{27}$$

where $H = \dot{a}/a$ is the Hubble parameter, $M_{\mathrm{pl}} = 1/\sqrt{8\pi G}$ is the Planck mass, ρ and p are the energy density and pressure of some matter/energy content[2]. Let us assume now a phase of accelerated expansion that we name inflation such that $\ddot{a} > 0$.

Question 1. [1 point] Show that inflation is necessarily driven by a matter/energy content with negative pressure.

Solution: *Let us examine the time derivative of the Hubble parameter. We have*

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2.$$

Substituting this expression in the second Friedmann equation, we obtain

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(\rho + 3p).$$

Inflation occurs when $\ddot{a} > 1$. Because the scale factor is always positive, we then have

$$\rho + 3p < 0 \Leftrightarrow \frac{p}{\rho} < -1/3.$$

The energy density being always positive, we see that successful inflation requires a negative pressure.

[2] It can be baryons, dark energy, photons, neutrinos, etc.

Let us now consider a scalar field $\phi(t)$ that we call the *inflaton*. Its evolution is described by the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{28}$$

where $V(\phi)$ is for now an arbitrary potential, and $V' = \frac{\mathrm{d}V}{\mathrm{d}\phi}$. The energy density and pressure of this scalar field are

$$\rho = \frac{1}{2}\dot{\phi} + V(\phi), \qquad p = \frac{1}{2}\dot{\phi} - V(\phi). \tag{29}$$

Question 2. [0.5 point] Show that when the inflaton potential dominates over its kinetic energy, the field ϕ can cause the universe acceleration.

Solution: *Using the formulae for the energy density and the pressure of a scalar field, we have*

$$\frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{30}$$

When the potential dominates over the kinetic energy of the inflaton $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$*, we have* $p/\rho \approx -1 < -1/3$*. Thus in a universe filled with a scalar field* ϕ*, inflation occurs when the field's energy is almost entirely contained in its potential. As a consequence, it has low velocity. The picture is that the inflaton then slowly rolls down its potential, hence the slow-roll approximation.*

This limit is called the *slow-roll* approximation because the inflaton field then slowly rolls down its potential. In the following, we define the slow-roll parameter $\varepsilon = -\dot{H}/H^2$ that measures the time variation of the Hubble parameter.

Question 3. [2 points] Show that inflation occurs when $\varepsilon < 1$. Under the slow-roll approximation, show that one can write

$$\varepsilon \approx \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2. \tag{31}$$

Solution: *By definition and using Eq. (2), we have*

$$\varepsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{a}}{aH}. \tag{32}$$

Because inflation occurs when $\ddot{a} > 0$*, and both* a *and* H *are positive*[3]*, we then obtain* $\varepsilon < 1$*. In order to prove Eq. (31), let us first start from the first Friedmann equation*

$$H^2 = \frac{1}{3M_{pl}^2}\rho = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \tag{33}$$

We then derive this equation with respect to time, yielding

$$2\dot{H}H = \frac{1}{3M_{pl}^2}\dot{\phi}\left[\ddot{\phi} + V'\right]. \tag{34}$$

Using now the Klein-Gordon equation (28), we obtain

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2}. \tag{35}$$

Under the slow-roll approximation $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$*, Eq. (33) can be written*

$$H^2 \approx \frac{V(\phi)}{3M_{pl}^2}. \tag{36}$$

[3] During inflation the scale factor increases as a function of time so the Hubble parameter is positive.

We note as well that for the slow-roll condition to hold, we need the acceleration of the scalar field to be small. This simplifies the Klein-Gordon equation (28)

$$3H\dot{\phi} \approx -V'. \tag{37}$$

In the end using all the previous equations, we obtain

$$\varepsilon = -\frac{\dot{H}}{H^2} \overset{(35)}{=} \frac{\dot{\phi}^2}{2M_{pl}^2 H^2} \overset{(36)}{\approx} \frac{3}{2}\frac{\dot{\phi}^2}{V} \overset{(37)}{\approx} \frac{1}{6}\frac{(V')^2}{V}\frac{1}{H^2} \overset{(36)}{\approx} \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{38}$$

Question 4. [3 points] For a quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$ and according to CMB observations, show that we need to consider super-Planckian values for the inflation $\phi_{\rm CMB} > M_{\rm pl}$.

Hint: Express the number of e-folds as an integral over the inflation field ϕ using the slow-roll approximation.

Solution: *It is useful to express the number of e-folds during inflation as an integral over time*

$$N_{\textit{inflation}} = \ln(a_f/a_i) = \int_{a_i}^{a_f} \frac{\mathrm{d}a}{a} = \int_{t_i}^{t_f} H(t)\mathrm{d}t. \tag{39}$$

Under the slow-roll approximation, we have

$$H\mathrm{d}t = \frac{H}{\dot{\phi}}\mathrm{d}\phi \overset{(37)}{\approx} -\frac{3H}{V'}H\mathrm{d}\phi \overset{(36)}{\approx} -\frac{1}{M_{pl}^2}\left(\frac{V}{V'}\right)\mathrm{d}\phi \approx -\frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}\phi}{M_{pl}}. \tag{40}$$

So we then have

$$N_{\textit{inflation}} \approx -\int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}\phi}{M_{pl}} > -\int_{\phi_{CMB}}^{\phi_f} \frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}\phi}{M_{pl}} \sim 60 \tag{41}$$

For a quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$, $\varepsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right) = \frac{2M_{pl}^2}{\phi^2}$, *so*

$$N_{\textit{inflation}} > -\int_{\phi_{CMB}}^{\phi_f} \frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}\phi}{M_{pl}} = -\frac{1}{2M_{pl}^2}\int_{\phi_{CMB}}^{\phi_f} \phi^2\mathrm{d}\phi = \frac{1}{4M_{pl}^2}\left[\phi_{CMB}^2 - \phi_f^2\right]. \tag{42}$$

At the end of inflation, the field reaches the minimum of its potential and we can neglect its value $\phi_f \approx 0$ *so that*

$$N_{\textit{inflation}} > \frac{\phi_{CMB}^2}{4M_{pl^2}} \sim 60, \tag{43}$$

and hence

$$\phi_{CMB} \sim \sqrt{4 \times 60}M_{pl} \sim 15M_{pl} > M_{pl}, \tag{44}$$

so that the inflaton has super-Planckian values.

This means that with cosmological observations, we can probe the laws of physics at the highest possible energy, *i.e.* close to the Planck mass, which is far above the energies involved in ground-based experiments such as that in the Large Hadron Collider!

3 Fluctuations during inflation

In the first part of the problem, we considered that the universe is perfectly homogeneous. However, this is not exactly the case. When looking around us, we see for example galaxy clusters. This means that we have to improve our previous description of the Universe by considering small perturbations on top of a homogeneous background. This can be done by writing:

$$\phi(t) = \bar{\phi}(t) + \delta\phi(\boldsymbol{x}, t), \text{ with } \delta\phi \ll \bar{\phi}, \tag{45}$$

where $\bar{\phi}$ is the background homogeneous part of the inflation and $\delta\phi$ is a space-varying small perturbation. We are then interested in understanding the behaviour of $\delta\phi$. One can show –

after tedious calculations that we are not going to do – that $\delta\phi$ satisfies the Mukhanov-Sasaki equation:

$$v''_{\boldsymbol{k}} + \left(k^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}} = 0, \text{ where } v = zH\frac{\delta\phi}{\dot{\phi}} \text{ and } z = a\sqrt{2\varepsilon M_{\rm pl}^2}; \tag{46}$$

the prime denotes a derivative with respect to conformal time[4] τ defined by $\mathrm{d}\tau = \mathrm{d}t/a$, and we have decomposed $v(\boldsymbol{x}, t)$ into Fourier modes:

$$v_{\boldsymbol{k}}(\tau) = \int \mathrm{d}^3x\, e^{-i\boldsymbol{k}.\boldsymbol{x}} v(\tau, \boldsymbol{x}). \tag{47}$$

Question 5. [1 point] Show that when the Universe is expanding exponentially $a(t) = a_0 e^{Ht}$ with an approximately constant Hubble parameter H, we get $z''/z = 2/\tau^2$.

Solution: ***Question 5.*** *Let us assume $a(t) = a_0 e^{Ht}$ with H approximately constant. We first look for the scale factor as a function of conformal time $a(\tau)$. This can be done by integrating the definition of conformal time*

$$\tau = \int \frac{\mathrm{d}t}{a_0 e^{Ht}} = -\frac{1}{aH}, \tag{48}$$

after what we find $a(\tau) = -1/(H\tau)$. We note that in the strict H = constant limit, ε is zero and that z''/z is undefined. However, we H is approximately constant and so we will consider that ε is in a good approximation constant but very small. In this case, we obtain that

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}. \tag{49}$$

Question 6. [0.5 point] Show that

$$v_{\boldsymbol{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right), \tag{50}$$

is a solution of the Mukhanov-Sasaki equation and that it satisfies[5]

Solution: *One just needs to replace $v_{\boldsymbol{k}}(\tau)$ in the Mukhanov-Sasaki equation. The initial condition is straighforwardly satisfied.*

$$\lim_{\tau\to-\infty} v_{\boldsymbol{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{51}$$

In actual measurements, the quantity $\delta\phi$ is not directly observable. However, we can measure the power spectrum of $\delta\phi$ that we denote $P_{\delta\phi}(k)$ such that

$$P_{\delta\phi}(k) = \lim_{k\tau\to 0} \delta\phi\, \delta\phi^\star. \tag{52}$$

Question 7. [2 points] Compute the primordial power spectrum $P_{\delta\phi}(k)$.

Solution: *Using the definition of the power spectrum, we have*

$$\begin{aligned} P_{\delta\phi}(k) &= \lim_{k\tau\to 0} \delta\phi\, \delta\phi^\star \\ &= \frac{\dot{\phi}^2}{z^2H^2} \lim_{k\tau\to 0} v_{\boldsymbol{k}}(\tau) v_{\boldsymbol{k}}(\tau)^\star \\ &= \frac{\dot{\phi}^2}{z^2H^2}\frac{1}{2k} \lim_{k\tau\to 0}\left(1 + \frac{1}{k^2\tau^2}\right) \\ &= \frac{\dot{\phi}^2}{4k^3\varepsilon M_{pl^2}} \end{aligned} \tag{53}$$

where in the last line we have used the definition of z and the fact that $\tau = -1/aH$.

This power spectrum matches exactly with the CMB observations. In the end, we have shown that all the large-scale structures – like galaxy clusters – come from fluctuations that were generated during inflation – a phase of accelerated expansion in the early universe.

[4] The conformal time τ runs from $-\infty$ to the end of inflation when $\tau = 0$.

[5] This initial condition is called the Bunch-Davis vacuum.

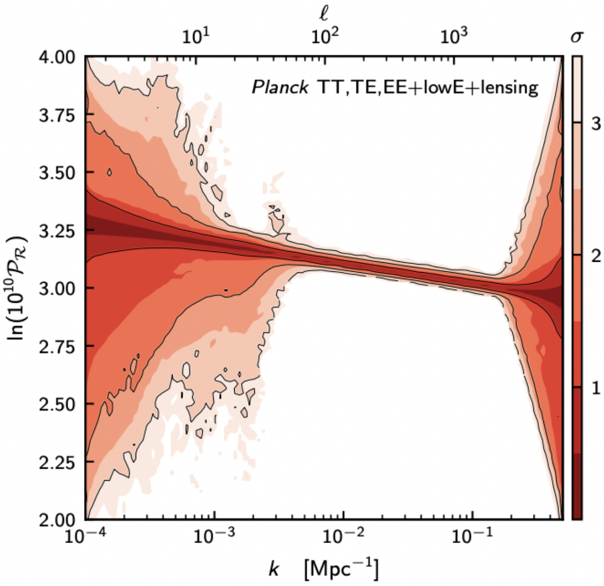

Figure 2: Reconstruction of the dimensionless primordial power spectrum $\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}P_\mathcal{R}(k)$ where $\mathcal{R} = H\frac{\delta\phi}{\phi}$ using the Cosmic Microwave Background data by Planck 2018. This Figure essentially tells us that the primordial power spectrum is almost scale-invariant $\mathcal{P}_\mathcal{R} \sim$ constant meaning that the primordial power is equally distributed over the various scales in the sky. This fact is far from being trivial.

Part IV

Introduction to a quantum Szilárd engine

In 1929, Leo Szilárd, a Hungarian physicist, proposed an imaginary device now known as *a Szilárd engine*, based on the concept of Maxwell's demon. The engine consisted of a box coupled to a thermal bath containing a single classical particle. A demon, knowing the position of the particle, could insert a moveable barrier (which can move without friction like a piston) in the box, separating the box into two smaller volumes, and keep the particle on one side of the barrier. The isothermal expansion of the ideal gas composed of the single particle could then result in the displacement of the barrier and be extracted as useful work (see Fig. 3). Thus it was apparently possible using this machine to transform heat from an isothermal environment into useful work, seemingly violating the Second law of Thermodynamics.

In this problem, we focus on a quantum model of a Szilárd engine, in which the particle is described using quantum mechanics. In the following, we will focus on the description of the different steps of a cycle of the engine.

1 Description of the initial state

The particle of mass m is prepared in a thermal state of a harmonic well. The Hamiltonian of the initial state is:

$$\hat{H}_i = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \tag{54}$$

with $\hat{q}$ and $\hat{p}$ the position and momentum operators. We introduce $|\psi_n\rangle$ the eigenstates of an usual harmonic oscillator, and we recall the expression of the corresponding energies $E_n = (n+1/2)\hbar\omega$, with $n \in \mathbb{N}$. In the following, we introduce $\beta = \frac{1}{k_B T}$, with k_B the Boltzmann constant and T the temperature of the thermal bath.

1.1 [0.75 point] We recall that the partition function of a system can be calculated using $Z = \mathrm{Tr}(\mathrm{e}^{-\beta\hat{H}})$, where Tr corresponds to the Trace operation. In our case, for an operator $\hat{A}$,

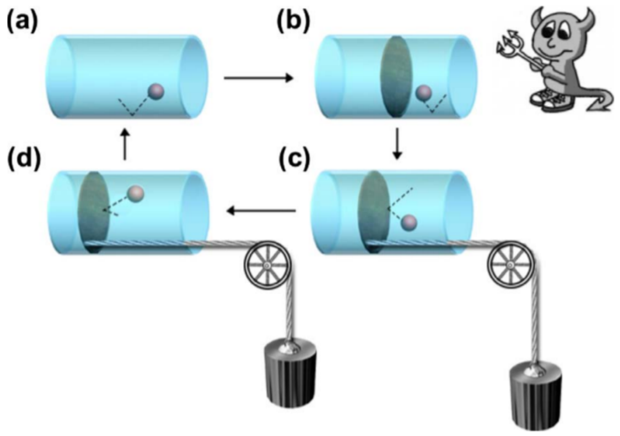

Figure 3: Schematic diagram of Szilard's heat engine. (a) Initially, the position of the molecule is unknown. (b) Maxwell's demon inserts a partition at the center and observes the molecule to determine whether it is in the right- or the left-hand side of the partition. (c) Depending on the outcome of the measurement, the demon connects a load to the partition. (d) The isothermal expansion of the gas does work upon the load. Adapted from Maruyama *et al.*, Rev. Mod. Phys., 81, 2009.

the trace of this operator is given by:

$$\mathrm{Tr}(\hat{A}) = \sum_{n=0}^{\infty} \langle \psi_n | \hat{A} | \psi_n \rangle. \tag{55}$$

Show that the partition function Z_i of the system can be expressed as:

$$Z_i = \frac{1}{2\sinh(\beta\hbar\omega/2)} \tag{56}$$

Solution: *Using the definition of the trace :*

$$\begin{aligned} Z_i &= \mathrm{Tr}(\mathrm{e}^{-\beta \hat{H}_i}) \\ &= \sum_{n=0}^{\infty} \langle \psi_n | \mathrm{e}^{-\beta \hat{H}_i} | \psi_n \rangle \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n} \langle \psi_n | \psi_n \rangle \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega(n+1/2)} \\ &= \mathrm{e}^{-\beta\hbar\omega/2} \frac{1}{1-\mathrm{e}^{-\beta\hbar\omega}} \end{aligned}$$

After factorizing the denominator, we obtain the desired expression of Z_i.

1.2 [1.5 points] Express the Helmholtz free energy F_i and the average energy E_i of the system. Interpret the low-temperature limit and the high-temperature limit of the expression of E_i.

We indicate that these quantities can be calculated using the following expressions:

$$F = -\frac{1}{\beta}\ln Z, \tag{57a}$$

$$E = -\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}\beta}. \tag{57b}$$

Solution: *Using the given formulae:*

$$F_i = \frac{1}{\beta}\ln\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right],$$
$$E_i = \frac{1}{2}\hbar\omega\frac{1}{\tanh(\beta\hbar\omega/2)}.$$

In the low-temperature limit ($x = \beta\hbar\omega \gg 1$), $\frac{1}{\tanh x/2} = \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} \approx \frac{e^{x/2}}{e^{x/2}} \approx 1$, thus $E_i \approx \hbar\omega/2$. The particle settles in the ground state when we tend towards zero temperature.
In the high-temperature limit ($x \ll 1$), $\frac{1}{\tanh x/2} \approx \frac{1+x/2+1-x/2}{1+x/2-1+x/2} = \frac{2}{x}$, thus $E_i \approx 1/\beta = k_BT$. This result is consistent with the equipartition theorem, since there are 2 degrees of freedom, which each contribute an energy of $k_BT/2$ in the classic high-temperature limit.

1.3 [1.5 points] Calculate the entropy S_i of the system, using the following expression:

$$S = -\frac{\mathrm{d}F}{\mathrm{d}T}. \tag{58}$$

Check that the entropy vanishes in the low-temperature limit and that in the high-temperature limit, one obtains $S_i = k_B\ln(k_BT/\hbar\omega)$.

Solution: *Using the given formula:*

$$S_i = k_B\left[\frac{\beta\hbar\omega}{2}\frac{1}{\tanh(\beta\hbar\omega/2)} - \ln\left(2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right)\right].$$

We have studied in the previous question the behaviour of the first term of the expression of S_i. For the second term:

$$\text{For } x \gg 1 \quad \ln(2\sinh(x/2)) \approx \ln(e^{x/2}) = x/2$$
$$\text{For } x \ll 1 \quad \ln(2\sinh(x/2)) \approx \ln(1 + x/2 - (1 - x/2)) = \ln(x).$$

Thus in the low-temperature limit, $S_i \to 0$, and in the high-temperature limit,

$$S_i \approx k_B(1 + \ln(\beta\hbar\omega)) \approx k_B\ln(\beta\hbar\omega).$$

2 Insertion of the barrier

An infinitely thin potential barrier is inserted at $q = 0$. The Hamiltonian thus becomes:

$$\hat{H}_{bar}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + \alpha(t)\delta(\hat{q}), \tag{59}$$

where $\alpha(t)$ gives the time-dependent strength of the barrier (modelled by a delta function), which satisfies $\alpha(-\infty) = 0$ and $\alpha(+\infty) = \infty$.
We use an adiabatic approximation ($|\dot{\alpha}/\alpha| \ll \omega$), ensuring that the system is in thermal equilibrium with the thermal bath at every instant. Let $|\psi_n(t)\rangle$ be the eigenstates of the system at instant t, and $E_n(t)$ be the corresponding energy.

2.1 [0.75 point] What should be the value of $\langle\psi_n|\hat{q}|\psi_n\rangle|_{t=+\infty}$ (i.e. the probability density) for $q = 0$? Comment on the effect of the barrier on the odd and the even eigenvectors (using Fig. 4).

Solution: *Since the potential barrier will have a magnitude of ∞ for $t = +\infty$, the wavefuntions will drop to 0 for $q = 0$, thus $\langle\psi_n|\hat{q}|\psi_n\rangle(q = 0, t = +\infty) = 0$.*
In a harmonic well, the odd-numbered eigenvectors have antisymetric associated wave functions, and thus vanish for $q = 0$. Therefore the addition of the delta-function barrier does not affect the eigenvectors, nor their associated energies. On the other hand, even-numbered eigenvectors will be sensitive to the amplitude of the barrier, and so will their energy (as can be seen in Figure 4). The presence of an infinite barrier forces the associated wavefunction to drop to 0, thus adding a node in the wave function, just like the odd-numbered eigenvectors. We can understand then why for $t = +\infty$ the energies of the even-numbered eigenvectors tend towards the energies of the odd-numbered eigenvectors.

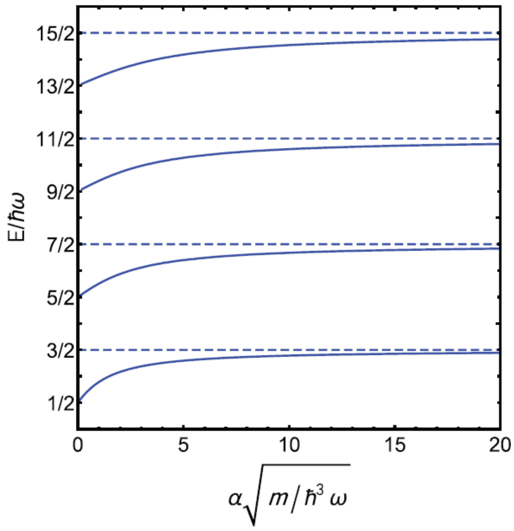

Figure 4: Evolution of the eigenenergies $E_n(t)$ with $\alpha(t)$ for n odd (dashed lines) and even (solid lines).

2.2 [1.5 points] In the end, for $t = +\infty$, the spectrum of the system is that of a harmonic oscillator of frequency 2ω, with the bottom of the well shifted up by $\hbar\omega/2$ and with each energy level with a degeneracy of 2. Show that

$$Z_\infty = \mathrm{e}^{-\beta\hbar\omega/2}\frac{1}{\sinh(\beta\hbar\omega)}, \tag{60}$$

where Z_∞ is the partition function of the system at $t = +\infty$, and compute the quantities F_∞ and S_∞, the Helmholtz free energy and the entropy of the system at $t = +\infty$.
Solution: *Using the definition of the trace :*

$$\begin{aligned}
Z_\infty &= \mathrm{Tr}(\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}) \\
&= \sum_{n=0}^{\infty}\langle\psi_n|\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}|\psi_n\rangle(t=+\infty) \\
&= \sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_n(t=+\infty)}\langle\psi_n|\psi_n\rangle(t=+\infty) \\
&= \sum_{n=0}^{\infty}\mathrm{e}^{-\beta(2\hbar\omega(n+1/2)+\hbar\omega/2)}\times 2 \\
&= 2\mathrm{e}^{-\frac{3}{2}\beta\hbar\omega}\frac{1}{1-\mathrm{e}^{-2\beta\hbar\omega}}
\end{aligned}$$

After factorizing the denominator, we obtain the desired expression of Z_∞.
Using the formulae, we derive:

$$\begin{aligned}
F_\infty &= \frac{1}{\beta}\ln\left[\sinh(\beta\hbar\omega)\right] + \frac{\hbar\omega}{2}, \\
S_\infty &= k_B\left[\beta\hbar\omega\frac{1}{\tanh(\beta\hbar\omega)} - \ln(\sinh(\beta\hbar\omega))\right].
\end{aligned}$$

2.3 [0.75 point] Show that $F_\infty - F_i > 0$, *i.e.* that the demon has to provide energy to the system with the insertion of the barrier. **Solution:** *We compute*

$$\begin{aligned}
F_\infty - F_i &= \frac{1}{\beta}\ln\left[\sinh(\beta\hbar\omega)\right] + \frac{\hbar\omega}{2} - \frac{1}{\beta}\ln\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right] \\
&= \frac{\hbar\omega}{2} + \frac{1}{\beta}\ln\left[\frac{1}{2}\frac{\mathrm{e}^{\beta\hbar\omega} - \mathrm{e}^{-\beta\hbar\omega}}{\mathrm{e}^{\beta\hbar\omega/2} - \mathrm{e}^{\beta\hbar\omega/2}}\right] \\
&= \frac{\hbar\omega}{2} + \frac{1}{\beta}\ln\left[\frac{1}{2}(\mathrm{e}^{\beta\hbar\omega/2} + \mathrm{e}^{\beta\hbar\omega/2})\right] \\
&= \frac{\hbar\omega}{2} + \frac{1}{\beta}\ln(\cosh(\beta\hbar\omega/2)).
\end{aligned}$$

Since $\forall x \in \mathbb{R}\ \cosh(x) > 1,\ F_\infty - F_i > 0.$

3 Quantum measurement by the demon

Let us introduce the *left* and *right* eigenstates of the system:

$$|L_n\rangle = \frac{1}{\sqrt{2}}(|\psi_{2n}\rangle(\infty) - |\psi_{2n+1}\rangle(\infty)), \tag{61a}$$

$$|R_n\rangle = \frac{1}{\sqrt{2}}(|\psi_{2n}\rangle(\infty) + |\psi_{2n+1}\rangle(\infty)), \tag{61b}$$

and the projectors:

$$\hat{P}_L = \sum_{n=0}^{\infty} |L_n\rangle\langle L_n|, \tag{62a}$$

$$\hat{P}_R = \sum_{n=0}^{\infty} |R_n\rangle\langle R_n|. \tag{62b}$$

3.1 [0.75 point] Justify that states $\{|L_n\rangle, |R_n\rangle\}$ correspond to eigenstates of the Hamiltonian H_{bar} and give the energies associated to these states. Explain the denomination *left* and *right* eigenstates.

Solution: *We find that:*

$$\begin{aligned}
\hat{H}_{bar}|L_n\rangle &= \hat{H}_{bar}\frac{1}{\sqrt{2}}(|\psi_{2n}\rangle(\infty) - |\psi_{2n+1}\rangle(\infty)) \\
&= \frac{1}{\sqrt{2}}(\hat{H}_{bar}|\psi_{2n}\rangle(\infty) - \hat{H}_{bar}|\psi_{2n+1}\rangle(\infty)) \\
&= \frac{1}{\sqrt{2}}(E_{2n}(\infty)|\psi_{2n}\rangle(\infty) - E_{2n+1}(\infty)|\psi_{2n+1}\rangle(\infty)) \\
&= E_{2n}(\infty)\frac{1}{\sqrt{2}}(|\psi_{2n}\rangle(\infty) - |\psi_{2n+1}\rangle(\infty)).
\end{aligned}$$

So $|L_n\rangle$ *is an eigenstate of* $\hat{H}_{bar}$ *with energy* $E_{2n} = 2\hbar\omega(2n+1/2)+\hbar\omega/2$. *A similar justification can be given for* $|R_n\rangle$.
$|L_n\rangle$ *and* $|R_n\rangle$ *correspond to states where the particle is exclusively on the left or right side of the barrier.*

When the demon determines the position of the particle, it acts as a projective measurement of $\hat{P}_L$ and $\hat{P}_R$ on the state of the particle. We recall that the partition function Z_P after projection by projector $\hat{P}$ can be calculated as $Z_P = \mathrm{Tr}(\hat{P}\,\mathrm{e}^{-\beta\hat{H}})$.

3.2 [1.5 points] Calculate the partition function Z_L and Z_R of the system after the particle has been measured on the left or on the right side of the barrier and show that:

$$Z_L = \mathrm{e}^{-\beta\hbar\omega/2}\frac{1}{2\sinh(\beta\hbar\omega)} \tag{63}$$

and compute F_L, F_R, S_L and S_R the associated free energy and entropy.

Solution: *Using the definition of the trace :*

$$\begin{aligned}
Z_L &= \mathrm{Tr}(\hat{P}_L \mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}) \\
&= \sum_{n=0}^{\infty}\langle\psi_n|\hat{P}_L\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}|\psi_n\rangle(\infty) \\
&= \sum_{n=0}^{\infty}\langle\psi_n|\sum_{m=0}^{\infty}|L_m\rangle\langle L_m|\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}|\psi_n\rangle(\infty) \\
&= \sum_{n=0}^{\infty}\langle\psi_{2n}|\sum_{m=0}^{\infty}|L_m\rangle\langle L_m|\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}|\psi_{2n}\rangle(\infty) + \sum_{n=0}^{\infty}\langle\psi_{2n+1}|\sum_{m=0}^{\infty}|L_m\rangle\langle L_m|\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}|\psi_{2n+1}\rangle(\infty) \\
&= \sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_{2n}(\infty)}\langle\psi_{2n}|L_n\rangle\langle L_n|\psi_{2n}\rangle(\infty) + \sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_{2n+1}(\infty)}\langle\psi_{2n+1}|L_n\rangle\langle L_n|\psi_{2n+1}\rangle(\infty) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_{2n}(\infty)} + \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_{2n+1}(\infty)} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{e}^{-\beta E_n(\infty)} \\
&= \frac{Z_\infty}{2} \\
&= \mathrm{e}^{-\frac{3}{2}\beta\hbar\omega}\frac{1}{1-\mathrm{e}^{-2\beta\hbar\omega}}
\end{aligned}$$

We could save some time by realizing that in the basis $\{|L_n\rangle, |R_n\rangle\}$, the matrix $\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}$ can be written in the same way as in the basis $\{|\psi_n\rangle(\infty)\}$ (given the degeneracy of levels $|\psi_{2n}\rangle(\infty)$ and $|\psi_{2n+1}\rangle(\infty)$). Therefore the trace of $\hat{P}_L\mathrm{e}^{-\beta\hat{H}_{bar}(\infty)}$ corresponds to the sum of the matrix elements corresponding to $|L_n\rangle$ states which account for exactly half of the eigenstates. In the end, $Z_L = Z_R = Z_\infty/2$.
We obtain directly from the results of the previous section:

$$\begin{aligned}
F_L = F_R &= \frac{1}{\beta}\ln\left[\sinh(\beta\hbar\omega)\right] + \frac{\hbar\omega}{2} - \frac{1}{\beta}\ln(2), \\
S_L = S_R &= k_B\left[\beta\hbar\omega\frac{1}{\tanh(\beta\hbar\omega)} - \ln(\sinh(\beta\hbar\omega))\right] - k_B\ln(2).
\end{aligned}$$

3.3 [1 point] Show that the entropy of the system decreases exactly by $|\Delta S| = k_B\ln(2)$ after the collapse of the wave function, and that the free energy of the system increased by a quantity $k_BT\ln(2)$. Interpret the value of ΔS given Boltzmann's formula $S = k_B\ln(\Omega)$, where Ω can be interpreted as the number of microstates (*i.e.* possible configurations of the system) associated to a macrostate of the system.

Solution: *We obtain directly that $S_L - S_\infty = -k_B\ln(2)$. This corresponds to the decrease of the entropy of the system by 1 bit of information. Using Boltzmann's formula, one sees that the number of possible states has been divided by 2 (before measurement, the particle was either in the left of the right side, while after measurement, the particle can only be in one of the sides), giving the value of ΔS obtained.*
Calculating $F_L - F_\infty$ gives directly the awaited result. This surplus of energy can be extracted as useful work by the demon. After the projection of the quantum system onto one of the sides, one can prove that the barrier is subject to a force oriented towards the opposite side, which

can be extracted as useful work. The demon seems to extract work from the heat bath, the Szilárd machine thus acting as a monothermal heat machine (forbidden by the Second law of thermodynamics)!

In order to solve this problem, we would have to actually take into account the demon in the description of the system. One can show that a demon cannot operate the machine for more than one cycle, unless the demon is “reset” (which would then require at least as much free energy as the amount extracted during one cycle of the machine).