SciPost

Proceedings of the 34th International Colloquium on Group Theoretical Methods in Physics

Group

ICGTMP

Editors: Prof. Rutwig Campoamor-Stursberg
Prof. Michel Rausch de Traubenberg
Prof. Mauricio Valenzuela
Prof. Marc de Montigny

Contents

1 The 34th International Colloquium on Group Theoretical Methods in Physics

2 Laudatio of Dr. Erik Panzer, 2020 Hermann Weyl Prize

3 The acceptance speech of the 2022 Weyl-Wigner award

4 Dark matter as a QCD effect in an anti de Sitter geometry: Cosmogonic implications of de Sitter, anti de Sitter and Poincaré symmetries

5 The prediction of anyons: Its history and wider implications

6 On the unexpected fate of scientific ideas: An archeology of the Carroll group

7 Emergent symmetries in atomic nuclei: Probing nuclear dynamics and physics beyond the standard model

8 Integrability for Feynman integrals

9 Jordan meets Freudenthal. A black hole exceptional story

10 Kurt Bernardo Wolf memorial lecture

11 Tchavdar Dimitrov Palev - In memoriam

12 Two entangled and scientifically impactful lives: Jiří Patera, Pavel Winternitz and the Montréal school of mathematical physics

13 Remembering David J. Rowe

14 Comments on the negative grade KdV hierarchy

15 The impact of the diffusion parameter on the passage time of the folding process

16 Irreducible representations of $\mathbb{Z}_2^2$-graded supersymmetry algebra and their applications

17 A general approach to noncommutative spaces from Poisson homogeneous spaces: Applications to (A)dS and Poincaré

18 Computation of entanglement entropy in inhomogeneous free fermions chains by algebraic Bethe ansatz

19 Renormalizable extension of the Abelian Higgs-Kibble model with a dim.6 derivative operator

20 The tower of Kontsevich deformations for Nambu-Poisson structures on $\mathbb{R}^d$: Dimension-specific micro-graph calculus

21 Hilbert space structure and classical limit of the low energy sector of U(N) quantum Hall ferromagnets

22 Generalisation of affine Lie algebras on compact real manifolds

23 Generalized Heisenberg-Weyl groups and Hermite functions

24 Newton mechanics, Galilean relativity, and special relativity in α-deformed binary operation setting

25 On old relations of Lie theory, classical geometry and gauge theory

26 Linking ladder operators for the Rosen-Morse and Pöschl-Teller systems

27 New perspectives in gravity-mediated supersymmetry breaking

28 An algebraic approach to intertwined quantum phase transitions in the Zr isotopes

29 Study of entanglement in symmetric multi-quDits systems through information diagrams

30 Hyperquaternions and physics

31 Group invariants for Feynman diagrams

32 Quantum cylindrical integrability in magnetic fields

33 SO(1,3) Yang-Mills solutions on Minkowski space via cosets

34 Spin degrees of freedom incorporated in conformal group: Introduction of an intrinsic momentum operator

35 Vinberg's T-algebras: From exceptional periodicity to black hole entropy

36 Mixed permutation symmetry quantum phase transitions of critical three-level atom models

37 Relativistic kinematics in flat and curved space-times

38 Unitary Howe dualities in fermionic and bosonic algebras and related Dirac operators

39 Wigner function analysis of finite matter-radiation systems

40 Group theoretical derivation of consistent particle theories

41 On Möbius gyrogroup and Möbius gyrovector space

42 Clocking mechanism from a minimal spinning particle model

43 Segal's contractions, AdS and conformal groups

44 Towards higher super-σ-model categories

45 Construction of matryoshka nested indecomposable N-replications of Kac-modules of quasi-reductive Lie superalgebras, including the sl(m/n) and osp(2/2n) series

46 The parastatistics of braided Majorana fermions

47 On the spin content of the classical massless Rarita-Schwinger system

48 Extensions of realisations for low-dimensional Lie algebras

The 34th International Colloquium on Group Theoretical Methods in Physics

Rutwig Campoamor-Stursberg[1], Marc de Montigny[2], Michel Rausch de Traubenberg[3⋆] and Mauricio Valenzuela[4]

1 Instituto de Matemática Interdisciplinar and
Dpto. Geometría y Topología, UCM, E-28040 Madrid, Spain
2 Faculté Saint-Jean, University of Alberta,
8406 91 Street, Edmonton, Alberta T6B 0M9, Canada
3 Université de Strasbourg, CNRS, IPHC UMR7178, F-67037 Strasbourg Cedex, France
4 Centro de Estudios Científicos (CECs), Av. Arturo Prat 514, Valdivia, Chile

⋆ michel.rausch@iphc.cnrs.fr

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Introduction to the proceedings of the 34th International Colloquium on Group Theoretical Methods in Physics.

Received 25-10-2023
Accepted 01-11-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.001

Contents

1 Preface 2

2 Description of the Proceedings 4

3 Committees 5

4 Acknowledgment 7

1 Preface

The 34$^{\text{th}}$ International Colloquium on Group Theoretical Methods in Physics ("Group 34" in short) took place in Strasbourg, France, from June 18 to June 22, 2022. The ICGTMP Series, which started in Marseille (France) in 1972, organized by H. Bacry and A. Janner, has developed since then to a regular meeting that reunites mathematicians and physicists sharing the same research interests. In this context, Group 34 was a milestone, as the 50th anniversary of the conference series was commemorated. In addition to the regular schedule, two special ceremonies and some lectures on group theory for PhD students were included in the program. After a four year interruption, the ICGTMP has been reactivated, with the novelty of being organised in a hybrid modality, with both the plenary lectures and the parallel sessions being accessible online.

The ICGTMP brings together physicists, mathematicians and other scientists working on related disciplines, but primarily using techniques associated to "group theoretical and geometric methods". This Colloquium has been shown to be an optimal forum for researchers to acquire knowledge on current developments as well as to be informed about applications to other domains. Historical information about the Series and its evolution are at the disposal of the public at the homepage of ICGTMP: https://icgtmp.blogs.uva.es/.

The conference was hosted by the University of Strasbourg, involving in the organisation the various research institutes devoted to mathematics and physics, such as the Institut de Physique et Chimie des Matériaux de Strasbourg (IPCMS), Institut Pluridisciplinaire Hubert Curien (IPHC), Institut Charles Sadron (ICS) and Institut de Recherche Mathématique Avancée (IRMA). The Group 34 Colloquium was located in different areas of Strasbourg University. The plenary sessions and the *In Memoriam* ceremonies took place at the Faculté de droit (Law School), while the parallel sessions, the two special ceremonies and the group theory lectures took place at the Faculté de physique et ingénierie. In the first special ceremony, a conference in French targeted to the general public was offered, while the second special ceremony, for the participants of the conference, was held in the evening, just before the Award ceremony. The conference in French, intended for a broad audience, was recorded, with the video being publicly accessible on the conference's website. The Wigner-Weyl and Hermann Weyl Prize Award Ceremonies were organised by the Strasbourg City Hall in its historical location.

The Strasbourg University, located in the central European city of Strasbourg, has a long tradition and experience in the organisation of scientific conferences. It should be mentioned that both the sanitary and geopolitical situation prevented some of the regular participants to attend this year, in spite of which the colloquium can be considered to have been very successful, with more than one hundred and forty participants coming from twenty-four different countries from Europe, Asia, Australia, North and South America, as well as Africa. All the inhabited continents were thus represented, which shows the importance of the scientific topics covered and the worldwide reputation that this series has gained among specialists. It should not remain unmentioned that the success of the colloquium was made possible by the grants offered by various international institutions, that we acknowledge in the following: International Association of Mathematical Physics (IAMP); International Union of Pure and Applied Physics (IUPAP); Multidisciplinary Digital Publishing Institute (Open Access Journal Symmetry); Theoretical Physics Institute, University of Alberta, Canada, as well as by grants of French Institutions: Doctoral School of Physics and Chemical Physics (ED182, Unistra); Quantum Science and Nanomaterials (QMat, Unistra); Institut de Physique et de Chimie des Matériaux de Strasbourg (IPCMS, Unistra); Institut Pluridisciplinaire Hubert Curien (IPHC, Unistra); Institut de Recherche Mathématique Avancée (IRMA, Unistra); Institut National de Physique Nucléaire et de Physique des Particules (IN2P3, CNRS); Mathématiques, Interactions et Applications (IRMIA++, Unistra), the University of Strasbourg (Unistra) and the Mairie de Strasbourg.

The scientific program of the conference was quite dense, consisting of twelve plenary talks, eighty parallel sessions, three poster sessions, three special ceremony lectures (two in English and one in French), twelve group theory lectures, two laudatory speeches in honor of the Wigner-Weyl Awardee and the Weyl Prize winner, respectively, as well as five *In Memoriam* talks. These honored our reputed colleagues David J. Rowe, Tchavdar Palev, Jiri Patera, Pavel Winternitz and Kurt Bernardo Wolf, who have recently passed away.

The Inauguration session of the Colloquium consisted of the welcome speeches, given by the Chairman of the ICGTMP Standing Committee, Mariano del Olmo, and Sandrine Courtin, head of the Institut Pluridisciplinaire Hubert Curien.

A remarkable innovation of Group 34, in contrast to the previous Colloquia, was the organisation of group theory lectures for Master and Ph.D. students, with the intention of promoting the active involvement of the new generation. It was also the first time where the conference combined both in-person and remote participation. Both the plenary and parallel sessions were offered online, with about 20 participants attending virtually. Two of the plenary lectures and seventeen of the parallel sessions were presented online.

The nowadays traditional Wigner-Weyl Award and Weyl Prize were held in the historically relevant building of the Strasbourg City Hall. The Wigner-Weyl Award recognises and awards outstanding contributions based on group theoretical and representation methods. The Selection Committee, chaired by Efim Zelmanov, awarded the seminal contributions of Nikolai Reshetikhin to Quantum Field Theory, as well as quantum groups and integrable systems applied to problems of statistical mechanics with the 2022 Wigner-Weyl Award. On the other hand, the Weyl Prize is conceived as a recognition for young scientists who have contributed significant scientific work to the area of physical phenomena through the use of symmetries. The Selection Committee, chaired by María Antonia Lledó, awarded the Hermann Weyl prize for 2020/22 to Erik Panzer for his outstanding achievements in the calculation of amplitudes in gauge theories and for the development of new mathematical techniques based on the notion of symmetry, as well as his description of relevant physical phenomena observed in Nature.

Michel Rausch de Traubenberg
Chairman of the Local Organising Committee

2 Description of the Proceedings

In this volume, we present most of the contributions that were presented as the 34$^{\text{th}}$ International Colloquium on Group Theoretical Methods in Physics, held on June 18–22, 2022 in Strasbourg, France.

Maintaining the multidisciplinary spirit of previous editions, a wide range of subjects of current research interest in mathematical and theoretical physics, as well as adjacent natural sciences, were covered in the conference. This variety is adequately reflected by the contents of the plenary sessions, whose objective was to provide a glimpse into the various disciplines and research lines considered in the conference.

During the colloquium, the two traditional 'Wigner-Weyl" and "Hermann Weyl" prizes were awarded. Nicolai Reshetikhin was awarded with the "Wigner-Weyl" medal, while Erik Panzer received the "Hermann Weyl prize". This volume opens with a short summary of the Laudatios and the research of the awardees.

In the first section, we present a selected number of the plenary talks. Contributions to this volume, which were limited to 15 printed pages, are organised alphabetically. All submissions, plenary and regular, were subjected to an independent refereeing process and editorial decisions.

The *In Memoriam* talks held to honor renowned colleagues can be found in the following section.

As for the regular talks, these can be found in the last section. Despite the fact that these talks were organised into subject-specific sessions during the conference, as well as into the various parallel sessions, we judged it convenient to organise these contributions alphabetically to simplify the localisation. As a general rule, these contributions were restricted to a maximum of 8 pages. However, some of the articles are longer, as they combine the contents of two different talks. In addition, a contribution to the Group 32 Conference, that had been omitted in those proceedings, has also been included in this volume.

The Editors
Rutwig Campamor-Stursberg, Marc de Montigny,
Michel Rausch de Traubenberg and Mauricio Valenzuela

3 Committees

Organising Committee

Michel Rausch de Traubenberg (Chair), IPHC/Strasbourg U.
Rutwig Campoamor-Strusberg, Complutense, Madrid.
Marc de Montigny, University of Alberta, Edmonton.
Idrish Huet, FCFM/UNACH, Mexico.
Kamila Sieja, IPHC, Strasbourg U.
Marcus Slupinski, IRMA/Strasbourg U.
Fabrice Thalmann, ICS/Strasbourg U
Mauricio Valenzuela, Centro de Estudios Científicos (CECs), Valdivia.
Joris Van der Jeugt, Ghent University.
Guillaume Weick, IPCMS/Strasbourg U.

The ICGTNP Standing Committee

Naruhiko Aizawa (Osaka Prefecture Univ., Japan)
Chengming Bai (Nankai Univ., Tianjin, China)
Cestmir Burdik (Czech Technical Univ., Prague, Czech Rep.)
Octavio Castaños (ICN, UNAM, Mexico D.F., Mexico)
Enrico Celeghini (INFN, Firenze Univ., Italy)
Giulio Chiribella (Oxford Univ., UK)
Viktor Dodonov (University of Brasilia, Brazil)
Heinz Dietrich Doebner (Honorary Member, Clausthal Univ., Germany)
Edward Frenkel (Univ. California, Berkeley, USA)
Jean-Pierre Gazeau (Honorary Member, Paris Diderot Univ., France)
Piotr Kielanowski (Cinvestav, Mexico D.F., Mexico)
Ian Marquette (University of Queensland, Australia)
Mariano del Olmo (Chairman, Valladolid Univ., Spain)
Michel Rausch de Traubenberg (IPHC, Strasbourg Univ., France)
Christoph Schweigert (Hamburg Univ., Germany)
Nedialka Ilieva Stoilova (INRNE, Bulgarian Acad. Sc., Bulgaria)
Anne Taormina (Durham Univ., UK)
Francesco Toppan (CBPF, Rio de Janeiro, Brazil)
Piet Van Isacker (GANIL, Caen, France)
Luc Vinet (Univ. de Montréal, Canada)
Apostol Vourdas (Bradford Univ., UK)
Katrin Wendland (Freiburg Univ., Germany)

The International Advisory Committee

Luigi Accardi (Rome)
Lars Brink (Gothenburg)
Sultan Catto (New York)
Sara Cruz y Cruz (Mexico)
Toka Diagana (Alabama)
Vladimir Dobrev (Sofia)
Michael Douglas (Stony Brook)
Charles Dunkl (Virginia)
Mama Foupouagnigni (Yaounde)

Gerald Goldin (Rutgers)
Jan Govaerts (Louvain)
Harald Grosse (Vienna)
Andrzej Horzela (Krakow)
Véronique Hussin (Montreal)
Alexey Isaev (Dubna)
Michio Jimbo (Tokyo)
Victor Kac (Cambridge)
Luis Kauffman (Chicago)
Richard Kerner (Paris)
Andrey Klimov (Guadalajara)
Tom Koonwinder (Amsterdam)
Şengül Kuru (Ankara)
Maria Antonia Lledó (Valencia)
Roberto de Mello Koch (Johannesburg)
Alexander Molev (Sydney)
Alexei Morozov (Moscow)
Zouhair Mouayn (Béni Mellal)
Werner Nahm (Dublin)
Hermann Nicolai (Postdam)
Anatol Odzijewicz (Bialystok)
George Pogosyan (Dubna, Guadalajara, Yerevan)
Andrei Slavnov (Moscow)
Alexander Sorin (Dubna)
Richard Szabo (Edinburgh)
Leon Takhtajan (Stony Brook)
Saliou Toure (Ivory Coast)
Piergulio Tempesta (Madrid)

The “cellule Congrès-Formation” of Strasbourg University

Caroline Jaclot
Marion Oswald

4 Acknowledgment

The organisation of a Colloquium of this size would not have been possible without the involvement and active support of Strasbourg University and its various institutes of physics and mathematics. An important part of the administrative part of the organisation was very efficiently executed by the “cellule Congrès-Formation” of Strasbourg University. The Local Organising Committee is in debt to Caroline Jaclot and Marion Oswald from the “cellule Congrès-Formation”, for benefiting from their advice, ability to organise, and availability.

The video recording of the French conference has been performed by Fanny Roubineau and David Surmin, whom we gratefully acknowledge.

The Mayor of Strasbourg, Mme Jeanne Barseghian and all her team, in particular Mesdames Chantal Kroffig and Christine Malecot, are acknowledged for making it possible to hold the Award Ceremony at the Hôtel de Ville de Strasbourg.

The faculty of Physics and Engineering and the Law School of Strasbourg University are kindly acknowledged for providing amphitheatres and rooms for all sessions of the conference free of charge.

The Local Organising Committee

SciPost Phys. Proc. 14, 002 (2023)

Laudatio of Dr. Erik Panzer, 2020 Hermann Weyl Prize

María Antonia Lledó Barrena⋆

Departament de Física Teòrica, Universitat de València and IFIC (CSIC-UVEG)
C/ Dr. Moliner, 50. 46100 Burjassot, SPAIN

⋆ maria.lledo@ific.uv.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Erik Panzer, from the University of Oxford, has been awarded the 2020 Hermann Weyl Prize of the International Colloquium on Group Theoretical Methods in Physics, for "his pioneering achievements in the calculation of amplitudes in gauge theories, for developing new mathematical structures that exploit the language of symmetries, and for his contribution to the description of important physical phenomena present in nature."

Received 24-11-2022
Accepted 11-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.002

Dear Colleagues,

It is a pleasure and an honor for me to introduce here today to Dr. Erik Panzer, who has been awarded de 2020 Hermann Weyl Prize for his "pioneering achievements in the calculations of amplitudes in gauge theories, for developing new mathematical structures that exploit the language of symmetries, and for his contribution to the description of important physical phenomena present in nature".

Erik is a brilliant mathematical physicist, working in the area of "Feynman amplitudes", which include quantum field theory, phenomenology related with the LHC experiments, string perturbation theory, algebraic geometry and number theory. It is only recently that the community started to realize that well posed mathematical questions are at the heart of the calculation of Feynman amplitudes. The area has been growing enormously in the last years and Erik, in spite of his youth, has become an indispensable reference in it.

His research is highly original and of exceptional quality. Besides, it is truly interdisciplinary, since he has made important contributions to both, mathematics and physics. He has generated entirely new problems in abstract mathematics which are of fundamental interest. On the other hand, he has succeeded in applying the most powerful tools in algebraic geometry to the solution of long standing problems in quantum field theory. These comprehend polylogarithms, iterated elliptic integrals, modular forms, K3 surfaces, Calabi-Yau manifolds etc. He has the rare skill of becoming a bridge between the two sectors of researchers interested in Feynman amplitudes, physicists and mathematicians.

I will talk only about a part of his work; perhaps the more representative.

Erik's earliest work, his doctoral thesis (2015), was groundbreaking and had a significant impact in the physics community. He brought the technique of parametric integration to a new level, showing rigorously how a certain class of Feynman integrals (previously inaccessible) are indeed multiple polylogarithms. This was a great progress in the analytic evaluation of Feynman periods, work that required a very careful study of the divergences of those functions. Moreover, his work has direct relevance in generating experimental predictions for the LHC. He even developed a versatile and efficient software for parametric integration, called "HyperInt", which is nowadays widely used in many applications. A large literature has emerged in this topic, and many claims on how to evaluate certain integrals are based in computational experimentation using Erik's software. In further work, he revealed Galois symmetries among the Feynman periods, in particular, he conjectured (with Schnetz) the possibility that the motivic periods of the Φ^4 theory are a comodule under the coaction of the Galois group. Multiple zeta values play an important role in the theory of periods and motives. This is one of the clearest examples of the "understanding physics through symmetries" principle, specially valued in the Weyl Prize.

In 1997 Kontsevich solved the long standing problem of the deformation quantization of Poisson manifolds. His formula is an expansion on polydifferential operators. Each operator is expressed in terms of a graph, weighted with certain universal coefficients, that is, coefficients that do not depend on the Poisson bracket. These are defined as integrals over configurations of points in the upper half-plane. The underlying techniques make heavy use of the theory of multiple polylogarithms on the moduli space of marked, genus 0 curves. Cataneo and Felder showed that these integrals correspond to the Feynman amplitudes of a topological string theory. Kontsevich conjectured that these integrals correspond indeed to integer linear combinations of multiple zeta values. Erik and collaborators (Banks and Pym) remarkably showed that this is indeed the case and that, appropriately normalized, they are integer numbers. Moreover, their proof included an algorithm to compute them and they created the first software for their symbolic calculation. This made, finally, Kontsevich's formula tractable, problem that had been standing for about 20 years. New avenues are opened for future research in this direction.

Erik's more recent work on tropical quantum field theory is the completely new study of the Feynman integral in terms of the Hep bound. The Hep bound is a simplified field theory that shares many symmetry properties with the original Feynmann integral, so it contains qualitative information about it. For example, it has the same asymptotic behavior than the original series. Another remarkable property of the perturbation series of the Hep bound is that it can actually be evaluated by numerical methods at large loop orders (Φ^4 theory). Being a bound, it is not the best approximation (it may be two orders of magnitude difference), but it correlates with the Feynmann integral in such a way that one can use it to predict numerically its value to a great degree of accuracy. In this way one can study properties of the summation of the original perturbation series, of which very few terms are exactly known. Moreover, Erik has advanced the radical conjecture that two Feynman integrals are equal if and only if their Hep bounds are equal. This has received numerical evidence to a large order of loops and then used to make interesting predictions at larger order. It implies the existence of a symmetry of the Feynman integrals that has eluded us so far.

Another of Erik's achievements was on the noncommutative Φ^4 theory in two dimensions, were he was able to resum a perturbation series and solve the nonlinear Dyson-Schwinger equation analytically, in terms of the Lambert W function. This is a remarkable achievement, which makes us hope to solve it in four dimensions.

Together with Bitoun, Bogner and Klaussen he studied the master integrals, that is, the Feynmann integrals that remain after applying the integration by parts procedure to reduce them. In this work, the number of such integrals is defined unambiguously as the dimension of a certain vector space. The amazing result of this study is that the number of master integrals is the Euler characteristic of the complement of a hypersurface, defined by a polynomial associated to each graph. This was proven using the theory of algebraic D-modules, and it is of central importance in its application to phenomenology problems.

Since he received the award, he has been very active and each new paper of his has been received with great expectation. In all his works, Erik reveals himself as a truly original and outstanding researcher, operating in the boundary between mathematical and theoretical physics. He works in different collaborations rather easily and he has made of the symmetries underlying Feynman integrals the leitmotiv of his work. This is particularly in accordance with the spirit of the Weyl Prize. As you have seen he is also an extraordinary speaker, of extreme clarity. This shows also in the big amount of seminars and conferences that he has given all around the world.

This year we had very competitive candidates for the Prize, but at the end he was elected unanimously by the Committee, who considered him as the most deserving recipient of the Prize. It is for all this that I invite you now to join me in recognizing the effort, intelligence and achievements of this young researcher by receiving him with a big applause.

Thank you very much.

The acceptance speech of the 2022 Weyl-Wigner award

Nicolai Reshetikhin⋆

YMSC, Tsinghua University, Beijing, China
Department of Mathematics, University of California, Berkeley, USA

⋆ reshetik@math.berkeley.edu

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

This is the acceptance speech for the 2022 Weyl-Wigner award.

Received 05-01-2023
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.003

Two books greatly influenced me in my early student years: the book by Herman Weyl "Symmetry" (1952, Princeton University Press) and the book by E. Wigner "Symmetry and Reflections" (1967, Indiana University Press). The notion of symmetry became the guideline in my studies and research. It adds greatly to the pleasure of receiving the 2022 Wigner-Weyl award.

When I learned that I was selected as a recipient of the Wigner-Weyl award, my first reaction was "Wow, it happened!". Finding myself continuing the list of imminent scientists who were awarded the Wigner medal was extremely rewarding.

To those who nominated me, who supported the nomination : thank you for your trust and confidence. To all my teachers: thank you!

This is a good time to reflect on my research career and on wonderful people whom I met, from whom I learned and with whom I worked with during all these years.

I was very lucky to meet many outstanding physicists and mathematicians. My studies at the Leningrad Polytechnic Institute in the nuclear engineering group. During the first year I realized that designing anti-radiation shields is infinitely far away from my interests in mathematics and physics. But this is where I met an outstanding teacher who became life long friend, Prof. S.P. Preobrazhensky. By the stroke of luck, as a sophomore, I came to a graduate course on path integrals given by V.N. Popov. This is how I got to Steklov mathematical institute and to Faddev's seminar. This seminar was an intellectual haven.

My first research success was the hierarchical algebraic Bethe ansatz construction that came out from a suggestion of P. Kulish to construct an algebraic version of Bethe ansatz vectors found in the work of C.N.Yang about for non-identical particles with δ-interaction in one dimensional space. The use of Schul-Weyl duality was essential. This work was my official arrival to the world of symmetries and integrable systems.

I was very lucky to start my research with V.N. Popov and P.P. Kulish from whom I learned a great deal. Soon I ended up working at the Laboratory of mathematical methods of theoretical physics headed by L. D. Faddeev who greatly influenced my research. One of his mottos was that one should do interesting mathematical physics, mathematics and physics. The other was to move on to new subjects, new ideas and not to get stuck in one theme.

During my earlier years at the Lab, I benefited greatly from working with my colleagues and friends, among whom are A. Kirillov, E.Sklyanin, M. Semenov-Tian-Shanski, F. Smirnov and L. Takhtajan.

In early 1980's I had an important collaboration with P. Wiegmann on principal chiral field theories. I got to know many people from Landau Institute in Chernogolovka which was an another intellectual gift. About the same time L.D. Faddeev and I worked on different aspects of the same chiral field model from a different perspective. This collaboration with L.D. continued later with the work on quantum groups. About the same time I was visiting A.M. Vershik seminar on representation theory which I started to appreciate more and more.

With the emergence of quantum groups V.G. Drinfeld came to Leningrad a number of times. Discussions with him were a treasure. About the same time, late-mid 1980's the collaboration with V. Turaev developed and brought some important results about invariants of knots and 3-manifolds. At the same time we wrote an important paper of quantum groups and solutions to the Yang-Baxter equation with L.D.Faddeev and L.A.Takhtajan. There many other important results, but now is not the time to discuss them.

Throughout Leningrad years contacts with researchers around the world were extremely important, it was done by mail and through discussions when international scientists were visiting Leningrad. Among. these people are M.Jimbo, T.Miwa, L. Kauffman, D. Gross and many others. I am still grateful to them and to others who are able distinguish people of a country from its government and science from politics.

In 1989 I moved from LOMI to Harvard and the whole new world of mathematics and physics opened up for me. There I met D. Kazhdan who seriously influenced my perception of mathematics. Among other first class mathematicians and physicists whom I met there and who affected my vision of mathematics and theoretical physics were J. Bernstein, R.Bott, V.Kac, I.Singer, C.Taubes, C.Vafa, E.Witten, S.T.Yau. The complete list would be too long. It was an intellectual feast. The invitation to Harvard came from A. Jaffe (via A. Beilinson) with whom we had many discussions about a "geometrization" of the constructive field theory program. I still think it is a potentially great direction which may eventually give a better mathematical understanding of path integrals in quantum field theory. E. Frenkel and B. Tsygan came to Harvard at the same time and became close friends. With Ed Frenkel we had a very productive collaboration later on. During my stay at Harvard we developed a wonderful collaboration with I. Frenkel which resulted in q-Knizhnik-Zamolodchikov equations.

In 1991 I moved from Harvard to Berkeley where I worked for 30 happy years. One of the great things about Berkeley was friendship with V. Jones. Another treasure was talking to R. Kirby. In Berkeley I had many enlightening discussions with E. Frenkel, A. Grunbaum, A. Okounkov, M. Rieffel, V. Serganova, A. Weinstein, and many others. Special thanks to my graduate students for keeping me in a good shape. I have great memories of all of them. Many of them became prominent mathematicians. In physics I was very lucky to have many discussions with B. Zumino.

In 2022 I retired from Berkeley and moved to Yau Mathematical Sciences Center at Tsinghua University in Beijing, starting a new fascinating chapter in my life.

Throughout all these years I was very lucky to work with many excellent people such as A. Cattaneo, C. De Concini, P. Mnev, C. Procesi, M. Rosso, and many others, again, the complete list would be too long.

My favorie research themes are:

- Integrable systems, Lie groups, Lie algebras.
- Quantum groups, their applications to integrable systems and topological invariants.
- Quantization of gauge theories.
- Statistical mechanics, integrable models.

My main goal is to retain optimism about humanity to continue with research, and with sharing the knowledge that I accumulated over the years.

Finally, special thanks to my family for continuing support over all these years.

Dark matter as a QCD effect in an anti de Sitter geometry: Cosmogonic implications of de Sitter, anti de Sitter and Poincaré symmetries

Gilles Cohen-Tanoudji[1] and Jean-Pierre Gazeau[2⋆]

1 Laboratoire de recherche sur les sciences de la matière, LARSIM CEA, Université Paris-Saclay, F-91190 Saint-Aubin, France
2 Université Paris Cité, CNRS, Astroparticule et Cosmologie, F-75013 Paris, France

⋆ gazeau@apc.in2p3.fr

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The ΛCDM standard model of cosmology involves two dark components of the universe, dark energy and dark matter. Whereas dark energy is usually associated with the (positive) cosmological constant Λ associated with a de Sitter geometry, we propose to explain dark matter as a pure QCD effect, namely a gluonic Bose Einstein condensate with the status of a *Cosmic Gluonic Background (CGB)*. This effect is due to the trace anomaly viewed as an effective negative cosmological constant determining an Anti de Sitter geometry and accompanying baryonic matter at the hadronization transition from the quark gluon plasma phase to the colorless hadronic phase. Our approach also allows to assume a ratio Dark/Visible equal to 11/2.

Received 15-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.004

Contents

1 Introduction **2**

2 Three maximal symmetries, Poincaré, dS, AdS **3**
2.1 The place of the cosmological constant 3
2.2 Two unique deformations of Poincaré symmetry 3
2.3 de Sitter and anti-de-Sitter Geometries 5
2.4 Compared classifications of Poincaré, dS and AdS UIR's for quantum elementary systems 5
2.5 Proper mass versus "at rest" energy in de Sitter and anti-de-Sitter 6

3 Dark matter from QCD: A relic of quark period **8**
3.1 Cosmology chronology: The salient stages 8

3.2 Quark-gluon plasma: Experimental evidence 9
3.3 Quark-gluon plasma and effective AdS geometry 10
3.4 Cold dark matter: Bose-Einstein condensation of (di-)gluons in effective anti-de Sitter geometry 11

4 Conclusion 12

A Complements: Facts of ΛCDM standard model 12

References 14

1 Introduction

Let us start out this group theoretical oriented contribution with three motivating quotes. The first one from Newton and Wigner (1949) [1] is about the concept of elementary system.

> *The concept of an "elementary system" requires that all states of the system be obtainable from the relativistic transforms of any state by superpositions. In other words, there must be no relativistically invariant distinction between the various states of the system which would allow for the principle of superposition. This condition is often referred to as irreducibility condition ...*
> *The concept of an elementary system (...) is a description of a set of states which forms, in mathematical language, an irreducible representation space for the inhomogeneous Lorentz (≃ Poincaré) group*

The second one from Fronsdal (1965) [2] is about curvature versus flatness of space-time.

> *A physical theory that treats spacetime as Minkowskian flat must be obtainable as a well-defined limit of a more general physical theory, for which the assumption of flatness is not essential.*

The third one from Sakharov [3], quoted by Adler in [4].

> *The presence of the action*

$$S_{\text{grav}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}(R-2\Lambda)\,, \tag{1}$$

> *leads to a metrical elasticity of space, i.e., to generalized forces which oppose the curving of space. Here we consider the hypothesis which identifies the action (1) with the change in the action of quantum fluctuations of the vacuum if space is curved.*

These statements are the leitmotiv guiding our interpretation of dark matter, as it will be exposed in the sequel. Section 2 is devoted to the description of three fundamental space-time symmetries, Poincaré group and its two deformations, de Sitter (dS) and Anti de Sitter (AdS) groups, and their respective significance in terms of invariants, spin, mass, and "energy at rest". Cosmology chronology is put in perspective in Section 3 with regard to our interpretation [5,6] of the dark matter as a gluonic Bose-Einstein condensate emerging at the end of the so-called quark period (see also [7–9] about the genesis of our common work). Following the short conclusion (Section 4), we give in Appendix A some insight in relation with the ΛCDM standard model.

2 Three maximal symmetries, Poincaré, dS, AdS

2.1 The place of the cosmological constant

Firstly let us observe that there exist two standpoints about the Einstein equation of general relativity [10]:

- **Standpoint 1**

$$\underbrace{\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\, g_{\mu\nu}}_{\text{geometrical content}} = \underbrace{-\kappa\, T_{\mu\nu} - \Lambda\, g_{\mu\nu}}_{\text{matter content}}, \qquad \kappa = \frac{8\pi G}{c^4}. \tag{2}$$

  Here, the fundamental state that contains the maximum number of symmetries is the Minkowskian geometry, and the cosmological term $\Lambda g_{\mu\nu}$ may be interpreted as an extra pressure, named world matter by de Sitter in his debate with Einstein:

$$\Lambda > 0 \sim \text{“dark energy”}\ , \qquad \Lambda < 0 \sim \text{“dark matter”}\,?$$

- **Standpoint 2**

$$\underbrace{\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\, g_{\mu\nu} + \Lambda\, g_{\mu\nu}}_{\text{geometrical content}} = \underbrace{-\kappa\, T_{\mu\nu}}_{\text{matter content}}. \tag{3}$$

  Here, the fundamental states that contain the maximum number of symmetries are the de-Sitter (dS) ($\Lambda \equiv \Lambda_{\mathrm{dS}} > 0$) and the Anti-de-Sitter (AdS) ($\Lambda \equiv \Lambda_{\mathrm{AdS}} < 0$) geometries.

Note that the split between these two standpoints should not be considered as absolute, since we could as well model situations in a mixed way:

- **Standpoint 3**

$$\underbrace{\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\, g_{\mu\nu} + \Lambda_L\, g_{\mu\nu}}_{\text{geometrical content}} = \underbrace{-\kappa\, T_{\mu\nu} - \Lambda_R\, g_{\mu\nu}}_{\text{matter content}}. \tag{4}$$

2.2 Two unique deformations of Poincaré symmetry

From the above di-or tri-lemna let us give present some points in favor of dS/AdS studies

- dS and AdS are *maximally symmetric* (remind that in a metric space of dimension n, the maximum number of metric preserving symmetries is $n(n+1)/2$, here 10 since $n=4$).
- Their symmetries are one-parameter deformations of Minkowskian symmetry with
  - negative curvature $-\varkappa_{\mathrm{dS}} = -\sqrt{\Lambda_{\mathrm{dS}}/3}$ ($= -H/c$, H: Hubble parameter)
  - positive curvature $\varkappa_{\mathrm{AdS}} = \sqrt{|\Lambda_{\mathrm{AdS}}|/3}$

  respectively
- As soon as a constant curvature is present, we lose some of our so familiar conservation laws like energy-momentum conservation!
- Then what is the physical meaning of a scattering experiment (“space” in dS is like the sphere $\mathbb{S}^3$, let alone the fact that time is ambiguous)?

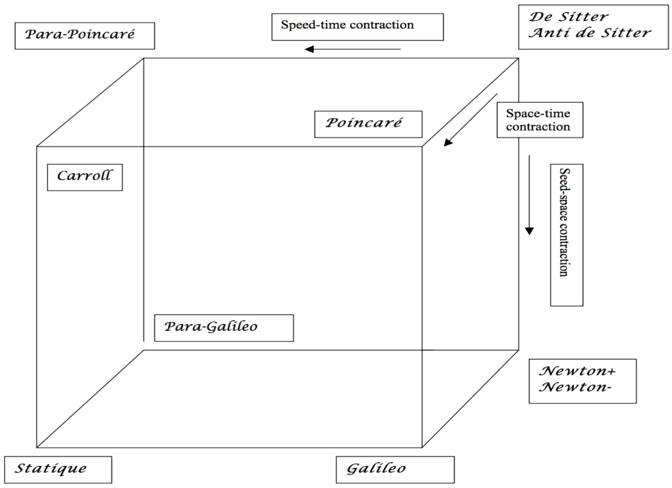


Figure 1: The eleven kinematics (Bacry & Levy-Leblond, JMP (1968)). From [12].

- Which relevant "physical" quantities are going to be considered as (asymptotically? contractively?) experimentally available?

In addition to the previous observations, we should insist on the fact that dS and AdS symmetries are the two unique deformations of the Poincaré symmetry. They occupy the extreme vertex of the cubic diagram in Figure 1 showing the eleven kinematics classified by Bacry & Levy-Leblond (1968) [11]. More precisely, under the assumptions that space is isotropic (rotation invariance), parity and time-reversal are automorphisms of the kinematical groups, and inertial transformations in any given direction form a noncompact subgroup, then there are eight types of Lie algebras for kinematical groups corresponding to eleven possible kinematics. These algebras are [11]:

*R*1 The two de Sitter Lie algebras isomorphic, respectively, to the Lie algebras of SO(4,1) and SO(3,2);

*R*2 The Poincaré Lie algebra;

*R*3 Two "para-Poincaré" Lie algebras, of which one is isomorphic to the ordinary Poincaré Lie algebra but physically different and the other is the Lie algebra of an inhomogeneous SO(4) group;

*R*4 The Carroll Lie algebra;

*A*1 The two "nonrelativistic cosmological" Lie algebras;

*A*2 The Galilei Lie algebra;

*A*3 The "para-Galilei" Lie algebra;

*A*4 The "static" Lie algebra.

While the Lie algebras of class *R* have no nontrivial central extensions by a one-parameter Lie algebra, those of class *A* each have one class of such extensions. Hence, with the requirements of kinematical rotation, parity, and time-reversal invariance, there exists only one way to deform the proper orthochronous Poincaré group $\mathbb{R}^{1,3} \rtimes \mathrm{SO}_0(1,3)$ (or $\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2,\mathbb{C})$), namely,

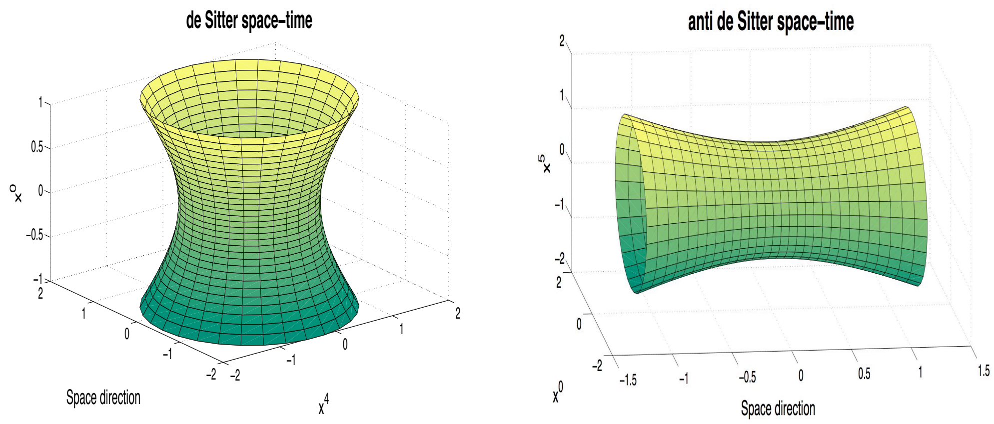


Figure 2: Left: the one-sheeted de Sitter hyperboloid as a manifold embedded in the $1+4$ Minkowski space-time. x^0 might be chosen as a time parameter, but there is no global time-like Killing vector. Right: the one-sheeted Anti de Sitter hyperboloid as a manifold embedded in the $2+3$ ambient space. Angular position along the central belt can be chosen as a local time coordinate, and it is in one-to-one correspondence with a global time-like Killing vector.

in endowing space-time with a certain curvature. This leads to the two simple Lie groups, namely the ten-parameter de Sitter group $SO_0(1,4)$ (or its universal covering $Sp(2,2)$) and the ten-parameter Anti de Sitter group $SO_0(2,3)$ (or its two-fold covering $Sp(4,\mathbb{R})$).

2.3 de Sitter and anti-de-Sitter Geometries

The de Sitter space may be viewed (on the left in Fig. 2) as a one-sheeted hyperboloid embedded in a five-dimensional Minkowski space with metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,-1)$ (but keep in mind that all points are physically equivalent):

$$M_{dS} \equiv \left\{ x \in \mathbb{R}^5;\ x^2 = \eta_{\alpha\beta}\ x^\alpha x^\beta = -\frac{3}{\Lambda_{\mathrm{dS}}} \right\}, \quad \alpha,\beta = 0,1,2,3,4\,. \tag{5}$$

The Anti de Sitter space may as well be viewed (on the right in Fig. 2) as a one-sheeted hyperboloid embedded in another five-dimensional space with metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1,1)$ (here too all points are physically equivalent):

$$M_{AdS} \equiv \left\{ x \in \mathbb{R}^5;\ x^2 = \eta_{\alpha\beta}\ x^\alpha x^\beta = \frac{3}{|\Lambda_{\mathrm{AdS}}|} \right\}, \quad \alpha,\beta = 0,1,2,3,5\,. \tag{6}$$

Note that the fifth dimension is space-like in dS whereas it is time-like in AdS.

2.4 Compared classifications of Poincaré, dS and AdS UIR's for quantum elementary systems

In a given unitary irreducible representation (UIR) of dS and AdS groups, ($\sim$ elementary system in Wigner's sense) their respective generators map to self-adjoint operators in Hilbert spaces of spinor-tensor valued fields on dS and AdS respectively:

$$\mathrm{K}_{\alpha\beta} \mapsto \mathrm{L}_{\alpha\beta} = \mathrm{M}_{\alpha\beta} + \mathrm{S}_{\alpha\beta}\,, \tag{7}$$

with orbital part $\mathrm{M}_{\alpha\beta} = -\mathrm{i}(x_\alpha\partial_\beta - x_\beta\partial_\alpha)$ and spinorial part $\mathrm{S}_{\alpha\beta}$ acting on the field components.

The physically relevant UIR's of the Poincaré, dS and AdS groups are denoted by $\mathcal{P}^{>}(m,s)$ (">" for positive energies), $U_{\mathrm{dS}}(\varsigma_{\mathrm{dS}},s)$, and $U_{\mathrm{AdS}}(\varsigma_{\mathrm{AdS}},s)$, respectively. These UIR's are specified by the spectral values $\langle\cdot\rangle$ of their quadratic and quartic Casimir operators. The latter define two invariants, the most basic ones being predicted by the relativity principle, namely proper mass m for Poincaré and ς_{dS}, ς_{AdS} for dS and AdS respectively, and spin s for the three cases (see [12] and references therein).

Poincaré

For Poincaré the Casimir operators are fixed as

$$\begin{aligned} Q^{(1)}_{\text{Poincaré}} &= P^\mu P_\mu = P^{0\,2} - \mathbf{P}^2 = m^2 c^2\,, \\ Q^{(2)}_{\text{Poincaré}} &= W^\mu W_\mu = -m^2 c^2 s(s+1)\hbar^2\,, \qquad W_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma\,. \end{aligned} \tag{8}$$

de Sitter

For de Sitter,

$$\begin{aligned} Q^{(1)}_{\mathrm{dS}} &= -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} = \varsigma^2_{\mathrm{dS}} - \left(s-\frac{1}{2}\right)^2 + 2 \equiv \langle Q^{(1)}_{\mathrm{dS}}\rangle\,, \\ Q^{(2)}_{\mathrm{dS}} &= -W_\alpha W^\alpha = \left(\varsigma^2_{\mathrm{dS}} + \frac{1}{4}\right) s(s+1)\,, \qquad W_\alpha := -\frac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}\,. \end{aligned} \tag{9}$$

Anti-de-Sitter

For anti-de Sitter,

$$\begin{aligned} Q^{(1)}_{\mathrm{AdS}} &= -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} = \varsigma_{\mathrm{AdS}}(\varsigma_{\mathrm{AdS}}-3) + s(s+1) \equiv \langle Q^{(1)}_{\mathrm{AdS}}\rangle\,, \\ Q^{(2)}_{\mathrm{AdS}} &= -W_\alpha W^\alpha = -(\varsigma_{\mathrm{AdS}}-1)(\varsigma_{\mathrm{AdS}}-2)s(s+1)\,, \qquad W_\alpha := -\frac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}\,. \end{aligned} \tag{10}$$

2.5 Proper mass versus "at rest" energy in de Sitter and anti-de-Sitter

While the proper mass is identified as the at rest energy, which means the energy spectrum infimum in Minkowski, these two quantities come apart in de Sitterian/anti-de Sitterian geometry. They have to be devised from a flat-limit viewpoint, *i.e.*, from the study of the contraction limit $\Lambda \to 0$ of these representations

Proper mass versus at rest energy in de Sitter: Garidi mass

In this respect, a mass formula for dS has been established by Garidi (2003) [13]:

$$m^2_{\mathrm{dS}} := \frac{\hbar^2 \Lambda_{\mathrm{dS}}}{3c^2}(\langle Q^{(1)}_{\mathrm{dS}}\rangle - 2) = \frac{\hbar^2 \Lambda_{\mathrm{dS}}}{3c^2}\left(\varsigma^2_{\mathrm{dS}} + \left(s-\frac{1}{2}\right)^2\right). \tag{11}$$

This definition should be understood through the contraction limit of representations:

$$\mathrm{dS\,UIR} \longrightarrow \text{Poincaré UIR}\,.$$

More precisely, with

$$\Lambda_{\mathrm{dS}} \to 0\,, \quad \varsigma_{\mathrm{dS}} \to \infty\,, \quad \text{while fixing} \quad \varsigma_{\mathrm{dS}}\hbar\sqrt{\Lambda_{\mathrm{dS}}}/\sqrt{3}c = m_{\text{Poincaré}} \equiv m\,, \tag{12}$$

we have

$$U_{\mathrm{dS}}(\varsigma_{\mathrm{dS}}, s) \underset{\substack{\Lambda_{\mathrm{dS}}\to 0\,,\,|\varsigma_{\mathrm{dS}}|\to\infty \\ |\varsigma_{\mathrm{dS}}|\sqrt{\Lambda_{\mathrm{dS}}}/\sqrt{3}=\frac{mc}{\hbar}}}{\longrightarrow} c_{>}\mathcal{P}^{>}(m,s)\oplus c_{<}\mathcal{P}^{<}(m,s)\,. \tag{13}$$

This result was proved in [14] and discussed in [15]. Note the breaking of dS irreducibility into a direct sum of two Poincaré UIR's with positive and negative energy respectively. To some extent the choice of the factors $c_<$, $c_>$, is left to a local tangent observer. The latter will naturally fix one of these factors to 1 and so the other one is forced to vanish. This crucial dS feature originates from the dS group symmetry mapping any point $(x^0, \mathsf{P}) \in \mathsf{H}_{\mathrm{dS}}$ into its mirror image $(x^0, -\mathsf{P}) \in \mathsf{H}_{\mathrm{dS}}$ with respect to the x^0-axis. Under such a symmetry the four dS generators L_{a0}, $a = 1,2,3,4$, (and particularly L_{40} which contracts to energy operator!) transform into their respective opposite $-\mathrm{L}_{a0}$, whereas the six L_{ab}'s remain unchanged. We think that the mathematical fact (13) should be carefully revisited with regard to the inflation scenario and the breaking of the matter-antimatter symmetry [16].

Proper mass versus at rest energy in Anti de Sitter

Concerning AdS a mass formula similar to that one for dS exists [10, 17]:

$$\begin{aligned} m^2_{\mathrm{AdS}} &= \frac{\hbar^2|\Lambda_{\mathrm{AdS}}|}{3c^2}\left(\langle Q^{(1)}_{\mathrm{AdS}}\rangle - \langle Q^{(1)}_{\mathrm{AdS}}|_{\varsigma_{\mathrm{AdS}}=s+1}\rangle\right) \\ &= \frac{\hbar^2|\Lambda_{\mathrm{AdS}}|}{3c^2}\left[\left(\varsigma_{\mathrm{AdS}}-\frac{3}{2}\right)^2-\left(s-\frac{1}{2}\right)^2\right]. \end{aligned} \tag{14}$$

One here deals with the AdS group representations $U_{\mathrm{AdS}}(\varsigma_{\mathrm{AdS}}, s)$ with $\varsigma_{\mathrm{AdS}} \geq s+1$ (discrete series and its lowest limit), and their contraction limit holds with no ambiguity:

$$U_{\mathrm{AdS}}(\varsigma_{\mathrm{AdS}}, s) \underset{\substack{\Lambda_{\mathrm{AdS}}\to 0\,,\,\varsigma_{\mathrm{AdS}}\to\infty \\ \varsigma_{\mathrm{AdS}}\sqrt{|\Lambda_{\mathrm{AdS}}|/3}=\frac{mc}{\hbar}}}{\longrightarrow} \mathcal{P}^{>}(m,s)\,. \tag{15}$$

Proper mass as an absolute invariant

Now, contraction formulae for both dS and AdS give us the freedom to write

$$m_{\mathrm{dS}} = m_{\mathrm{AdS}} = m\,.$$

This agrees with the Einstein position that the proper mass of an elementary system should be independent of the geometry of space-time, or equivalently it should not exist any difference between inertial and gravitational mass.

Rest energy of a free particle in AdS versus dS and Poincaré

Each Anti-deSitterian quantum elementary system (in the Wigner sense) has a discrete energy spectrum bounded below by its rest energy [18–20]

$$E^{\mathrm{rest}}_{\mathrm{AdS}} = \left[m^2c^4+\hbar^2c^2\frac{|\Lambda_{\mathrm{AdS}}|}{3}\left(s-\frac{1}{2}\right)^2\right]^{1/2}+\frac{3}{2}\hbar\sqrt{\frac{|\Lambda_{\mathrm{AdS}}|}{3}}\,c\,. \tag{16}$$

Hence, to the order of $\hbar$, a "massive" AdS elementary system is a deformation of both a relativistic free particle with rest energy mc^2 and a 3d isotropic quantum harmonic oscillator with ground state energy $3/2\hbar\sqrt{|\Lambda_{\mathrm{AdS}}|/3}\,c \equiv 3/2\hbar\omega_{\mathrm{AdS}}$ [21, 22].

In contrast to AdS, energy is ill-defined for dS. However a local tangent observer will naturally choose the invariant with positive sign:

$$E_{\mathrm{dS}}^{\mathrm{rest}} = \left[m^2 c^4 - \hbar^2 c^2 \frac{\Lambda_{\mathrm{dS}}}{3} \left(s - \frac{1}{2} \right)^2 \right]^{1/2} . \tag{17}$$

Noticeable simplification in both AdS and dS for **fermions** $s = 1/2$:

$$\text{for dS :} \quad E_{\mathrm{dS}}^{\mathrm{rest}} = mc^2 \,, \tag{18}$$

$$\text{for AdS:} \quad E_{\mathrm{AdS}}^{\mathrm{rest}} = mc^2 + \frac{3}{2}\hbar\omega_{\mathrm{AdS}} \,. \tag{19}$$

In the massless case and spin s, we have

$$\text{for dS :} \quad E_{\mathrm{dS}}^{\mathrm{rest}} = \pm i\hbar \sqrt{\frac{\Lambda_{\mathrm{dS}}}{3}} c \left(s - \frac{1}{2} \right) , \tag{20}$$

$$\text{for AdS:} \quad E_{\mathrm{AdS}}^{\mathrm{rest}} = \hbar \sqrt{\frac{|\Lambda_{\mathrm{AdS}}|}{3}} c(s+1) \,. \tag{21}$$

Therefore, while for dS the energy at rest makes sense only for massless fermionic systems and is just zero, for AdS the energy at rest makes sense for any spin, and in particular for spin 1 massless bosons we get

$$E_{\mathrm{AdS}}^{\mathrm{rest}} = 2\hbar\omega_{\mathrm{AdS}} \,, \tag{22}$$

and for scalar massless bosons

$$E_{\mathrm{AdS}}^{\mathrm{rest}} = \hbar\omega_{\mathrm{AdS}} \,. \tag{23}$$

3 Dark matter from QCD: A relic of quark period

We now explain the rôle of the above material in our interpretation of Dark Matter.

3.1 Cosmology chronology: The salient stages

Let us start out with the cosmology chronology depicted in Figures 3 and 4 (see for instance [25] for a comprehensive account of early cosmology versus particle physics). In Figure 4, the cosmic evolution is schematized on the thick line, on which the cosmic time, that is proportional to the logarithm of the scale factor, is made implicit, by replacing all dimensioned quantities depending on the local time t, by "effective co-moving densities" that are scaled by the scale factor depending on a global time.

In Figure 4 Greek letters represent noticeable events, to be understood as phase transitions for γ (electroweak symmetry breaking), δ (hadronization or color confinement), ϵ (dominance of matter over radiation), as Universe temperature ($\sim$ thermal time) is decreasing from the "Planck epoch" to ours. Futhermore, one should not omit the neutrino decoupling, lying between δ and ϵ, at a temperature $T \approx 1$ MeV, as shown in Figure 3 (electroweak phase transition). Now, the cosmic microwave background (CMB) is the relic of the photon decoupling, *i.e.*, when photons started to travel freely through space rather than constantly being scattered by electrons and protons in plasma. This represents a pure QED effect, and one of its outcome is precisely that we see or experience those photons. Similarly, the cosmic neutrino background (CNB) is the relic of the neutrino decoupling when the rate of weak interactions between neutrinos and other forms of matter dropped below the rate of expansion of the Universe, which produced a cosmic neutrino background of freely streaming neutrinos. In turn,

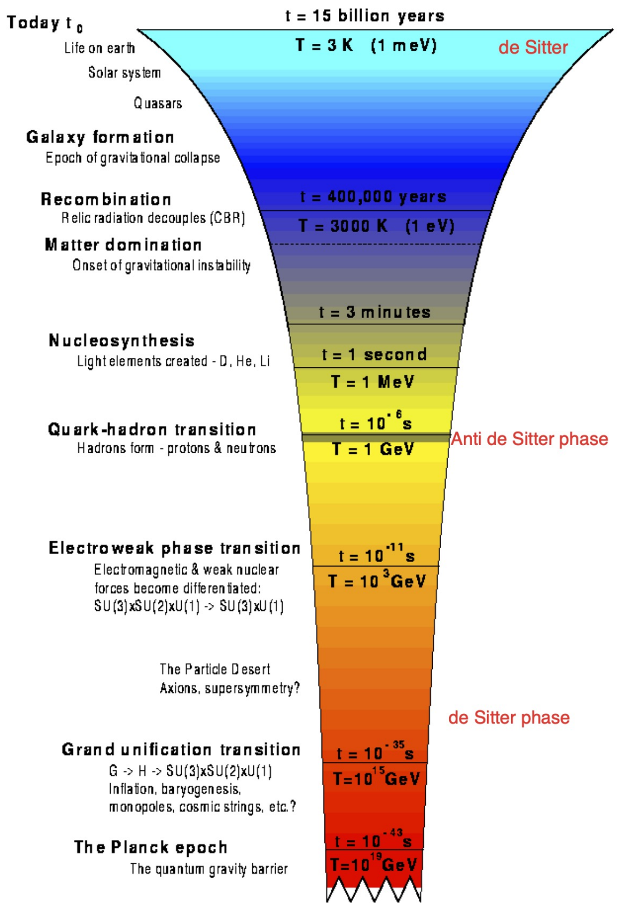


Figure 3: Cosmology chronology (from http://zebu.uoregon.edu/images/bb_history.gif).

this represents a pure electroweak effect. Our interpretation of Dark Matter is based on a similar scenario: gluonic component of the quark epoch (quark-gluon plasma) freely subsists after hadronization within an effective AdS environment. This represents a pure QCD effect, and we do not observe those gluons but we observe their gravitational effects. Hence, dark matter could be as well named cosmic gluonic background (CGB)... But let us tell more about dark matter.

According to the Planck 2018 analysis [26] of the CMB power spectrum, our Universe is spatially flat, accelerating, and composed of 5% baryonic matter, 27% cold dark matter (CDM, non baryonic) and 68% dark energy (Λ) [27]. (Cold) dark matter is observed by its gravitational influence on luminous, baryonic matter The dark matter mass halo and the total stellar mass are coupled through a function that varies smoothly with mass (with controversial exception(s)). One can notice that, up to now, all hypothetical particle models (WIMP, Axions, Neutrinos ...) failed direct or indirect detection tests. Similarly, alternative theories (e.g. MOND) for dark matter have failed to explain clusters and the observed pattern in the CMB, despite recurrent propitious announcements...

3.2 Quark-gluon plasma: Experimental evidence

The main physical ingredient of our interpretation [5] is the specific state of matter Quark-Gluon Plasma (QGP), *e.g.*, see Figure 5, characteristic of the Quark Epoch quark, *i.e.* from 10^{-12}s to 10^{-6}s, with temperature $T > 10^{12}$K (point δ in Figure 4). Theories predicting the existence of quark-gluon plasma were developed in the late 1970s and early 1980s (Satz,

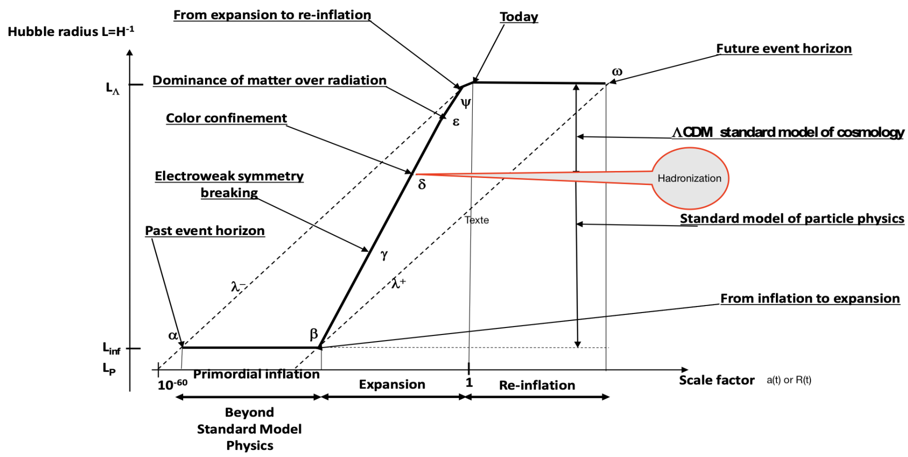


Figure 4: Cosmology chronology: Hubble radius $L(a) \equiv H^{-1}(a)$ ($c = 1$) is plotted versus the scale factor $a(t) \equiv R(t)$ in logarithmic scale (from [7]).

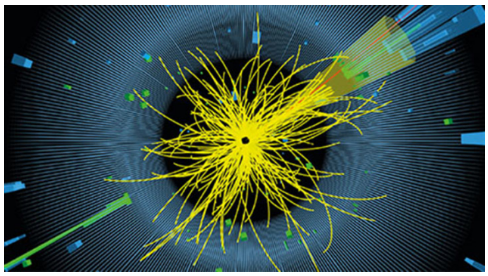


Figure 5: From Strong interactions News: *Protons probe quark-gluon plasma at CMS*, 13 January 2017.

Rafelsky, Kapusta, Müller, Letessier...), and the quark-gluon plasma was detected for the first time at CERN (2000). Lead and gold nuclei have been used for collisions yielding QGP at CERN SPS and BNL RHIC, respectively. The current estimate of the hadronization temperature for light quarks is $T_{cf} = 156.5 \pm 1.5\,\text{MeV} \approx 1.8 \times 10^{12}\,\text{K}$ ("chemical freeze-out temperature"). See for instance [28–30].

3.3 Quark-gluon plasma and effective AdS geometry

Our scenario [5] is that the colorless gluonic component (*e.g.*, digluons) of the quark epoch which freely subsists after hadronization within an effective AdS environment (QCD effect) is the dark matter. As a matter of fact the contribution of the so-called di-gluons through what is called by Adler [4] the *gluon pairing amplitude* to the QCD trace anomaly reads as

$$\left\langle T^{\mu}{}_{\mu} \right\rangle_0 = -\frac{1}{8}\left[11N_c - 2N_f\right]\left\langle \frac{\alpha_s}{\pi}\left(F^a_{\mu\nu}F^{a\mu\nu}\right)^r \right\rangle_0 , \tag{24}$$

where N_c is the number (=3) of colors, and N_f the effective number of quark flavors which was put at 3 as a first guess, but will rather be considered as an adjustable parameter in [16], with the purpose of matching the two standard models, the one of particle phyics and the one of comology.. As asserted by G. Cohen-Tannoudji [8]

> *The minus sign in the right hand side shows that when the factor $\left(11N_c - 2N_f\right)$ is positive, all the QCD condensates contribute negatively to the energy density, which*

means that the QCD world-matter is globally an AdS world-matter (dominance of an AdS world-matter over a smaller dS world-matter),

and so

- the bosonic (gluon) loops, proportional to N_c, contribute to the AdS world matter,
- the fermionic (quark) loops, proportional to N_f, contribute to the normal dS world matter.

Compare the ratio $\frac{11}{2}\frac{N_c}{N_f} \sim 5.5$ with the estimate [dark matter]/[visible matter] $\sim 27/5 = 5.4$.

3.4 Cold dark matter: Bose-Einstein condensation of (di-)gluons in effective anti-de Sitter geometry

We now explain the mechanism which makes the remaining gluonic component the dark matter or CGB. First we remind that in an AdS geometry:

$$E^{\text{rest}}_{\text{AdS}} = \left[m^2c^4 + \hbar^2c^2\frac{|\Lambda_{\text{AdS}}|}{3}\left(s-\frac{1}{2}\right)^2\right]^{1/2} + \frac{3}{2}\hbar\sqrt{\frac{|\Lambda_{\text{AdS}}|}{3}}\,c\,. \tag{25}$$

As an assembly of N_G non-interacting (*i.e.*, colorless) scalar bosonic di-gluons with individual energies $E_n = E^{\text{rest}}_{\text{AdS}} + n\hbar\omega_{\text{AdS}}$ with $E^{\text{rest}}_{\text{AdS}}$ is mc^2 ($m = m_G$ can be zero or negligible) and degeneracy $g_n = (n+1)(n+3)/2$, those remnant components, analogous to isotropic harmonic oscillators in 3-space, are assumed to form a grand canonical Bose-Einstein ensemble whose chemical potential μ is, at temperature T, fixed by the requirement that the sum over all occupation probabilities at temperature T yields

$$N_G = \sum_{n=0}^{\infty}\frac{g_n}{\exp\left[\frac{\hbar\omega_{\text{AdS}}}{k_BT}(n+\nu_0-\mu)\right]-1}\,, \qquad \nu_0 := \frac{E^{\text{rest}}_{\text{AdS}}}{\hbar\omega_{\text{AdS}}}\,. \tag{26}$$

The number N_G is very large and so the gas condensates at temperature

$$T_c \approx \frac{\hbar\omega_{\text{AdS}}}{k_B}\left(\frac{N_G}{\zeta(3)}\right)^{1/3}\,, \quad \zeta(3)\approx 1.2 \quad \text{(Riemann zeta function)}\,, \tag{27}$$

to become the currently observed dark matter. The above formula involving the value $\zeta(3)\approx 1.2$ of the Riemann function is standard for all isotropic harmonic traps (see for instance [31]). Actually there is no harmonic trap here, it is the AdS geometry due to QCD trace anomaly which originates the harmonic spectrum on the quantum level. To support this scenario it is known from ultra-cold atoms physics that Bose Einstein condensation can occur in non-condensed matter but also in gas, and that this phenomenon is not linked to *interactions* but rather to the *correlations implied by quantum statistics.*

Although we do not precisely know at which stage beyond the hadronization phase transition does take place the gluonic Bose Einstein condensation, let us see if our estimate on T_c yields reasonable orders of magnitude. Take T_c equal to the current CMB temperature, $T_c = 2.78$K, and $|\Lambda_{\text{AdS}}| \approx \frac{5.5}{6.5}\times\frac{11}{24}\times\Lambda_{\text{dS}} = 0.39\times\Lambda_{\text{dS}}$ (an estimate based on the ΛCDM model, see complements), with $\Lambda_{\text{dS}} \equiv$ present $\Lambda = 1.1\times 10^{-52}\text{m}^{-2}$. We then get the estimate on the number of di-gluons in the condensate:

$$N_G \approx 5\times 10^{88}\,. \tag{28}$$

This seems reasonable since the gluons are around 10^9 times the number of baryons, and the latter is estimated to be around 10^{80}.

4 Conclusion

We have tentatively explained dark matter by actually asking a simple question (!): what becomes the huge amount of gluons after the transition from QGP period to hadronization? Similarly to the emergence of the two validated CMB (QED effect) and CNB (electroweak effect), we propose to consider Dark Matter, observed through its gravitational effects, as a pure QCD effect. From our viewpoint it would legitimate to replace the puzzling expression “Dark Matter” with the realistic “Cosmic Gluonic Background”.

A Complements: Facts of ΛCDM standard model

Let us recall the cosmological formalism ($c = 1$) based on the Robertson metric. In an isotropic and homogeneous cosmology, the Einstein's equation reads as

$$\mathsf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathsf{R} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}\,, \tag{A.1}$$

where the stress energy momentum stands for a perfect fluid with density ρ and isotropic pressure P, *i.e.*,

$$T_{\mu\nu} = -P g_{\mu\nu} + (P+\rho) u_\mu u_\nu\,. \tag{A.2}$$

Its solution is the Robertson metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right), \tag{A.3}$$

where k is the curvature index, and $R(t)$ is the time-dependent radius of the universe. It is the cosmological scale factor (also noted $a(t)$) which determines proper distances in terms of the comoving coordinates. The radial variable r is dimensionless.

The radius R, the density ρ, and the pressure P obey the Friedmann-Lemaître (FL) equations of a perfect fluid modelling the material content of the universe.

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3}\,, \tag{A.4}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)\,, \tag{A.5}$$

$$\dot{\rho} = -3H(\rho + P) \qquad \text{(Conservation of the energy)}\,. \tag{A.6}$$

Note that the cosmological term $\Lambda g_{\mu\nu}$ is taken to the right-hand side of the Einstein's equation and may be interpreted as an extra pressure, named world matter by de Sitter in his debate with Einstein:

$$\mathsf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathsf{R} = 8\pi G (P+\rho) u_\mu u_\nu + (\Lambda - 8\pi G P) g_{\mu\nu}\,. \tag{A.7}$$

According to the sign of this extra pressure one talks of a de Sitter world matter (Λ positive, pressure negative) or an anti-de Sitter world matter (Λ negative, pressure positive). From the first FL equation at $\Lambda \approx 0$ one derives

$$\frac{k}{R^2} = \frac{8\pi G}{3}\rho - H^2 \equiv \frac{8\pi G}{3}\rho - \frac{8\pi G}{3}\rho_{\mathrm{c}}\,, \qquad \rho_{\mathrm{c}} := \frac{3H^2}{8\pi G_{\mathrm{N}}}\,, \tag{A.8}$$

where ρ_{c} is the so-called critical density. Since the ($\sim$ observed) flatness rule $k = 0$ expresses the vanishing of the spatial curvature one can write

$$\rho - \rho_{\mathrm{c}} \equiv \rho_{\mathrm{vis}} + \rho_{\mathrm{DM}} + \rho_{\mathrm{DE}} - \rho_{\mathrm{c}} = 0\,, \tag{A.9}$$

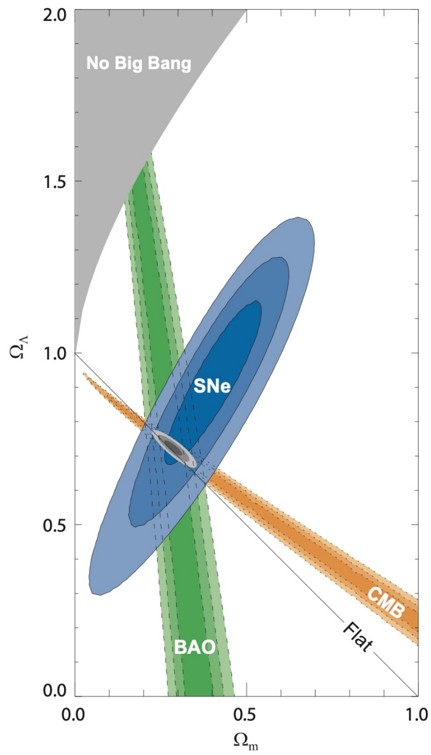


Figure 6: From [32]: 68.3 %, 95.4 % and 99.7% Confidence level contours on $\Omega_\Lambda \equiv \Omega_{DE}$ and Ω_m obtained from CMB, BAO and the Union SN set ($P/\rho \equiv w = -1$). This is "Concordance Cosmology": The contributions of the cosmological constant Ω_Λ and of the (ordinary + dark) matter Ω_m to the ratio total density/critical density, *i.e.*, the density for which the Universe is spatially flat, are yielded (modulo their uncertainty ranges) through Supernovae (SNe), Baryonic Acoustic Oscillations (BAO), and Cosmic Microwave Radiation (CMB). One sees that alternative models to Big Bang (No Nig Bang) are excluded. The straight line $\Omega_\Lambda + \Omega_\mathrm{m} = 1$ which is marked "flat" corresponds to a spatially flat Universe.

with

$$\rho_\mathrm{vis} = \rho_\mathrm{bar} + \rho_\mathrm{rad}\,, \qquad \rho_\mathrm{DE} = \frac{\Lambda}{8\pi G_N}\,. \tag{A.10}$$

Hence ρ_c is the energy density at the boundaries in the far past and in the far future of the Hubble horizon in the absence of any "integration constant" Λ and any spatial curvature ($k=0$). Next, from the second FL equation

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G_N}{3}(\rho + 3P) \equiv -\frac{4\pi G_N}{3}(\rho - 2\rho_\mathrm{DE} + 3P) \equiv -\frac{4\pi G_N}{3}(\rho_\mathrm{effective} + 3P)\,, \tag{A.11}$$

one infers that at the inflection points $\ddot{R} = 0$ one has the "equation of state" (EoS)

$$w_\mathrm{inflexion} \equiv P/\rho_\mathrm{effective} = -1/3\,.$$

Inside the "confidence area" of the figure 6 in which $\Omega_\Lambda = \rho_\mathrm{DE}/\rho_\mathrm{c}$ is expressed versus $\Omega_\mathrm{M} = \rho_\mathrm{m}/\rho_\mathrm{c}$ one finds the points

- $(\Omega_\mathrm{DM}, \Omega_\mathrm{DE} + \Omega_\mathrm{vis})$,
- $(\Omega_\mathrm{m} = 1/3, \Omega_\mathrm{DE} = 2/3)$.

The value $\Omega_{\mathrm{DE}} = 2/3$ results from our assumption completing the flatness sum rule as which the total energy vanishes (from the Robertson metric):

$$\rho_{\mathrm{vis}} + \rho_{\mathrm{DM}} + \rho_{\mathrm{DE}} = \rho_{\mathrm{c}} = \frac{3}{2}\rho_{\mathrm{DE}}\,. \tag{A.12}$$

References

[1] T. D. Newton and E. P. Wigner, *Localized states for elementary systems*, Rev. Mod. Phys. **21**, 400 (1949), doi:10.1103/RevModPhys.21.400.

[2] C. Fronsdal, *Elementary particles in a curved space*, Rev. Mod. Phys. **37**, 221 (1965), doi:10.1103/RevModPhys.37.221.

[3] A. D. Sakharov, *Vacuum quantum fluctuations in curved space and the theory of gravitation*, Dok. Akad. Nauk. SSSR **177**, 70 (1967).

[4] S. L. Adler, *Einstein gravity as a symmetry-breaking effect in quantum field theory*, Rev. Mod. Phys. **54**, 729 (1982), doi:10.1103/RevModPhys.54.729.

[5] G. Cohen-Tannoudji and J.-P. Gazeau, *Cold dark matter: A gluonic Bose-Einstein condensate in anti-de Sitter space time*, Universe **7**, 402 (2021), doi:10.3390/universe7110402.

[6] G. Cohen-Tannoudji and J.-P. Gazeau, *Scientific cosmogony, the time in quantum relativistic physics*, in *Time in science*, World Scientific, Singapore, ISBN 9781800613867 (2023), doi:10.1142/9781800613775_0010.

[7] G. Cohen-Tannoudji, *Lambda, the fifth foundational constant considered by Einstein*, Metrologia **55**, 486 (2018), doi:10.1088/1681-7575/aac10b.

[8] G. Cohen-Tannoudji, *The de Broglie universal substratum, the Lochak monopoles and the dark universe*, Ann. Fond. Louis **44**, 187 (2019).

[9] J.-P. Gazeau, *Mass in de Sitter and anti-de Sitter universes with regard to dark matter*, Universe **6**, 66 (2020), doi:10.3390/universe6050066.

[10] J.-P. Gazeau and M. Novello, *The nature of Λ and the mass of the graviton: A critical view*, Int. J. Mod. Phys. A **26**, 3697 (2011), doi:10.1142/S0217751X11054176.

[11] H. Bacry and J.-M. Lévy-Leblond, *Possible kinematics*, J. Math. Phys. **9**, 1605 (1968), doi:10.1063/1.1664490.

[12] M. Enayati, J.-P. Gazeau, H. Pejhan and A. Wang, *The de Sitter (dS) group and its representations*, Springer, Cham, Switzerland, ISBN 9783031160448 (2023), doi:10.1007/978-3-031-16045-5.

[13] T. Garidi, *What is mass in de Sitterian physics?*, (arXiv preprint) doi:10.48550/arXiv.hep-th/0309104.

[14] J. Mickelsson and J. Niederle, *Contractions of representations of de Sitter groups*, Commun. Math. Phys. **27**, 167 (1972), doi:10.1007/BF01645690.

[15] T. Garidi, E. Huguet and J. Renaud, *de Sitter waves and the zero curvature limit*, Phys. Rev. D **67**, 124028 (2003), doi:10.1103/PhysRevD.67.124028.

[16] G. Cohen-Tannoudji and J.-P. Gazeau, *In preparation*.

[17] J. P. Gazeau and M. Novello, *The question of mass in (anti-) de Sitter spacetimes*, J. Phys. A: Math. Theor. **41**, 304008 (2008), doi:10.1088/1751-8113/41/30/304008.

[18] C. Fronsdal, *Elementary particles in a curved space. II*, Phys. Rev. D **10**, 589 (1974), doi:10.1103/PhysRevD.10.589.

[19] C. Fronsdal and R. B. Haugen, *Elementary particles in a curved space. III*, Phys. Rev. D **12**, 3810 (1975), doi:10.1103/PhysRevD.12.3810.

[20] C. Fronsdal, *Elementary particles in a curved space. IV. Massless particles*, Phys. Rev. D **12**, 3819 (1975), doi:10.1103/PhysRevD.12.3819.

[21] J.-P. Gazeau and J. Renaud, *Relativistic harmonic oscillator and space curvature*, Phys. Lett. A **179**, 67 (1993), doi:10.1016/0375-9601(93)90652-G.

[22] M. Enayati, J.-P. Gazeau, M. A. del Olmo, and H. Pejhan, *Anti-de Sitterian "massive" elementary systems and their Minkowskian and Newtonian limits*, (arXiv preprint) doi:10.48550/arXiv.2307.06690.

[23] B. Binegar, C. Fronsdal and W. Heidenreich, *De Sitter QED*, Ann. Phys. **149**, 254 (1983), doi:10.1016/0003-4916(83)90197-5.

[24] J.-P Gazeau, M. Hans and R. Murenzi, *Invariant bilinear forms on 3+2 de Sitter space*, Class. Quantum Gravity **6**, 329 (1989), doi:10.1088/0264-9381/6/3/012.

[25] V. A. Rubakov and D. S. Gorbunov, *Introduction to the theory of the early Universe*, World Scientific, Singapore, ISBN 9789813209879 (2017), doi:10.1142/10447.

[26] N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, Astron. Astrophys. **641**, A6 (2020), doi:10.1051/0004-6361/201833910.

[27] A. Arbey and F. Mahmoudi, *Dark matter and the early Universe: A review*, Prog. Part. Nucl. Phys. **119**, 103865 (2021), doi:10.1016/j.ppnp.2021.103865.

[28] R. Pasechnik and M. Šumbera, *Phenomenological review on quark-gluon plasma: Concepts vs. observations*, Universe **3**, 7 (2017), doi:10.3390/universe3010007.

[29] A. Andronic, P. Braun-Munzinger, K. Redlich and J. Stachel, *Decoding the phase structure of QCD via particle production at high energy*, Nature **561**, 321 (2018), doi:10.1038/s41586-018-0491-6.

[30] J. Rafelski, *Discovery of quark-gluon plasma: Strangeness diaries*, Eur. Phys. J. Spec. Top. **229**, 1 (2020), doi:10.1140/epjst/e2019-900263-x.

[31] W. J. Mullin, *Bose-Einstein condensation in a harmonic potential*, J. Low Temp. Phys. **106**, 615 (1997), doi:10.1007/BF02395928.

[32] M. Kowalski et al., *Improved cosmological constraints from new, old, and combined supernova data sets*, Astrophys. J. **686**, 749 (2008), doi:10.1086/589937.

The prediction of anyons: Its history and wider implications

Gerald A. Goldin⋆

Department of Mathematics, Department of Physics & Astronomy,
Rutgers University, New Brunswick NJ, USA

⋆ geraldgoldin@dimacs.rutgers.edu

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Prediction of "anyons," often attributed exclusively to Wilczek, came first from Leinaas & Myrheim in 1977, and independently from Goldin, Menikoff, & Sharp in 1980-81. In 2020, experimentalists successfully created anyonic excitations. This paper discusses why the possibility of quantum particles in two-dimensional space with intermediate exchange statistics eluded physicists for so long after bosons and fermions were understood. The history suggests ideas for the preparation of future researchers. I conclude by addressing failures to attribute scientific achievements accurately, both inadvertent and intentional. Such practices disproportionately hurt women and minorities in physics.

Received 18-12-2022
Accepted 11-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.005

Contents

1 Introduction **2**

2 The idea of the anyon: Why so long? **2**
2.1 Anyons and nonabelian anyons 2
2.2 Historical and psychological barriers in physics 3
2.3 Epistemological obstacles to the prediction of anyons 3

3 Ideas antecedent to the prediction of anyons **5**

4 Intermediate statistics: Three independent predictions **6**

5 Implications for the education of future physicists **8**

6 A sequel to discovery: Citation omission and its consequences **9**
6.1 Anyons: A case study in systematic citation omission 9
6.2 Citation inequity in physics: Untruth and its consequences 11
6.3 A personal note 11

7 Conclusion 12

References 12

1 Introduction

"Anyons" are quantum particles or excitations, theoretically possible in two space dimensions, with exchange statistics intermediate between bosons and fermions. They are associated with surface phenomena in the presence of magnetic flux. Theoretical applications include explaining the quantum Hall effect, describing quantum vortices in superfluids, and their relevance to quantum computing. In 2020, more than forty years after they were first suggested [1], experimentalists succeeded in creating anyonic excitations. The experimental confirmation of their prediction attracted considerable new attention to these fascinating possibilities.

Predicting the anyon required basic changes in our understanding of quantum statistics. The prediction is often attributed exclusively (and incorrectly) to Frank Wilczek, while the first clear predictions were by Leinaas and Myrheim in 1977 [1] and by Menikoff, Sharp, and myself in 1980-1981 [2,3], from different theoretical perspectives. Wilczek's 1982 work [4,5] took still a third path to the prediction. He also coined the name "anyons" to describe such particles. This article describes the early history of intermediate quantum statistics, including the predecessor ideas that led to those predictions. Some immediately subsequent insights, often overlooked in citing early published results, are also described [6–8].

One may ask why the possibility of intermediate statistics took physicists so long to discern, from the time when bosons and fermions were well understood. And why, after fifty years, did three independent predictions occur within such a relatively short time interval? Examining these questions suggests some interesting implications for the teaching of mathematics and physics, relevant to the preparation of future researchers.

Finally, I discuss the systemic failure of scientists' and journalists' to attribute scientific achievements accurately. With respect to anyons, breaches of integrity are current as I write this article, though there is no dispute as to the history. But the "anyon" case is not unique. It provides a case study illustrating a far wider problem. Acknowledgment failure, and its tacit acceptance by the scientific community, does damage far beyond disappointing or hurting a few individuals. Non-recognition disproportionately creates career obstacles for women, Black and other minorities, and scientists in developing countries. Young scientists experience disillusion, even intimidation, and have much to lose by speaking out. And potentially fruitful directions of investigation cannot be pursued when researchers are unaware of their existence.

2 The idea of the anyon: Why so long?

2.1 Anyons and nonabelian anyons

The idea behind intermediate quantum statistics in two-dimensional space (three-dimensional spacetime) is extraordinarily easy. Let us imagine a pair of indistinguishable particles moving on a two-dimensional surface – constrained, for instance, to the surface of some material. Suppose they exchange positions, but without actually passing "through" each other. They must have done so by moving either clockwise or counterclockwise. One can characterize any exhange, then, by a winding number: the net number (positive or negative) of counterclockwise windings occurring during the exchange. This feature is specific to two-dimensional space.

If the system is described by a complex-valued wave function ψ, one wants $|\psi|^2$ to be invariant under any such exchange. For a single clockwise exchange, ψ might then be multiplied by a complex number of modulus one: $\psi \mapsto [\exp i\theta]\psi$. With $\theta = \pi$ we obtain fermions, and with $\theta = 2\pi$ we have bosons. But in two-space, two counterclockwise exchanges in succession are *inequivalent* to no exchange. Hence we need not require $\exp 2i\theta = 1$. For "any" value of θ, the predicted outcomes of all physical measurements remain invariant under the exchange.

For three or more indistinguishable particles in two-space, path-dependent exchanges performed in succession no longer commute. Then nonabelian representations of the group describing exchanges, acting by unitary operators on multicomponent wave functions, become possible. Therefore quantum mechanics also allows "nonabelian anyons".

With these ideas so simple to describe in an elementary way, why did so many brilliant physicists overlook them for so long? Bosons and fermions were understood in 1924-25; the intermediate statistics of anyons was not explicitly proposed until 1977-82. What led to this idea, so long deferred, becoming one "whose time had come"?

2.2 Historical and psychological barriers in physics

The concept of an *epistemological obstacle*, introduced by Bachelard in 1938 [9] and discussed by Schneider [10], is well-known in science and mathematics education research. It refers to prior conceptions that impede the understanding necessary for a breakthrough. As elaborated by Brousseau such obstacles are not a "result of ignorance [...] or chance", but an "effect of prior knowledge that was relevant and had its success, but which now proves to be false, or simply inadequate" [11]. The term suggests *inevitable* barriers in historical paths of discovery, evolution of conceptual schemes, and ascribing meanings to mathematical representations.

In the psychology of an individual thinker or learner, the parallel notion of a *cognitive obstacle* refers to prior knowledge limiting the person's development of new conceptions. As students overcome cognitive obstacles, their stages of learning often recapitulate historical processes. There is an analogy with biology, where the "ontogeny" of developing organisms seems to recapitulate the "phylogeny" of the species' evolution.

Examples abound of epistemological obstacles in physics and mathematics. To understand that objects without applied forces continue in rectilinear motion; to see space and time as not absolute; to embrace wave-particle duality and the uncertainty principle – each required physicists to overcome universal categories of experience and abandon previously-successful explanations of observed phenomena. Seeing axioms and propositions as "self-evident truths" impeded mathematicians' taking them as arbitrary assumptions characterizing abstract structures. Acceptance of negative numbers, "imaginary" numbers, non-Euclidean geometries, and transfinite cardinals, all required overcoming beliefs that these did not "really exist".

In physics an empirical source of epistemological obstacle can be the inaccesibility of relevant domains of experiment. Thus frictionless dynamical systems, high velocities approaching light-speed, and observability at the subatomic level, remained difficult or impossible to access for centuries; no driving force from experiment yet demanded conceptual change.

With these ideas in mind, I think it is possible to identify five major epistemological/cognitive obstacles that impeded the prediction of anyon statistics, and all that followed. We can also see how antecedent ideas gradually dismantled those obstacles.

2.3 Epistemological obstacles to the prediction of anyons

Index vs. value permutations. The first obstacle was the use of index permutations to describe particle exchange. In a configuration of N indistinguishable particles, their positions were labeled with subscripts (indices) $1, ..., N$; with the wave function written $\psi(x_1, ..., x_N)$.

A permutation $\sigma_{(ij)}$ exchanged "particle i" with "particle j" – an index permutation. With this meaning of exchange, there is conceptually no physical path of exchange – the particles are relabeled abstractly. Writing $|\psi(x_1,...,x_N)|^2 = |\psi(x_{\sigma(1)},...,x_{\sigma(N)})|^2$ asserts invariance under index exchange – reintroducing indistinguishability after labeling the particles distinguishably.

Value permutations, in contrast, make reference to the coordinates locating the particles being exchanged; i.e., their positions in the physical space. The permutation $\sigma_{(ij)}$ exchanges the particle in the i-th location with the one in the j-th location. This description requires an ordering of points in physical space. The arbitrariness of that (which is actually no stronger objection than the arbitrariness of indexing) is one reason for conceptual difficulty in letting permutations act on coordinate values rather than indices. Nevertheless, exchanging actual particle positions allows one to focus on possible paths of exchange.

Note that introducing labels or adopting a coordinate system is *always* arbitrary. By itself, it leaves the physics invariant. A symmetry property of coordinates is not a physical symmetry, and in principle provides no new physical information. Physical insight comes when the symmetry group is understood to act on the system itself, with some system properties identified as invariant under the symmetry. This distinction, while essential and obvious, is often easily overlooked in the language we use to discuss group theoretical applications in physics.

Coordinate space vs. configuration space. A second obstacle inheres in considering ordered N-tuples at all. A configuration of indistinguishable particles is actually an *unordered* N-point subset of the physical space. To see statistics as arising from paths of exchange requires a focus on configuration space topology. But an unordered set allows no description of particle exchange – hence the introduction of indices and imposition of invariance under exchange, obscuring that focus. A different idea was needed. Note that the expression $\Delta_N^{(d)} = [(\mathbb{R}^d)^{\times N} - D] \, mod \, S_N$ for N-particle configuration space in $\mathbb{R}^d$, where D is the "diagonal" consisting of N-tuples for which $x_i = x_j$ for some $i \neq j$, and S_N is the symmetric group, seems to reflect the historical conception. It is much simpler to write $\Delta_N^{(d)} = \{\gamma \subset \mathbb{R}^d \,|\, \mathrm{card}(\gamma) = N\}$.

Continuous single-valued wave functions. A third obstacle was the assumption over many years that wave functions on coordinate space must be continuous and single-valued. This perhaps reaches as far back as the Bohr atom, where electrons were posited to occupy fixed circular "orbits". A continuous wave describing this would have an appropriate period, forbidding self-interference and leading naturally to quantization. In subsequent models based on the Schrödinger equation, single-valuedness and continuity provide a natural framework for quantization of energy and momentum. The conception of ψ as a kind of physical field modeled on points in physical space (albeit configurations of such points) also seemed to demand single-valuedness. Thus if $\psi \mapsto [\exp i\theta]\psi$ under a single exchange, of necessity we need $\exp 2i\theta = 1$. The "simple" idea provided above is then easily dismissed as fallacious.

Established empirical knowledge: fermions and bosons. The dramatic achievements of quantum theory with just two types of particles posed a fourth obstacle. Successes included the Pauli exclusion principle for fermions, standing behind the periodic table of elements and explaining in principle chemical reactions. They included the phenomenon of Bose-Einstein condensation; also local quantum field theories where bosons are the quanta carrying fundamental forces of nature. No experiments compelled inquiry into more exotic possibilities.

Axiomatic quantum theory. Finally, a fifth obstacle inhered in the understanding, achieved through axiomatic relativistic quantum field theory, of how some fundamental physical laws follow from basic assumptions. The Wightman axioms [12] included the proposition that space-like separated fields either commute or anticommute. These axioms led to rigorous proofs of PCT-invariance and the spin-statistics connection. Of course, the axioms encoded widely-shared beliefs about the properties quantum fields should have. But working from fixed axioms does create an intellectual context where they are no longer questioned; only their implications are explored. Bose and Fermi statistics exclusively were thus firmly embedded in the foundations of physics.

3 Ideas antecedent to the prediction of anyons

Important antecedent ideas, developed over several decades, eventually overcame these obstacles to intermediate statistics.

Intermediate occupation number statistics. As early as 1940-1942, Gentile [13,14] explored the possibility of hypothetical occupation number statistics other than those of fermions or bosons, where an intermediate, finite number of particles could be permitted to occupy the same quantum state. He drew some possible consequences for the theory of superfluidity.

Topology in quantum mechanics. In 1959 Aharonov and Bohm [15] considered charged particles excluded by an infinite potential barrier from a cylindrical region, behind which a current in a tightly-wound infinitely-long solenoid sustains a magnetic flux. Quantum theory then predicts shifts in the energy and kinetic angular momentum spectrum of the particles, despite the absence of a physical magnetic field in the accessible region. Their paper, conceived as a *Gedanken* experiment, evoked much controversy as to its meaning. Though known as the "Aharonov-Bohm effect", a similar but much less noticed proposal had actually been published in 1949 by Ehrenberg and Siday [16], as Hiley describes in a historical article [17]. The earlier work is highlighted in the Wikipedia entry on the subject [18], which is how I learned of it just last year. The (Ehhrenberg-Siday)-Aharonov-Bohm effect pointed to the role of the topology of the space in which quantum particles move – particularly, but not exclusively, when charged particles circle excluded regions of magnetic flux. The possibility of multivalued wave functions needed to be entertained – though challenged, it began to achieve legitimacy.

Following on these ideas, in the context of the Feynman path-integral fomulation, physicists considered homotopy classes of trajectories from initial to final particle configurations. Here one moves from the topology of physical space to that of configuration space. Schulman [19] in 1968 proposed a model for the topological origin of particle spin. In 1971, Laidlaw and (Cécile) DeWitt [20] deduced the topological origin of Fermi and Bose exchange statistics. In a footnote to their result, they remarked that in two space dimensions there seemed to be additional possibilites for quantum statistics, but they did not pursue this. In 1972, Dowker [21] provided a more general discussion of quantum theory on multiply-connected spaces.

Parastatistics, kinks, etc. During the 1950s and 1960s, other paths led to some generalizations of exchange statistics. In 1953, Green [22] obtained parastatistics from trilinear brackets of quantum fields (combining canonical commutation and anticommutation relations). This work, with resulting investigations of symmetrization in 1964-1965 by Messiah and Greenberg [23] and Girardeau [24], brought in higher-dimensional, nonabelian representations of S_N. As concrete alternatives to Bose and Fermi statistics, parastatistics evoked unfulfilled conjectures that fundamental particles such as quarks might satisfy them. In 1968 Finkelstein and Rubinstein [25] suggested more general possibilities for the spin/statistics relation for "kinks" in the context of quantized nonlinear fields, by admitting double-valued state functionals.

Braid groups and Yang-Baxter relations. As different models in quantum field theory were invented and studied, braid groups began to enter into consideration in the late 1960s and 1970s. In two-dimesional models with soliton fields they found a place through the Yang-Baxter equation in articles by Streater and Wilde [26] and Fröhlich [27, 28]. Related work was published by Klaiber [29], Souriau [30], Kadanoff and Ceva [31], and Wegner [32].

Group representations and current algebras. In parallel with these developments, unitary group representations came into their own as pillars of quantum theory. Wigner, Mackey, and numerous others established their fundamental role [33, 34]. In strong interaction physics and the theory of fundamental particles, dramatic findings confirmed the predictive power of $SU(3)$ [35]. This led eventually to unification of the electroweak and strong forces via $SU(2) \times U(1) \times SU(3)$ gauge theory in the "Standard Model".

Moving from Lie groups to Lie algebras, Adler and Dashen [36] made a strong case in the 1960s for current algebras in fundamental particle physics. In 1968 Dashen and Sharp [37] proposed a certain highly singular, local current algebra describing nonrelativistic quantum field theory. Their goal was to describe hadrons by gauge-invariant quantities such as densities and currents, rather than gauge-dependent quantum fields. Behind this work stood earlier ideas of Haag and Kastler [38] and others on algebras of local observables.

In the late 1960s, I was able to regularize and exponentiate Dashen and Sharp's algebra to obtain an infinite-dimensional group, and establish a framework for studying its unitary representations [39]. The group is the natural semidirect product of a diffeomorphism group $\mathit{Diff}_0(\mathbb{R}^3)$ of the physical space (describing flows generated by momentum density operators), with an additive group of scalar functions on M (describing exponentiated mass density operators). Then in collaboration with Grodnik, Powers, and Menikoff [40–44], Sharp and I found applications confirming the fundamental role of this group and its Lie algebra. Across the 1970s, Menikoff, Sharp, and I successfully extended Mackey's method of induced representations of locally compact Lie groups, to obtain a class of unitary representations of diffeomorphism groups [2]. It followed from our work that Bose and Fermi exchange statistics could be understood as inequivalent unitary representations of our semidirect group, induced by representations of S_N. And S_N entered naturally as the fundamental group of N-particle configuration space. This provided a new, wholly kinematical perspective on the earlier work of Laidlaw and DeWitt [20], which had been based on Feynmann paths; and it led to intermediate quantum statistics in two-space.

The roles of $SO(3)$ and $SU(2)$ in describing orbital and spin angular momentum were of course long known at the time of all this work. A 1976 paper by Martin [45] proposed a model for the Aharonov-Bohm effect based on rotation generators in two-dimensinal space. Here she presaged group-theoretically the fractional statistics subsequently predicted for anyons.

4 Intermediate statistics: Three independent predictions

By the mid- to late 1970s requisite ideas were in place, the obstacles mostly removed.

In 1977, Leinaas and Myrheim [1] presented the first clear prediction of quantum exchange statistics interpolating Bose and Fermi statistics in two space dimensions. They based their analysis on Schrödinger quantization of particle dynamics, using the topology of Feynman paths. They drew a connection with electromagnetism, noting the singularity in configuration-space associated with the coincidence points of particles. In 1978 Leinaas [46] suggested a model based on charged-particle/monopole composites.

This picture did leave open the issue of whether Feynman paths might "cross" – i.e., can two particles "pass through" each other as the quantum configuration evolves? One might then need a hard-core, singular repulsive potential for intermediate statistics. In the Aharonov-Bohm setup, the nontrivial topology was established by introducing an infinite barrier to exclude the charged particles from a region of space; would such exclusion be necessary here?

Menikoff, Sharp, and I published our prediction of intermediate statistics in 1980-81, not yet aware of the Leinaas-Myrheim papers. Our findings confirmed theirs but assumed less, being kinematical rather than dynamical [2, 3]. Studying the Aharonov-Bohm setup with our local current algebra meant representing the group of diffeomorphisms of a non-simply connected space. We had already established a foundational role for unitary representations of

$Diff_0(\mathbb{R}^3)$ in classifying quantum systems; now we found – to our surprise at the time – that the unitary representations of $Diff_0(\mathbb{R}^2)$ included intermediate exchange statistics. Thus we did not obtain our results by quantizing a classical system, but by rigorously pursuing fundamental group-theoretic methods. Discovering intermediate statistics culminated 15 years of research.

A number of things became clear from our work. One obtains directly the shifted spectrum of self-adjoint kinetic angular momentum operators associated with anyons. Wave functions are single-valued on the true configuration space for indistinguishable particles; the exchange statistics is established by the operators describing local observables. Coincidence points are excluded necessarily and not arbitrarily; no repulsive potential is necessary. The (illusory) multi-valuedness of wave functions reflects an equivalent representation on the Hllbert space of *equivariant* wave functions on the universal covering space of configuration space. Equivariance is with respect to its fundamental group (first homotopy group). The inner product is defined by integration on configuration space, not on the covering space – an essential idea, because there are in general infinitely many sheets to the covering space.

In 1982 Wilczek, who by his second article that year knew of the earlier articles, published his own, independent prediction [4, 5]. His systematic investigation of fractional quantum numbers suggested fractional-spin particles in two dimensions. He modeled this with charged particles bound to units of magnetic flux orthogonal to the surface confining the particles – like miniature "Aharonov-Bohm solenoids" with net charge. Wave functions describing such particle/flux tube composites pick up the intermediate phase $\exp i\theta$ in a single counterclockwise exchange. His name "anyons" expresses that θ can take "any" value between 0 and 2π.

Wilczek recognized and advocated for anyons' theoretical importance, especially as applications were found to understanding the quantum Hall effect. Twenty-three of the most impactful articles from 1983 to 1990, including seven by Wilczek and his collaborators, are reprinted in his 1990 book, *Fractional Statistics and Anyon Superconductivity* [47]. Space here does not permit my citing them; the reader is referred to [47]. This influential volume provided a valuable resource for researchers and reviewers of the field across the next decades.

Some immediate, fundamental consequences of my work with Menikoff and Sharp at Los Alamos were also published during the 1980s. In 1983, we presented a rigorous kinematical framework for the fractional spin of anyons, including the first (as far as I can determine) explicit identification of the braid group B_N as the homotopy group whose unitary representations govern N-anyon exchange statistics [6, 7]. In 1985 we first predicted nonabelian anyons [8], described by wave functions equivariant under higher-dimensional unitary representations of B_N. This parallels the earlier idea of parastatistics for particles in $\mathbb{R}^3$, where the homotopy group is S_N. We also pointed out in 1985 that systems of distinguishable particles in two-space are described by wave functions equivariant for the group of colored braids. These wave functions can pick up intermediate phases as particles fully circle each other, without exchange. Our conclusion about nonabelian anyons was contrary to the expectation expressed by Wu in 1984 [48] that a "general theory" would include only one-dimensional representations of the braid group; we published it as a response to Wu's paper.

Many further applications of anyon theory followed across the decades; not only in physics (e.g., to the quantum Hall effect and to quantum vortices), but also in the burgeoning field of quantum computing. Two years ago, more than four decades after their first prediction, two groups of experimental physicists announced success in creating and observing anyonic excitations [49, 50], stimulating wide interest.

5 Implications for the education of future physicists

This retrospective on the first predictions of anyons, the historical obstacles that delayed them and the antecedent research that removed those obstacles, suggests some more general considerations for future (or current) physicists. I would like to offer a few thoughts on the topic of education, before turning to the issue of citation integrity in physics. Epistemological and cognitive obstacles are most likely present now, though we may be unaware of them. In teaching university-level theory – classical mechanics, electricity and magnetism, optics, quantum mechanics, relativity, thermodynamics – one goal should be to facilitate scientifically sound exploration that can penetrate or even overturn prevailing conceptions.

How can we foster students' questioning of established constructs and encourage generation of new ones? What best enables a student to identify tacit assumptions and make them explicit? What tools help sudents overcome personal cognitive obstacles? I think that answers consist, in part, in exploring what it means for a student (or researcher) to "really understand" a newly-studied concept in physics or mathematics.

Consider the skills normally emphasized (or not) in teaching theory to physics students:

Textbook problems. Solving progressively more complex problems based in established theory, using standard techniques, is central to most physics courses. Students develop their understanding of theoretical constructs by applying them to situations where they fit directly.

Nonroutine problems. Many physics courses incorporate some problems that are less routine. Valuable heuristic methods and general strategies typically apply in such contexts. The best students gradually acquire them, though they may or may not be discussed explicitly. Examples include examining special cases, testing limiting cases or idealized cases, creating multiple representations, establishing and using insightful notation, choosing a helpful coordinate system, finding hidden symmetry, exploiting units of measurement, carefully distinguishing what is happening physically from its mathematical description, and so forth. Most such strategies pertain to mathematics as well as to physics.

But sometimes we bypass the thinking process: presenting students with notation, suggesting the desired representation, providing the coordinate system, and pointing the way to insight rather than allowing the experience of discovery. If we consistently dismantle obstacles, we may limit students' development of powerful methods for breaking through them.

History of discoveries. Physics teaching typically includes stories about discoveries and breakthroughs, how historic experiments forced reconsideration of previously accepted theories. I think there is much more we can do. We can explore the *thinking process* that led to a new theory. We can identify critical epistemological/cognitive obstacles delaying its invention. We can ask what philosophical or metaphysical assumptions may have impeded the idea, and how one might have noticed these earlier. We can explore alternative theories that did not pan out.

Students as inventive theorists. In introducing the foundational ideas behind a new concept, we might seek to engage students in thinking as original theorists. Before presenting established theory, students can offer their own conjectures and explore their consequences. The goal is not to see who is "right" and who is "wrong", but to foster students' creative theorizing. We can study rival theories and abandoned theories, to consider how one evaluates a scientific idea as valid or worth pursuing. I favor posing the following question to students and to ourselves: "If no one had ever seen this idea before, or if you had never previously encountered it, can you imagine how you personally might have invented it?"

6 A sequel to discovery: Citation omission and its consequences

Scientific research is an intimate activity. Disappointments and frustrations are interspersed with occasions of insight and satisfaction. One rarely achieves in full what one aspires to, but strives to fulfill the ideals of one's teachers and mentors. One values their words of encouragement, their belief in one's ability to contribute to understanding the natural world. One develops friendships and shared memories of collaborative success. One does what one loves.

A young mathematical physicist might realistically hope to create some interesting new models, to place some well-established physics on a more rigorous foundation, to unify previously disparate phenomena, or to show how some physical effects result from more fundamental laws. But perhaps the highest aspiration of the young theorist might be to predict a wholly new, unsuspected phenomenon – and to see that prediction confirmed.

Four decades after their prediction, anyonic excitations were observed by two experimental groups [49, 50]. This should be a deep source of satisfaction to all of us who put forth the prediction. Developing the theory that led to my own group's prediction took 15 years, during difficult times professionally. But both groups of experimentalists, apparently unaware of our early published work, cited only the articles by others. Shortly thereafter, widely circulated science magazines repeatedly attributed the prediction of anyons exclusively to Wilczek [51–54], who had shared the Nobel Prize in 2004 for his earlier work on quark confinement and asymptotic freedom. And the related fundamental insights about anyons that we were first to publish remain wholly unacknowledged.

How this came about, and how it continues today, are also part of the history of anyons. We have learned that the problem is systemic, stemming from omissions inadvertent and intentional, and magnified in impact by open journalistic dishonesty. But citation inequity does more than disappoint a few individual researchers. It has far wider implications for science.

It should not be necessary for me to emphasize that in describing this history, there is no intent whatsoever to diminish the fact or importance of Wilczek's substantial contributions to the physics of anyons. I am aware of no dispute among any of us regarding the authorship, the priority, the content, or the originality of the published results.

6.1 Anyons: A case study in systematic citation omission

Between 1982 and 1989, omission of proper reference to our articles was widespread. Most of these omissions were inadvertent, as researchers relied in their bibliographic research on the citations in earlier papers. But some, beginning with the most often-cited article [5], were intentional and consistently maintained. The effects were immediate and enduring. Among all the numerous citations in the 23 articles from 1983 to 1990 reprinted in Wilczek's influential 1990 book [47], there is just one citation of an article of ours. In the period that followed, this greatly limited awareness of our work by those for whom the book served as a major resource.

Acknowledgment as it relates to one's own research is a highly personal and difficult subject to address. It is a remarkable experience to be written out of history before one's eyes, year after year. Our work appeared in leading refereed journals. Our approach to intermediate statistics was novel – wholly group-theoretical in its foundation – and independently developed. Some findings that became widely known we were first to publish. As the omissions persisted, my colleagues and I made every effort to acquaint others with our published articles. At the time, internet and email did not exist. Correspondence was through letters – slow, difficult to write, always polite, and frequently unanswered. It made no difference.

In retrospect, our efforts in the 1980s were not sufficiently aggresssive. We relied on the good will and integrity of the research community. We sought to be scrupulous in acknowledging the work of others, and expected proper reference to our findings. But we saw how fame and power relationships influenced acknowledgment. The absence of information availability closed off interest in some research directions. Theoretical methods pursued were those people knew of – which sometimes led to insights we had already obtained by other means.

Then in 1989, *Physics Today* published an extensive article about anyons [55] attributing their discovery to Wilczek, mentioning Leinaas and Myrheim's article briefly, and omitting all reference to our work. My colleagues and I requested a correction. Somewhat to our surprise, there ensued an extraordinary battle – at heavy personal cost – before we achieved publication of the proper attributions. Eventually, the error was corrected. A detailed letter, authored by Biedenharn, Lieb, Simon, and Wilczek [56], described our contributions accurately and succinctly. Wilczek included a sentence in his book [47] (p. 105) citing our three earliest papers. He invited our contribution to a special issue of *International Journal of Modern Physics B* he was then editing [57]. There had never been an actual dispute regarding the sequence of discoveries, and it seemed that after eight years the acknowledgment issue was resolved.

But in 1991, an article about anyons in *Scientific American*, authored by Wilczek, credited him with the discovery without reference to any prior predictions [58]. This was corrected by letter from Sharp and me, after some struggle [59].

Over the next thirty years, citations appeared only sporadically. During this period, I came to believe that had the internet existed back in the 1980s, such consistent failures of attribution could never have occurred. Since 2020, Sharp and I have learned how untrue that is.

With the recent experimental findings, fame triumphed decisively over journalistic integrity in science reporting. Omissions, no longer inadvertent, are openly deliberate. Efforts at correction are ignored or refused. In two successive articles, *Discover Magazine* credited only Wilczek with the prediction [51, 53]. They disregarded detailed communications, refusing to post a simple update to their featured on-line articles [60, 61]. *The editors chose to violate explicit canons of ethics in journalism.* For example, the policy of the *Washington Post* states:

> "Fairness results from a few simple practices: No story is fair if it omits facts of major importance or significance. Fairness includes completeness. ... No story is fair if it consciously or unconsciously misleads or even deceives the reader. Fairness includes honesty — leveling with the reader." [62]

Quanta Magazine also published two features, crediting only Wilczek with the prediction of intermediate statistics [52,54]. Its editors simply disregarded repeated efforts to communicate our request for a correction. Thus in major journalistic outlets, it is as if neither Leinaas and Myrheim nor our group had participated at all in the research.

Researchers identifying appropriate citations in a specialized domain of physics often consult the relevant Wikipedia entry, and follow up with academic sources. I do so myself. Though not always 100% accurate or complete in describing the physics, Wikipedia is important.

At Wikipedia, the "anyon" entry [63] (discussing anyons, nonabelian anyons, topology, the braid group, and fractional spin) omits all indication of my colleagues' and my correct predictions about these topics. An anonymous editor (screen name "HouseofChange") has expunged every correct citation entered by at least two independent experts. This editor claims that the absence of earlier citations proves our work to be irrelevant. It is the opposite stance to Wikipedia's informative entry about the "Aharonov-Bohm effect" mentioned above, acquainting readers with the prior work of Ehrenberg and Siday. When removing the citations by other editors, "HouseofChange" leveled false and malicious accusations against them, and asserted untruths about Sharp and myself. Most or all exchanges are on Wikipedia's "talk" feature. One unknown person ensures that those interested in the physics of anyons should never know of our prediction, or of the group-theoretical and current-algebraic methods that led to it.

To sum up, in the nearly two years since the experimental confirmation of anyonic statistics, the most strenuous efforts possible have produced not one additional correction or footnote in the information generally available to the public.

6.2 Citation inequity in physics: Untruth and its consequences

Why does this matter? If one is not directly affected, one might be a little amused by the importance accorded here to brief mention and a few footnotes, and the fierce resistance to it. But there is a wider perspective. Denial of information slows research and its attendant benefits – an intangible loss to the entire scientific community. But this is far from the only damage. Acknowledgment failure profoundly undermines fairness and equity.

Egregious examples of overlooking women in physics for top honors are well-known: Chien-Shiung Wu's pioneering work on parity conservation violation, disregarded in the 1957 Nobel Prize award to Lee and Yang; Jocelyn Bell's crucial role in discovering pulsars, unrecognized in the 1974 Nobel to Hewish and Ryle. But for every such inequity at the highest levels, countless examples occur in less dramatic contexts. Substantial citation inequity toward women in phyiscs occurs [64] while our field remains overwhelimingly male.

Unfairness toward women is not the only form of discrimination citation inequity takes. It hurts Black and other minority scientists, those in developing countries, and early career researchers who have much to lose by speaking out – all who lack influential connections. In his 2021 book *Fear of a Black Universe* [66], physicist Stephon Alexander highlights the experiences of Black physicists who are marginalized. Interviewing with the *The Guardian* [65], he notes how citation inequity affected his mentor, physicist Jim Gates, whose contribution to supersymmetry together with Nishino was disregarded for at least ten years:

> "I was there when Jim realized that the work was not cited and he wrote one of the authors directly. Then they cited it, but it was kind of too late. That's why I wrote about it in this book, to celebrate that it was [Gates and Nishino's] discovery. ... That is exactly the phenomenon that Black people experience in other fields where we're not supposed to occupy these spaces."

Non-recognition of scientific achievement and the resulting career obstacles may be affecting younger readers of this article, even as I write. Success in pursuing a long-term, original project is never assured. Less-traveled paths may mean more limited opportunites. To predict a new, unsuspected phenomenon in one's early career risks evoking skepticism, particularly if the time for the prediction is not yet ripe. When the best results go unrecognized and unacknowledged, consequences can be serious. Disillusionment, alienation, and discouragement about risk-taking set in, and may call young physicists' love of science into question. And citations matter greatly in university tenure and promotion decisions.

Most important of all is the issue of our community's commitment to scientific integrity. Does the physics community (or more generally, the wider scientific community) recognize excellent research through quiet hard work, or do we value fame and promotional ability more? Honest and thorough acknowledgment of prior research should be a standard for every scientific publication, taken as seriously as we now take other standards of integrity such as truthful representation of data and authenticity of authorship. Genuinely inadvertent omissions, while probably inevitable, are easily corrected with on-line updates. When untruth is deemed to be minor and correcting it unimportant, or when intentional dishonesty of any kind goes unchallenged, the slope is slippery. In science reporting, dishonesty must be denounced. Tolerance for it endangers the very value we place as scientists on truthfulness.

6.3 A personal note

Sharp and I have been extremely privileged. Encouraged to pursue science, we received superb educations and graduate fellowships at elite U.S. universities. We had successful careers in research. We received honors and awards in physics. Research risks taken led to challenging obstacles in the 1970s, but we were able to see these through and continue as active scientists.

In retrospect it is distressingly clear that only our status and connections in 1989-91 allowed us to achieve the corrections published in *Physics Today* and *Scientific American*. Less connected researchers would have had no chance. And today the obstacles to correcting untruths seem vastly greater! With all our credentials and refereed publications, we cannot achieve passing mention by journalists drawn to a more famous person, or a few footnotes in Wikipedia for undisputedly original findings. What can others with fewer resources do?

Thus Sharp and I came to feel it is our obligation to speak out, not just for ourselves but for science. The outcome will not benefit us materially, nor will continued disregard harm us furher. A public stand is necessary to address a systemic problem of integrity in physics.

7 Conclusion

We have reviewed the history of ideas leading to the prediction of anyons. Epistemological obstacles stood in the way; research across decades helped break through them, leading to three independent predictions. Close study suggests the value of educating future theorists in the history of discoveries, encouraging them to question assumptions and invent alternate explanations. We have also described systemic citation inequity, and ongoing dishonest journalism. Such practices have adverse consequences not only for individuals but for science.

My faith is that the physics community fundamentally values truth and integrity in scholarship, and that inequitable practices contrary to these values will become unacceptable. Hopefully this article contributes meaningfully toward that goal.

Acknowledgments

Jürg Fröhlich (ETH Zürich) and Douglas Lundholm (Uppsala University) suggested valuable references antecedent to the prediction of anyons. I am grateful to the organizers of the Group34 Colloquium for the opportunity to speak in the 50th Anniversary Special Session. Finally, I would like to acknowledge David H. Sharp for his seminal contributions to the research reviewed here, and for our life-long collaboration.

References

[1] J. M. Leinaas and J. Myrheim, *On the theory of identical particles*, Nuovo Cimento B **37**, 1 (1977), doi:10.1007/BF02727953.

[2] G. A. Goldin, R. Menikoff and D. H. Sharp, *Particle statistics from induced representations of a local current group*, J. Math. Phys. **21**, 650 (1980), doi:10.1063/1.524510.

[3] G. A. Goldin, R. Menikoff and D. H. Sharp, *Representations of a local current algebra in nonsimply connected space and the Aharonov-Bohm effect*, J. Math. Phys. **22**, 1664 (1981), doi:10.1063/1.525110.

[4] F. Wilczek, *Magnetic flux, angular momentum, and statistics*, Phys. Rev. Lett. **48**, 1144 (1982), doi:10.1103/PhysRevLett.48.1144.

[5] F. Wilczek, *Quantum mechanics of fractional-spin particles*, Phys. Rev. Lett. **49**, 957 (1982), doi:10.1103/PhysRevLett.49.957.

[6] G. A. Goldin and D. H. Sharp, *Rotation generators in two-dimensional space and particles obeying unusual statistics*, Phys. Rev. D **28**, 830 (1983), doi:10.1103/PhysRevD.28.830.

[7] G. A. Goldin, R. Menikoff and D. H. Sharp, *Diffeomorphism groups, gauge groups, and quantum theory*, Phys. Rev. Lett. **51**, 2246 (1983), doi:10.1103/PhysRevLett.51.2246.

[8] G. A. Goldin, R. Menikoff and D. H. Sharp, *Comments on "general theory for quantum statistics in two dimensions"*, Phys. Rev. Lett. **54**, 603 (1985), doi:10.1103/PhysRevLett.54.603.

[9] G. Bachelard *La formation de l'esprit scientifique*, Vrin, Paris, France, ISBN 9782711611508 (1938).

[10] M. Schneider, *Epistemological obstacles in mathematics education*, in *Encyclopedia of mathematics education*, Springer, Dordrecht, Netherlands, ISBN 9789400749788 (2014), doi:10.1007/978-94-007-4978-8_57.

[11] G. Brousseau, *Les obstacles épistémologiques et les problèmes en mathématique*, Rech. Didact. Math. **4**, 165 (1983), https://revue-rdm.com/1983/les-obstacles-epistemologiques-et/.

[12] R. F. Streater, A. S. Wightman, *PCT, spin and statistics and all that*, Princeton University Press, Princeton, USA, ISBN 978140088420 (1984), http://www.jstor.org/stable/j.ctt1cx3vcq.

[13] G. Gentile, *Osservazioni sopra le statistiche intermedie*, Nuovo Cimento **17**, 493 (1940), doi:10.1007/BF02960187.

[14] G. Gentile, *Le statistiche intermedie e le proprietà dell'elio liquido*, Nuovo Cimento **19**, 109 (1942), doi:10.1007/BF02960192.

[15] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Phys. Rev. **115**, 485 (1959), doi:10.1103/PhysRev.115.485.

[16] W. Ehrenberg and R. E. Siday, *The refractive index in electron optics and the principles of dynamics*, Proc. Phys. Soc. B **62**, 8 (1949), doi:10.1088/0370-1301/62/1/303.

[17] B. J. Hiley, *The early history of the Aharonov-Bohm effect*, (arXiv preprint) doi:10.48550/arXiv.1304.4736.

[18] *Aharonov-Bohm effect*, Wikipedia, https://en.wikipedia.org/wiki/Aharonov-Bohm_effect.

[19] L. Schulman, *A path integral for spin*, Phys. Rev. **176**, 1558 (1968), doi:10.1103/PhysRev.176.1558.

[20] M. G. G. Laidlaw and C. M. DeWitt, *Feynman functional integrals for systems of indistinguishable particles*, Phys. Rev. D **3**, 1375 (1971), doi:10.1103/PhysRevD.3.1375.

[21] J. S. Dowker, *Quantum mechanics and field theory on multiply connected and on homogeneous spaces*, J. Phys. A: Gen. Phys. **5**, 936 (1972), doi:10.1088/0305-4470/5/7/004.

[22] H. S. Green, *A generalized method of field quantization*, Phys. Rev. **90**, 270 (1953), doi:10.1103/PhysRev.90.270.

[23] A. M. L. Messiah and O. W. Greenberg, *Symmetrization postulate and its experimental foundation*, Phys. Rev. **136**, B248 (1964), doi:10.1103/PhysRev.136.B248.

[24] M. D. Girardeau, *Permutation symmetry of many-particle wave functions*, Phys. Rev. **139**, B500 (1965), doi:10.1103/PhysRev.139.B500.

[25] D. Finkelstein and J. Rubinstein, *Connection between spin, statistics, and kinks*, J. Math. Phys. **9**, 1762 (1968), doi:10.1063/1.1664510.

[26] R. F. Streater and I. F. Wilde, *Fermion states of a boson field*, Nucl. Phys. B **24**, 561 (1970), doi:10.1016/0550-3213(70)90445-1.

[27] J. Fröhlich, *New super-selection sectors ("soliton-states") in two dimensional Bose quantum field models*, Commun. Math. Phys. **47**, 269 (1976), doi:10.1007/BF01609844.

[28] J. Fröhlich, *Quantum sine-Gordon equation and quantum solitons in two space-time dimensions*, in *Renormalization theory*, Springer, Dordrecht, Netherlands, ISBN 9789401014922 (1976), doi:10.1007/978-94-010-1490-8_12.

[29] B. Klaiber, *The Thirring model*, Lect. Theor. Phys. A **10**, 141 (1968).

[30] J.-M. Souriau, *Structure des systèmes dynamiques. Maîtrises de mathématiques*, Dunod, Paris, France (1970).

[31] L. P. Kadanoff and H. Ceva, *Determination of an operator algebra for the two-dimensional Ising model*, Phys. Rev. B **3**, 3918 (1971), doi:10.1103/PhysRevB.3.3918.

[32] F. J. Wegner, *Duality in generalized Ising models and phase transitions without local order parameters*, J. Math. Phys. **12**, 2259 (1971), doi:10.1063/1.1665530.

[33] E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40**, 149 (1939), doi:10.2307/1968551.

[34] V. S. Varadarajan, *George Mackey and his work on representation theory and foundations of physics*, Contemp. Math. **449**, 417 (2008), doi:10.1090/conm/449/08722.

[35] M. Gell-Mann, *The eightfold way: A theory of strong interaction symmetry*, Synchrotron lab, Pasadena, USA (1961), doi:10.2172/4008239.

[36] S. L. Adler, R. F. Dashen, *Current algebras and applications to particle physics*, W. A. Benjamin, New York, USA (1968).

[37] R. F. Dashen and D. H. Sharp, *Currents as coordinates for hadrons*, Phys. Rev. **165**, 1857 (1968), doi:10.1103/PhysRev.165.1857.

[38] R. Haag and D. Kastler, *An algebraic approach to quantum field theory*, J. Math. Phys. **5**, 848 (1964), doi:10.1063/1.1704187.

[39] G. A. Goldin, *Nonrelativistic current algebras as unitary representations of groups*, J. Math. Phys. **12**, 462 (1971), doi:10.1063/1.1665610.

[40] G. A. Goldin and D. H. Sharp, *Lie algebras of local currents and their representations*, in *Group representations in mathematics and physics*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540053101 (1970), doi:10.1007/3-540-05310-7_31.

[41] G. A. Goldin, J. Grodnik, R. T. Powers and D. H. Sharp, *Non-relativistic current algebra in the 'N/V' limit*, J. Math. Phys. **15**, 88 (1974), doi:10.1063/1.1666513.

[42] R. Menikoff, *The Hamiltonian and generating functional for a nonrelativistic local current algebra*, J. Math. Phys. **15**, 1138 (1974), doi:10.1063/1.1666764.

[43] R. Menikoff and D. H. Sharp, *Representations of a local current algebra: Their dynamical determination*, J. Math. Phys. **16**, 2341 (1975), doi:10.1063/1.522495.

[44] R. Menikoff and D. H. Sharp, *A gauge invariant formulation of quantum electrodynamics using local currents*, J. Math. Phys. **18**, 471 (1977), doi:10.1063/1.523291.

[45] C. Martin, *A mathematical model for the Aharonov-Bohm effect*, Lett. Math. Phys. **1**, 155 (1976), doi:10.1007/BF00398379.

[46] J. M. Leinaas, *Statistics of charge-monopole composites*, Nuovo Cimento A **47**, 19 (1978), doi:10.1007/BF02896224.

[47] F. Wilczek, *Fractional statistics and anyon superconductivity*, World Scientific, Singapore, ISBN 9789810200480 (1990), doi:10.1142/0961.

[48] Y.-S. Wu, *General theory for quantum statistics in two dimensions*, Phys. Rev. Lett. **52**, 2103 (1984), doi:10.1103/PhysRevLett.52.2103.

[49] H. Bartolomei et al., *Fractional statistics in anyon collisions*, Science **368**, 173 (2020), doi:10.1126/science.aaz5601.

[50] J. Nakamura, S. Liang, G. C. Gardner and M. J. Manfra, *Direct observation of anyonic braiding statistics*, Nat. Phys. **16**, 931 (2020), doi:10.1038/s41567-020-1019-1.

[51] S. Ornes, *Physicists prove anyons exist, a third type of particle in the Universe*, Discov. Mag. (2020), https://www.discovermagazine.com/the-sciences/physicists-prove-anyons-exist-a-third-type-of-particle-in-the-universe.

[52] D. Najjar, *'Milestone' evidence for anyons, a third kingdom of particles*, Quanta Mag. (2020), https://www.quantamagazine.org/milestone-evidence-for-anyons-a-third-kingdom-of-particles-20200512/.

[53] C. Cottier, *Bosons, fermions and anyons: What are the three particle kingdoms in the quantum world?*, Discov. Mag. (2021), https://www.discovermagazine.com/the-sciences/bosons-fermions-and-anyons-what-are-the-three-particle-kingdoms-in-the.

[54] C. Dreifus, *A prodigy who cracked open the cosmos*, Quanta Mag. (2021), https://www.quantamagazine.org/frank-wilczek-cracked-open-the-cosmos-20210112/.

[55] A. Khurana, *Bosons condense and fermions 'exclude', but anyons. . . ?*, Phys. Today **42**, 17 (1989), doi:10.1063/1.2811205.

[56] L. Biedenharn, E. Lieb, B. Simon and F. Wilczek, *The ancestry of the 'anyon'*, Phys. Today **43**, 90 (1990), doi:10.1063/1.2810672.

[57] G. A. Goldin and D. H. Sharp, *The diffeomorphism group approach to anyons*, Int. J. Mod. Phys. B **05**, 2625 (1991), doi:10.1142/S0217979291001048.

[58] F. Wilczek, *Anyons*, Sci. Am. **264**, 58 (1991), https://www.jstor.org/stable/e24936887.

[59] G. A. Goldin, D. H. Sharp, *Anyons*, Sci. Am. **265**, 12 (1991), https://www.jstor.org/stable/24938702.

[60] G. A. Goldin, D. H. Sharp, *Acknowledging the physicists who first predicted "anyons"* (2021), https://wgmp.uwb.edu.pl/wgmp39/slides/Goldin/.

[61] G. A. Goldin, D. H. Sharp, *Certified mail to discover magazine editorial director Steve George* (2021), https://wgmp.uwb.edu.pl/wgmp39/slides/Goldin/.

[62] *Policies and standards*, The Washington Post (2021), https://www.washingtonpost.com/policies-and-standards/.

[63] *Anyon*, Wikipedia, https://en.wikipedia.org/wiki/Anyon.

[64] E. G. Teich et al., *Citation inequity and gendered citation practices in contemporary physics*, Nat. Phys. **18**, 1161 (2022), doi:10.1038/s41567-022-01770-1.

[65] S. Sewell, *'Discomfort can break ground': Physicist Stephon Alexander on the value of difference*, The Guardian (2021), https://www.theguardian.com/us-news/2021/sep/01/stephon-alexander-physics-science-brown-university.

[66] S. Alexander, *Fear of a black universe*, Basic Books, New York, USA, ISBN 9781541699632 (2021).

A. Ananthaswamy, *Who is allowed to have wild ideas in physics?*, Nature **597**, 471 (2021), doi:10.1038/d41586-021-02526-2.

On the unexpected fate of scientific ideas: An archeology of the Carroll group

Jean-Marc Lévy-Leblond[⋆]

Université de Nice, France

⋆ jmll@unice.fr

Group ICGTMP
34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

In 1965, I published a paper, exhibiting a hitherto unknown limit of the Lorentz group, which I christened "Carroll group" due to its seemingly paradoxical physical contents. Since I saw it as more curious than relevant, I published it in French in a journal somewhat afar from the mainstream of theoretical physics at that time. It was most gratifying to witness the quite unexpected favour this paper started to enjoy half a century later, so much that a so-called "Carrollian physics" is now developing, with applications in various domains of forefront theoretical physics, such as quantum gravitation, supersymmetry, string theory, etc. I offer this narrative as an example of the very diverse time scales with which scientific ideas may develop — or not.

Received 22-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.006

...notwithstanding the sagacious advice
by Lewis Carroll himself, who wrote:
«It's no use going back to yesterday,
because I was a different person then.» [1]

1 Paying tributes

Let me start by diving in a more remote time than my own in order to acknowledge my debt to some of our predecessors who stressed the importance of group structure as one of the pillars of theoretical physics. A prominent figure in my personal Pantheon is Paul Langevin (1872-1946), whose 150th birthday we recently celebrated. Not only did he contribute to crucial developments of Einsteinian relativity, but in his most resolute and successful endeavour for clarifying and popularizing it, he insisted as early as 1911, on the group theoretical perspective, writing

"It is an experimental fact that the equations between physical quantities by which we translate the laws of the outside world, must have exactly the same form for different groups of observers, for various systems of reference in uniform translation relative to each other.

Figure 1: Paul Langevin and Albert Einstein in 1922.

This requires, in the language of mathematics, that these equations admit of a group of transformations corresponding to a change of reference system to another moving relative to it. The equations of physics must be preserved for all transformations of this group. In such a transformation, when one moves from one reference system to another, measures of various magnitudes, especially those that are related to space and time, are changed in a manner that corresponds to the structure of these notions." [2]

It is worthwhile recalling here what Einstein, who maintained a lifelong friendly relationship with Langevin (Figure 1), wrote in his funeral tribute:

"It appears to me as a foregone conclusion that he would have developed the special relativity theory, had not that been done elsewhere." [3]

One may well admire the elegance of the last words... On a more private level, I wish to salute the memory of my group-theoretical masters Louis Michel (1923-1999) and François Lurçat (1927-2012), as well as of my friend and collaborator Henri Bacry (1928-2010).

2 Siring the Carroll group

After my PhD, dedicated to the Galilei group and its representations in the line of Wigner's epochal paper on the Lorentz group [4], I soon started to teach Einsteinian relativity, stressing both its analogies and its differences with Galilean relativity. While preparing my lectures, I stumbled upon an unexpected difficulty, concerning the validity of the Galilean approximation to the Lorentz transformations. Choosing the natural system of units where the limit velocity c is taken as unity, the Lorentz transformations relating spacetime intervals in two equivalent reference frames with relative velocity v take the form:

$$\begin{cases} \Delta x' = \gamma(\Delta x - v\Delta t)\,, \\ \Delta t' = \gamma(\Delta t - v\Delta x)\,, \quad \text{where} \quad \gamma = \sqrt{1-v^2}\,. \end{cases} \tag{1}$$

If we now wish to obtain approximate formulas in the situation where $v \ll 1$, the Galilean group law transformations

$$\Delta x' = \Delta x - v\Delta t\,, \tag{2}$$

$$\Delta t' = \Delta t\,, \tag{3}$$

do not arise obviously unless we require the additional condition $\Delta x \ll \Delta t$. It thus appears that the validity of the Galilean approximation requires not only velocities small with respect

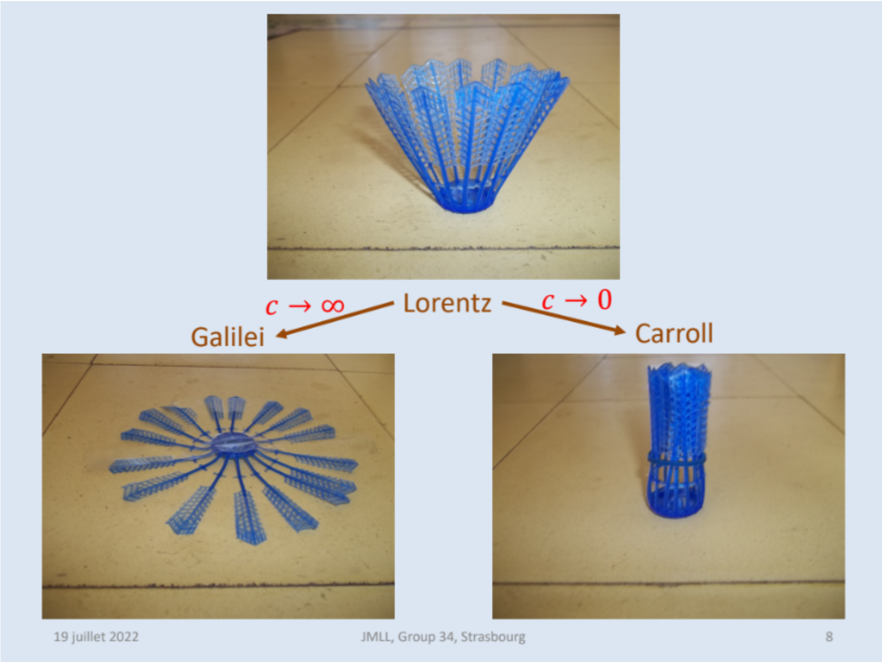


Figure 2

to the limit velocity, but also large time-like intervals, a condition which has stood implicit in most (not to say all) treatments of the subject. It is then natural to inquire what happens in the opposite case of large space-like intervals, that is for $\Delta t \ll \Delta x$, yielding the transformation laws

$$\Delta x' = \Delta x\,, \tag{4}$$
$$\Delta t' = \Delta t - v\Delta x\,. \tag{5}$$

These transformations obviously form a group, which, as well as the Galilei group, is a contraction of the Lorentz group [5] and thus seemed as well worth of recognition [6]. In a world governed by such an invariance group, causality almost completely disappears, since time-ordering is only preserved along the timelines at given space points. For this reason, I ventured to propose the name "Carroll group" for this alternate degenerate limit of the Lorentz group [1].

An illustrative way of expressing the situation is that, while the Galilei group appears as the limit of the Lorentz group when a rescaling of Minkowski space flattens the light cone on the constant time hyperplane, the Carroll group emerges when the light cone closes up on the time axis (Figure 2). This leads to another manner of considering the Carroll group, by recovering dimensional velocities. Since the limit velocity is but the slope of light rays in Minkowski spacetime, it results that, whereas the Galilei group corresponds to the well-known limit $c \to \infty$, the Carroll group may be seen as resulting from the inverse limit, that is $c \to 0$, weird as this limit may appear at first glance.

At the time, I was convinced that, because of the acausal nature of a universe obeying Carrollian invariance, the usefulness of the Carroll group was very low, and I apologized for begetting it, arguing that my purpose was mainly pedagogical. So little did I believe in its fate that I published my result in French, in a journal which did not belong to the mainstream publications of theoretical physics, concluding my paper by stating with some cheekiness that "theoretical physics has recently shown itself to be friendly enough for many groups with a limited physical interest; this is why I have not too much scruple in bringing to light this degenerate cousin of the Poincaré group".

It took me a long time to learn that a paper quite similar to mine had been written independently by an Indian colleague, N. D. Sen Gupta, working at the Bombay Tata Institute for Fundamental Research [7]. Rather unexpectedly, his paper had been published very soon after mine, and, as far as I know, might even had been written before. While it appeared in a journal at that time more prominent in theoretical physics, the long delay in my recognition of Sen Gupta's work and the paucity of any references to it in the literature for many years bear witness to the little interest elicited by our almost simultaneous small discovery.

3 A terminological excursus

"Relativity" ?

With the benefit of half a century of personal and collective maturation, I wish to indulge here in a piece of self-criticism concerning the use of the term "non-relativistic" in the title and text of my old paper. The word was routinely used until late in the XXth century to mean, as many dictionaries still propose, "not based on or not involving the special relativity of Einstein". However, it has become clear today that Einsteinian relativity is not the only consistent one and that the kinematical structure of classical mechanics obeys a relativity theory of its own, now known as Galilean relativity. One might object to this denomination that it is somewhat anachronistic, as the general notion of space-time symmetries would take almost three centuries after Galileo to emerge. Nonetheless, a most important paragraph in Galileo's seminal Dialogo shows quite clearly that he had fully understood the invariance of physical laws with respect to changes of reference frames with uniform velocity [8].

But we might well take one more step in the revision of the current terminology. Indeed, the term "relativity" itself, which, concerning its use in physics, dates back to Poincaré [9], may be considered as a misnomer. As early as 1948, an undisputable authority, namely A. Sommerfeld, wrote:

"[The theory of space-time] is an Invariantentheorie of the Lorentz group: The relativity of space and time is not the essential thing, which is the independence of the laws of nature from the point of view of the observer." [10]

This independence/invariance is characterized by the intrinsic structure of spacetime, that is, what I believe natural to name a **chronogeometry**, exactly as we call elementary geometry the theoretical structure of Euclidean space, with the term "geometry" having been generalised to describe the structure of variously defined spaces, according to Klein's Erlangen program [11].

A proposal:
— Replace "relativity" by "chronogeometry".
— Replace "non-relativistic" by "Galilean" (... or "Carrollian").

"Speed of light" ?

Einstein's derivation of Lorentz transformations (1905) was based on the so-called second postulate, that of the invariance of light velocity. But Einsteinian chronogeometry is not intrinsically linked to the properties of light: Indeed, as a universal structure of Minkowskian spacetime, it rules as well non-electromagnetic phenomena, such as strong interactions. As a matter of fact, it was realized as early as 1911 that the Lorentz group may be constructed without any appeal to the second postulate (Ignatowsky, Frank & Rothe), as many authors have

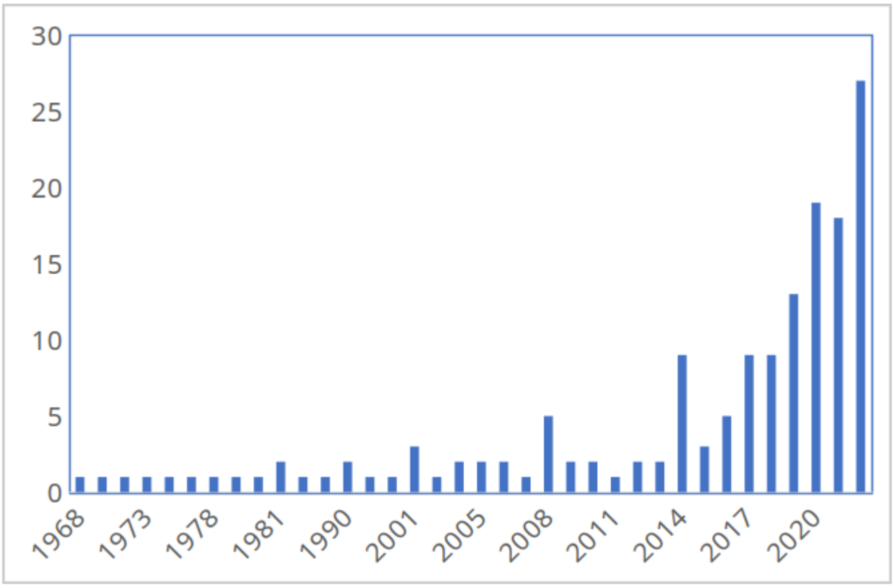


Figure 3: Yearly quotations of the birth certificate for the Carroll group [6]. (Thanks to Yves Gingras for providing me with this graph).

rediscovered since.[1] Furthermore, suppose the photon has a non-zero mass, however small, a case which, obviously, cannot be excluded; then light would not travel with a non-invariant velocity...

A proposal:
— Replace "speed of light" by "speed limit" or better: "Einstein constant".

"Group contraction"

The notion of group contraction, which formalize the limiting process leading from Einsteinian chronogeometry to the Galilean (or Carrollian) one, was introduced just a century ago by Inönu and Wigner in the following terms:

"We shall call the operation of obtaining a new group by a singular transformation of the infinitesimal elements of the old group a contraction of the latter. The reason for this term will become clear below. (...) In the limit $e = 0$ (if such a limit exists), one will have contracted the whole group to an infinitesimally small neighborhood of the group."

But this is hardly a proper description. In fact, the process goes rather the other way, that is, extending the structure of an infinitesimal neighbourhood of the group to that of a fully-fledged new group. More than a simple extension, this change of structure in fact deserves to be considered as a distension.

A proposal:
— Replace "group contraction" by "group distension".

Such discussions about the terminology of physics are by many considered as futile nit-picking: Why, do they ask, should we care about words since we have the formulas to rely on? This is not the place to develop a detailed answer [13]. Let me only state that paying attention to our linguistic choices and assessing their relevance, may be of great significance for

[1] For a recent example with bibliography (incomplete), see [12].

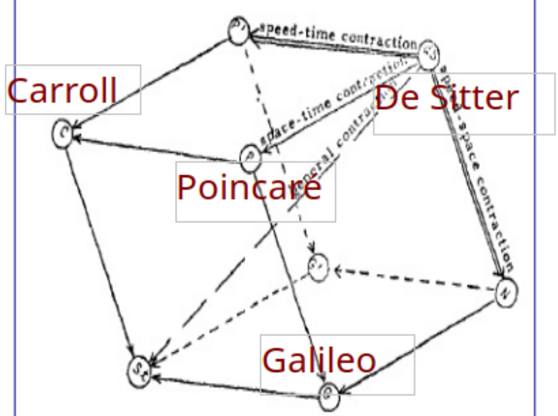


FIG. 1. The contraction scheme for the relativity groups.

Figure 4: Possible chronogeometries [16].

research, as it should go beyond formalism, for teaching, as it should go beyond technicalities, for popularisation, as it should go beyond catchwords.

4 The late blooming of Carrollian physics

As I had suspected right from the beginning, the decades following the appearance of the Carroll group on the theoretical scene, viewed very little references to my paper or Sen Gupta's.

From 1965 to the early 2000s, there were one or two quotations per year in journals as varied as *J. Math.Phys., Ann. Der Phys., J. of Phys. Comm., Nuovo Cimento, Int. J. Theor. Phys., Bull. Acad. R. Belg., Phys. Lett., J. Geometry & Phys.*, etc., dealing mainly with general considerations about abstract Group theory, Special relativity, Electromagnetism. But, to my utmost surprise, from the 2000s onwards, with a notable acceleration after 2010, more and more papers appeared dealing with Carrollian chronogeometry, with a concentration in *J. Math. Phys., J. of High Energy Phys., Class. & Qu. Gravity, Phys. Rev. D, Phys. Rev. Lett., General Relat. & Gravit., J. of Cosmology & Astroparticle, Phys. Lett.* (Figure 3). These works are now mainly concerned with General relativity, Field Theory, Gravitation, etc. Some keywords characterizing them are: Conformal structures, Asymptotic flat spacetimes, Nonrelativistic SUSY, Symplectic spacetime, Non-Riemannian isometries, Bondi-Metzner-Sachs group, Null manifolds, Cartan geometry, Anti-deSitter symmetry, BMS field theories, Tachyon cosmology, Flat holography, Chern-Simons supergravity, etc.[2]

While this renewal of interest may at first seem puzzling, the reason in fact is rather easy to understand. Indeed, if Galilean chronogeometry yields a simple approximate way to explore the portion of Minkowski spacetime interior to the lightcone, Carrollian chronogeometry furnishes a similar simple approximate way to explore the portion of Minkowski spacetime exterior to the lightcone. Even though the latter is an acausal region, it is a constitutive portion of spacetime and plays a significant role in many physical phenomena. This argument may be strengthened by considering how Lorentz transformations may be generated by a combination of Galilean and Carrollian ones[3]

A few years after the appearance of the Carroll group, it was shown by Henri Bacry and

[2]See the recent Carroll workshops [14].
[3]See a recent and long overdue annex to [6]: J.-M. Lévy-Leblond (2023) [15].

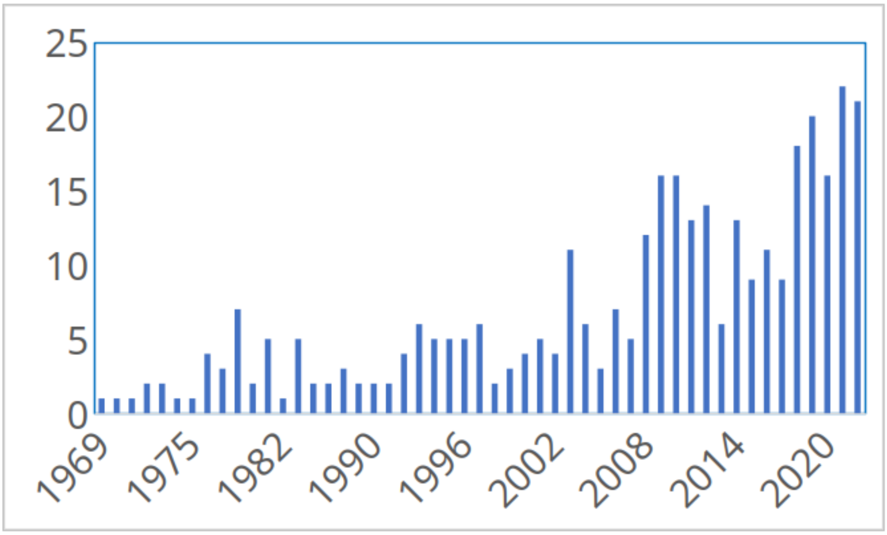


Figure 5: Yearly quotations of [16]. (Thanks to Yves Gingras for providing me with this graph).

myself to take place in an overall description and classification of logically and physically possible chronogeometries [16], as summed up by an elegant diagram (Figure 4). This paper had the good fortune to be remarked by F. Dyson who wrote the following lines in a wonderful article about various "missed opportunities" in theoretical physics:

"The eight groups can then be visualized as the vertices of a cube. P and G are the only kinematical groups that correspond to orthodox physical universes. But the other five groups are just as good, mathematically speaking. The most interesting of the heterodox groups are N and C. N describes a Newtonian universe with curved space-time. C describes a universe in which space is absolute, in contrast to the Galilei group G which has time absolute. The group C was discovered by Lévy-Leblond and called by him the Carroll group. In the Carroll universe, all objects have zero velocity although they may have nonzero momentum. Carroll was a pure mathematician who had already foreseen this possibility in 1871: 'A slow sort of country,' said the Queen, "Now, here, you see, it takes all the running you can do, to keep in the same place." But his mathematical colleagues once again missed an opportunity by failing to take him seriously." [17]

Our paper had a career quite similar, although a bit more favourable, to the birth act of the Carroll group, in that it has known a fast increasing number of quotations after 2000 (Figure 5).

5 A few conclusions

- The pace of contemporary science is not necessarily "fast and furious":

  "Torniamo all'antica. Sarà un progresso." ["Let us go back to antiquity. It will be a progress."]. [18]

- Sharing knowledge may help developing it:

  "L'homme ne peut jouir de ce qu'il sait qu'autant qu'il peut le communiquer à quelqu'un (et ainsi l'enrichir)." ["One cannot enjoy one's knowledge but by sharing it (and thereby enriching it)."]. [19]

- Language should be taken seriously:

  "Le parole (...) non presentano la sola idea dell'oggetto significato, ma quando piu o quando meno immagini accessorie. Ed è pregio sommo della lingua l'aver di queste parole. Le voci scientifiche presentano la nuda e circoscritta idea di quel tale oggetto e percio si chiamano termini perche determiniano e definiscono la cosa da tutte le parti." ["Words do not convey only the sheer idea of the object signified, but also a more or less important number of related meanings and pictures. It is the utmost value of language to be thus made of words. Most scientific terms present but the bare and limited idea of the object: They may indeed be called terms, as they determine and confine the thing."]. [20]

- It is worthwhile exploring neglected opportunities:

  "Undoubtedly, there exist many more missed opportunities to create new branches of pure mathematics out of old problems of applied science." [17]

  But no less undoubtedly, reciprocally and more generally, there exist many missed opportunities to solve new problems of science out of old branches of it.

References

[1] L. Carroll, *Alice's adventures in wonderland*, Everyman Paperbacks, New York, USA, ISBN 9780460873598 (1993).

[2] P. Langevin, *L'évolution de l'espace et du temps*, Scientia **10**, 31 (1911), [English translation: https://en.wikisource.org/wiki/Translation:The_Evolution_of_Space_and_Time].

[3] A. Einstein, *Paul Langevin*, La Pensée, **12**, 13 (1947).

[4] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40**, 149 (1939).

[5] E. Inönu and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Natl Acad. Sci. **39**, 510 (1953).

[6] J.-M. Lévy-Leblond, *Une nouvelle limite non-relativiste du groupe de Poincaré*, Ann. Inst. Henri Poincaré **3**, 1 (1965).

[7] N. D. Sen Gupta, *On an analogue of the Galilei group*, Nuov. Cim. **44**, 512 (1966).

[8] G. Galileo, *Dialogo sopra i due massimi sistemi del mondo, seconda giornata*, Florence (1632).

[9] H. Poincaré, *La Science et l'hypothèse*, Flammarion, Paris, France, ISBN 9782081404366 (1917).

[10] A. Sommerfeld, *Philosophie und Physik seit 1900*, Naturwis. Rundschau **1**, 97 (1948).

[11] F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Math. Ann. **43**, 63 (1893), doi:10.1007/BF01446615.

[12] J. Llosa, *Yet another derivation of special relativity transformations without the second postulate*, (arXiv preprint) doi:10.48550/arXiv.1401.6568.

[13] J.-M. Lévy-Leblond, *La langue tire la science*, in *La Pierre de touche*, Éditions Gallimard, Paris, France (1996)

J.-M. Lévy-Leblond, *Quantum words for a quantum world*, in *Epistemological and experimental perspectives on quantum physics*, Vienna Circle Institute Yearbook 7, 75 (1999).

[14] Carrol workshop, Vienna (February 2022), http://quark.itp.tuwien.ac.at/~carroll/; 2nd Carrol workshop, Mons (September 2022), http://web.umons.ac.be/pucg/event/carroll-workshop-2/.

[15] J.-M. Lévy-Leblond, *Quand Galilée et Carroll engendrent Lorentz*, Ann. Henri Poincaré **24**, 3209 (2023), doi:10.1007/s00023-023-01321-1.

[16] H. Bacry and J.-M. Lévy-Leblond, *Possible kinematics*, J. Math. Phys. **9**, 1605 (1968).

[17] F. Dyson, *Missed opportunities*, Bull. Am. Math. Soc. **78**, 635 (1965).

[18] G. Verdi, *A letter to F. Florimo* (1871).

[19] G. Casanova, L'Icosaméron, Prague, Bohemia (1788) [with [...] added by myself].

[20] G. Leopardi, *Zibaldone di pensieri*, Le Monnier, Florence, Italy (1898).

Emergent symmetries in atomic nuclei: Probing nuclear dynamics and physics beyond the standard model

Kristina D. Launey[1⋆], K. S. Becker[1], G. H. Sargsyan[1,2], O. M. Molchanov[1], M. Burrows[1], A. Mercenne[1], T. Dytrych[1,3], D. Langr[4] and J. P. Draayer[1]

1 Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA
2 Lawrence Livermore National Laboratory, Livermore, California 94550, USA
3 Nuclear Physics Institute of the Czech Academy of Sciences, 250 68 Řež, Czech Republic
4 Department of Computer Systems, Faculty of Information Technology, Czech Technical University in Prague, 16000 Praha, Czech Republic

⋆ klauney@lsu.edu

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Dominant shapes naturally emerge in atomic nuclei from first principles, thereby establishing the shape-preserving symplectic Sp(3, ℝ) symmetry as remarkably ubiquitous and almost perfect symmetry in nuclei. We discuss the critical role of this emergent symmetry in enabling machine-learning descriptions of heavy nuclei, *ab initio* modeling of α clustering and collectivity, as well as tests of beyond-the-standard-model physics. In addition, the Sp(3, ℝ) and SU(3) symmetries provide relevant degrees of freedom that underpin the *ab initio* symmetry-adapted no-core shell model with the remarkable capability of reaching nuclei and reaction fragments beyond the lightest and close-to-spherical species.

Received 17-12-2022
Accepted 21-07-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.007

Contents

1 Introduction **2**

2 Emergent symmetries in nuclei: Sp(3, ℝ) and SU(3) **3**
2.1 SU(3) scheme 3
2.2 Sp(3, ℝ) scheme 4
2.3 *Ab initio* symmetry-adapted no-core shell model 7

3 Critical role of symmetries for studies and predictions of nuclear properties **8**
3.1 Machine learning pattern recognition with the SA-NCSM 8
3.2 Probing clustering and physics beyond the standard model 9
3.3 Optical potential in the symmetry-adapted framework for nuclear reactions 10

4 Conclusion 11

References 12

1 Introduction

Dominant shapes, often very few in number, naturally emerge in atomic nuclei.[1] This remarkable result has been recently shown by large-scale nuclear simulations from first principles [2]. Indeed, each nuclear shape respects an exact symmetry, namely, the symplectic $\mathrm{Sp}(3,\mathbb{R})$ symmetry [3, 4]. Thereby the outcome of these simulations establishes the symplectic $\mathrm{Sp}(3,\mathbb{R})$ symmetry as remarkably ubiquitous and almost perfect symmetry in nuclei up through the calcium region (anticipated to hold even stronger in heavy nuclei [5]). This outcome also exposes for the first time the fundamental role of the $\mathrm{Sp}(3,\mathbb{R})$ symmetry and suggests that its origin is rooted in the strong nuclear force, in the low-energy regime.

This builds upon a decades-long research, starting with the pivotal work of Draayer [4,6–8] and that of Rowe and Rosensteel [3, 5, 9, 10], who have successfully harnessed group theory as a powerful tool for understanding and computing the intricate structure of nuclei. This pioneering work has been instrumental in designing the theory that underpins many highly ordered patterns unveiled amidst the large body of experimental data [11–13]. In addition, it has explained phenomena observed in energy spectra, $E2$ transitions and deformation, giant resonances (GR), scissor modes and $M1$ transitions, electron scattering form factors, as well as the interplay of pairing with collectivity. The new developments and insights have provided the critical structure raised upon the very foundation laid by Elliott [14–16] and Hecht [17,18], and opened the path for large-scale calculations feasible today on supercomputers. And while these earlier algebraic models have been very successful in explaining dominant nuclear patterns, they have assumed symmetry-based approximations and have often neglected symmetry mixing. This establishes $\mathrm{Sp}(3,\mathbb{R})$ as an *effective symmetry*[2] for nuclei, which may or may not be badly broken in realistic calculations. It is then imperative to probe if this symmetry naturally arises within an *ab initio* framework, which will, in turn, establish its fundamental role.

Indeed, within an *ab initio* framework without *a priori* symmetry assumptions, the symmetry-adapted no-core shell model (SA-NCSM) [8, 20, 21] with chiral effective field theory (EFT) interactions [22–24] has recently confirmed the goodness of the symplectic $\mathrm{Sp}(3,\mathbb{R})$ symmetry that is only slightly broken. With no parameters to adjust, the SA-NCSM is capable then not only to explain but also to predict the emergence of nuclear shapes and collectivity across nuclei, even in close-to-spherical nuclear states without any recognizable rotational properties.

Within an *ab initio* framework, the emergent symmetries play a critical role, as they can inform relevant degrees of freedom. In particular, a symmetry-adapted many-body basis can be employed, as in the SA-NCSM, thereby providing solutions for drastically reduced sizes of the spaces in which particles reside (referred to as "model spaces") compared to the corresponding ultra-large model spaces, without compromising the accuracy of results for various nuclear observables. By exploiting symplectic symmetry, *ab initio* descriptions of spherical and

[1]This publication reuses some material from [1] under the terms of its CC BY license.

[2]A familiar example for an effective symmetry is SU(3). While the Elliott model with a single SU(3) irrep explains ground-state rotational states in deformed nuclei, the SU(3) symmetry is, in general, largely mixed, mainly due to the spin-orbit interaction (nonetheless, SU(3) has been shown to be an excellent quasi-dynamical symmetry, that is, each rotational state has almost the same SU(3) content [19]).

deformed nuclei up through the calcium region are now possible without the use of effective charges [8, 21, 25–27]. This allows the SA-NCSM to accommodate even larger model spaces and to reach heavier nuclei, such as ^{20}Ne [2], ^{21}Mg [28], ^{22}Mg [29], ^{28}Mg [30], as well as ^{32}Ne and ^{48}Ti [31].

In this paper, we briefly outline the SU(3) and Sp(3, $\mathbb{R}$) schemes utilized by the *ab initio* SA-NCSM. We overview the critical role of the emergent Sp(3, $\mathbb{R}$) symmetry in enabling machine-learning descriptions of heavy nuclei [32], *ab initio* modeling of α clustering and collectivity, along with tests of beyond-the-standard-model physics [33]. In addition, we show that with the help of the SA-NCSM, which expands *ab initio* applications up to medium-mass nuclei by using the dominant symmetry of nuclear dynamics, one can provide solutions to reaction processes in this region, with a focus on elastic neutron scattering.

2 Emergent symmetries in nuclei: Sp(3, $\mathbb{R}$) and SU(3)

2.1 SU(3) scheme

It is well known that SU(3) [6, 14, 18, 34, 35] is the symmetry group of the spherical harmonic oscillator (HO) that underpins the valence-shell model and the valence-shell SU(3) (Elliott) model [14–16] (for technical details of SU(3), see Ref. [36]). The Elliott model has been shown to naturally describe rotations of a deformed nucleus without the need for breaking rotational symmetry. But even beyond the valence shell, the SU(3) scheme provides a classification of the complete shell-model space in multiple shells, and is related to the LS-coupling and jj-coupling schemes via a unitary transformation. It divides the space into basis states of definite $(\lambda\,\mu)$ quantum numbers of SU(3) that are linked to the intrinsic quadrupole deformation according to the established mapping [37–39]. For example, the simplest cases, $(0\,0)$, $(\lambda\,0)$, and $(0\,\mu)$, describe spherical, prolate, and oblate deformation, respectively,[3] while a general nuclear state is typically a superposition of several hundred various triaxially deformed configurations. Note that, in this respect, basis states can have little to no deformation, and, e.g., about 60% of the ground state of the closed-shell ^{16}O is described by a single SU(3) basis state, the spherical $(0\,0)$.

Specifically, in the SU(3) scheme, in place of the spherical quantum numbers $|\eta l m_l\rangle$, one can consider the single-particle HO basis $\left|\eta_z \eta_x \eta_y\right\rangle$, the HO quanta in the three Cartesian directions, z, x, and y, with $\eta_x + \eta_y + \eta_z = \eta$ ($\eta = 0, 1, 2, \ldots$ for s, p, sd, ... shells). For a given HO major shell, the complete shell-model space is then specified by all distinguishable distributions of η_z, η_x and η_y. E.g., for $\eta = 2$, there are 6 different distributions, $(\eta_z, \eta_x, \eta_y) = (2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1)$ and $(0,0,2)$. The number of these configurations is $\Omega_\eta = (\eta+1)(\eta+2)/2$ (spatial degeneracy) and the associated symmetry is described by the $U(\Omega_\eta)$ unitary group. Each of these (η_z, η_x, η_y) configurations can be either unoccupied or has maximum of two particles with spins $\uparrow\downarrow$.

As a simple example for an SU(3)-scheme basis state, consider $A = 2$ protons in the sd shell ($\eta = 2$) with a particle in the $(2,0,0)$ level with spin $\uparrow$ and another in the $(1,1,0)$ level with spin $\uparrow$. The total number of quanta in each direction is $(\eta_z^{\text{tot}}, \eta_x^{\text{tot}}, \eta_y^{\text{tot}},) = (3,1,0)$, or equivalently, $\eta^{\text{tot}}(\lambda\,\mu) = 4(2\,1)$, where $\eta^{\text{tot}} = \eta_x^{\text{tot}} + \eta_y^{\text{tot}} + \eta_z^{\text{tot}}$, together with $\lambda = \eta_z^{\text{tot}} - \eta_x^{\text{tot}}$ and $\mu = \eta_x^{\text{tot}} - \eta_y^{\text{tot}}$ labeling an SU(3) irrep, in addition to the total intrinsic spin and its projection SM_S. For given $(\lambda\,\mu)$, the quantum numbers κ, L and M_L are given by Elliott [14, 15], according to the SU(3) $\overset{\kappa}{\supset}$ SO(3)$_L \supset$SO(2)$_{M_L}$, where the label κ distinguishes multiple occur-

[3] Following this mapping, quadrupole moments of $(0\,0)$, $(\lambda\,0)$, and $(0\,\mu)$ configurations – in a simple classical analogy to rotating spherical, prolate, and oblate spheroids in the lab frame [40] – are zero, negative, and positive, respectively.

rences of the same orbital angular momentum L in the parent irrep $(\lambda\,\mu)$. For our example, $(\lambda\,\mu) = (2\,1)$ with $\kappa = 1$, $L = 1,2,3$, and $M_L = -L, -L+1, \dots, L$. Hence, the set $\{\eta^A(\lambda\,\mu)\kappa(LS)JM\}$ completely labels a 2-proton SU(3)-scheme basis state (with $\eta^{\text{tot}} = A\eta$). A basis state in this scheme for a 2-particle system is given by, $\{a^\dagger_{(\eta\,0)st_z} \times a^\dagger_{(\eta'0)s't'_z}\}^{(\lambda\,\mu)\kappa(LS)JM}\,|0\rangle$, which is an SU(3)-coupled product, provided that $a^\dagger$ is a proper SU(3) tensor; incidentally, the SU(3) tensor $a^\dagger$ of rank $(\lambda\,\mu) = (\eta\,0)$ coincides with the familiar particle creation operator, $a^\dagger_{(\eta\,0)lms\sigma t_z} \equiv a^\dagger_{\eta lms\sigma t_z}$, while the particle annihilation SU(3) tensor of rank $(\lambda\,\mu) = (0\,\eta)$ is given as $\tilde{a}_{(0\,\eta)l-ms-\sigma t_z} = (-1)^{\eta+l-m+s-\sigma} a_{\eta lms\sigma t_z}$. Note that for $\eta = \eta' = 2$, e.g., there are only a few 2-proton configurations $(\lambda\,\mu) = (4\,0)$ with $L = 0,2,4$, $(2\,1)$ with $L = 1,2,3$, and $(0\,2)$ with $L = 0,2$. Furthermore, these basis states are related to LS-coupled basis states (similarly, to jj-coupled basis states) via a simple unitary transformation,

$$\left\{a^\dagger_{(\eta\,0)st_z} \times a^\dagger_{(\eta'0)s't'_z}\right\}^{(\lambda\,\mu)\kappa(LS)JM}|0\rangle = \sum_{l,l'} \langle(\eta\,0)l;(\eta'0)l'\|(\lambda\,\mu)\kappa L\rangle \{a^\dagger_{\eta lst_z} \times a^\dagger_{\eta'l's't'_z}\}^{(LS)JM}|0\rangle\,, \quad (1)$$

where $\langle\dots;\dots\|\dots\rangle$ is the SU(3) analog of the familiar reduced Clebsch-Gordan coefficient [note that there is no dependence on the particle orbital angular momenta, l and l', in the SU(3)-scheme basis states].

An important feature of the SU(3) scheme is that all possible configurations within a major HO shell η (for protons or neutrons) are not constructed using the tedious procedure of coupling of creation operators referenced above, but are readily available based on the $U(\Omega_\eta)$ unitary group of the many-body three-dimensional HO. In particular, the basis construction is implemented according to the reduction [41]

$$\begin{array}{ccc} \mathrm{U}(\Omega_\eta) & & \times \;\; \mathrm{SU}(2) \\ \left[f_1, f_2, \dots f_{\Omega_\eta}\right] & & S_\eta \\ \cup & \alpha_\eta & \\ \mathrm{SU}(3) & & \\ (\lambda_\eta\,\mu_\eta) & & \end{array} \quad (2)$$

with $\mathrm{SU}(3)_{(\lambda_\eta\,\mu_\eta)} \overset{\kappa_\eta}{\supset} \mathrm{SO}(3)_{L_\eta} \supset \mathrm{SO}(2)_{M_{L_\eta}}$ [14, 15], where a multiplicity index α_η distinguishes multiple occurrences of an SU(3) irrep $(\lambda_\eta\,\mu_\eta)$ in a given $\mathrm{U}(\Omega_\eta)$ irrep labeled by Young tableaux, $[\mathbf{f}] = [f_1, f_2, \dots, f_{\Omega_\eta}]$, with $f_1 \geq f_2 \geq \dots \geq f_{\Omega_\eta}$ and $f_i = 0$ (unoccupied), 1 (occupied by a particle), or 2 (occupied by 2 particles of spins $\uparrow\downarrow$). An illustrative example for 4 particles in the pf shell ($\eta = 3$) is shown in Table 1.

2.2 Sp(3, ℝ) scheme

The key role of deformation in nuclei and the coexistence of low-lying quantum states in a single nucleus characterized by configurations with different quadrupole moments [11] makes the quadrupole moment a dominant fundamental property of the nucleus. Hence, the quadrupole moment Q (or deformation) and the monopole moment r^2 (or "size" of the nucleus), along with nuclear masses, establishes the energy scale of the nuclear problem. Indeed, the nuclear monopole and quadrupole moments underpin the essence of symplectic $\mathrm{Sp}(3,\mathbb{R})$ symmetry.

Specifically, for A particles in three-dimensional space, the complete basis for the shell model is described by $\mathrm{Sp}(3A,\mathbb{R})\times\mathrm{U}(4)$ [10], where $\mathrm{Sp}(3A,\mathbb{R})$ is the group of all linear canonical transformations of the $3A$-particle phase space and Wigner's supermultiplet group U(4) describes the complementary spin-isospin space. A complete translationally invariant shell-

Table 1: SU(3)×SU(2)$_S$ configurations for 4 protons (neutrons) in the pf shell ($\eta = 3$ with $\Omega_\eta = 10$). Note that a spatial symmetry represented by a Young tableau $\left[f_1, \ldots, f_{\Omega_\eta}\right]$ is uniquely determined by its complementary spin symmetry of a given intrinsic spin S_η (conjugate Young tableaux) ensuring the overall antisymmetrization of each $U(\Omega_\eta)\times\mathrm{SU}(2)_{S_\eta}$ configuration with respect to spatial and spin degrees of freedom (d.o.f.) [41].

Spatial d.o.f.		Spin d.o.f.
U(10) $\supset$	SU(3)	SU(2)
$[f_1 f_2 \ldots f_{10}]$	$(\lambda\,\mu)$	S
$[2^2]$	$(8\,2),(7\,1),(4\,4)^2,(5\,2),(0\,6),(6\,0),(3\,3)$ $(1\,4),(4\,1),(2\,2)^2,(1\,1)$	$S=0$
$[21^2]$	$(9\,0),(6\,3),(7\,1),(4\,4),(2\,5),(5\,2)^2,(3\,3)^2$ $(1\,4)^2,(4\,1)^2,(2\,2),(0\,3),(3\,0)^2,(1\,1)$	$S=1$
$[1^4]$	$(5\,2),(0\,6),(3\,3),(2\,2),(3\,0)$	$S=2$

model basis is classified according to (see, e.g., [5, 10]),

$$\begin{array}{ccc} \mathrm{Sp}(3(A-1),\mathbb{R}) & \times & \mathrm{U}(4) \\ \cup & & \cup \\ \mathrm{Sp}(3,\mathbb{R})\times \mathrm{O}(A-1) & & \mathrm{SU}(2)_S \times \mathrm{SU}(2)_T\,. \end{array} \tag{3}$$

The Sp(3, $\mathbb{R}$) scheme utilizes the symplectic group Sp(3, $\mathbb{R}$). It consists of all *particle-independent* linear canonical transformations of the single-particle phase-space observables, the positions $\vec{r}_i$ and momenta $\vec{p}_i$ (with particle index $i = 1,\ldots,A$ and spacial directions $\alpha, \beta = x, y, z$)

$$r'_{i\alpha} = \sum_\beta A_{\alpha\beta} r_{i\beta} + B_{\alpha\beta} p_{i\beta}\,, \tag{4}$$

$$p'_{i\alpha} = \sum_\beta C_{\alpha\beta} r_{i\beta} + D_{\alpha\beta} p_{i\beta}\,, \tag{5}$$

that preserve the Heisenberg commutation relations $[r_{i\alpha}, p_{j\beta}] = i\hbar\delta_{ij}\delta_{\alpha\beta}$ [5, 8, 42]. Generators of these transformations, symbolically denoted as matrices **A**, **B**, **C**, and **D**, are constructed as "quadratic coordinates" in phase space, $\vec{r}_i$ and $\vec{p}_i$, and, most importantly, sum over all the particles and act on the space orientation. Hence, the generators include physically relevant operators: the total kinetic energy ($\frac{p^2}{2} = \frac{1}{2}\sum_i \vec{p}_i\cdot\vec{p}_i$), the monopole moment ($r^2 = \sum_i \vec{r}_i\cdot\vec{r}_i$), the quadrupole moment ($Q_{2M} = \sqrt{16\pi/5}\sum_i r_i^2 Y_{2M}(\hat{r}_i)$), the orbital angular momentum $\left(\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i\right)$, and the many-body harmonic oscillator Hamiltonian $\left(H_0 = \frac{p^2}{2} + \frac{r^2}{2}\right)$.

In addition, other generators describe multi-shell collective vibrations and vorticity degrees of freedom for a description from irrotational to rigid rotor flows.

On the contrary, the generators of the complementary O(A) sum over the three spatial directions and act on the particle index, with a growing complexity with increasing particle number. One can then organize the A-particle model space according to the dual group O($A-1$), with $O(A) \supset \mathrm{O}(A-1) \supset S_A$. The O($A$) is the group of orthogonal transformations that act on the "particle-index" space (transformations of nucleon coordinates, $r_{i\alpha} \to \sum_{j=1}^{A} r_{j\alpha} P_{ji}$, that leave the O(A) scalars $r_\alpha \cdot r_\beta = \sum_{i=1}^{A} r_{i\alpha} r_{i\beta}$ invariant for $\alpha, \beta = x, y, z$). This scheme is reviewed in detail in Refs. [5, 10]. O($A-1$) is the subgroup of O(A) which leaves center-of-mass coordinates invariant (note that center-of-mass coordinates are symmetric with respect to nucleon indices and, therefore, invariant under S_A permutations) and has as a subgroup the permutation group S_A, which permutes the spatial coordinates of a system of A particles.

The Sp($3,\mathbb{R}$) scheme utilizes an important group reduction to classify many-particle basis states $|\sigma n \rho \omega \kappa L M\rangle$ of a symplectic irrep,

$$\begin{array}{ccccccc} \mathrm{Sp}(3,\mathbb{R}) & \supset & U(3) & \supset & \mathrm{SO}(3) & \supset & SO(2) \\ \sigma & n\rho & \omega & \kappa & L & & M \end{array} \tag{6}$$

where $\sigma \equiv N_\sigma(\lambda_\sigma \mu_\sigma)$ labels the Sp($3,\mathbb{R}$) irrep, $n \equiv N_n(\lambda_n \mu_n)$, $\omega \equiv N(\lambda_\omega \mu_\omega)$, and $N = N_\sigma + N_n$ is the total number of HO quanta (ρ and κ are multiplicity labels) [5]. The relation of these symplectic basis states to M-scheme states of the NCSM is provided in Ref. [43]. Importantly, a single-particle Sp($3,\mathbb{R}$) irrep spans *all* positive-parity (or negative-parity) states for a particle in a three-dimensional spherical or triaxial (deformed) harmonic oscillator.

The translationally invariant (intrinsic) symplectic Sp($3,\mathbb{R}$) generators can be written as SU(3) tensor operators in terms of the harmonic oscillator raising, $b_{i\alpha}^{\dagger(10)} = \frac{1}{\sqrt{2}}(r_{i\alpha} - i p_{i\alpha})$, and lowering $b^{(01)}$ dimensionless operators (with $\mathbf{r}$ and $\mathbf{p}$ the laboratory-frame position and momentum coordinates and $\alpha = 1,2,3$ for the three spatial directions),

$$A^{(20)}_{\mathfrak{L}M} = \frac{1}{\sqrt{2}} \sum_{i=1}^{A} \{b_i^\dagger \times b_i^\dagger\}^{(20)}_{\mathfrak{L}M} - \frac{1}{\sqrt{2}A} \sum_{s,t=1}^{A} \{b_s^\dagger \times b_t^\dagger\}^{(20)}_{\mathfrak{L}M}, \tag{7}$$

$$C^{(11)}_{\mathfrak{L}M} = \sqrt{2} \sum_{i=1}^{A} \{b_i^\dagger \times b_i\}^{(11)}_{\mathfrak{L}M} - \frac{\sqrt{2}}{A} \sum_{s,t=1}^{A} \{b_s^\dagger \times b_t\}^{(11)}_{\mathfrak{L}M},$$

$$H^{(00)}_{00} = \sqrt{3} \sum_{i} \{b_i^\dagger \times b_i\}^{(00)}_{00} - \frac{\sqrt{3}}{A} \sum_{s,t} \{b_s^\dagger \times b_t\}^{(00)}_{00} + \frac{3}{2}(A-1), \tag{8}$$

together with $B^{(02)}_{\mathfrak{L}M} = (-)^{\mathfrak{L}-M}(A^{(20)}_{\mathfrak{L}-M})^\dagger$ ($\mathfrak{L} = 0,2$), where the sums run over all A particles of the system. Equivalently, the symplectic generators, being one-body-plus-two-body operators can be expressed in terms of the fermion creation operator $a^\dagger_{(\eta\, 0)}$ and its SU(3)-conjugate annihilation operator, $\tilde{a}_{(0\,\eta)}$. This is achieved by using the known matrix elements of the position and momentum operators in a HO basis, and hence, e.g., the first sum of $A^{(20)}_{\mathfrak{L}M}$ in Eq. (7) becomes, $\sum_\eta \sqrt{\frac{(\eta+1)(\eta+2)(\eta+3)(\eta+4)}{12}} \left\{a^\dagger_{(\eta+2\,0)} \times \tilde{a}_{(0\,\eta)}\right\}^{(20)}_{\mathfrak{L}M}$ [44]. Note that this operator describes excitations of a nucleon from the η shell to the $\eta+2$ shell, which corresponds to creating two single-particle HO excitation quanta, as manifested in the first term of Eq. (7). The eight $0\hbar\Omega$ operators $C^{(11)}_{\mathfrak{L}M}$ ($\mathfrak{L} = 1,2$) generate the SU(3) subgroup of Sp($3,\mathbb{R}$). They realize the angular momentum operator (dimensionless), $L_{1M} = C^{(11)}_{1M}$, and the Elliott "algebraic" quadrupole moment tensor $\mathcal{Q}^a_{2M} = \sqrt{3} C^{(11)}_{2M}$.

The many-body basis states of an Sp($3,\mathbb{R}$) irrep are built over a bandhead $|\sigma\rangle$ (defined by the usual requirement that the symplectic lowering operators $B^{(02)}_{\mathcal{L}M}$ annihilate it) by $2\hbar\Omega$ 1p-1h monopole or quadrupole excitations, realized by the first term in $A^{(20)}_{\mathfrak{L}M}$ of Eq. (7), together with a smaller $2\hbar\Omega$ 2p-2h correction for eliminating the spurious center-of-mass (CM) motion, realized by the second term in $A^{(20)}_{\mathfrak{L}M}$:

$$|\sigma n\rho\omega\kappa(LS_\sigma)JM\rangle = \sum_{M_L M_S}\langle LM_L;S_\sigma M_S|JM\rangle\{\{A^{(20)}\times A^{(20)}\cdots\times A^{(20)}\}^n\times|\sigma;S_\sigma M_S\rangle\}^{\rho\omega}_{\kappa LM_L}. \quad (9)$$

States within a symplectic irrep have the same spin value, which are given by the spin S_σ of the bandhead $|\sigma;S_\sigma\rangle$. Symplectic basis states span the entire shell-mode space. A complete set of labels includes additional quantum numbers $|\{\alpha\}\sigma\rangle$ that distinguish different bandheads with the same $N_\sigma\,(\lambda_\sigma\,\mu_\sigma)$. Remarkably, these Sp$(3,\mathbb{R})$ basis states are in one-to-one correspondence with a coupled product of the states of the Bohr vibrational model (realized in terms of giant monopole-quadrupole resonance states with irrotational flows), $\{\{A^{(20)}\times A^{(20)}\cdots\times A^{(20)}\}^n\times|N_\sigma(0\,0)\rangle\}^{(\lambda_n\,\mu_n)}$, and $(\lambda_\sigma\,\mu_\sigma)$ deformed states of an SU(3) model [42].

2.3 *Ab initio* symmetry-adapted no-core shell model

Not surprisingly, the symplectic Sp$(3,\mathbb{R})$ symmetry, the underlying symmetry of the symplectic rotor model [3,5], has been found to play a key role across the nuclear chart – from the lightest systems [45,46], through intermediate-mass nuclei [4,8,47], up to strongly deformed nuclei of the rare-earth and actinide regions [5, 19, 48, 49]. The results agree with experimental evidence that supports formation of enhanced deformation and clusters in nuclei, as well as vibrational and rotational patterns, as suggested by energy spectra, electric monopole and quadrupole transitions, radii and quadrupole moments [11,29,50].

The symmetry-adapted no-core shell model [2, 8, 20] capitalizes on these findings and presents solutions in terms of a physically relevant basis of nuclear shapes. It exploits both the SU(3) and Sp$(3,\mathbb{R})$ schemes. Indeed, since the symplectic symmetry does not mix nuclear shapes, the SA-NCSM provides important insight from first principles into the physics of nuclei and their low-lying excitations as dominated by only a few (typically one or two) collective shapes – equilibrium shapes with their vibrations – that rotate (Fig. 1).

By exploiting this almost perfect symmetry, the SA framework resolves the scale explosion problem in nuclear structure calculations, *i.e.*, the explosive growth in computational resource demands with increasing number of particles and model spaces size. We note that the SA-NCSM uses the complete model space (that is, all possible shapes) as usually done in

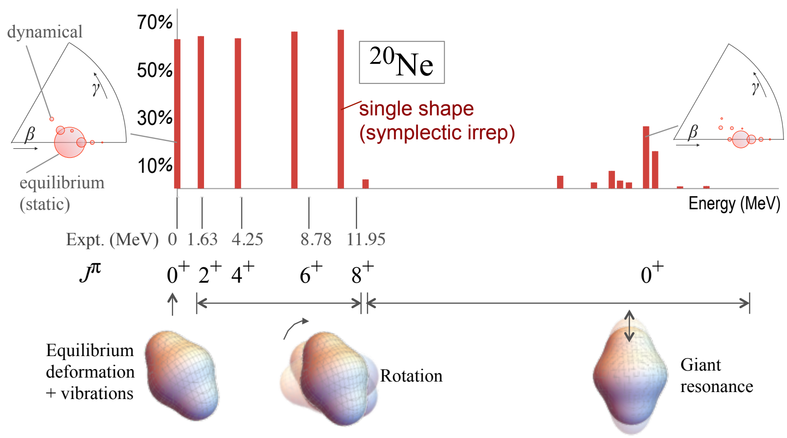


Figure 1: Emergence of almost perfect symplectic Sp$(3,\mathbb{R})$ symmetry in nuclei from first principles, enabling *ab initio* descriptions of collectivity and clustering. *Source:* Figure from [2] @ APS; reproduced with permission.

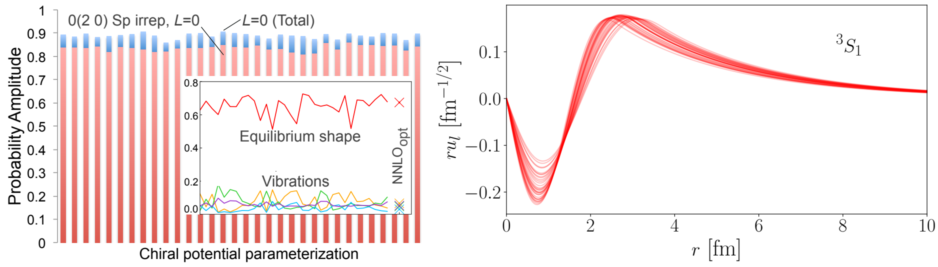


Figure 2: Chiral parameterization independence for nuclear shapes and cluster formation: (Left) Probability amplitude of the predominant Sp(3, $\mathbb{R}$) irrep $N_\sigma(\lambda_\sigma\,\mu_\sigma) = 0(2\,0)$ $(L = 0)$ in the ^{6}Li 1^+ ground state. Inset: Contributions from the equilibrium shape (symplectic bandhead) and its vibrations (the case for the NNLO$_{\text{opt}}$ is also shown). (Right) $\alpha + d$ $^3S_1$-wave vs. the relative distance r. Calculated from the ^{6}Li 1^+ ground state, computed with the SA-NCSM in the Sp(3, $\mathbb{R}$) scheme with NNLO chiral potential for 10 HO shells and $\hbar\Omega$=15 MeV. The ±10% variation in the LECs of the chiral potential is shown (left) on the horizontal axis and (right) by the spread of the curve. *Source:* Figures adapted/reused from [52] @ Frontiers; reproduced under the terms of its CC BY license.

conventional shell models, but expands, in a prescribed way, only for those deformed configurations with vibrations that lie outside of the complete model space. This is critical for enhanced prolate deformation, since spherical and less deformed or oblate shapes easily develop in comparatively small model-space sizes.

The SA-NCSM, when combined with a high-precision realistic inter-nucleon interaction, provides *ab initio* predictions of nuclear observables. We often adopt the NNLO$_{\text{opt}}$ chiral potential [51] that is used without 3N forces, which have been shown to contribute minimally to the 3- and 4-nucleon binding energy [51]. Chiral potentials are typically parameterized by two-nucleon (and three-nucleon) data, whereas the parameters, called the low-energy constants (LECs), remain unchanged and are not adjusted from one many-body system to another. This ensures a predictive power. At the next-to-next-to-leading order (NNLO), there are 14 LECs that enter into the chiral nucleon-nucleon (NN) potential. Our recent findings reveal the remarkable result that the chiral potential parameterizations have no significant effect on the dominant nuclear features, such as nuclear shape and the associated Sp(3, $\mathbb{R}$) symmetry, along with cluster formation (Fig. 2), but only slightly vary details in the nuclear wave functions, such as the contributions of the equilibrium deformation and its vibrations within the predominant nuclear shape (Fig. 2, left, inset) [52].

3 Critical role of symmetries for studies and predictions of nuclear properties

3.1 Machine learning pattern recognition with the SA-NCSM

Machine learning approaches are ideal for pattern recognition, thereby providing a suitable framework to detect and utilize the highly organized patterns in atomic nuclei governed by the symplectic Sp(3, $\mathbb{R}$) symmetry.

Specifically, Ref. [32] introduces a novel machine learning approach to provide further insight into atomic nuclei and to detect orderly patterns amidst a vast data of large-scale calculations. The method utilizes a physics-informed neural network that is trained on *ab initio* results from the SA-NCSM for light nuclei. Indeed, the SA-NCSM, which expands *ab initio*

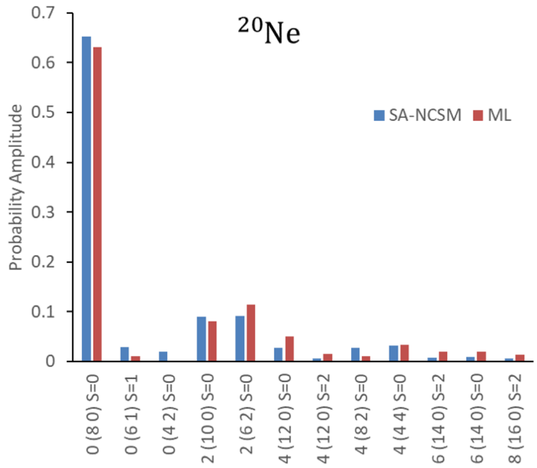


Figure 3: A novel machine learning approach coupled with the *ab initio* SA-NCSM is capable to detect orderly patterns amidst a vast data of large-scale calculations and to describe sd-shell nuclei, such as ^{20}Ne (shown), ^{24}Si, ^{40}Mg, and even the extremely heavy nuclei such as ^{166,168}Er and ^{236}U, by training only on nuclei up to ^{16}O. *Source:* Figure from [32] @ APS; reproduced with permission.

applications up to medium-mass nuclei, can reach even heavier nuclei when coupled with the machine learning approach. In particular, we find that a neural network trained on probability amplitudes for s-and p-shell nuclear wave functions not only predicts dominant configurations for heavier nuclei but in addition, when tested for the ^{20}Ne ground state, it accurately reproduces the probability distribution (Fig. 3).

The nonnegligible configurations predicted by the network provide an important input to the SA-NCSM for reducing ultra-large model spaces to manageable sizes that can be, in turn, utilized in SA-NCSM calculations to obtain accurate observables. The neural network is capable of describing nuclear deformation and is used to track the shape evolution along the $^{20-42}$Mg isotopic chain, suggesting a shape-coexistence that is more pronounced toward the very neutron-rich isotopes [32]. Furthermore, the neural network provides first descriptions of the structure and deformation of ^{24}Si and ^{40}Mg of interest to x-ray burst nucleosynthesis, and even of the extremely heavy nuclei such as ^{166,168}Er and ^{236}U, that build upon first principles considerations [32].

3.2 Probing clustering and physics beyond the standard model

The left-handed vector minus axial-vector (V—A) structure of the weak interaction was postulated in late 1950's and early 1960's guided in large part by a series of beta-decay experiments, and later was incorporated in the Standard Model of particle physics. However, in its most general form, the weak interaction can also have scalar, tensor, and pseudoscalar terms as well as right-handed currents. The β decay of ^{8}Li to ^{8}Be, which subsequently breaks up into two α particles, has long been recognized as an excellent testing ground to search for new physics (e.g. see [53]) due to the high decay energy and the ease of detecting the β and two α particles. These experiments have achieved remarkable precision (e.g., see [54, 55]) that now requires confronting the systematic uncertainties that stem from the higher-order corrections in nuclear beta decay that are difficult to measure experimentally.

As a remarkable result, the *ab initio* SA-NCSM has recently determined the size of the recoil-order form factors in the β decay of ^{8}Li (Fig. 4). It has shown that states of the $\alpha + \alpha$

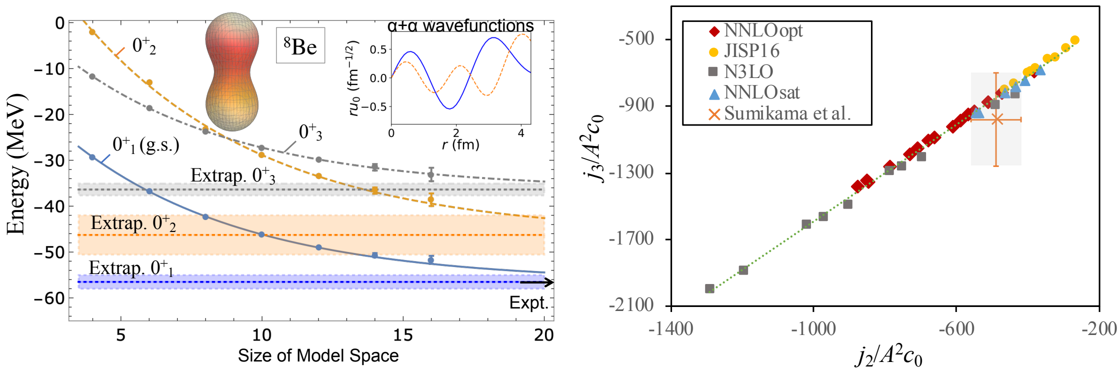


Figure 4: The *ab initio* SA-NCSM places unprecedented constraints on higher- (recoil) order corrections (j_2/A^2c_0 and j_3/A^2c_0) in the β decay of ^{8}Li→^{8}Be by addressing the challenging $\alpha+\alpha$ structure of ^{8}Be. The results are essential for largely improving the sensitivity of high-precision experiments that probe the weak interaction theory and test physics beyond the Standard Model [33,55]. Calculations performed on the NERSC and Frontera HPC systems. *Source:* Figures from [33] @ APS; reproduced with permission.

system not included in the evaluated ^{8}Be energy spectrum have an important effect on all $j_{2,3}/A^2c_0$, b/Ac_0 and d/Ac_0 recoil-order terms, and can explain the elusive M_{GT} discrepancy in the $A=8$ systems common to all other *ab initio* approaches.

The SA-NCSM outcomes of Ref. [33] reduce – by over 50% – the uncertainty on these recoil-order corrections. These results help improve the sensitivity of high-precision β-decay experiments that probe the V—A structure of the weak interaction in the most stringent limit on tensor current contribution to the weak interaction theory to date, established in Ref. [55]. Furthermore, the SA-NCSM predicted b/Ac_0 and d/Ac_0 values are important for other investigations of the Standard Model symmetries, such as the conserved vector current hypothesis and the existence of second-class currents in the weak interaction.

3.3 Optical potential in the symmetry-adapted framework for nuclear reactions

In recent years there has been a significant interest in describing nuclear reactions from *ab initio* approaches, and especially in constructing from first principles effective inter-cluster interactions, often referred to as optical potentials. *Ab initio* optical potentials for elastic scattering at low energy are of particular interest for experiments at rare isotope beams. To utilize the efficacy of the symmetry-adapted basis, we combine the *ab initio* symmetry-adapted no-core shell model with the Green's function technique (SANCSM/GF) and construct non-local optical potentials rooted in first principles [56,57]. Using the Green's function technique ensures that all relevant cluster partitionings are included in the effective potential between the two reaction fragments (clusters) that are typically in their ground state in the entrance channel. With the view toward studying neutron and proton elastic scattering from deformed and heavy targets, we first examine a target of ^{4}He (Fig. 5a), where the effect of the spurious center-of-mass motion is most evident.

In a complementary symmetry-adapted resonating group method (SA-RGM) framework [58], one starts from an *ab initio* description of all particles involved and derives the effective potential for localized clusters, which are properly normalized and orthogonalized in the particle sector, which yields non-local effective nucleon-nucleus interactions for the cluster partitioning or channel under consideration. For a single channel, if the effects of the target excitations are neglected, the non-local effective nucleon-nucleus interaction can be calculated

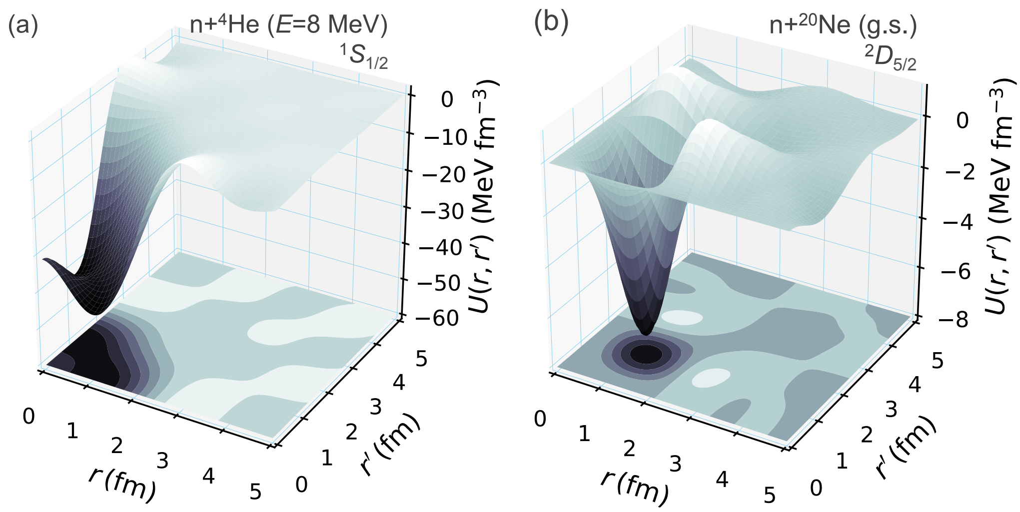


Figure 5: (a) Translationally invariant non-local optical potential for elastic neutron scattering for a ^{4}He target at $E = 8$ MeV center-of-mass energy, calculated in the SA-NCSM with the Green's function technique (10 shells, $\hbar\Omega$=17MeV). Figure from [56]. (b) Effective neutron-nucleus non-local potential (translationally invariant) for the ^{20}Ne ground state, where effects of the target excitations and antisymmetrization involving three nucleons are neglected (based on *ab initio* SA-NCSM calculations of ^{20}Ne with NNLO$_{\rm opt}$ in a model space of 11 shells and $\hbar\Omega$=15 MeV inter-shell distance). *Source:* Figure from [1] @ Annual Reviews; reproduced under the terms of its CC BY license.

for each partial wave, as illustrated for n+^{20}Ne($0^+_{\rm g.s.}$) with NNLO$_{\rm opt}$ in 11 shells (Fig. 5b). While these calculations limit the antisymmetrization to two nucleons only, this is a first step toward constructing effective nucleon-nucleus potentials for light and medium-mass nuclei for the astrophysically relevant energies [59, 60].

4 Conclusion

We have discussed the critical role of the emergent Sp(3, $\mathbb{R}$) symmetry in atomic nuclei and the associated subgroup SU(3), which in turn underpin the Sp(3, $\mathbb{R}$) and SU(3) schemes. By exploiting these schemes, the *ab initio* SA-NCSM has enabled machine-learning pattern recognition and descriptions of heavy nuclei, *ab initio* modeling of α clustering and collectivity, along with tests of beyond-the-standard-model physics. In addition, we show that with the help of the SA-NCSM, which expands *ab initio* applications up to medium-mass nuclei by using the dominant symmetry of nuclear dynamics, one can provide solutions to reaction processes in this region, with a focus on elastic neutron scattering.

Acknowledgements

We are grateful to J. E. Escher, A. Ekström, D. Rowe, G. Rosensteel, and J. Wood for useful discussions, as well as to N.D. Scielzo, M.T. Burkey, A.T. Gallant, G. Savard, and collaborators for motivating and providing important insights on the ^{8}Li decay study.

Funding information This work was supported in part by the U.S. National Science Foundation (PHY-1913728, PHY-2209060), the U.S. Department of Energy (DE-SC0019521,DE-SC0023532) and the Czech Science Foundation (22-14497S). This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344 and the National Nuclear Security Administration through the Center for Excellence in Nuclear Training and University Based Research (CENTAUR) under Grant No. DE-NA0003841. This work benefited from high performance computational resources provided by LSU (www.hpc.lsu.edu), the National Energy Research Scientific Computing Center (NERSC), a U.S. Department of Energy Office of Science User Facility operated under Contract No. DE-AC02-05CH11231, as well as the Frontera computing project at the Texas Advanced Computing Center, made possible by National Science Foundation award OAC-1818253.

References

[1] K. D. Launey, A. Mercenne and T. Dytrych, *Nuclear dynamics and reactions in the ab initio symmetry-adapted framework*, Annu. Rev. Nucl. Part. Sci. **71**, 253 (2021), doi:10.1146/annurev-nucl-102419-033316.

[2] T. Dytrych, K. D. Launey, J. P. Draayer, D. J. Rowe, J. L. Wood, G. Rosensteel, C. Bahri, D. Langr and R. B. Baker, *Physics of nuclei: Key role of an emergent symmetry*, Phys. Rev. Lett. **124**, 042501 (2020), doi:10.1103/PhysRevLett.124.042501.

[3] G. Rosensteel and D. J. Rowe, *Nuclear Sp(3,R) model*, Phys. Rev. Lett. **38**, 10 (1977), doi:10.1103/PhysRevLett.38.10.

[4] J. P. Draayer, K. J. Weeks and G. Rosensteel, *Symplectic shell-model calculations for* ^{20}*Ne with horizontal configuration mixing*, Nucl. Phys. A **413**, 215 (1984), doi:10.1016/0375-9474(84)90371-3.

[5] D. J. Rowe, *Microscopic theory of the nuclear collective model*, Rep. Prog. Phys. **48**, 1419 (1985), doi:10.1088/0034-4885/48/10/003.

[6] J. P. Draayer and Y. Akiyama, *Wigner and Racah coefficients for SU(*3*)*, J. Math. Phys. **14**, 1904 (1973), doi:10.1063/1.1666267.

[7] A. L. Blokhin, C. Bahri and J. P. Draayer, *Origin of pseudospin symmetry*, Phys. Rev. Lett. **74**, 4149 (1995), doi:10.1103/PhysRevLett.74.4149.

[8] K. D. Launey, T. Dytrych and J. P. Draayer, *Symmetry-guided large-scale shell-model theory*, Prog. Part. Nucl. Phys. **89**, 101 (2016), doi:10.1016/j.ppnp.2016.02.001.

[9] G. Rosensteel and D. J. Rowe, *On the algebraic formulation of collective models III. The symplectic shell model of collective motion*, Ann. Phys. **126**, 343 (1980), doi:10.1016/0003-4916(80)90180-3.

[10] D. J. Rowe, *Dynamical symmetries of nuclear collective models*, Prog. Part. Nucl. Phys. **37**, 265 (1996), doi:10.1016/0146-6410(96)00058-0.

[11] K. Heyde and J. L. Wood, *Shape coexistence in atomic nuclei*, Rev. Mod. Phys. **83**, 1467 (2011), doi:10.1103/RevModPhys.83.1467.

[12] J. L. Wood, *Nuclear collectivity – Its emergent nature viewed from phenomenology and spectroscopy*, in *Emergent phenomena in atomic nuclei from large-scale modeling*, World Scientific, Singapore, ISBN 9789813146068 (2017), doi:10.1142/9789813146051_0001.

[13] D. J. Rowe and J. L. Wood, *A relationship between isobaric analog states and shape coexistence in nuclei*, J. Phys. G: Nucl. Part. Phys. **45**, 06LT01 (2018), doi:10.1088/1361-6471/aabc1b.

[14] J. P. Elliott, *Collective motion in the nuclear shell model. I. Classification schemes for states of mixed configurations*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **245**, 128 (1958), doi:10.1098/rspa.1958.0072.

[15] J. P. Elliott, *Collective motion in the nuclear shell model II. The introduction of intrinsic wave-functions*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **245**, 562 (1958), doi:10.1098/rspa.1958.0101.

[16] J. P. Elliott, *Collective motion in the nuclear shell model III. The calculation of spectra*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **272**, 557 (1963), doi:10.1098/rspa.1963.0071.

[17] K. T. Hecht and A. Adler, *Generalized seniority for favored $J \neq 0$ pairs in mixed configurations*, Nucl. Phys. A **137**, 129 (1969), doi:10.1016/0375-9474(69)90077-3.

[18] K. T. Hecht and W. Zahn, *An SU(3) approach to nuclear multi-cluster problems*, Nucl. Phys. A **318**, 1 (1979), doi:10.1016/0375-9474(79)90465-2.

[19] C. Bahri and D. J. Rowe, *SU(3) quasi-dynamical symmetry as an organizational mechanism for generating nuclear rotational motions*, Nucl. Phys. A **662**, 125 (2000), doi:10.1016/S0375-9474(99)00394-2.

[20] T. Dytrych, K. D. Sviratcheva, C. Bahri, J. P. Draayer and J. P. Vary, *Evidence for symplectic symmetry in ab initio no-core shell model results for light nuclei*, Phys. Rev. Lett. **98**, 162503 (2007), doi:10.1103/PhysRevLett.98.162503.

[21] T. Dytrych et al., *Collective modes in light nuclei from first principles*, Phys. Rev. Lett. **111**, 252501 (2013), doi:10.1103/PhysRevLett.111.252501.

[22] P. F. Bedaque and U. van Kolck, *Effective field theory for few-nucleon systems*, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002), doi:10.1146/annurev.nucl.52.050102.090637.

[23] E. Epelbaum, A. Nogga, W. Glöckle, H. Kamada, U.-G. Meißner and H. Witała, *Three-nucleon forces from chiral effective field theory*, Phys. Rev. C **66**, 064001 (2002), doi:10.1103/PhysRevC.66.064001.

[24] D. R. Entem and R. Machleidt, *Accurate charge-dependent nucleon-nucleon potential at fourth order of chiral perturbation theory*, Phys. Rev. C **68**, 041001 (2003), doi:10.1103/PhysRevC.68.041001.

[25] T. Dytrych et al., *Efficacy of the SU(3) scheme for ab initio large-scale calculations beyond the lightest nuclei*, Comput. Phys. Commun. **207**, 202 (2016), doi:10.1016/j.cpc.2016.06.006.

[26] T. Dytrych, A. C. Hayes, K. D. Launey, J. P. Draayer, P. Maris, J. P. Vary, D. Langr and T. Oberhuber, *Electron-scattering form factors for ^{6}Li in the ab initio symmetry-guided framework*, Phys. Rev. C **91**, 024326 (2015), doi:10.1103/PhysRevC.91.024326.

[27] R. B. Baker, K. D. Launey, S. Bacca, N. N. Dinur and T. Dytrych, *Benchmark calculations of electromagnetic sum rules with a symmetry-adapted basis and hyperspherical harmonics*, Phys. Rev. C **102**, 014320 (2020), doi:10.1103/PhysRevC.102.014320.

[28] P. Ruotsalainen et al., *Isospin symmetry in b(e2) values: Coulomb excitation study of ^{21}Mg*, Phys. Rev. C **99**, 051301 (2019), doi:10.1103/PhysRevC.99.051301.

[29] J. Henderson et al., *Testing microscopically derived descriptions of nuclear collectivity: Coulomb excitation of ^{22}Mg*, Phys. Lett. B **782**, 468 (2018), doi:10.1016/j.physletb.2018.05.064.

[30] J. Williams et al., *Structure of ^{28}Mg and influence of the neutron pf shell*, Phys. Rev. C **100**, 014322 (2019), doi:10.1103/PhysRevC.100.014322.

[31] K. D. Launey, A. Mercenne, G. H. Sargsyan, H. Shows, R. B. Baker, M. E. Miora, T. Dytrych and J. P. Draayer, *Emergent clustering phenomena in the framework of the ab initio symmetry-adapted no-core shell model*, AIP Conf. Proc. **2038**, 020004 (2018), doi:10.1063/1.5078823.

[32] O. M. Molchanov, K. D. Launey, A. Mercenne, G. H. Sargsyan, T. Dytrych and J. P. Draayer, *Machine learning approach to pattern recognition in nuclear dynamics from the ab initio symmetry-adapted no-core shell model*, Phys. Rev. C **105**, 034306 (2022), doi:10.1103/PhysRevC.105.034306.

[33] G. H. Sargsyan et al., *Impact of clustering on the ^{8}Li β decay and recoil form factors*, Phys. Rev. Lett. **128**, 202503 (2022), doi:10.1103/PhysRevLett.128.202503.

[34] M. Moshinsky, *Wigner coefficients for the SU_3 group and some applications*, Rev. Mod. Phys. **34**, 813 (1962), doi:10.1103/RevModPhys.34.813.

[35] M. Moshinsky, J. Patera, R. T. Sharp and P. Winternitz, *Everything you always wanted to know about SU(3) $\supset$ O(3)*, Ann. Phys. **95**, 139 (1975), doi:10.1016/0003-4916(75)90048-2.

[36] V. K. B. Kota, *SU(3) symmetry in atomic nuclei*, Springer, Singapore, ISBN 9789811536021 (2020), doi:10.1007/978-981-15-3603-8.

[37] G. Rosensteel and D. J. Rowe, *On the shape of deformed nuclei*, Ann. Phys. **104**, 134 (1977), doi:10.1016/0003-4916(77)90048-3.

[38] Y. Leschber and J. P. Draayer, *Algebraic realization of rotational dynamics*, Phys. Lett. B **190**, 1 (1987), doi:10.1016/0370-2693(87)90829-X.

[39] O. Castaños, J. P. Draayer and Y. Leschber, *Shape variables and the shell model*, Z. Phys. At. Nucl. **329**, 33 (1988), doi:10.1007/BF01294813.

[40] D. J. Rowe and J. L. Wood, *Fundamentals of nuclear models*, World Scientific, Singapore, ISBN 9789812569554 (2010), doi:10.1142/6209.

[41] J. P. Draayer, Y. Leschber, S. C. Park and R. Lopez, *Representations of U(3) in U(N)*, Comput. Phys. Commun. **56**, 279 (1989), doi:10.1016/0010-4655(89)90024-6.

[42] D. J. Rowe, *The fundamental role of symmetry in nuclear models*, AIP Conf. Proc. **1541**, 104 (2013), doi:10.1063/1.4810815.

[43] T. Dytrych, K. D. Sviratcheva, J. P. Draayer, C. Bahri and J. P. Vary, *Ab initio symplectic no-core shell model*, J. Phys. G: Nucl. Part. Phys. **35**, 123101 (2008), doi:10.1088/0954-3899/35/12/123101.

[44] J. Escher and J. P. Draayer, *Fermion realization of the nuclear Sp(6,R) model*, J. Math. Phys. **39**, 5123 (1998), doi:10.1063/1.532562.

[45] D. J. Rowe, G. Thiamova and J. L. Wood, *Implications of deformation and shape coexistence for the nuclear shell model*, Phys. Rev. Lett. **97**, 202501 (2006), doi:10.1103/PhysRevLett.97.202501.

[46] A. C. Dreyfuss, K. D. Launey, T. Dytrych, J. P. Draayer and C. Bahri, *Hoyle state and rotational features in carbon-12 within a no-core shell-model framework*, Phys. Lett. B **727**, 511 (2013), doi:10.1016/j.physletb.2013.10.048.

[47] G. K. Tobin, M. C. Ferriss, K. D. Launey, T. Dytrych, J. P. Draayer, A. C. Dreyfuss and C. Bahri, *Symplectic no-core shell-model approach to intermediate-mass nuclei*, Phys. Rev. C **89**, 034312 (2014), doi:10.1103/PhysRevC.89.034312.

[48] O. Castaños, P. O. Hess, J. P. Draayer and P. Rochford, *Pseudo-symplectic model for strongly deformed heavy nuclei*, Nucl. Phys. A **524**, 469 (1991), doi:10.1016/0375-9474(91)90280-J.

[49] M. Jarrio, J. L. Wood and D. J. Rowe, *The SU(3) structure of rotational states in heavy deformed nuclei*, Nucl. Phys. A **528**, 409 (1991), doi:10.1016/0375-9474(91)90096-O.

[50] M. Freer, H. Horiuchi, Y. Kanada-En'yo, D. Lee and U.-G. Meißner, *Microscopic clustering in light nuclei*, Rev. Mod. Phys. **90**, 035004 (2018), doi:10.1103/RevModPhys.90.035004.

[51] A. Ekström et al., *Optimized chiral nucleon-nucleon interaction at next-to-next-to-leading order*, Phys. Rev. Lett. **110**, 192502 (2013), doi:10.1103/PhysRevLett.110.192502.

[52] K. S. Becker, K. D. Launey, A. Ekström and T. Dytrych, *Ab initio symmetry-adapted emulator for studying emergent collectivity and clustering in nuclei*, Front. Phys. **11**, 1064601 (2023), doi:10.3389/fphy.2023.1064601.

[53] B. R. Holstein, *Recoil effects in allowed beta decay: The elementary particle approach*, Rev. Mod. Phys. **46**, 789 (1974), doi:10.1103/RevModPhys.46.789.

[54] M. G. Sternberg et al., *Limit on tensor currents from ^{8}Li β decay*, Phys. Rev. Lett. **115**, 182501 (2015), doi:10.1103/PhysRevLett.115.182501.

[55] M. T. Burkey et al., *Improved limit on tensor currents in the weak interaction from ^{8}Li β decay*, Phys. Rev. Lett. **128**, 202502 (2022), doi:10.1103/PhysRevLett.128.202502.

[56] M. B. Burrows et al., *Ab initio low-energy optical potentials from the symmetry-adapted no-core shell model (in preparation)*.

[57] M. B. Burrows et al., *Ab initio optical potentials for elastic scattering at low energies using the symmetry-adapted no-core shell model*, 2021 fall meeting of the division of nuclear physics, American Physical Society, online (2021), https://meetings.aps.org/Meeting/DNP21/Session/KM.3.

[58] A. Mercenne, K. D. Launey, T. Dytrych, J. E. Escher, S. Quaglioni, G. H. Sargsyan, D. Langr and J. P. Draayer, *Efficacy of the symmetry-adapted basis for ab initio nucleon-nucleus interactions for light- and intermediate-mass nuclei*, Comput. Phys. Commun. **280**, 108476 (2022), doi:10.1016/j.cpc.2022.108476.

[59] A. Mercenne, K. D. Launey, J. E. Escher, T. Dytrych and J. P. Draayer, *New ab initio approach to nuclear reactions based on the symmetry-adapted no-core shell model*, in *Recent progress in few-body physics*, Springer, Cham, Switzerland, ISBN 9783030323561 (2020), doi:10.1007/978-3-030-32357-8_44.

[60] A. Mercenne, K. D. Launey, J. E. Escher, T. Dytrych and J. P. Draayer, *New symmetry-adapted ab initio approach to nuclear reactions for intermediate-mass nuclei*, in *Compound-nuclear reactions*, Springer, Cham, Switzerland, ISBN 9783030580810 (2020), doi:10.1007/978-3-030-58082-7_9.

Integrability for Feynman integrals

Florian Loebbert⋆

Bethe Center for Theoretical Physics, University of Bonn,
Nussallee 12, 53115 Bonn, Germany

⋆ loebbert@uni-bonn.de

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We give a brief overview of the Yangian symmetry of Feynman integrals. After a short introduction to the Yangian and integrability, we motivate the emergence of integrable structures for Feynman integrals via the fishnet limit of AdS/CFT. We discuss the resulting Yangian differential equations for massless fishnets in four dimensions as well as generalizations to massive propagators and generic dimensions. We also comment on the relation to momentum space conformal symmetry and on examples in dimensional regularization. Finally we sketch the recent application to fishnet integrals in two spacetime dimensions and the curious identification of Yangian invariants with period integrals of Calabi–Yau geometries.

Received 18-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.008

Contents

1 Introduction 2

2 Integrability and the Yangian 2

3 AdS/CFT and Fishnet theories 4

4 Integrability for Feynman integrals 5

5 Generalizations: Dimensions, masses, momentum space, . . . 7

6 Yangian symmetry and Calabi–Yau geometry 10

7 Conclusion 12

References 13

1 Introduction

Integrable models appear in all areas of physics, from classical mechanics to quantum field theory. While they have a natural home in two-dimensional systems like spin chains and 2d field theories, applications in four-dimensional particle physics have long been limited to special high-energy limits in QCD [1]. From the integrability viewpoint an important (still formal) step into the direction of particle phenomenology was the discovery of integrable structures in a four-dimensional quantum field theory, i.e. planar $\mathcal{N}=4$ super Yang–Mills (SYM) theory [2]. While also here a clear connection to two-dimensional physics in the AdS/CFT-dual string theory is present, this finding opened a new door to apply and extend the toolbox of integrability. Here we discuss another step into this direction, namely the appearance of integrable structures for Feynman integrals, which constitute the building blocks of generic quantum field theories including those realized in nature. On the one hand, this detaches integrability in four dimensions from the special nature of $\mathcal{N}=4$ SYM theory. On the other hand, an explanation for these mathematical structures of Feynman integrals can still be found in the integrability of the planar AdS/CFT correspondence via its so-called fishnet limits [3]. The simplified fishnet theories arise as particular double-scaling limits of the so-called gamma-deformed $\mathcal{N}=4$ SYM theory and they have the particular feature that their correlation functions are in one-to-one correspondence with individual Feynman integrals [4, 5]. Notably, some of the integrability structures which underly the planar AdS/CFT duality are inherited by these elementary Feynman integrals as discussed below. Here we will focus on the so-called Yangian symmetry of these integrals, which is an infinite dimensional extension of a Lie algebra and can be understood as the algebraic foundation of certain classes of integrable models. The need to study the mathematical properties and symmetry structures of Feynman integrals is underlined by the fact that their computation still represents a bottle neck for phenomenological predictions, see [6] for a recent review. At lower loop orders they are governed by the class of multiple polylogarithms which are defined as iterated integrals with rational integration kernels. In more general cases elliptic integrals and worse geometric structures (e.g. Calabi–Yau geometries) appear. These are typically characterized by roots of higher order polynomials in the denominator of the integrand. The exploration of multi-loop Feynman integrals and their mathematical structure is currently a very active field of research, which makes the appearence of integrable structures even more fascinating.

2 Integrability and the Yangian

As indicated above, integrability is rooted in two-dimensional physics where it is often identified with the concept of *factorized scattering*. The latter denotes the phenomenon that the n-body scattering matrix of a given theory factorizes into two-body scattering events.[1] In fact, the idea of factorized scattering may be considered the closest to a proper definition of quantum integrability which to date is still lacking, see e.g. [7,8]. On the one hand, the implications of factorized scattering include the applicability of powerful solution techniques such as the celebrated Bethe Ansatz. On the other hand, the reason for factorized scattering is found in the existence of a tower of conserved charges or higher symmetries. In spacetime dimensions greater than two, higher symmetries are believed to imply a trivial S-matrix via the Coleman–Mandula theorem [9]. Two-dimensional models, however, provide a fascinating loop-hole and allow for non-trivial scattering and higher symmetries at the same time. In fact, one can make the connection between these higher symmetries and the factorization of the S-matrix more precise and identify a set of symmetry generators $\widehat{\mathrm{J}}$ such that $\left[\widehat{\mathrm{J}}, S\right]=0$ implies the factoriza-

[1] Of course, this is only possible in models which feature a notion of scattering.

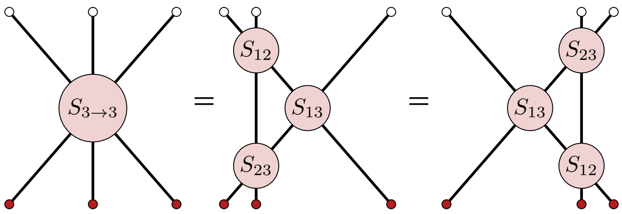


Figure 1: Factorized scattering in two dimensions. For consistency the quantum Yang–Baxter equation (right equality) has to hold: $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$. A similar equation holds for the generating function $T(u)$ of the Yangian generators together with the quantum R-matrix: $RTT = TTR$.

tion of S, cf. [10]. The notion of factorized scattering reduces the integrable S-matrix in two dimensions to the two-body scattering matrix as its fundamental building block, cf. Figure 1. In the simplest situations, this two-body matrix is a rational function of the rapidity parameter. This case of rational quantum integrable models is related to the so-called Yangian, an infinite dimensional extension of a Lie algebra $\mathfrak{g}$ that was introduced by Drinfeld in 1985 [11], see also the reviews [12–14]. The Yangian $Y[\mathfrak{g}]$ in its so-called *first realization* is defined by two sets of generators which are characterized by their tensor product actions of the following form:[2]

$$\text{Level 0}: \qquad \mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a\,, \tag{1}$$

$$\text{Level 1}: \qquad \widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c\, \mathrm{J}_k^b + \sum_{k=1}^{n} s_k \mathrm{J}_k^a\,, \tag{2}$$

$$\text{Serre relations:} \qquad \big[\widehat{\mathrm{J}}_a, \big[\widehat{\mathrm{J}}_b, \mathrm{J}_c\big]\big] - \big[\mathrm{J}_a, \big[\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c\big]\big] = \mathcal{O}\big(\mathrm{J}^3\big)\,. \tag{3}$$

Here the local level-0 operators generate the underlying Lie algebra and have a trivial coproduct. Their densities J_k^a are also employed to construct the bilocal level-one generators $\widehat{\mathrm{J}}_a$ as specified above. All higher generators of this infinite dimensional algebra can be obtained by iterative commutation. Here the Serre relations, a quantum algebra generalization of the Lie algebra's Jacobi identity, provide additional constraints on the representation. Note that the Serre relations for the case of the differential operator representation of the conformal algebra, which becomes relevant below, have been discussed in [15, 16]. The so-called evaluation parameters s_k in the above definition parametrize an external automorphism of the Yangian algebra which is realized by the Lorentz boost symmetry in relativistic models.

The Yangian has been studied in the context of various physical setups; examples include the Heisenberg spin chain with $\mathfrak{g} = \mathfrak{su}(2)$, cf. [14]; the AdS/CFT duality with $\mathfrak{g} = \mathfrak{psu}(2,2|4)$, cf. [17]; or Euclidean fishnet integrals in four dimensions with $\mathfrak{g} = \mathfrak{so}(1,5)$, cf. [5].

In fact, the above Yang–Baxter equation depicted in Figure 1 can be lifted to a purely algebraic structure relating the so-called quantum R-matrix and the monodromy matrix T via the RTT-relations:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{21}(u-v)\,. \tag{4}$$

Here the commutation of two monodromy matrices $T(u)$ acting on spaces 1 and 2, is encoded in the Yangian R-matrix. The expansion of $T(u)$ in the spectral parameter u gives rise to the different levels of the Yangian generators. This so-called *RTT-realization* of the Yangian represents an alternative way to define the same algebra, which is closely related to the concept of factorized scattering in physical models.

[2] In the following these tensor products will be identified with the external legs of planar Feynman graphs.

3 AdS/CFT and Fishnet theories

One of the physical setups where the Yangian algebra plays a crucial role is the planar AdS/CFT correspondence. Here both sides of the duality, i.e. $\mathcal{N}=4$ SYM theory and IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ feature the Lie algebra symmetry $\mathfrak{psu}(2,2|4)$. In the planar limit, this symmetry extends to the Yangian $Y[\mathfrak{psu}(2,2|4)]$ with a natural supersymmetric generalization of the above algebra definitions. The Yangian has been investigated in many different setups on the gauge and string side of the duality. It first appeared in this context as a symmetry of the dilatation generator of $\mathcal{N}=4$ SYM theory [18]. Here a consequence of Yangian symmetry is that the spectrum of the dilatation operator can be computed via appropriate generalizations of the Bethe Ansatz technique that was originally designed for the $\mathfrak{su}(2)$ Heisenberg spin chain [2]. Notably, $\mathcal{N}=4$ SYM theory represents the first four-dimensional quantum field theory which is believed to be completely integrable[3] in the planar limit. It is defined by the following schematic Lagrangian with $\mathrm{SU}(N)$ gauge symmetry:

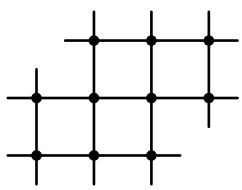
$$\mathcal{L}_{\mathcal{N}=4} \simeq \mathrm{Tr}\Big(-\mathcal{F}\mathcal{F}-\mathcal{D}\Phi\mathcal{D}\Phi+\bar{\Psi}\mathcal{D}\Psi-g^2[\Phi,\Phi]^2-g\Psi[\Phi,\Psi]-g\bar{\Psi}\big[\Phi,\bar{\Psi}\big]\Big). \tag{5}$$

It includes six matrix-valued scalars Φ_m, four Dirac spinors Ψ_A and $\bar{\Psi}^A$ and the gauge field strength $\mathcal{F}$. Remarkably, the β-function of $\mathcal{N}=4$ SYM theory is zero which makes the theory quantum conformal. Integrability arises in the large-N limit with fixed t'Hooft coupling $\lambda = g^2 N$, where non-planar Feynman diagrams are suppressed by factors of $1/N$.

In the above model an additional set of three parameters γ_1, γ_2 and γ_3 can be introduced via the so-called gamma-deformation of $\mathcal{N}=4$ SYM theory. All products of fields in the Lagrangian $\mathcal{L}_{\mathcal{N}=4}$ are replaced by non-commutative products, which leads to phase factors $e^{i\gamma_j(\dots)}$ in front of the different terms in the resulting $\mathcal{L}^{\gamma}_{\mathcal{N}=4}$. Here the $(\dots)$ in the phase factor depends on the different SU(4) Cartan charges of the fields. The additional parameters in the gamma-deformed Lagrangian now allow for interesting double-scaling limits as noted in [3]. After rescaling the fields by $\sqrt{N}$, one takes $\lambda \to 0$ and sends the gamma-parameters to imaginary infinity, i.e. $\gamma_j \to i\infty$, while keeping the new couplings $\xi_j = \sqrt{\lambda} e^{-i\gamma_j/2}$ constant. In the simplest case only one of the couplings, e.g. $\xi = \xi_3$, is non-vanishing and most of the fields decouple. The result is the following Lagrangian of the bi-scalar fishnet model for two complex matrix-valued fields denoted by X and Z:[4]

$$\mathcal{L}_{\mathrm{F}} = N\,\mathrm{Tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\big). \tag{6}$$

One of the remarkable properties of this model is that a large class of its planar correlation functions is computed by single Feynman integrals of fishnet structure. The correlator $\langle XXXZZ\bar{X}Z\bar{X}\bar{X}\bar{X}\bar{Z}\bar{Z}X\bar{Z}\rangle$ for instance corresponds to the following position space Feynman graph:



(7)

Here blobs represent integration vertices for the spacetime coordinates x^μ with $\mu = 0,\dots,3$

[3] A precise definition of this statement is hard to provide.

[4] We note that in general this bi-scalar Lagrangian is not complete and requires additional double-trace terms for conformality [19, 20]. For the Feynman graphs (alias correlators) considered in this paper, these double-trace terms do not play a role.

and lines stand for propagators as summarized in the following Feynman rules:

$$j \longrightarrow k \;\rightarrow\; \frac{1}{x_{jk}^2}\,, \qquad\qquad x_j \text{-vertex} \;\rightarrow\; \int \mathrm{d}^4 x_j\,. \tag{8}$$

Note that we write $x_{jk}^\mu = x_j^\mu - x_k^\mu$ for differences of the Euclidean spacetime vectors and we omit the color structure of the graph. Any graph that can be cut out of a square fishnet lattice thus represents a correlator in the above fishnet theory. This drastic limitation to a single contributing Feynman graph is attributed to the chiral four-point vertex in the fishnet Lagrangian.[5] The fact that the four-point vertex in (6) is chiral implies a particular admissible orientation of the color flow in the diagram.

While the fishnet model has been investigated from various perspectives, in the following we will focus on a particular implication, namely that Feynman integrals possess a higher Yangian symmetry.

4 Integrability for Feynman integrals

A conformal Lie algebra (or Yangian level-0) symmetry of the above Euclidean Feynman integrals is realized via the following differential operator representation of $\mathfrak{so}(1,5)$:

$$\mathrm{J}^a = \sum_{j=1}^n \mathrm{J}_j^a\,, \qquad \text{with} \qquad \mathrm{J}_j^a \in \begin{cases} \mathrm{D}_j = -i x_{j\mu}\partial_{x_j}^\mu - i\Delta_j\,, \\ \mathrm{L}_j^{\mu\nu} = i x_j^\mu \partial_{x_j}^\nu - i x_j^\nu \partial_{x_j}^\mu\,, \\ \mathrm{P}_j^\mu = -i\partial_{x_j}^\mu\,, \\ \mathrm{K}_j^\mu = i x_j^2 \partial_{x_j}^\mu - 2i x_j^\mu x_{j\nu}\partial_{x_j}^\nu - 2i\Delta_j x_j^\mu\,. \end{cases} \tag{9}$$

Here derivatives act on the external legs of the Feynman graph and the scaling dimensions Δ_j are set to 1 for the scalar particles considered. Invariance under the above conformal generators implies that an n-point fishnet integral can be written as a product of a prefactor V_n, which carries the conformal weight of the integral, and a conformal function that only depends on a set of conformal variables defined in terms of cross ratios:

$$I_n = V_n\, \phi(u_1, u_2, \ldots)\,. \tag{10}$$

Here the precise number of cross ratios u_k depends on the number of external points n of the Feynman graph.

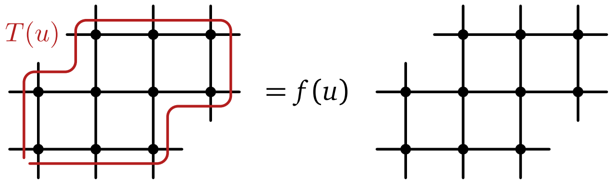
The level-1 Yangian symmetry on the other hand can be constructed from the Lie algebra generator densities given above with the prescription dictated by (2). Given full level-0 invariance of a Feynman integral I_n, the Yangian commutation relations imply that invariance under a single level-1 generator yields full Yangian symmetry of the graph. Here one typically works with the level-1 momentum generator whose explicit form is given by

$$\widehat{\mathrm{P}}^\mu = \tfrac{i}{2} \sum_{j,k=1}^n \mathrm{sign}(k-j)\big(\mathrm{P}_j^\mu \mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}_k^{\mu\nu}\big) + \sum_{j=1}^n s_j \mathrm{P}_j^\mu\,. \tag{11}$$

The invariance of the above fishnet integrals under the Yangian can be proven with the so-called lasso method and the help of the monodromy matrix $T(u)$, whose expansion coefficients around $u = \infty$ are essentially the Yangian generators, cf. [5]. At order k of the $1/u$ expansion these correspond to k-local generators, while the whole monodromy $T(u)$ acts on all n legs of

[5] This chirality is also responsible for the model being non-unitary.

a given graph and is constructed as a product of n Lax operators. Schematically, this action of $T(u)$ can be depicted by a line encircling the Feynman diagram similar to a lasso. A set of commutation relations and identities for the Lax operators and propagators allow to disentangle the monodromy from this graph, which results in an eigenvalue equation depicted by (see [5] for details)



(12)

Here $f(u)$ denotes a polynomial eigenvalue function of the spectral parameter. Remembering that $T(u) \simeq 1 + 1/u\,\mathrm{J} + 1/u^2\,\widehat{\mathrm{J}} + \ldots$, this eigenvalue equation implies the level-0 and level-1 invariance of the given integral for a judicious choice of the evaluation parameters s_k, see the prescription in [21].

While level-0 conformal invariance of a given Feynman graph implies the form (10), the level-1 symmetry yields additional constraints for the function ϕ of the conformal cross ratios. The invariance under the level-1 momentum generator can be rewritten in the following form [22]:

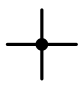
$$0 = \widehat{\mathrm{P}}^{\mu} I_n = V_n \sum_{j<k=1}^{n} \frac{x_{jk}^{\mu}}{x_{jk}^2} D_{jk} \phi \,. \tag{13}$$

Here at least for lower numbers of external points one can argue that the vectors x_{jk}^{μ}/x_{jk}^2 are in fact independent, which implies that the Yangian invariance can be translated into a system of partial differential equations (PDEs) in the cross ratios:

$$D_{jk}\,\phi = 0\,, \qquad 1 \leq j < k \leq n\,. \tag{14}$$

These coupled differential equations are highly constraining and their solutions provide a set of building blocks, whose linear combination represents the Feynman integral.

Example: 4D cross (or box) integral. The simplest four-dimensional example is the Euclidean cross (or box) integral, whose two names refer to the two alternative representations in x- or p-space:

$$I_4^{4D} = $$  $$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} = \int \frac{\mathrm{d}^4 \ell}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}\,. \tag{15}$$

Here the coordinates x_j^{μ} can alternatively be interpreted as positions, or dual momenta via the relation to the ordinary momenta $p_j^{\mu} = x_j^{\mu} - x_{j+1}^{\mu}$. The level-0 Yangian, or (dual) conformal symmetry, of the four-point integral I_4 implies the form $I_4 = \frac{1}{x_{13}^2 x_{24}^2}\phi(z,\bar{z})$, with the conformal variables z and $\bar{z}$ defined via the following relation to the cross ratios u and v:

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u\,, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v\,. \tag{16}$$

The additional Yangian level-1 symmetry yields two coupled differential equations of the form (cf. (14) and [22])

$$[D_j(z) - D_j(\bar{z})]\phi(z,\bar{z}) = 0\,, \qquad j = 1,2\,, \tag{17}$$

with the second order differential operators D_j given by

$$D_1(z) = z(z-1)^2\partial_z^2 + (3z-1)(z-1)\partial_z + z\,, \tag{18}$$

$$D_2(z) = z^2(z-1)\partial_z^2 + (3z-2)z\partial_z + z\,. \tag{19}$$

One finds four solutions to these equations which are of the form $f_j(z,\bar{z})/(z-\bar{z})$ with [22]

$$\begin{aligned} f_1 &= 1\,, \\ f_2 &= \log(\bar{z}) - \log(z)\,, \\ f_3 &= \log(1-\bar{z}) - \log(1-z)\,, \\ f_4 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log\frac{1-z}{1-\bar{z}}\log(\bar{z}z)\,. \end{aligned} \tag{20}$$

In addition to the Yangian symmetry the cross integral is invariant under all permutations of its external legs. This permutation invariance singles out the Bloch–Wigner dilogarithm f_4 as the correct Yangian invariant which represents this integral. While this solution has been known since the early works [23], it is remarkable that the cross can be bootstrapped using integrability and permutation symmetry only [22]. We note that the overall constant prefactor remains to be fixed by other means, e.g. numerics or a coincidence limit of external points.

While one might expect that integrability fixes an observable completely, here we have employed the additional permutation symmetry to single out the correct Yangian invariant of the four solutions (20). It turns out that all four building blocks are required to express the integral when going from Euclidean to Minkowski kinematics [24]. As opposed to a single Euclidean region, in the Minkowski case one distinguishes 64 kinematic regions depending on the signs of the x_{jk}^2. Here, in a given region the integral is typically invariant under a subset of all permutations and f_4 is no longer the only admissible Yangian invariant. Via similar reasoning as above one can bootstrap the Minkowski integral using its Yangian symmetry modulo a small number of constant coefficients [25].

5 Generalizations: Dimensions, masses, momentum space, ...

In this section we indicate some generalizations, which go beyond massless square fishnet integrals in four spacetime dimensions.

Fishnet structures and dimensions. The above bi-scalar fishnet model represents the simplest double-scaling limit of gamma-deformed $\mathcal{N}=4$ SYM theory. More involved models are obtained by combining different limits of the three parameters γ_j and the coupling constant. In particular, models exist where also some of the fermions survive, which results e.g. in Yangian-invariant Feynman graphs of brick wall structure [26]. Here the non-scalar particles require non-scalar representations of the conformal algebra to construct the Yangian generators.

Apart from generalizing the field content, the above fishnet theories have natural generalizations to different spacetime dimensions. In particular, there exists a double-scaling limit of Aharony–Bergman–Jafferis–Maldacena (ABJM) theory which results in a three-dimensional fishnet model with triangular Feynman graphs [4]. Also in six spacetime dimensions one can write down a similar fishnet theory with hexagonal graph structure [27]. All of the associated Feynman graphs feature the above Yangian symmetry over the conformal algebra in the respective spacetime dimension [26]. This even generalizes to graphs with deformed propagator powers a_j as long as these sum up to the spacetime dimension at each integration vertex [21, 22, 26]:

$$\frac{1}{x_{jk}^2} \quad\to\quad \frac{1}{(x_{jk}^2)^{a_j}}\,, \qquad \sum_{j\in\text{vertex}} a_j = D\,. \tag{21}$$

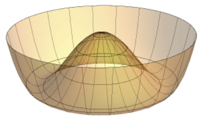
If the latter conformal condition does not hold, one still finds an invariance under the level-one momentum generator, which relates to the below discussion of momentum space conformal symmetry. Finally, the square fishnet model was generalized to a D-dimensional Lagrangian of the form [28]

$$\mathcal{L}_{\mathrm{F}}^{D} = N\,\mathrm{Tr}\Big[X(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{X} + Z(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{Z} + \xi^2 XZ\bar{X}\bar{Z}\Big]. \tag{22}$$

Here the operators $(\partial_\mu\partial^\mu)^{\frac{D}{4}}$ are understood as integral operators for the case $D \neq 4$. The fact that integrability properties are preserved by all the above modifications demonstrates the universality of these mathematical structures.

Yangian symmetry for the masses. In the massless case, we argued that Feynman integrals can be interpreted as correlators in the fishnet theory, which is connected to the integrable $\mathcal{N}=4$ SYM theory by means of the above double scaling limit. It is well known how to introduce masses in the latter theory via the Higgs mechanism, but no integrable structures are known that survive this process. Hence, a priori one might not expect to find integrable structures in massive Feynman integrals following the same logic. Still it is instructive to look at the massive version of $\mathcal{N}=4$ SYM theory on the so-called Coulomb branch. Masses are introduce by giving a vacuum expectation value (VEV) to one of the scalar fields in the Lagrangian (5) [29]:

$$\hat{\Phi} = \langle\Phi\rangle + \Phi \tag{23}$$



This leads to the so-called Coulomb phase of the theory and implies the appearence of massive propagators of the form

$$\frac{1}{\hat{x}_{jk}^2} = \frac{1}{x_{jk}^2 + (m_j - m_k)^2}\,. \tag{24}$$

Importantly, here the masses enter in differences in analogy to the spacetime coordinates. In particular, for a judicious choice of the above VEV only planar Feynman integrals with masses on the boundaries survive the large-N limit [29, 30]. These have been shown to posess a massive extension of the dual conformal symmetry with generators of the following form [29]:

$$\mathrm{J}^a = \sum_{j=1}^{n} \mathrm{J}_j^a\,, \qquad \text{with} \qquad \mathrm{J}_j^a \in \begin{cases} \mathrm{D}_j = -i\big(x_{j\mu}\partial_{x_j}^{\mu} + m_j\partial_{m_j} + \Delta_j\big)\,, \\ \mathrm{L}_j^{\mu\nu} = i x_j^{\mu}\partial_{x_j}^{\nu} - i x_j^{\nu}\partial_{x_j}^{\mu}\,, \\ \mathrm{P}_j^{\mu} = -i\partial_{x_j}^{\mu}\,, \\ \mathrm{K}_j^{\mu} = -2i x_j^{\mu}\big(x_{j\nu}\partial_{x_j}^{\nu} + m_j\partial_{m_j} + \Delta_j\big) + i(x_j^2 + m_j^2)\partial_{x_j}^{\mu}\,. \end{cases} \tag{25}$$

Here the mass can be interpreted as the $(D+1)$th component of the spacetime vector: $x_j^D = m_j$. Unfortunately to date no massive Yangian symmetry has been detected in the full Coulomb phase of $\mathcal{N}=4$ SYM theory. Still we can employ the definition of the level-1 generators in terms of the Lie algebra generator densities as given in (2). Applying these to Feynman integrals with massive propagators, it turns out that planar one- and two-loop integrals in generic kinematics are annihilated for an appropriate choice of evaluation parameters [21]. This statement can be proven explicitly for generic number of external legs and propagator powers obeying the conformal condition $\sum_{j\in\text{vertex}} a_j = D$ at each integration vertex [31]. The only criterion for this symmetry is that the internal loop-to-loop propagator of the two-loop diagrams remains massless. Numerical tests at higher loop orders suggest that in fact all Feynman graphs cut out of a regular tiling of the plain feature this massive Yangian symmetry as long as all internal propagators remain massless [31]. Also a massive extension of the fishnet Lagrangian can be defined via a double-scaling limit of a gamma-deformed version of massive Coulomb branch $\mathcal{N}=4$ SYM theory [30]:

$$\mathcal{L}_{\mathrm{MF}} = N\,\mathrm{Tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\big) - N(m_a-m_b)^2 X^a{}_b \bar{X}^b_a - N(m_a-m_b)^2 Z^a{}_b \bar{Z}^b_a\,. \quad (26)$$

The planar off-shell amplitudes in this theory correspond to the massive versions of Yangian invariant Feynman integrals described above.[6]

Momentum space conformal symmetry. Notably, the level-1 momentum generator remains a symmetry of the above Feynman integrals even if the conformal condition for the propagator powers in (21) does not hold, i.e. if there is no level-0 (dual) conformal symmetry. This statement applies to the massless and massive version of the Yangian symmetry and implies powerful constraints [21]. Translating the x-space level-one momentum $\widehat{\mathrm{P}}^\mu$ to the dual momentum variables defined via $p_j^\mu = x_j^\mu - x_{j+1}^\mu$, one finds a massive generalization of the special conformal generator $\bar{\mathrm{K}}_j^\mu$ in momentum space which forms part of the following set of operators obeying the conformal algebra relations:

$$\bar{\mathrm{J}}^a = \sum_{j=1}^{n} \bar{\mathrm{J}}^a_j\,, \qquad \text{with} \qquad \bar{\mathrm{J}}^a_j \in \left\{ \begin{aligned} \bar{\mathrm{D}}_j &= p_{j\nu}\partial^\nu_{p_j} + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\,, \\ \bar{\mathrm{L}}^{\mu\nu}_j &= p^\mu_j \partial^\nu_{p_j} - p^\nu_j \partial^\mu_{p_j}\,, \\ \bar{\mathrm{P}}^\mu_j &= p^\mu_j\,, \\ \bar{\mathrm{K}}^\mu_j &= p^\mu_j \partial^2_{p_j} - 2\Big[p_{j\nu}\partial^\nu_{p_j} + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\Big]\partial^\mu_{p_j}\,. \end{aligned} \right.$$

Note that these generator densities feature a nearest-neighbor action on the masses of the external legs of the Feynman graph. In the context of $\mathcal{N}=4$ SYM theory it is a well known statement that ordinary and dual conformal symmetry of scattering amplitudes close into the Yangian algebra [33]. Similar statements hold for the above Feynman integrals in the case of the ordinary (bosonic) conformal algebra [5,31]. The constraints from $\widehat{\mathrm{P}}^\mu$ or equivalently $\bar{\mathrm{K}}^\mu$ can thus be exploited independently of the dual conformal symmetry for massless or massive integrals. Here one representation or the other may be more convenient depending on whether one works in x- or p-space. Solving the momentum space conformal Ward identities in the massless case has recently been subject of great interest, see e.g. [34,35] and follow-ups.

Dimensional regularization. Since the above symmetries can be formulated in generic spacetime dimension D, one can also employ them in the context of dimensionally regulated integrals. An interesting example is the following family of Euclidean three-point integrals with half integer propagator powers a_j in $D=3-2\epsilon$ dimensions:

$$I_3^D[a_1,a_2,a_3] := \int \frac{\mathrm{d}^D x_0}{(x_{01}^2)^{a_1}(x_{02}^2)^{a_2}(x_{03}^2)^{a_3}}\,. \tag{27}$$

Here we assume that $a_1+a_2+a_3 \leq D/2$. The motivation to study these integrals comes from gravitational physics. The Post-Newtonian (PN) expansion with velocities $v \ll c$ of the 3-body effective potential in general relativity requires these integrals as an input, e.g. in the following

[6] Alternatively one can introduce masses via spontaneous symmetry breaking in the bi-scalar fishnet theory which, however, leads to different propagators and seems not to allow for integrable symmetries [32].

contribution (in red) to the 3PN potential [36]:

$$S^{3\text{PN}} = \dots + \sum_{\substack{j\neq i\\ k\neq i,j}} \frac{G^2 m_i m_j m_k}{4\pi} \Big\{ \Big[(6v_i^2+v_k^2-8v_i\cdot v_j)(v_{ki}\cdot\partial_{x_i})(v_{kj}\cdot\partial_{x_j}) \\ +(8v_{ik}^2-4v_k^2)(v_{ji}\cdot\partial_{x_i})(v_{ij}\cdot\partial_{x_j})\Big] I_3[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}] \\ +\left(v_k\cdot\partial_{x_k}\right)^2 \Big[(v_{ki}\cdot\partial_{x_i})(v_{kj}\cdot\partial_{x_j})+2(v_{ik}\cdot\partial_{x_k})(v_{ij}\cdot\partial_{x_j}) \\ +4(v_{ji}\cdot\partial_{x_i})(v_{ij}\cdot\partial_{x_j})+8(v_{jk}\cdot\partial_{x_k})(v_{kj}\cdot\partial_{x_j})\Big] I_3[\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}] \Big\}. \tag{28}$$

To bootstrap these integrals one can employ the Yangian level-1 momentum generator, which yields two second order PDEs in the variables $r_{jk}=|x_j-x_k|$ (see [36] for explicit expressions):

$$D_j I_3 = 0\,, \qquad j=1,2\,. \tag{29}$$

These differential equations are solved by the ansatz

$$\mu^{-2\epsilon} I_3^{3-2\epsilon}[a_1,a_2,a_3] = \frac{A}{2\epsilon} + B + C\log\left(\tfrac{r_{12}+r_{13}+r_{23}}{\mu}\right) + \mathcal{O}(\epsilon)\,,$$

with a scale μ and given polynomials A,B,C, which is inspired by the leading integral in this family as presented in [37]. Plugging the resulting integrals into the effective potential via (28), this leads to new $v^{2n}G^2$ contributions to the Post-Newtonian expansion of the 3-body effective potential. The same approach can be applied to higher PN integrals $I_3^{3-2\epsilon}[a_1,a_2,a_3]$ in this family (cf. [36]), which can then be inserted into the higher order analogues of (28).

Soft and collinear anomalies. Notably, the above Yangian or momentum space conformal symmetry does not commute with coincidence or lightlike limits of the external kinematic variables x_j^μ or their differences, respectively. These correspond to soft or on-shell limits in the dual momentum variables p_j^μ:

$$\text{Coincidence/soft limit:} \qquad x_j^\mu \to x_{j+1}^\mu \qquad \longleftrightarrow \qquad p_j^\mu \to 0\,, \tag{30}$$

$$\text{Lightlike/on-shell limit:} \qquad x_{j,j+1}^2 \to 0 \qquad \longleftrightarrow \qquad p_j^2 \to 0\,. \tag{31}$$

These situations lead to anomalies even for finite integrals, see [38] for examples of coincidence limits and [39,40] for the on-shell case.

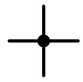
6 Yangian symmetry and Calabi–Yau geometry

In order to make progress on understanding fishnet integrals at higher loop orders, we consider Feynman graphs in $D=2$ spacetime dimensions, see [41] for more details. For square fishnets with propagators $(x^{-2})^{a_j}$ to obey the conformal condition $\sum_{j=1}^4 a_j = D$ we set $a_j=1/2$ for all propagators. The respective integrals represent correlators in the D-dimensional version of the square fishnet theory defined by the Lagrangian in (22) [28]. Similar to the conformal Lie algebra in two dimensions, the conformal Yangian splits into a holomorphic and an anti-holomorphic part: $Y[\mathfrak{sl}(2,\mathbb{R})]\oplus\overline{Y[\mathfrak{sl}(2,\mathbb{R})]}$. We thus expect the following double copy form for the respective fishnet integrals:

$$\phi(z,\bar z) = \vec{\Pi}^\dagger(\bar z)\cdot\Sigma\cdot\vec{\Pi}(z)\,. \tag{32}$$

Here $\vec{\Pi}(z)$ denotes a vector of solutions to the Yangian invariance equations for $Y[\mathfrak{sl}(2,\mathbb{R})]$ with z indicating the conformal cross ratios and $\bar z$ their complex conjugates. The symbol Σ represents a constant matrix which defines the precise linear combination of the Yangian invariants representing the Feynman integral.

Example: 2D cross (or box) integral. Consider the simplest example of the 2D cross integral which, using conformal symmetry, can be written in the form

$$I_4^{2D} = \quad$$  $$= \frac{1}{|x_{12}||x_{34}|}\phi(z,\bar{z})\,. \tag{33}$$

The cross integral depends on a single conformal variable z (and its conjugate $\bar{z}$). Imposing invariance of the holomorphic part of the integral under $Y[\mathfrak{sl}(2,\mathbb{R})]$, the respective Yangian differential equation takes the form

$$\left(1+4(2z-1)\partial_z+4z(z-1)\partial_z^2\right)\vec{\Pi}(z)=0\,. \tag{34}$$

One finds two solutions $K(z)$ and $K(1-z)$ to this equation, which indeed combine into a double copy of Yangian invariants for the 2D cross integral [38, 42]:

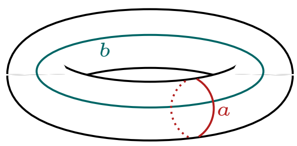
$$\phi(z,\bar{z})=\frac{4}{\pi}i\begin{pmatrix}K(z) & iK(1-z)\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix}K(\bar{z})\\ -iK(1-\bar{z})\end{pmatrix}. \tag{35}$$

The two solutions $K(z)$ and $K(1-z)$ of the Yangian equation are identified with the complete elliptic integral of the first kind evaluated at different arguments:

$$K(z)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}\,. \tag{36}$$

This integral has an interesting relation to geometry via the elliptic curve defined by the polynomial in the denominator of the integrand:

$$y^2=\left(1-t^2\right)\left(1-zt^2\right) \tag{37}$$



The elliptic curve is isomorphic to a torus with two distinguished cycles denoted by a and b. The two period integrals associated to these two cycles are precisely the two Yangian invariants $K(z)$ and $K(1-z)$ with (34) representing the so-called Picard–Fuchs differential equation associated with the elliptic curve.

For higher loop fishnet integrals in 2D it turns out that the above double copy pattern continues to hold. Also these integrals can be written in the form (32) with a vector of Yangian invariant functions $\vec{\Pi}$ of the conformal cross ratios [41].[7] The geometric interpretation of these Yangian invariants generalizes to a family of Calabi–Yau's with the above elliptic curve representing a Calabi–Yau 1-fold. We note that a Calabi–Yau ℓ-fold is defined as an ℓ-dimensional complex Kähler manifold with vanishing first Chern class. Here the last condition relates to the conformal nature of the fishnet integrals defined in terms of four-point vertices only. As for the elliptic curve, also the higher Calabi–Yau's posses an associated set of period integrals which are annihilated by a Picard–Fuchs ideal of differential operators. The conformal cross ratios of the fishnet integrals correspond to the Calabi–Yau moduli. At 3 loops for instance, the respective 2D integral depends on 5 cross ratios and corresponds to a Calabi–Yau 3-fold with a 12-dimensional vector $\vec{\Pi}$ of Yangian-invariant period integrals. The conjecture of [41]

[7] To obtain the complete Picard–Fuchs ideal of differential operators one has to take the permutation symmetries of the integrals into account, see also the example of the 4D cross integral given in Section 4 where these were used in a different way.

suggests that for generic 2D fishnet integrals the Picard–Fuchs ideal is obtained by combining Yangian and permutation symmetries of the respective Feynman graph. This represents a curious new relation between integrability and Calabi–Yau geometries!

Notably, the geometric interpretation of fishnet integrals goes even further [41]. The fishnet expression of (32) corresponds to the so-called quantum volume of a particular Calabi–Yau geometry, which is obtained from the original fishnet Calabi–Yau via the celebrated mirror symmetry. This finding extends the volume interpretation of Feynman integrals, which was well known at one loop order (see e.g. [43]), to higher loops — at least in two spacetime dimensions. Another notable fact is that the conformal fishnet function ϕ computes the Kähler potential V of the Calabi–Yau via the relation $\phi = \exp(-V)$. More details on this elaborate construction are given in [41].

7 Conclusion

It is fascinating that integrable structures appear for rather broad classes of Feynman integrals, which constitute the building blocks of generic quantum field theories. The two-dimensional nature of integrability is satisfied with the ordering of the external legs of planar Feynman graphs. Here we have focussed on the Yangian invariance of Feynman integrals, which provides systems of partial differential equations originating in a spacetime symmetry. We note that integrability in the context of fishnet integrals can be considered from various different angles and already in 1980 A.B. Zamolodchikov investigated integrable structures for these integrals in the context of statistical vertex models [44]. More recent applications include the extraction of Feynman integrals from elements of the fishnet dilatation operator [4], the Basso–Dixon correlators of [42,45], or the interesting number-theoretic relation to Q-functions of [46].

There are numerous intriguing directions to further explore the Yangian symmetry of Feynman integrals. In particular, the connection to Calabi–Yau geometries promises interesting novel insights and new connections to mathematics.

Another curious appearance of integrability in the context of higher dimensional conformal field theory is the interpretation of conformal blocks as eigenfunctions of integrable Calogero–Sutherland or Gaudin Hamiltonians [47,48]. It would be fascinating to relate these structures to the Yangian symmetry of conformal correlation functions discussed above.

Going beyond the realm of flat space, there are curious new relations to correlation functions in Anti-de-Sitter space. In [49] it was noted that Witten contact diagrams are identical with one-loop Feynman integrals featuring Yangian symmetry [21]. If this connection generalizes to other classes of Witten diagrams, this may open a new playground for applications of integrability in AdS spaces.

Acknowledgements

First, I would like to thank the organizers of the *34th International Colloquium on Group Theoretical Methods in Physics* in Strasbourg for the invitation to this stimulating interdisciplinary conference. I am very grateful to the collaborators who worked with me on the interesting topic discussed here, that is Dima Chicherin, Luke Corcoran, Claude Duhr, Volodya Kazakov, Albrecht Klemm, Julian Miczajka, Dennis Müller, Hagen Münkler, Christoph Nega, Jan Plefka, Franziska Porkert, Canxin Shi, Matthias Staudacher, Tianheng Wang and De-liang Zhong.

Funding information The work of FL is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)-Projektnummer 363895012, and by funds of the Klaus Tschira Foundation gGmbH.

References

[1] L. N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models*, J. Exp. Theor. Phys. Lett. **59**, 596 (1994).

[2] N. Beisert et al., *Review of AdS/CFT integrability: An overview*, Lett. Math. Phys. **99**, 3 (2011), doi:10.1007/s11005-011-0529-2.

[3] Ö. Gürdoğan and V. Kazakov, *New integrable 4d quantum field theories from strongly deformed planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory*, Phys. Rev. Lett. **117**, 201602 (2016), doi:10.1103/PhysRevLett.117.201602.

[4] J. Caetano, Ö. Gürdoğan and V. Kazakov, *Chiral limit of $\mathcal{N} = 4$ SYM and ABJM and integrable Feynman graphs*, J. High Energy Phys. **03**, 077 (2018), doi:10.1007/JHEP03(2018)077.

[5] D. Chicherin, V. Kazakov, F. Loebbert, D. Müller and D.-l. Zhong, *Yangian symmetry for bi-scalar loop amplitudes*, J. High Energy Phys. **05**, 003 (2018), doi:10.1007/JHEP05(2018)003.

[6] J. L. Bourjaily et al., *Functions beyond multiple polylogarithms for precision collider physics*, (arXiv preprint) doi:10.48550/arXiv.2203.07088.

[7] B. Sutherland, *Beautiful models: 70 years of exactly solved quantum many-body problems*, World Scientific, Singapore, ISBN 9789813102149 (2004), doi:10.1142/5552.

[8] J.-S. Caux and J. Mossel, *Remarks on the notion of quantum integrability*, J. Stat. Mech.: Theory Exp. P02023 (2011), doi:10.1088/1742-5468/2011/02/P02023.

[9] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Phys. Rev. **159**, 1251 (1967), doi:10.1103/PhysRev.159.1251.

[10] F. Loebbert and A. Spiering, *Nonlocal symmetries and factorized scattering*, J. Phys. A: Math. Theor. **51**, 485202 (2018), doi:10.1088/1751-8121/aae7ff.

[11] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, in *Yang-Baxter equation in integrable systems*, World Scientific, Singapore, ISBN 9789814507066 (1990), doi:10.1142/9789812798336_0013.

[12] D. Bernard, *An introduction to Yangian symmetries*, Int. J. Mod. Phys. B **07**, 3517 (1993), doi:10.1142/S0217979293003371.

[13] N. J. Mackay, *Introduction to Yangian symmetry in integrable field theory*, Int. J. Mod. Phys. A **20**, 7189 (2005), doi:10.1142/S0217751X05022317.

[14] F. Loebbert, *Lectures on Yangian symmetry*, J. Phys. A: Math. Theor. **49**, 323002 (2016), doi:10.1088/1751-8113/49/32/323002.

[15] J. Miczajka, *The Yangian bootstrap for massive Feynman diagrams*, PhD thesis, Humboldt Universität, Berlin, Germany (2022), doi:10.18452/23855.

[16] N. Dokmetzoglou and L. Dolan, *Properties of the conformal Yangian in scalar and gauge field theories*, J. High Energy Phys. **02**, 137 (2023), doi:10.1007/JHEP02(2023)137.

[17] A. Torrielli, *Review of AdS/CFT integrability. Chapter VI.2: Yangian algebra*, Lett. Math. Phys. **99**, 547 (2011), doi:10.1007/s11005-011-0491-z.

[18] L. Dolan, C. R. Nappi and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, J. High Energy Phys. **10**, 017 (2003), doi:10.1088/1126-6708/2003/10/017.

[19] C. Sieg and M. Wilhelm, *On a CFT limit of planar γ_i-deformed $\mathcal{N} = 4$ SYM theory*, Phys. Lett. B **756**, 118 (2016), doi:10.1016/j.physletb.2016.03.004.

[20] D. Grabner, N. Gromov, V. Kazakov and G. Korchemsky, *Strongly γ-deformed $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as an integrable conformal field theory*, Phys. Rev. Lett. **120**, 111601 (2018), doi:10.1103/PhysRevLett.120.111601.

[21] F. Loebbert, J. Miczajka, D. Müller and H. Münkler, *Massive conformal symmetry and integrability for Feynman integrals*, Phys. Rev. Lett. **125**, 091602 (2020), doi:10.1103/PhysRevLett.125.091602.

[22] F. Loebbert, D. Müller and H. Münkler, *Yangian bootstrap for conformal Feynman integrals*, Phys. Rev. D **101**, 066006 (2020), doi:10.1103/PhysRevD.101.066006.

[23] N. I. Ussyukina and A. I. Davydychev, *An approach to the evaluation of three- and four-point ladder diagrams*, Phys. Lett. B **298**, 363 (1993), doi:10.1016/0370-2693(93)91834-A.

[24] L. Corcoran and M. Staudacher, *The dual conformal box integral in Minkowski space*, Nucl. Phys. B **964**, 115310 (2021), doi:10.1016/j.nuclphysb.2021.115310.

[25] L. Corcoran, F. Loebbert, J. Miczajka and M. Staudacher, *Minkowski box from Yangian bootstrap*, J. High Energy Phys. **04**, 160 (2021), doi:10.1007/JHEP04(2021)160.

[26] D. Chicherin, V. Kazakov, F. Loebbert, D. Müller and D.-l. Zhong, *Yangian symmetry for fishnet Feynman graphs*, Phys. Rev. D **96**, 121901 (2017), doi:10.1103/PhysRevD.96.121901.

[27] O. Mamroud and G. Torrents, *RG stability of integrable fishnet models*, J. High Energy Phys. **06**, 012 (2017), doi:10.1007/JHEP06(2017)012.

[28] V. Kazakov and E. Olivucci, *Biscalar integrable conformal field theories in any dimension*, Phys. Rev. Lett. **121**, 131601 (2018), doi:10.1103/PhysRevLett.121.131601.

[29] L. F. Alday, J. Henn, J. Plefka and T. Schuster, *Scattering into the fifth dimension of $\mathcal{N} = 4$ super Yang-Mills*, J. High Energy Phys. **01**, 077 (2010), doi:10.1007/JHEP01(2010)077.

[30] F. Loebbert and J. Miczajka, *Massive fishnets*, J. High Energy Phys. **12**, 197 (2020), doi:10.1007/JHEP12(2020)197.

[31] F. Loebbert, J. Miczajka, D. Müller and H. Münkler, *Yangian bootstrap for massive Feynman integrals*, SciPost Phys. **11**, 010 (2021), doi:10.21468/SciPostPhys.11.1.010.

[32] G. K. Karananas, V. Kazakov and M. Shaposhnikov, *Spontaneous conformal symmetry breaking in fishnet CFT*, Phys. Lett. B **811**, 135922 (2020), doi:10.1016/j.physletb.2020.135922.

[33] J. Drummond, J. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory*, J. High Energy Phys. **05**, 046 (2009), doi:10.1088/1126-6708/2009/05/046.

[34] C. Corianò, L. Delle Rose, E. Mottola and M. Serino, *Solving the conformal constraints for scalar operators in momentum space and the evaluation of Feynman's master integrals*, J. High Energy Phys. **07**, 011 (2013), doi:10.1007/JHEP07(2013)011.

[35] A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space*, J. High Energy Phys. **03**, 111 (2014), doi:10.1007/JHEP03(2014)111.

[36] F. Loebbert, J. Plefka, C. Shi and T. Wang, *Three-body effective potential in general relativity at second post-Minkowskian order and resulting post-Newtonian contributions*, Phys. Rev. D **103**, 064010 (2021), doi:10.1103/PhysRevD.103.064010.

[37] T. Ohta, H. Okamura, T. Kimura and K. Hiida, *Physically acceptable solution of Einstein's equation for many-body system*, Prog. Theor. Phys. **50**, 492 (1973), doi:10.1143/PTP.50.492.

[38] L. Corcoran, F. Loebbert and J. Miczajka, *Yangian Ward identities for fishnet four-point integrals*, J. High Energy Phys. **04**, 131 (2022), doi:10.1007/JHEP04(2022)131.

[39] D. Chicherin and E. Sokatchev, *Conformal anomaly of generalized form factors and finite loop integrals*, J. High Energy Phys. **04**, 082 (2018), doi:10.1007/JHEP04(2018)082.

[40] D. Chicherin and G. P. Korchemsky, *The SAGEX review on scattering amplitudes Chapter 9: Integrability of amplitudes in fishnet theories*, J. Phys. A: Math. Theor. **55**, 443010 (2022), doi:10.1088/1751-8121/ac8c72.

[41] C. Duhr, A. Klemm, F. Loebbert, C. Nega and F. Porkert, *Yangian-invariant fishnet integrals in two dimensions as volumes of Calabi-Yau varieties*, Phys. Rev. Lett. **130**, 041602 (2023), doi:10.1103/PhysRevLett.130.041602.

[42] S. Derkachov, V. Kazakov and E. Olivucci, *Basso-Dixon correlators in two-dimensional fishnet CFT*, J. High Energy Phys. **04**, 032 (2019), doi:10.1007/JHEP04(2019)032.

[43] J. L. Bourjaily, E. Gardi, A. J. McLeod and C. Vergu, *All-mass n-gon integrals in n dimensions*, J. High Energy Phys. **08**, 029 (2020), doi:10.1007/JHEP08(2020)029.

[44] A. B. Zamolodchikov, *"Fishing-net" diagrams as a completely integrable system*, Phys. Lett. B **97**, 63 (1980), doi:10.1016/0370-2693(80)90547-X.

[45] B. Basso and L. J. Dixon, *Gluing ladder feynman diagrams into fishnets*, Phys. Rev. Lett. **119**, 071601 (2017), doi:10.1103/PhysRevLett.119.071601.

[46] Ö. Gürdoğan, *From integrability to the Galois coaction on Feynman periods*, Phys. Rev. D **103**, L081703 (2021), doi:10.1103/PhysRevD.103.L081703.

[47] M. Isachenkov and V. Schomerus, *Superintegrability of d-dimensional conformal blocks*, Phys. Rev. Lett. **117**, 071602 (2016), doi:10.1103/PhysRevLett.117.071602.

[48] I. Burić, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *From Gaudin integrable models to d-dimensional multipoint conformal blocks*, Phys. Rev. Lett. **126**, 021602 (2021), doi:10.1103/PhysRevLett.126.021602.

[49] K. C. Rigatos and X. Zhou, *Yangian symmetry in holographic correlators*, Phys. Rev. Lett. **129**, 101601 (2022), doi:10.1103/PhysRevLett.129.101601.

Jordan meets Freudenthal. A black hole exceptional story

Alessio Marrani*

Instituto de Física Teorica, Dep.to de Física, Universidad de Murcia,
Campus de Espinardo, E-30100, Spain

* alessio.marrani@um.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Within the extremal black hole attractors arising in ungauged $\mathcal{N} \geqslant 2$-extended Maxwell Einstein supergravity theories in $3+1$ space-time dimensions, we provide an overview of the stratification of the electric-magnetic charge representation space into "large" orbits and related "moduli spaces", under the action of the (continuous limit of the) non-compact *U*-duality Lie group. While each "large" orbit of the *U*-duality supports a class of attractors, the corresponding "moduli space" is the proper subspace of the scalar manifold spanned by those scalar fields on which the Attractor Mechanism is inactive. We present the case study concerning $\mathcal{N}=2$ supergravity theories with symmetric vector multiplets' scalar manifold, which in all cases (with the exception of the minimally coupled models) have the electric-magnetic charge representation of *U*-duality fitting into a reduced Freudenthal triple system over a cubic (simple or semi-simple) Jordan algebra.

Received 02-07-2023
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.009

Check for updates

Contents

1 Introduction **2**

2 $\mathcal{N}=2$ *symmetric* supergravities **5**
2.1 "Large" *U*-duality orbits 7
2.2 "Moduli spaces" of attractors 8
2.3 "Moduli spaces" of the ADM mass 12

References **12**

1 Introduction

Within the theory of dynamical (dissipative) systems, an *attractor* is defined by a *fixed point* of the evolution flow of the system itself, describing the equilibrium state and its stability features. In general, when approaching an attractor, the orbits of the dynamical evolution lose all memory of their initial conditions, but nonetheless the overall dynamics remains strictly deterministic.

Within Maxwell-Einstein-scalar theories endowed with local supersymmetry in 3+1 space-time dimensions, attractors were firstly discovered within the class of extremal black hole solutions preserving half of the supersymmetries, in presence of $\mathcal{N}=2$ spinor supercharges. This led to the discovery of the so-called *Attractor Mechanism* (AM), governing the dynamics of evolution of the scalar fields in the black hole background [1]- [4]. We will now review the basics of AM in such a framework.

As far as propagating (i.e., dynamical) massless fields are concerned, linearly realized $\mathcal{N}=2$ local supersymmetry in $3+1$ space-time dimensions admits three multiplet representations (see *e.g.* [6] for a general treatment and a list of Refs.):

1. one *gravity* multiplet, whose maximal helicity is 2, given by

$$\left(V_\mu^a, \psi^A, \psi_A, A^0\right), \tag{1}$$

where the *Vielbein* one-form V^a (together with the spin-connection one-form ω^{ab}) relates to the graviton ($a=0,1,2,3$), ψ^A, ψ_A are $SU(2)$-doublets of spinor one-forms (usually named gravitinos; $A=1,2$, with the upper and lower indices respectively denoting right and left chirality, *i.e.* $\gamma_5\psi_A=-\gamma_5\psi^A$), and A^0 denotes the Maxwell gauge boson 1-form potential usually named *graviphoton*.

2. n_V *vector* multiplets, whose maximal helicity is 1, given by $(I, i=1,...,n_V)$

$$\left(A^I, \lambda^{iA}, \overline{\lambda}^{\bar{i}}_A, z^i\right), \tag{2}$$

each containing a gauge boson one-form A^I , a $SU(2)$-doublet of zero-form spinors $\lambda^{iA}, \overline{\lambda}^{\bar{i}}_A$ (usually named gauginos), and a complex scalar field (zero-form) z^i. The z^i's coordinatize a complex manifold M_{n_V}, of complex dimension n_V, which is endowed with a projective *special Kähler* structure by supersymmetry.

3. n_H *hypermultiplets*, whose maximal helicity is 1/2, given by $(\alpha=1,...,2n_H)$

$$\left(\zeta_\alpha, \zeta^\alpha, q^u\right), \tag{3}$$

each containing a pair of zero-form spinors $\zeta_\alpha, \zeta^\alpha$ (named hyperinos), and four real scalar fields q^u ($u=1,...,4n_H$), which coordinatize a quaternionic manifold $\mathcal{Q}_{n_H}$ (of quaternionic dimension n_H).

When there is no gauging of any global isometry of M_{n_V} and/or Q_{n_H}, the n_H hypermultiplets are not involved in the AM, and they can be completely decoupled from the attractor dynamics in the black hole background. This is a direct consequence of the supersymmetry transformation properties of the zero-form spinor fields: the hyperinos ζ_α's transformations do not depend on the graviphoton A^0 nor on A^I's (i.e., on the Maxwell 1-form potentials), whereas gauginos λ^{iA}'s ones do depend on the Maxwell potentials. More precisely, when disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of hyperinos read [6]

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta}\,, \tag{4}$$

implying that the values of the quaternionic scalar fields q^u in the asymptotical(ly flat,) spacial background are unconstrained, and thus they can vary continuously within Q_{n_H}. In other words, the hyperscalars q^u's are *moduli* of the system in absence of gauging.

Consequently, in order to keep the framework as simple as possible, we can totally disregard hypermultiplets, and this actually does not imply any loss of generality, at least when ungauged theories are considered. Thus, we consider asymptotically flat, sphedrically symmetric, static, dyonic extremal black hole solutions of $\mathcal{N} = 2$-extended supergravity, in which the gravity multiplet (1) is coupled to n_V vector multiplets (2). Since there is no dependence of the black hole metric on time and azimuthal and polar angles, the unique coordinate characterizing the dynamical evolution of the n_V complex scalar fields (one for each vector multiplet) is the radial coordinate: the AM states that, when approaching the event horizon of the black hole, one can always find a solution of the scalar flow such that the scalars dynamically run into fixed points, acquiring values which only depend on (the ratios of) the electric and magnetic charges of the black hole (respectively denoted by q_Λ and p^Λ, with $\Lambda = 0, 1, ..., n_V$), which are conserved quantities due to the overall $U(1)^{n_V}$ gauge symmetry of the system itself and are arranged into the symplectic vector

$$\mathcal{Q} := (p^\Lambda, q_\Lambda)^T\,. \tag{5}$$

Such near-horizon configurations of the scalar fields are completely independent on the boundary conditions of the corresponding dynamics, namely on the spacial asymptotical values of the scalars. Consequently, the dynamical system describing the scalar flow completely loses memory of its initial data, because the dynamical evolution is "attracted" to some fixed configuration points, depending on the electric and magnetic charges *only*. Note that there are no attractors in "pure" $\mathcal{N} = 2$ supergravity, since the $\mathcal{N} = 2$ gravity multiplet (1) has no scalar fields (in fact, the Reissner-Nordström extremal black hole background is scalarless).

In presence of (linearly realized) local supersymmetry, extremal black holes can be interpreted as BPS (Bogomol'ny-Prasad-Sommerfeld)-saturated [7] solutions, in the low-energy, effective field theory limit of higher-dimensional, UV-complete theories, such as $(9+1)$-dimensional superstrings or $(10+1)$-dimensional M-theory [8]. As class of solutions to the Maxwell-Einstein equations of motion, the extremal black holes under considerations are determined by their (asymptotical) ADM mass [9], by the electrical and magnetic charges (defined by integrating the fluxes of related field strengths' 2-forms over a two-sphere at infinity), and by the asymptotical values of the n_V complex scalar fields. Thus, the AM implies that the extremal black holes become "bald", i.e. they lose all their "scalar hair" in the near-horizon limit; in other words, when the extremal black hole metric approaches the conformally flat Bertotti-Robinson $AdS_2 \otimes S^2$ metric [10, 11], it is completely characterized only by electric and magnetic charges, but *not* by the continuously-varying asymptotical values of the scalar fields.

A major breakthrough in the study of AM was achieved in [5], in which the fixed points of the scalar dynamics in the extremal black hole background were characterized as critical points of a suitably defined "black hole effective potential" V_{BH}, in general being a strictly positive definite function of the $2n_V$ real scalars ϕ^a (corresponding to n_V complex scalar fields) and of the $2n_V$ electric and magnetic (real) charges: $V_{BH} = V_{BH}(\phi, \mathcal{Q})$ For a fixed set of e.m. charges $\mathcal{Q}$ (5), the non-degenerate critical points of V_{BH} in M_{n_V}, i.e. those points in M_{n_V} such that

$$\frac{\partial V_{BH}}{\partial \phi^a} = 0: \quad V_{BH}|_{\frac{\partial V_{BH}}{\partial \phi}=0} > 0\,, \quad \forall\ a = 1, ..., 2n_V\,, \tag{6}$$

completely determine the values of the scalar fields in the near-horizon limit, which depend on the electric and magnetic charges of the black hole only. The (semi)classical Bekenstein-Hawking entropy (S_{BH}) - area (A_H) formula [12]- [15] yields the extremal black hole entropy S_{BH} to be given by (π times) the critical value of V_{BH} itself:

$$S_{BH} = \pi \frac{A_H}{4} = \pi \, V_{BH}|_{\frac{\partial V_{BH}}{\partial \phi}=0} \, . \tag{7}$$

These result reduce the study of extremal black hole attractors to the study and classification of the various classes of critical points of V_{BH} which yield a non-vanishing critical value of V_{BH} itself; as we will see below, each of these classes is in $1:1$ correspondence with a U-orbit supporting an attractor, and thus to an attractor "moduli space".

The fluxes (over S^2_∞, which exists because of the spherical symmetry of the black hole metric) of the Maxwell 2-form field strengths (and of their Lagrangian duals) determine the electric-magnetic charges $\mathcal{Q}$ (5) of the black hole itself, which are $2(n_V+1)$ conserved quantities, where n_V is the number of vector multiplets. The "+1" corresponds to the contribution of the graviphoton Maxwell field. In the limit of real values (which is customarily taken within supergravity, thus disregarding charge quantization, and in particular the Dirac-Schwinger-Zwanzinger quantizations condition for dyons), the $2(n_V+1)$ e.m. charges coordinatize a vector space which is the representation space $\mathcal{Q} \equiv \mathbf{R}_\mathcal{Q}$ of the U-duality Lie group G. Onto $\mathbf{R}_\mathcal{Q}$, G acts as a (maximal, non-symmetric) subgroup of $Sp(2(n_V+1), \mathbb{R})$, the split real form of the Lie group whose Lie algebra is $\mathfrak{c}_{n_V+1}$:

$$G \overset{\mathbf{R}_\mathcal{Q}}{\subsetneq} Sp(2(n_V+1), \mathbb{R}) \, . \tag{8}$$

Equivalently, one can state that the embedding (8), whose relevance in field theory was firstly studied by Gaillard and Zumino [16], is a consequence of the fact that the (not necessarily irreducible) G-representation $\mathbf{R}_\mathcal{Q}$ is anti-self-conjugated (i.e., symplectic), by applying a general theorem of Dynkin [17]. Moreover, it should be pointed out that what we are naming as U-duality Lie group $G \equiv G_\mathbb{R}$ is actually the (unquantized,) continuous version of the actual U-duality, stringy group $G_\mathbb{Z}$ [18]. This is consistent with the aforementioned (semi-)classical limit of real charges, also taken into account by the fact that we consider $Sp(2(n_V+1), \mathbb{R})$, and not $Sp(2(n_V+1), \mathbb{Z})$.

Since the action of G onto $\mathbf{R}_\mathcal{Q}$ is in general *non-transitive*, the linear representation vector space $\mathbf{R}_\mathcal{Q}$ gets stratified into disjoint classes of orbits under the action of G itself [19–21]: in general, a G-orbit $\mathcal{O}$ is a (usually non-symmetric) homogeneous space of G,

$$\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_\mathcal{Q} \, , \tag{9}$$

where the isotropy Lie group $\mathcal{H}$ is a (generally non-maximal nor compact) subgroup of G itself, and it is named *stabilizer* of $\mathcal{O}$.

A remarkable fact, stemming from the classical invariant theory applied to the mathematical structure of Maxwell-Einstein-scalar theories, is the following: in all[1] $\mathcal{N}=2$ supergravity theories with homogeneous *symmetric* (vector multiplets') scalar manifolds in $3+1$ space-time dimensions, the pair $\left(G, \mathbf{R}_\mathcal{Q}\right)$ is (a suitable real form of) a θ-group *à la Vinberg* [22], namely the number of nilpotent G-orbits in $\mathbf{R}_\mathcal{Q}$ is finite, and the ring of G-invariant polynomials on $\mathbf{R}_\mathcal{Q}$ is *finitely generated* (with no *syzygies*) by a unique primitive, homogeneous polynomial $\mathcal{I} \equiv \mathcal{I}(\mathcal{Q})$, of degree two or four in $\mathcal{Q}$ (which we will denotes as $\mathcal{I}_2$ resp. $\mathcal{I}_4$); see e.g. Table II of [23], and Refs. therein. In all these cases, the formula (7) acquires a manifestly G-invariant

[1]The same holds for $\mathcal{N} \geqslant 3$-extended supergravity theories, which however we will not treat here.

form,

$$S_{BH} = \pi \frac{A_H}{4} = \pi \begin{cases} |\mathcal{I}_2(\mathcal{Q})| \,, \\ or \\ \sqrt{|\mathcal{I}_4(\mathcal{Q})|} \,. \end{cases} \tag{10}$$

Interestingly, formula (10) relates the Bekenstein-Hawking entropy of extremal black holes to the theory of the aforementioned distinguished class of θ-groups, which can actually be identified as Lie groups of type E_7 *à la Brown* of non-degenerate (when $\mathcal{I} = \mathcal{I}_4$) [24] or degenerate (when $\mathcal{I} = \mathcal{I}_2$) [25, 26] type. After Brown [24], non-degenerate groups of type E_7 can always be characterized as automorphism groups of *Freudenthal triple systems* (which in turn can be of reduced or non-reduced type; see below).

Clearly, the value acquired by $\mathcal{I}$ is constant along any G-orbit. When $\mathcal{I} \neq 0$, the corresponding (generic, open, non-nilpotent) G-orbit supports a "large" extremal black hole, which has $S_{BH} \neq 0$, and thus $A_H \neq 0$, at the two-derivative (Einstein) level; on the other hand, when $\mathcal{I} = 0$, the corresponding (nilpotent) G-orbit supports a "small" extremal black hole, which has $S_{BH} = 0$, and thus $A_H = 0$, at the two-derivative (Einstein) level: thus, such a "small" black hole is intrinsically quantum, since it needs of an higher-derivative theory of gravity (such as the ones occurring in string effective actions) for a sensible description as solution within a Lagrangian theory.

Moreover, in all the above cases, a manifestly G-invariant presentation of the G-orbit stratification of $\mathbf{R}_{\mathcal{Q}}$ is given by the $1:1$ correpondence between G-invariant sets of algebro-differential constraints on $\mathcal{I}(\mathcal{Q})$ and the various (classes of isomorphic) G-orbits $\mathcal{O}$'s. Over $\mathbb{C}$, all "large" G-orbits, which are level hypersurfaces in $\mathbf{R}_{\mathcal{Q}}$, are isomorphic, defining the generic, open orbit; however, over $\mathbb{R}$, different real forms (of Riemannian or pseudo-Riemannian type) exist, distinguished by sign($\mathcal{I}$), but possibly (when G is non-degenerate and non-split) also by further G-invariant "finer" constraints on $\mathcal{I}$. On the other hand, when G is non-degenerate, the stratification of "small" (i.e. nilpotent) G-orbits over $\mathbb{C}$ may involve G-invariant differential constraints on $\mathcal{I}$, and, when G is non-split, finer splittings of the G-orbit stratification may occur over $\mathbb{R}$. For instance, when $\mathcal{I} = \mathcal{I}_4$ (i.e., for G being non-degenerate of type E_7), the stratification of nilpotent G-orbits is given by [27]

$$\begin{array}{lll} \text{nilp. } G\text{-orbit} & G\text{-inv. constraint} & \text{rank}_{FTS}(\mathcal{Q}) \\ \mathcal{O}_3 : & \mathcal{I}_4 = 0 \,, & 3 \,, \\ \mathcal{O}_2 : & \partial \mathcal{I}_4 = 0 \,, & 2 \,, \\ \mathcal{O}_1 : & \partial^2 \mathcal{I}_4 \big|_{\mathbf{Adj}(G)} = 0 \,, & 1 \,, \end{array} \tag{11}$$

where $\text{rank}_{FTS}(\mathcal{Q})$ indicates the G-invariant rank[2] of $\mathcal{Q} \equiv \mathbf{R}_{\mathcal{Q}}$ as element of a (reduced) *Freudenthal triple system* [28, 29], which in turn is constructed over a rank-3 Jordan algebra (which, for $\mathcal{N} = 2$ *symmetric* supergravities, are simple or semi-simple; see table 2). Over $\mathbb{R}$, when G is split, the stratification of nilpotent orbits is still given by (11), whereas when G is minimally non-compact, each of the $\mathcal{O}_3$ and $\mathcal{O}_2$ split into two G-orbits. Note that $\mathcal{O}_1$, which is the minimal, highest weight G-orbit, has the *largest* stabilizer and it is always *unique*.

2 $\mathcal{N} = 2$ *symmetric* supergravities

$\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories [33]- [35] with homogeneous symmetric special Kähler vector multiplets' scalar manifolds will henceforth be shortly referred to as *symmetric* Maxwell-Einstein supergravities. The Riemannian, non-compact, symmetric

[2] If $\mathcal{Q}$ belongs to a "large" G-orbit, i.e. when it is such that $\mathcal{I}_4(\mathcal{Q}) \neq 0$, then $\text{rank}_{FTS}(\mathcal{Q}) = 4$.

Table 1: Riemannian symmetric non-compact special Kähler spaces (*alias* vector multiplets' scalar manifolds of the *symmetric* $\mathcal{N}=2$, $D=4$ Maxwell Einstein supergravity theories). r denotes the geodesic rank of the manifold, whereas n_V stands for the number of vector multiplets.

	$\frac{G}{H_0\times U(1)}$	r	$\dim_{\mathbb{C}} \equiv n_V$
minimal coupling $n\in\mathbb{N}$	$\overline{\mathbb{CP}}^n \equiv \frac{SU(1,n)}{U(1)\times SU(n)}$	1	n
$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1},\ n\in\mathbb{N}$	$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,n)}{SO(2)\times SO(n)}$	$2\ (n=1)$ $3\ (n\geqslant 2)$	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3)\times U(3))}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6
$\mathbb{R}$	$\frac{SL(2,\mathbb{R})}{U(1)}$	1	1

special Kähler manifolds have the general coset structure

$$M_{n_V} := \frac{G}{H_0\times U(1)}\,, \tag{12}$$

where $H_0\times U(1)$ is the maximal compact subgroup (mcs) of the U-duality group G. They have been classified in [30, 31] (see *e.g.* [32] for a quite recent account), and they are reported in Table 1. All the corresponding supergravity theories actually have a five-dimensional origin, since they can be obtained from "parent" (minimally supersymmetric) $\mathcal{N}=2$ supergravities in $4+1$ space-time dimensions, by compactifying *à la Kaluza-Klein* on S^1, and retaining the massless sector. This is reflected in the fact that all such theories are endowed with a holomorphic prepotential function which, after projectivization of the coordinates, is a homogeneous cubic polynomial [33]- [35]. The unique exception is provided by the so-called *Luciani theories* [36], which do not have a five-dimensional origin and correspond to the *minimal coupling* of vector multiplets to $\mathcal{N}=2$ supergravity. The corresponding special Kähler manifolds are all symmetric spaces, all with geodesic rank one, and they are nothing but the Riemannian non-compact versions of the n_V-dimensional complex projective spaces $\overline{\mathbb{C}P}^{n_V}$ (see e.g. [37, 38]); in these theories, the prepotential is a homogeneous quadratic polynomial, and thus the trilinear coupling of $\mathcal{N}=2$ supergravity, expressed by the so-called C-tensor of special geometry, vanishes: $C_{ijk}=0$.

As unraveled for the first time in [33]- [35], the cubic prepotentials of *symmetric* Maxwell-Einstein supergravities are all related to the degree-3 (cubic) norm defined in the corresponding rank-3 Jordan algebra. The sequence of factorized spaces in the third row of Table 1, which is usually referred to as the *generic Jordan family*, is related to the semi-simple rank-3 Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, where $\mathbf{\Gamma}_{1,n-1}$ stands for the degree-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n-1)$ (*spin factors*) [39]. The complex dimension of the corresponding special Kähler manifold[3]

$$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}, \tag{13}$$

is $n+1$, and its geodesic rank is 1+min$(2, n)$. On the other hand, the four isolated *"magic"* theories are based on the four simple rank-3 Jordan algebras $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$, which can be realized as generalized matrix algebras of 3×3 Hermitian matrices over the four Huwitz's normed division algebras $\mathbb{O}$ (octonions), $\mathbb{H}$ (quaternions), $\mathbb{C}$ (complex numbers) and $\mathbb{R}$ (real numbers) [33–35, 39, 41–43]. The name *"magic"* is due to the fact that the Lie algebras of their U-duality groups in $D = 2{+}1$, $3{+}1$ and $4{+}1$ space-time dimensions fit into the celebrated *Magic Square* of Freudenthal and Tits [44–46]. By defining $A \equiv \dim_{\mathbb{R}} \mathbb{A}$ $(= 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively), the complex dimension of the symmetric cosets of the "magic" supergravities is $3(A+1)$. *Last but not least*, the special Kähler scalar manifold of the so-called T^3-model is the rank-1 coset $\frac{SL(2,\mathbb{R})}{U(1)}$ based on the cubic prepotential $F = T^3$ and related to the simplest cubic Jordan algebra, given by the real numbers, with the cubic norm simply given by the cube power. This model has a unique vector multiplet coupled to $\mathcal{N} = 2$ supergravity, and it can be obtained by dimensional reduction of five-dimensional minimal "pure" supergravity.

2.1 "Large" U-duality orbits

The classification of U-duality orbits supporting "large" extremal black holes in symmetric Maxwell-Einstein supergravities in $3+1$ space-time dimensions has been carried out in [37], and it is reported in[4] Table 2.

Given the scalar manifold (12), the U-duality orbits which support ($\frac{1}{2}$-)BPS-saturated black holes , i.e. which preserve the maximal ($\frac{1}{2}$) amount of supersymmetry, has structure

$$\mathcal{O}_{BPS} = \frac{G}{H_0}, \text{ with } H_0 \times U(1) \overset{\text{mcs}}{\subsetneq} G. \tag{14}$$

As discovered in [37], there are other two non-isomorphic classes of U-duality orbits, both supporting extremal black hole attractors which are non-supersymmetric (i.e., which dor not saturate the BPS bound [7]). The first non-supersymmetric (non-BPS) orbit has non-vanishing $\mathcal{N} = 2$ central charge at the horizon $(Z_H \neq 0)$, with coset structure

$$\mathcal{O}_{nBPS, Z_H \neq 0} = \frac{G}{\widehat{H}}, \text{ with } \widehat{H} \times SO(1,1) \subsetneq G, \tag{15}$$

where $\widehat{H}$ denotes the U-duality group of the corresponding parent theory in $4+1$ space-time dimensions, and $SO(1,1)$ corresponds to the radius of the circle S^1 in the Kaluza-Klein reduction from five to four dimensions. The second class of non-BPS U-duality orbits has vanishing central charge at the black hole horizon: $Z_H = 0$, with coset structure

$$\mathcal{O}_{nBPS, Z_H = 0} = \frac{G}{\widetilde{H}}, \text{ with } \widetilde{H} \times U(1) \subsetneq G. \tag{16}$$

[3] This is the unique special Kähler manifold which is the product of two irreducible spaces, as proved in [40].

[4] In the present report, we will not consider the highly-degenerate case given by the so-called T^3-model, for which the reader is addressed to [47], and to Refs. therein.

Table 2: Large G-orbits of *symmetric* $\mathcal{N}=2$, $D=4$ Maxwell-Einstein supergravities. They all support extremal black hole attractors, with different supersymmetry-preserving features.

	$\frac{1}{2}$-BPS orbit $\mathcal{O}_{\frac{1}{2}-BPS}=\frac{G}{H_0}$	nBPS $Z_H\neq 0$ orbit $\mathcal{O}_{nBPS,Z_H\neq 0}=\frac{G}{\widehat{H}}$	nBPS $Z_H=0$ orbit $\mathcal{O}_{nBPS,Z_H=0}=\frac{G}{\widetilde{H}}$
minimal coupling $n\in\mathbb{N}$	$\frac{SU(1,n)}{SU(n)}$	$-$	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}$ $n\in\mathbb{N}$	$SU(1,1)\times\frac{SO(2,n)}{SO(2)\times SO(n)}$	$SU(1,1)\times\frac{SO(2,n)}{SO(1,1)\times SO(1,n-1)}$	$SU(1,1)\times\frac{SO(2,n)}{SO(2)\times SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\times SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Note that $\widehat{H}$ and $\widetilde{H}$ are the only two non-compact forms of H_0 embedded (with rank-1 commutant) into G itself. Thus, the group embedding in the r.h.s. of (15) and (16) are both maximal and symmetric (see *e.g.* [48–50]).

While H_0 is a real compact Lie group (stabilizing the BPS "large" orbit (14)), the groups $\widehat{H}$ and $\widetilde{H}$, respectively stabilizing the non-BPS "large" orbits with $Z_H\neq 0$ (15) and $Z_H=0$ (16), are non-compact, and thus they will admit a proper maximal compact subgroup, which we denote with $\widehat{h}$ resp. $\widetilde{h}$:

$$\widehat{h}=\mathrm{mcs}\left(\widehat{H}\right), \qquad \widetilde{h}=\mathrm{mcs}\left(\widetilde{H}\right). \tag{17}$$

2.2 "Moduli spaces" of attractors

For symmetric $\mathcal{N}=2$ supergravities, general results on the rank $\mathfrak{r}$ of the $2n_V\times 2n_V$ Hessian matrix $\mathbf{H}$ of the effective black hole potential $V_{BH}(\phi,\mathcal{Q})$ at its critical points are known (see e.g. [37] and [51]).

The BPS (non-degenerate) critical points of V_{BH} are stable, and thus the Hessian matrix at BPS critical points $\mathbf{H}_{BPS}$ has no massless modes [5], and its rank is maximal: $\mathfrak{r}_{BPS}=2n_V$. Furthermore, the analysis carried out in [37] showed that for the other two classes of non-BPS critical points of V_{BH}, the rank of $\mathbf{H}$ is model-dependent:

$$\overline{\mathbb{CP}}^n : \ \mathfrak{r}_{nBPS,Z_H=0} = 2\,, \tag{18}$$

$$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1} : \begin{cases} \mathfrak{r}_{nBPS,Z_H\neq 0} = n+2\,, \\ \mathfrak{r}_{nBPS,Z_H=0} = 6\,, \end{cases} \tag{19}$$

$$J_3^{\mathbb{A}} : \begin{cases} \mathfrak{r}_{nBPS,Z_H\neq 0} = 3A+4\,, \\ \mathfrak{r}_{nBPS,Z_H=0} = 2A+6\,. \end{cases} \tag{20}$$

Correspondingly, the number $\sharp$ of massless Hessian modes for the various models is given by

$$\sharp := 2n_V - \mathfrak{r}\,, \tag{21}$$

and thus

$$\overline{\mathbb{CP}}^n : \ \sharp_{nBPS,Z_H=0} = 2\left(n_V-1\right), \tag{22}$$

$$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1} : \begin{cases} \sharp_{nBPS,Z_H\neq 0} = n\,, \\ \sharp_{nBPS,Z_H=0} = 2n-4\,, \end{cases} \tag{23}$$

$$J_3^{\mathbb{A}} : \begin{cases} \sharp_{nBPS,Z_H\neq 0} = 3A+2\,, \\ \sharp_{nBPS,Z_H=0} = 4A\,. \end{cases} \tag{24}$$

From previous statements, it also holds that

$$\sharp_{BPS} = 0\,, \tag{25}$$

for all $\mathcal{N}=2$ theories, regardless the properties of the special Kähler vector multiplets' scalar manifold.

Let us start by recalling that V_{BH} is defined as

$$V_{BH}\left(\phi,\mathcal{Q}\right) := -\frac{1}{2}\mathcal{Q}^T\mathbf{M}(\phi)\mathcal{Q}\,, \tag{26}$$

where ϕ denotes the $2n_V$ real scalar fields parametrising the special Kähler scalar manifold $\frac{G}{H_0\times U(1)}$, and $\mathcal{Q}$ is the symplectic vector of e.m. black hole charges sitting in the G-irrepr. $\mathbf{R}_{\mathcal{Q}}$ of the U-duality group G. Moreover, $\mathbf{M}(\phi)$ is the $2\left(n_V+1\right)\times 2\left(n_V+1\right)$ real, symmetric and symplectic matrix defined as [52–54]

$$\mathbf{M}(\phi) = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1}\,, \tag{27}$$

where $\mathbf{L}=\mathbf{L}(\phi)$ is coset representative of $\frac{G}{H_0\times U(1)}$, *i.e.* a local section of the principal G-bundle over the special Hodge-Kähler scalar manifold $\frac{G}{H_0\times U(1)}$, with structure group $H_0\times U(1)$.

The action of an element $g\in G$ on V_{BH} (26) is such that

$$G : V_{BH}\left(\phi,\mathcal{Q}\right)\mapsto V_{BH}\left(\phi_g,\mathcal{Q}^g\right) = V_{BH}\left(\phi_g,\left(g^{-1}\right)^T\mathcal{Q}\right), \tag{28}$$

thus, V_{BH} is not G-invariant, because its coefficients (given by the components of $\mathcal{Q}$) do not in general remain the same. The situation changes if one restricts g to $g_{\mathcal{Q}}\in\mathcal{H}$, i.e. if one restricts to the stabilizer $\mathcal{H}$ of the "large" G-orbits $\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_{\mathcal{Q}}$ (cf. (9)) to which $\mathcal{Q}$ belongs. In such a case, by definition of $\mathcal{H}$:

$$\mathcal{Q}^{g_{\mathcal{Q}}} = \mathcal{Q} \tag{29}$$

$$\Downarrow$$

$$\mathcal{H} : V_{BH}(\phi, \mathcal{Q}) \mapsto V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \mathcal{Q}^{g_{\mathcal{Q}}}\right) = V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \left(g_{\mathcal{Q}}^{-1}\right)^T \mathcal{Q}\right) = V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \mathcal{Q}\right) \simeq V_{BH}(\phi, \mathcal{Q}) \,. \tag{30}$$

Then, it is natural to split the $2n_V$ real scalar fields ϕ as $\phi = \left\{\phi_{\mathcal{Q}}, \breve{\phi}_{\mathcal{Q}}\right\}$, where

- $$\phi_{\mathcal{Q}} \in \frac{\mathcal{H}}{\mathrm{mcs}(\mathcal{H})} \subsetneq M_{n_V} \,, \tag{31}$$

  where we denote
  $$\frac{\mathcal{H}}{\mathrm{mcs}(\mathcal{H})} =: \mathcal{M}_{\mathcal{Q}} \,, \tag{32}$$

- $\breve{\phi}_{\mathcal{Q}}$ coordinatize the complement of $\mathcal{M}_{\mathcal{Q}}$ in M_{n_V}:
  $$\breve{\phi}_{\mathcal{Q}} \in M_{n_V} \backslash \mathcal{M}_{\mathcal{Q}} \,. \tag{33}$$

One can then define

$$V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right) := V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} (\neq 0) \,, \tag{34}$$

as the values of V_{BH} along the equations of motion for the scalars $\breve{\phi}_{\mathcal{Q}}$. Thus, (30) implies the invariance of $V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right)$ under $\mathcal{H}$:

$$\mathcal{H} : V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right) \mapsto V_{BH,crit}\left(\left(\phi_{\mathcal{Q}}\right)_{g_{\mathcal{Q}}}, \mathcal{Q}\right) \simeq V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right) . \tag{35}$$

Finally, it is crucial to observe that $\mathcal{H}$, except for the ($\frac{1}{2}$-)BPS "large" G-orbit, is generally a *non-compact* real Lie group. This implies that V_{BH} at its critical points is independent on the subset of scalar fields

$$\phi_{\mathcal{Q}} \in \mathcal{M}_{\mathcal{Q}} \subsetneq M_{n_V} \,, \tag{36}$$

i.e. on those scalar fields belonging to the homogeneous symmetric submanifold $\mathcal{M}_{\mathcal{Q}} \subsetneq M_{n_V}$, which thus be regarded as the "moduli space" of the attractor solutions supported by the charge orbit $\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_{\mathcal{Q}}$. Thus,

$$\frac{\partial V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right)}{\partial \phi_{\mathcal{Q}}} = 0 \Rightarrow V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} =: V_{BH,crit}(\mathcal{Q}) \,, \tag{37}$$

or, equivalently:

$$V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \phi}=0} = V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} =: V_{BH,crit}(\mathcal{Q}) \,. \tag{38}$$

By using this line of reasoning, in [55] (see also [56]) it was proved that, remarkably, the rank of **H** corresponds to all positive eigenvalues (i.e., stable directions in the scalar manifold), and also that the massless modes of **H** are actually "flat" directions of V_{BH} at the corresponding classes of its critical points. Thus, such "flat" directions of the critical values of V_{BH} span some "moduli spaces" of the attractor solutions [55], corresponding to those scalar degrees of

Table 3: "Moduli spaces" of non-BPS $Z_H \neq 0$ extremal black hole attractors in $\mathcal{N}=2$, $D=4$ *symmetric* Maxwell-Einstein supergravities. They are the real special (vector multiplets') scalar manifolds of the corresponding $\mathcal{N}=2$, $D=5$ symmetric "parent" supergravity theory.

	$\frac{\widehat{H}}{mcs(\widehat{H})}$	r	$\dim_{\mathbb{R}}$
$\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1},\ n\in\mathbb{N}$	$SO(1,1)\times\frac{SO(1,n-1)}{SO(n-1)}$	$1\ (n=1)$ $2\ (n\geqslant 2)$	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	26
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3,\mathbb{C})}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

freedom which are not stabilized by the AM at the horizon of the extremal black hole. The general coset structure of such "moduli spaces" has the orbit stabilizer as global isometry, and its corresponding mcs as isotropy group; thus, by virtue of the treatment above (cf. (32)), one can generally write that

$$\mathcal{M}_{BPS} = \frac{H_0}{\mathrm{mcs}\,(H_0)} \simeq \varnothing\,, \tag{39}$$

$$\mathcal{M}_{nBPS,Z\neq 0} = \frac{\widehat{H}}{\mathrm{mcs}\left(\widehat{H}\right)} = \frac{\widehat{H}}{\widehat{h}}\,, \quad \dim_{\mathbb{R}}\left(\mathcal{M}_{nBPS,Z\neq 0}\right) = \sharp_{nBPS,Z\neq 0}\,, \tag{40}$$

$$\mathcal{M}_{nBPS,Z=0} = \frac{\widetilde{H}}{\mathrm{mcs}\left(\widetilde{H}\right)} = \frac{\widetilde{H}}{\widetilde{h}}\,, \quad \dim_{\mathbb{R}}\left(\mathcal{M}_{nBPS,Z=0}\right) = \sharp_{nBPS,Z=0}\,, \tag{41}$$

where the non-existence of $\mathcal{M}_{BPS}$ follows from (25). This means that in $\mathcal{N}=2$ symmetric supergravities all critical points of V_{BH} supported by "large" U-duality orbits are *stable*, up to a (possibly vanishing) certain number $\sharp$ of "flat" directions, spanning some proper subspace of the scalar manifold itself:

$$\mathcal{M}_{nBPS,Z\neq 0} \subsetneq M_{n_V}\,, \tag{42}$$

$$\mathcal{M}_{nBPS,Z=0} \subsetneq M_{n_V}\,. \tag{43}$$

Tables 3 and 4 report spaces $\mathcal{M}_{nBPS,Z\neq 0}$ and $\mathcal{M}_{nBPS,Z=0}$, respectively [55].

Interestingly, the "moduli space" $\mathcal{M}_{nBPS,Z\neq 0}$ of non-BPS $Z_H \neq 0$ attractors is the scalar manifold of the corresponding "parent" theory in $4+1$ space-time dimensions [55] (see also [57] and [58] for a result holding for generic special d-geometries).

Table 4: "Moduli spaces" of non-BPS $Z_H = 0$ extremal black hole attractors in $\mathcal{N} = 2$, $D = 4$ *symmetric* Maxwell-Einstein supergravities. They are (non-special) symmetric Kähler manifolds.

	$\frac{\widetilde{H}}{mcs(\widetilde{H})} \equiv \frac{\widetilde{H}}{\widetilde{h}' \times U(1)}$	r	$\dim_{\mathbb{C}}$
minimal coupling $n \in \mathbb{N}$	$\frac{SU(1,n-1)}{U(1)\times SU(n-1)}$	1	$n-1$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, $n \in \mathbb{N}$	$\frac{SO(2,n-2)}{SO(2)\times SO(n-2)}$, $n \geqslant 3$	1 $(n=3)$ 2 $(n \geqslant 4)$	$n-2$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10)\times U(1)}$	2	16
$J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}$	2	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2)\times U(1)} \times \frac{SU(1,2)}{SU(2)\times U(1)}$	2	4
$J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2)\times U(1)}$	1	2

2.3 "Moduli spaces" of the ADM mass

Remarkably, by the thumb rule of orbit stabilizer modded by its mcs, one can associate "moduli spaces" also to "small" U-duality orbits, which do not attractor black holes: indeed, as mentioned above, the corresponding black hole has vanishing entropy in the Einstein, two-derivative approximation, and no AM (at least in the sense pointed out in the previous section; see [59]) holds [47, 60–62]. For "small" U-duality orbits there exists a "moduli space" also when the semi-simple part of the orbit stabilizer is a compact real Lie group: in such cases, the "moduli space" is spanned by the non-reductive, translational part of the orbit stabilizer itself [47, 62]. *Ça va sans dire* that for "small" orbits, there is no event horizon of the extremal black hole at which the N=2 central charge should be evaluated and no AM holds: in these cases, one may consider the asymptotical, spacial limit of the black hole, and put forward the interpretation of the "moduli spaces" associated to "small" orbits as "moduli spaces"of the ADM mass [9] of the "small" black hole itself.

References

[1] S. Ferrara, R. Kallosh and A. Strominger, *N = 2 extremal black holes*, Phys. Rev. D **52**, R5412 (1995), doi:10.1103/PhysRevD.52.R5412.

[2] A. Strominger, *Macroscopic entropy of N = 2 extremal black holes*, Phys. Lett. B **383**, 39 (1996), doi:10.1016/0370-2693(96)00711-3.

[3] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, Phys. Rev. D **54**, 1514 (1996), doi:10.1103/PhysRevD.54.1514.

[4] S. Ferrara and R. Kallosh, *University of supersymmetric attractors*, Phys. Rev. D **54**, 1525 (1996), doi:10.1103/PhysRevD.54.1525.

[5] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. B **500**, 75 (1997), doi:10.1016/S0550-3213(97)00324-6.

[6] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, *$N=2$ supergravity and $N=2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance gaugings and the momentum map*, J. Geom. Phys. **23**, 111 (1997), doi:10.1016/S0393-0440(97)00002-8.

[7] G. W. Gibbons and C. M. Hull, *A Bogomolny bound for general relativity and solitons in $N=2$ supergravity*, Phys. Lett. B **109**, 190 (1982), doi:10.1016/0370-2693(82)90751-1.

[8] G. W. Gibbons and P. K. Townsend, *Vacuum interpolation in supergravity via super p-branes*, Phys. Rev. Lett. **71**, 3754 (1993), doi:10.1103/PhysRevLett.71.3754.

[9] R. Arnowitt, S. Deser and C. W. Misner, *The dynamics of general relativity*, Gen. Relativ. Gravit. **40**, 1997 (2008), doi:10.1007/s10714-008-0661-1.

[10] B. Bertotti, *Uniform electromagnetic field in the theory of general relativity*, Phys. Rev. **116**, 1331 (1959), doi:10.1103/PhysRev.116.1331.

[11] I. Robinson, *A solution of the Maxwell-Einstein equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **7**, 351 (1959).

[12] J. D. Bekenstein, *Black holes and entropy*, Phys. Rev. D **7**, 2333 (1973), doi:10.1103/PhysRevD.7.2333.

[13] S. W. Hawking, *Gravitational radiation from colliding black holes*, Phys. Rev. Lett. **26**, 1344 (1971), doi:10.1103/PhysRevLett.26.1344.

[14] S. W. Hawking, *Black hole explosions?*, Nature **248**, 30 (1974), doi:10.1038/248030a0.

[15] S. W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. **43**, 199 (1975), doi:10.1007/BF02345020.

[16] M. K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, Nucl. Phys. B **193**, 221 (1981), doi:10.1016/0550-3213(81)90527-7.

[17] E. B. Dynkin, *Five papers on algebra and group theory*, American Mathematical Society, Providence, USA, ISBN 9780821817063 (1957), doi:10.1090/trans2/006.

[18] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438**, 109 (1995), doi:10.1016/0550-3213(94)00559-W.

[19] S. Ferrara and M. Günaydin, *Orbits of exceptional groups, duality and BPS states in string theory*, Int. J. Mod. Phys. A **13**, 2075 (1998), doi:10.1142/S0217751X98000913.

[20] R. D'Auria, S. Ferrara and M. A. Lledó, *Central charges and Hamiltonians for 0-brane dynamics*, Phys. Rev. D **60**, 084007 (1999), doi:10.1103/PhysRevD.60.084007.

[21] S. Ferrara and J. Maldacena, *Branes, central charges and U-duality invariant BPS conditions*, Class. Quantum Gravity **15**, 749 (1998), doi:10.1088/0264-9381/15/4/004.

[22] È. B. Vinberg, *The Weyl group of a graded Lie algebra*, Math. USSR Izv. **10**, 463 (1976), doi:10.1070/IM1976v010n03ABEH001711.

[23] V. G. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64**, 190 (1980), doi:10.1016/0021-8693(80)90141-6.

[24] R. B. Brown, *Groups of type E_7*, J. Reine Angew. Math. **236**, 79 (1969).

[25] R. S. Garibaldi, *Groups of type E_7 over arbitrary fields*, Commun. Algebra **29**, 2689 (2001), doi:10.1081/AGB-100002415.

[26] S. Ferrara, R. Kallosh and A. Marrani, *Degeneration of groups of type E_7 and minimal coupling in supergravity*, J. High Energy Phys. **06**, 074 (2012), doi:10.1007/JHEP06(2012)074.

[27] W. Kraskiewicz and J. Weyman, *Geometry of orbit closures for the representations associated to gradings of Lie algebras of types E_7*, (arXiv preprint) doi:10.48550/arXiv.1301.0720.

[28] S. Krutelevich, *Jordan algebras, exceptional groups, and higher composition laws*, (arXiv preprint) doi:10.48550/arXiv.math/0411104.

[29] S. Krutelevich, *Jordan algebras, exceptional groups, and Bhargava composition*, J. Algebra **314**, 924 (2007), doi:10.1016/j.jalgebra.2007.02.060.

[30] E. Cremmer and A. Van Proeyen, *Classification of Kähler manifolds in $N = 2$ vector multiplet-supergravity couplings*, Class. Quantum Gravity **2**, 445 (1985), doi:10.1088/0264-9381/2/4/010.

[31] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. B **400**, 463 (1993), doi:10.1016/0550-3213(93)90413-J.

[32] S. Ferrara and A. Marrani, *Symmetry in mathematics and physics*, American Mathematical Society, Providence, USA, ISBN 9780821847312 (2009), doi:10.1090/conm/490.

[33] M. Günaydin, G. Sierra and P. K. Townsend, *Exceptional supergravity theories and the magic square*, Phys. Lett. B **133**, 72 (1983), doi:10.1016/0370-2693(83)90108-9.

[34] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras*, Nucl. Phys. B **242**, 244 (1984), doi:10.1016/0550-3213(84)90142-1.

[35] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the $d = 5$ Maxwell/Einstein supergravity theories: More on Jordan algebras*, Nucl. Phys. B **253**, 573 (1985), doi:10.1016/0550-3213(85)90547-4.

[36] J. F. Luciani, *Coupling of $O(2)$ supergravity with several vector multiplets*, Nucl. Phys. B **132**, 325 (1978), doi:10.1016/0550-3213(78)90123-2.

[37] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge orbits of symmetric special geometries and attractors*, Int. J. Mod. Phys. A **21**, 5043 (2006), doi:10.1142/S0217751X06034355.

[38] S. Ferrara, A. Gnecchi and A. Marrani, *$d = 4$ attractors, effective horizon radius, and fake supergravity*, Phys. Rev. D **78**, 065003 (2008), doi:10.1103/PhysRevD.78.065003.

[39] P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29 (1934), doi:10.2307/1968117.

[40] S. Ferrara and A. Van Proeyen, *A theorem on N = 2 special Kähler product manifolds*, Class. Quantum Gravity **6**, L243 (1989), doi:10.1088/0264-9381/6/12/002.

[41] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society, Providence, usa, ISBN 9780821846407 (1968), doi:10.1090/coll/039.

[42] M. Günaydin, *Exceptional realizations of the Lorentz group: Supersymmetries and leptons*, Nuovo Cimento A **29**, 467 (1975), doi:10.1007/BF02734524.

[43] M. Günaydin, C. Piron and H. Ruegg, *Moufang plane and octonionic quantum mechanics*, Commun. Math. Phys. **61**, 69 (1978), doi:10.1007/BF01609468.

[44] H. Freudenthal, *Beziehungen der $\mathfrak{E}_7$ und $\mathfrak{E}_8$ zur Oktavenebene. VIII*, Indag. Math. (Proc.) **62**, 447 (1959), doi:10.1016/S1385-7258(59)50052-9.

[45] B. Rosenfeld, *Geometry of Lie froups*, Springer, Boston, USA, ISBN 9781441947697 (1997), doi:10.1007/978-1-4757-5325-7.

[46] J. Tits, *Sur certaines classes d'espaces homogè nes de groupes de Lie*, in *Jacques Tits, Œuvres – Collected works*, European Mathematical Society, Zurich, Switzerland, ISBN 9783985470303 (2013), doi:10.4171/126.

[47] L. Borsten, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Small orbits*, Phys. Rev. D **85**, 086002 (2012), doi:10.1103/PhysRevD.85.086002.

[48] R. Gilmore, *Lie groups, Lie algebras, and some of their applications*, Dover Publications, Mineola, USA, ISBN 9780486445298 (2006).

[49] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, USA, ISBN 9780821828489 (1978).

[50] R. Slansky, *Group theory for unified model building*, Phys. Rep. **79**, 1 (1981), doi:10.1016/0370-1573(81)90092-2.

[51] S. Bellucci, S. Ferrara, R. Kallosh, A. Marrani, *Extremal black hole and flux vacua attractors*, in *Supersymmetric mechanics - vol. 3*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540795223 (2008), doi:10.1007/978-3-540-79523-0_3.

[52] L. Andrianopoli, R. D'Auria, S. Ferrara, M. Trigiante, *Extremal black holes in supergravity*, in String theory and fundamental interactions, Springer, Berlin, Heidelberg, Germany, ISBN 9783540742326 (2008), doi:10.1007/978-3-540-74233-3_22.

[53] P. Aschieri, S. Ferrara, B. Zumino, *Duality rotations in nonlinear electrodynamics and in extended supergravity*, Riv. Nuovo Cimento **31**, 625 (2008), doi:10.1393/ncr/i2008-10038-8.

[54] A. Ceresole, S. Ferrara, A. Gnecchi and A. Marrani, *d-geometries revisited*, J. High Energy Phys. **02**, 059 (2013), doi:10.1007/JHEP02(2013)059.

[55] S. Ferrara and A. Marrani, *On the moduli space of non-BPS attractors for $\mathcal{N} = 2$ symmetric manifolds*, Phys. Lett. B **652**, 111 (2007), doi:10.1016/j.physletb.2007.07.001.

[56] S. Ferrara and A. Marrani, *$\mathcal{N} = 8$ non-BPS attractors, fixed scalars and magic supergravities*, Nucl. Phys. B **788**, 63 (2008), doi:10.1016/j.nuclphysb.2007.07.028.

[57] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d correspondence for the black hole potential and its critical points*, Class. Quantum Gravity **24**, 5651 (2007), doi:10.1088/0264-9381/24/22/023.

[58] S. Nampuri, P. K. Tripathy and S. P. Trivedi, *On the stability of non-supersymmetric attractors in string theory*, J. High Energy Phys. **08**, 054 (2007), doi:10.1088/1126-6708/2007/08/054.

[59] M. Trigiante, T. Van Riet and B. Vercnocke, *Fake supersymmetry versus Hamilton-Jacobi*, J. High Energy Phys. **05**, 078 (2012), doi:10.1007/JHEP05(2012)078.

[60] B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, *Duality, entropy, and ADM mass in supergravity*, Phys. Rev. D **79**, 125010 (2009), doi:10.1103/PhysRevD.79.125010.

[61] L. Borsten, D. Dahanayake, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Observations on integral and continuous U-duality orbits in $\mathcal{N} = 8$ supergravity*, Class. Quantum Gravity **27**, 185003 (2010), doi:10.1088/0264-9381/27/18/185003.

[62] B. L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, *Charge orbits of extremal black holes in five-dimensional supergravity*, Phys. Rev. D **82**, 085010 (2010), doi:10.1103/PhysRevD.82.085010.

Kurt Bernardo Wolf memorial lecture

George Pogosyan[1] and Mariano A. del Olmo[2⋆]

1 ICAS, Yerevan State University, Yerevan, Armenia
2 Departamento de Física Teórica, Atómica y Optica and IMUVA,
Universidad de Valladolid, 47011 Valladolid, Spain

⋆ marianoantonio.olmor@uva.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

A personal view of the Mexican mathematical physicist Kurt Bernardo Wolf (1942-2022) is presented here.

Received 23-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.010

To the memory of our friend and colleague Bernardo.

Introduction by M. A. del Olmo

Our colleague and friend Kurt Bernardo Wolf Bogner passed away on 25th May 2022 in Cuernavaca, Mexico. I received the sad news of his death a couple of days later and it shocked me. Just ten days before we had exchanged some emails. I was aware that he did not have good health, but I didn't think it was too serious for such a quick fatal outcome.

I met Bernardo in 1981 in Canterbury at Group10. It was my first participation in this series of Colloquia. We have been friends ever since. I have always been charmed by his ease in engaging in conversation on any subject, his boundless curiosity, his touch of eccentricity and his affability. We did not collaborate scientifically, although in recent years we have coincided on some topics we have discussed when we have met.

I do not want to be much longer, because our friend George Pogosyan, who has worked actively with him for many years, has prepared a memorial lecture. Unfortunately George is unable to be here with us due to health problems. Let me send a warm *abrazo* (hug) to George with our best wishes for a quick recovery.

Memorial lecture by G. Pogosyan

Kurt Bernardo Wolf, was a great friend to all of us, as great humanist and brilliant scientist. As a theoretical physicist deeply influenced the Mexican science, and not only. He left us on the 25th of May 2022 in Cuernavaca, Mexico.

In science he is well known for his contribution to mathematical physics, in particular in the application of group theory and symmetry methods to fundamental problems in atomic and molecular physics, classical and quantum optics, in Fourier integral transforms (where he wrote a book titled *Integral Transforms in Science and Engineering*), in theory of differential equations, special functions, integrable and superintegrable systems. He is the author of more than 200 articles and two scientific books [1,2].

Bernardo Wolf was encouraged by his parents from a young age to pursue his studies in sciences. After finishing the high school in 1960, being 16 years old, he started his undergraduate studies at the Universidad Nacional Autónoma de México (UNAM) to become a theoretical physicist, where his PhD advisor was Marcos Moshinsky. To continue his PhD studies he moved to the Weizmann Institute of Science and the Tel-Aviv University to finish his PhD thesis [3]. After obtaining his PhD in 1970, he lived in Gothenburg, Sweden, where he was a post-doctoral associate at Chalmers University. Finally, in 1971 he returned to Mexico where he became a principal investigator at the Instituto de Ciencias Físicas at UNAM.

In 2022 he was elected a Fellow by Optica for his outstanding and numerous contributions to mathematical optics, including signal analysis, by employing symmetry methods known as group theory. In the same year, the Mexican Government's National System of Researchers (SNI) honoured him as Emeritus National Researcher.

Along with a selfless love for theoretical physics, Bernardo's soul was craving for great travels. In his youth, he travelled through Ethiopia, Kenya, Uganda, Rwanda, Burundi, and

Tanzania, including Kilimanjaro, and South Asia, including Persia, Afghanistan, Pakistan and India. Bernardo Wolf authored a book about his travels [4].

Among many of his talents, he especially stood out as a brilliant organizer. In 1986, Wolf became the founding director of the Centro Internacional de Ciencias (CIC) at his alma mater UNAM. He remained director at CIC until 1993, where he organized more than a dozen symposiums and conferences. For a long time he was a member of the Standing Committee of ICGTMP and QTS series of conferences and main organizer of Group25 in Cocoyoc, Mexico in 2004. By 1987, he helped found the Mexican Academy of Optics (Academia Mexicana de Óptica).

I first met Bernardo Wolf almost thirty years ago in 1995 at Dubna, Russia, when he attended our Conference on Symmetry Methods in Physics. We immediately found many common scientific interests, especially in the field of the theory of superintegrable systems. The next time we met was in Cuernavaca, in 2000, where I was invited for two years as researcher in the Centro de Ciencias Fisicas. In Cuernavaca, under the patronage of Bernardo Wolf, we had a good scientific group, which, in addition to me, also included Natig Atakishiyev. In the past few years, Alexander Yakhno from the University of Guadalajara has also joined us. We managed to publish two extensive rounds of scientific articles (about 20) concerning the definition of Wigner distribution functions on n-dimensional spaces of constant curvature (spheres and hyperboloids) and superintegrable systems of Zernike type. By the way, recent studies (the Zernike system) have been continued in the work of Mariano A. del Olmo and Francisco Hérranz.

Bernardo was one of the brightest and original-thinking persons I have ever met. I was amazed by his knowledge. He understood well of Christianity or Islam, and as orientalist had a deep understanding of the Indian philosophy. He was fluent not only in Spanish and English which allowed him to brilliantly write articles in both languages, but also could read in many other languages such as Russian, Hebrew, or Swedish. I have to say that the beauty was his god and simplicity his ideal. It manifested in everything, whether in buying a car, decorating his house or his office. I remember when I sat in his office, he always played classical music before starting to work on his computer.

He knew how to be a great friend. Two weeks before his death, he wrote to me:

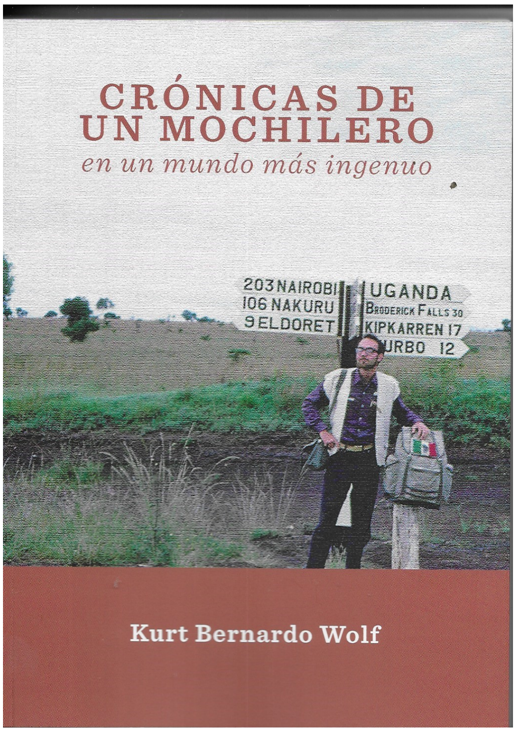
Dearest George!

It is good to know that you are there, even with a lot of health problems. Unfortunately, I am not so healthy either. Since 2020 I have been using oxygen 24/7 due to lung problems, and I am also confined at home. I have been working as far as I can on-line: Kenan graduated with *honors, 3 articles on Bessels and* 2×2 *matrices written, etc., Also my "crónicas de un hitchhiker in a more amicable world" (in Spanish) was published (292 pages, 120 of my photos) but it is becoming increasingly difficult to focus on things. I cannot return to Yerevan in this condition, but I would like to accompany you spiritually in our joint effort to stay alive, well and with bright future. About Natig, earlier this year he underwent a heart operation, and his state of health is also not great. ...such is life, as we often said...*

Still, may I send you a warm abrazo, such as only good old friends can give!

Bernardo

I am blessed to remember the time spent in Cuernavaca. These memories warm my heart.



Acknowledgements

The authors would like to thank Gunnar Wolf for permission to use the photos.

References

[1] K. B. Wolf, *Integral transforms in science and engineering*, Springer, New York, USA, ISBN 9781475708721 (1979), doi:10.1007/978-1-4757-0872-1.

[2] K. B. Wolf, *Geometric optics on phase space*, Springer, Belrin, Heidelberg, Germany, ISBN 9783642060366 (2004).

[3] K. B. Wolf, *Group theory and Regge poles*, PhD thesis, Tel Aviv University, Israel (1969).

[4] K. B. Wolf, *Crónicas de un mochilero ingenuo*, Tinta En Pié, Cuernavaca, México, ISBN 9786079178420 (2022).

Tchavdar Dimitrov Palev – In memoriam

N. I. Stoilova⋆

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciencies, Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria

⋆ stoilova@inrne.bas.bg

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Received 07-11-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.011

Our colleague and friend Tchavdar Palev passed away in Sofia, Bulgaria on November 19, 2021, after a long fight with a terrible disease.

Palev was an active member of the Standing Committee of the “International Colloquium of Group Theoretical Methods in Physics” from 1988 until December 2008.

As a scientist

- Tchavdar was successful as an outstanding researcher in theoretical and mathematical physics;
- he was an excellent lecturer;
- he was a promotor of academic freedom, international collaboration and tolerance.

Tchavdar Dimitrov Palev was born in Sofia on April 15, 1936. At the age of 6 months his family moved to Paris. Soon after, his father took part in the Spanish Civil War, where he was killed in action. The mother returned with her infant son to Sofia, where she raised and educated him in a spirit imbued with the pursuit of knowledge, hard work, honesty and integrity.

Tchavdar graduated from secondary school in Sofia in 1955 and dreamed of building airplanes, but as often happens at turning points in life, fate took him in another direction. He studied physics at Moscow State University "M.V. Lomonosov", where defended his diploma thesis in 1961. Then returning to Bulgaria, he won a competition to become a research assistant at the then existing Institute of Physics of the Bulgarian Academy of Sciences, and later went to work at one of its successors, the Institute for Nuclear Research and Nuclear Energy.

Palev was one of the first physicists to specialize at the International Center for Theoretical Physics in Trieste, Italy, which opened in 1964. Working for a year at the Center, he was one of the lucky ones who were in daily contact with its Director, Nobel Laureate Abdus Salam. In 1968, he defended his Ph.D. thesis in mathematical physics at the University of Marburg, (West) Germany on "Realization of Lie algebras as functions of Heisenberg algebra generators. General theory and applications to finite- and infinite-component field equations". His supervisor was Prof. Dr. H. D. Doebner.

Being back in Sofia he received his Habilitation degree and was elected as a Professor at the Institute for Nuclear Research and Nuclear Energy at the Bulgarian Academy of Sciences. In recognition of his achievements in the field of mathematical and theoretical physics, Palev was successively elected as a corresponding member and academician in 2004 and 2008.

As a human, Palev was a sports personality, he jogged every day. On a cold, snowy day he was jogging along a river in Sofia when he noticed something strange in the water. Tchavdar realized it was a 7-8 year old drowning child. Without a second thought, he entered the icy water, pulled him out, gave him first aid and took him to the hospital. This is how Tchavdar saved the life of the child, who is now a young man.

Each year from 1984 to 1994 Palev was a guest professor at the Arnold Sommerfeld Institute for Mathematical Physics, TU Clausthal, Germany giving lectures on associative algebras, Lie algebras, Lie superalgebras and Hopf algebras. He was a guest professor at the Department of Physics, University of Naples, Italy in 1990 and at the Department of Mathematics, Concordia University, Montreal in 1992. In addition Tchavdar was a visiting professor at the Department of Mathematics, University of Queensland, Brisbane, Australia, at the Research Institute for Fundamental Physics, Kyoto, Japan, at the Department of Physics and Astronomy, University of Rochester, USA, at the Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Belgium, at the Department of Mathematics, University of Southampton, UK and he was a honourary associate in ICTP, Trieste from 1992 to 1998.

In 1987, Tchavdar organized the XVI International Colloquium on Group Theoretical Methods in Physics, which took place in Varna, Bulgaria. The following year, 1988, he was elected as a member of the Standing Committee. Palev was an active contributor to the organization of our Colloquia until 2008 when he had to retire, along with 7 other established members.

A large part of Palev's scientific activity is devoted to the algebraic approach he introduced for the generalization of quantum statistics, i.e. the Bose-Einstein and Fermi-Dirac statistics, as well as their generalizations, the Green's parastatistics, which he proved to correspond to different representations of Lie algebras or superalgebras of class B. These results give a certain algebraic sense to the known statistics and become the starting point for many interesting generalizations. Palev introduced the notion of a Wigner quantum system and studied the physical properties of such systems. It turns out that the geometry of these systems is non-commutative, which is one of the important directions in particle physics today. The A statistics he introduced is known and quoted today as "Palev's statistics". Also, of interest are the mathematical problems solved by Tchavdar – he constructed explicit finite-dimensional representations of Lie superalgebras, defining an analogue of the Gelfand-Zetlin basis for them. He generalized these results to their infinite-dimensional analogues and to quantum deformations of these superalgebras.

At the celebration of the 75th anniversary of T. Palev, his co-authors said:

Ronald King: “My impressions of Tchavdar Palev were that he was immensely stimulating, with a great breadth of knowledge and expertise in group theoretical methods and their application to physics, characterised by great attention to detail. But perhaps above all he was great fun to work with, and it is both a delight and a privilege to contribute here to the celebration of his 75th birthday.”

Joris Van der Jeugt: “T.D. Palev laid the foundations of the investigation of Wigner quantum systems through representation theory of Lie superalgebras. His work has been very influential, in particular on my own research. It is quite remarkable that the study of Wigner quantum systems has had some impact on the development of Lie superalgebra representations.”

I was a Ph.D. student of Tchavdar and after defending my thesis I had the privilege of working with him for about 10 years. He was a dedicated mentor, a stimulating scientist, a splendid person and a great inspiration to me.

Acknowledgements

Funding information The author was supported by the Bulgarian National Science Fund, grant KP-06-N28/6.

Two entangled and scientifically impactful lives: Jiří Patera, Pavel Winternitz and the Montréal school of mathematical physics

Luc Vinet★

IVADO and Centre de Recherches Mathématiques,
Université de Montréal, Montréal, QC, Canada

★ vinet@crm.umontreal.ca

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

This text offers a personal account of the scientific legacy of two giants of mathematical physics at the turn of the Millenium and their heritage in Canada, their land of adoption.

Received 13-11-2022
Accepted 28-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.012

To the memory of my mentors and friends, Jiří and Pavel, with admiration and gratitude.

1 Introduction

Within a year, sadly, Pavel Winternitz and Jiří Patera passed away in 2021 and 2022. Each one of them stands tall and deserves separate praise. Fate has had it that they have often been celebrated together even though in fact, most of their scientific work has been done separately. I will also indulge in this conflation that will not fully do them justice. One reason is that they cannot be dissociated on the occasion of the 50th anniversary of the International Colloquium on Group Theoretical Methods in Physics (ICGTMP); another, is that they leave a truly joint heritage in Montreal.

Like quantum interacting particles that become entangled and yield "magic" [1], through the vicissitudes of History, these brilliant individuals were set on a colliding course which they steered to have profound influences on various fronts. This is the story that I will try to tell as a tribute to their accomplishments.

2 Early life and education in the eastern bloc

Born in Czechoslovakia, both in 1936, Jiří and Pavel have been educated as theoretical physicists in the great Russian tradition of the Soviet era.

Figure 1: Jiří and Pavel.

Jiří was a native of Zdice, a small town in central Bohemia near Prague. He attended high school in Děčín in northwestern Bohemia after which he studied at Moscow State University and Dubna as well. In 1964, he obtained his Doctorate from Charles University in Prague. One of his first papers published in 1963 in Nuclear Physics studied the production of Λ-hyperons in $\pi^- - p$ interactions [2] and was written in collaboration with the prominent physicist Blokhintsev, a student of Tamm, who founded the Joint Institute for Nuclear Research (JINR) in Dubna and was its first director. In 1965, Jiří took a leave from the Physical Institute of the Czechoslovak Academy of Sciences to hold a postdoctoral fellowship from the National Research Council (NRC) of Canada within the developing theory group of the Physics Department of the Université de Montréal that included Asok Bose, Guy Paquette and soon thereafter Jean-Robert Derome as well as Robert Brunet in the Mathematics Department. This is when, with Bose, Jiří began his work on group theoretical methods and as you appreciate, this stay in Montreal was to have in many other ways a determining effect on the future. It should be recalled that a memorable world fair the "Expo 67" took place in Montreal during that period and that the Tchechoslovak pavilion was one of the most popular. At the end of his NRC award in December 1966, Jiří came back to Prague.

Pavel was born in Prague. He spent the war years in England from where came his fluency in English. He pursued graduate studies at the Leningrad University where Fock was teaching and then in Dubna where he obtained his doctorate in 1966 under the supervision of Smorodinsky, a student of Landau, who Blokhintsev had invited in 1956 to become the head of the Theoretical Group of the JINR. In 1967, he took a leave from the JINR and from the position he had obtained at the Nuclear Research Institute of the Czechoslovak Academy of Sciences in Řež to spend time at the International Center for Theoretical Physics in Trieste. In 1968, he returned to his home country.

3 Prague and Montreal

Czechoslovakia was an exciting place to be at the beginning of 1968 with the election in January of Dubček as First Secretary. Followed the Prague Spring with its waves of proposed reforms. This unfortunately displeased the Soviet leaders to the extent that in late August troops from four Warsaw Pact countries invaded and controlled Czechoslovakia. For some days, it was possible to flee and so did Jiří and Pavel. With a visa in hand to attend a scientific meeting in Vienna, Jiří, his wife Tania and their baby daughter Sacha left with what they could pack in their small car. After a stay in London, they headed to a city that was familiar to Jiří, namely Montreal, where he was appointed Researcher in the nascent CRM in 1969. More hesitant to cut ties with Europe, Pavel initially went to England with his wife Milada and their twin boys Michael and Peter and spent some time at the Rutherford High Energy Laboratory in Chilton where Roger Philips a student of Dirac was his host. A year later he crossed the Rubicon and at the invitation of Wolfenstein, Pavel moved to Pennsylvania with his family first to Carnegie-Mellon and subsequently to the University of Pittsburgh. This is when he was encouraged to come to Montreal by Jiří.

In 1968, the Rector of the Université de Montréal Roger Gaudry, together with Maurice Labbé, the first Vice-Rector Research of the university and, Jacques St-Pierre who created the Computer Science Department, had the vision to establish a national institute for research in the mathematical sciences: the Centre de Recherches Mathématiques that was to become internationally known as the CRM. They obtained sizable funding from the NRC to that end. Jacques St-Pierre acted as Interim Director and hired Jiří. Three years later, in 1972, Pavel was also joining the CRM as Researcher. And this is how the stage was set for two young Czechs in their mid-thirties to shape the course of mathematical physics in Canada. In retrospect, the Czech diaspora generated by the 1968 events had a profound effect on the scientific life of Montreal. I shall expand on the roles that Jiří and Pavel played but there is another striking example that I wish to mention. Montreal has two university hospital systems attached to the Université de Montréal and McGill University. Quite strikingly at the same time in the 90s and 2000s, the research institutes of both these university hospitals were led by two Czechs: Pavel Hamet and Emil Skamene who had come to Montreal in circumstances similar to those of Jiří and Pavel and who all became friends of course.

I started my undergraduate studies at the Université de Montréal in 1970. Little was I suspecting that the Prague Spring demise that I had watched unfold with distress two years before was to have a defining impact on my life. In the Fall of 1972, totally oblivious to the creation of the CRM, I wished to enroll in the Master's program in Physics and was looking for a supervisor. It is Robert Brunet from whom I had taken a class who informed me of the existence of the CRM and of the fact that it had recruited two outstanding theoretical physicists with one, Pavel, that had just arrived. He thought they would be interested in taking graduate students and suggested that I approach them. I followed up and still recall the enthusiasm I felt when together Jiří and Pavel presented their research programs to me. This was obviously a personal defining moment.

4 CRM: The early period and the collaborative years

When the Centre de Recherches Mathématiques or CRM was created the plan was to develop research groups in mathematics and statistics, theoretical physics and computer science. While quality hires were made in all three sectors, somehow the group in computer science dispersed. The physics division had a statistical mechanics section that proved less cohesive and very much thanks to Jiří and Pavel a tradition in mathematical physics developed along with other more mathematical areas that also benefited from the presence of Czech scientists like Anton Kotzig and Ivo Rosenberg.

In the Summer of 1974, when I reported to begin my Master's, the CRM was located in the Jésus-Marie Pavilion and moved that very Summer to the location on Côte Ste-Catherine that many of you have visited and where it stayed until 1994. For starters, I was asked to read the book of Naimark on the representations of the Lorentz group. Eventually, I collaborated more closely with Pavel on superintegrable models. By then, Jiří and Pavel had hired a postdoc whose name was Ernie Kalnins that I had a hard time understanding because of his New-Zealander accent. The group was already very lively. Bob Sharp who was on Faculty at McGill University had found like-minded colleagues in Jiří and Pavel at the CRM and was already collaborating with them. Marcel Perroud who had completed a thesis with Derome and John Harnad became soon regular members of the team whom Jiří and Pavel have much supported. It is during that period in 1973, that Jiří together with David Sankoff another CRM researcher, produced, in book format [3], his first set of tables collecting data on representations of simple Lie algebras (others were produced later [4–7]). Thanks to particle physicists such as Pierre Ramond and Dick Slansky who made them largely known, these tables found their ways to the bookshelves of many scientists making practical use of representation theory.

André Aisenstadt, a Montreal philanthropist who held a doctorate in theoretical physics from the University of Zurich was a benefactor of the CRM and endowed a distinguished lectureship at the CRM known as the Aisenstadt Chair. In 1973-74, Jiří and Pavel arranged for Marcos Moshinsky to hold this Chair and to spend an extended period in Montreal. Charles Boyer and Kurt Bernardo Wolf came with him from Mexico on this occasion. Willard Miller Jr. was spending a sabbatical at the CRM (that led to his career-long partnership with Kalnins). You can imagine the intellectual intensity that such a concentration of visiting collaborators was generating and this came to be the rule with Jiří and Pavel acting as extraordinary magnets. A continuous flow of distinguished speakers would participate in the weekly seminar. Viktor Kac for example who in 1977 had recently completed the classification of Lie superalgebras visited the CRM very soon after he arrived at MIT. In 1979, the Aisenstadt Chair was attributed to Yuval Ne'eman and so on. Eugen Dynkin was also among the numerous distinguished people to hold that chair; Jiří had early on carefully studied his works in Russian and I always thought that this had a big influence on him.

After having completed my Master's, I embarked on a Ph.D. Together with John Harnad and Steven Shnider a differential geometer from McGill and under the benevolent eye of Pavel, I began investigations bearing on gauge field theories and their geometrical and topological properties that were generating much interest at the time. Yvan Saint-Aubin started his Ph.D. two years after I did and other talented students kept coming to this exceptional environment that Jiří and Pavel were animating.

Between 1973 and 1980, Jiří and Pavel wrote an astonishing average of 7 papers per year together. The Journal of Mathematical Physics or JMP which was then one of their favorite venues could as well have been called the JJP for journal of Jiří and Pavel! Their first paper published in 1973 [8] brought Heun polynomials into the realm of the rotation group representations and 50 years later is still inspiring new results. Jiři and Pavel had the knack for developing lasting collaborations with truly distinguished researchers. In the 70s and the 80's one of those was Hans Zassenhaus who is known as a pioneer of computer algebra. He has had an illustrious career that began by being an assistant to Emil Artin in Hamburg in 1936. After having occupied positions at various universities with McGill among those, in 1965 Zassenhaus settled at Ohio State University for the rest of his career and from there visited Montreal regularly even taking a sabbatical at the CRM in 1977-78. As an illustration of the fruitful collaboration he enjoyed with Jiři and Pavel I will recall the program they initiated in 1975 aiming to determine the continuous subgroups of the fundamental groups of physics [9]. This involved many additional collaborators (among them Guy Burdet and Martine Perrin from Marseille for work on the optical group), and culminated with the study of the conformal group from that perspective. The results found many applications among which the classification by Beckers (from Liège), Harnad, Perroud and Winternitz of the tensor fields invariant under conformal transformations [10]. My thesis work on solutions of the Yang-Mills equations has also roots in these foundational studies. And thus Jiří and Pavel thrived and drew many into their wake.

From 1973 to 1982, the Director of the CRM was Anatole Joffe. At the end of his term, the funding model of the CRM was modified to one where its researchers would hold Faculty positions in university departments and it is then that Jiří and Pavel became professors in the Department of Mathematics and Statistics of the Université de Montréal. In 1984, Francis Clarke was appointed Director and the CRM blazed new trails under his leadership. It played a pioneering role together with the MSRI in defining in the 80s the modern organization of research in the mathematical sciences around institutes organizing visitors and thematic programs.

5 ICGTMP and Outreach

The International Colloquium on Group Theoretical Methods in Physics was initiated by Henri Bacry from Marseille and Aloysio Jenner from Nijmegen and oscillated between these two cities from 1972 to 1975. In 1974, even though I was a rather young graduate student then, I had the privilege to attend the third edition of the colloquium in Marseille. There was a good contingent from Montreal led of course by Jiří and Pavel. I could thus observe firsthand the interest they were generating and how many interactions they were having. During the event, they proposed that the conference be held at the CRM in 1976, the year Montreal was to have the Olympic Games. This proposal was adopted and after returning to Nijmegen in 1975, the ICGTMP was held outside Europe for the first time and, thanks to the leadership of Jiří and Pavel, became truly an international series. Before the colloquium, they also organized a three-week summer school in the framework of the yearly Séminaire de Mathématiques Supérieures (SMS) initiated by Maurice Labbé in 1962 at the Université de Montréal. Renowned scientists such as Feza Gürsey, Sigurdur Helgason, Peter Lax, Louis Michel, Willard Miller, Marcos Moshinsky, and many others lectured at these meetings. Jiří and Pavel showed us the way. Following in their footsteps Yvan Saint-Aubin and I organized the ICGTMP again in Montreal in 1988 together with an edition of the SMS. Because of the glasnost and perestroika taking place in the Soviet Union, it proved possible to host for the first time in the West many distinguished Russian scientists such as Yuri Manin or Sacha Zamolodchikov who were playing key roles in the development of quantum groups and conformal field theories. I shall mention another ICGTMP story. In 1978-79, having mostly finished my Ph.D. work, I accompanied Pavel for a year in Paris as he was spending a sabbatical at Saclay. I shall come back to this later. During that year, André Aisenstadt offered me a scholarship that made it possible to attend the Jerusalem Einstein Centennial as well as the ICGTMP that followed in Kyriat Anavim. This offered memorable experiences with Jiří and Pavel who was kind enough to share a room with me. Pavel was for many years a member of the ICGTMP Standing Committee and in 2018 he received the Wigner medal.

In addition to the ICGTMP, Jiří and Pavel organized numerous meetings over the years thus influencing research directions broadly and much contributing to the animation of the international scientific community and the visibility of the CRM. One striking example is the workshop on Symmetries and Integrability of Difference Equations that I ideated with Pavel in 1994 and which Decio Levi helped put in place; as you know, this led to the ongoing SIDE series of biennial conferences. Another example involving Jiří this time is the thematic program entitled Aperiodic Long Range Order that he and Bob Moody organized albeit at the Fields Institute in 1995 [11].

Jiří and Pavel have also been very active in developing international collaborations. To that end they made good use of agreements between France and Belgium; this led in particular to the appointment of Véronique Hussin at the Université de Montréal. Over time Jiří concentrated more in North America. He developed a very fruitful and long lasting collaboration with Bob Moody who was based in Saskatchewan and Alberta. In 1983-84, he spent a sabbatical at Caltech. Around that time, he began collaborating with Gordon Shaw and became involved in the MIND Research Institute which was created in 1998 in Irvine. Jiří has also been a regular participant in the Aspen Center for Physics program and often visited the MSRI. From the mid 90s onward, with Eliza Shahbazian, Jiří also pursued collaborative projects with Lockheed Martin Canada and OODA technologies. The international collaborations of Pavel were concentrated mostly in Europe, more precisely in Italy, with Decio Levi and others and in Spain, especially with Miguel Angel Rodriguez and Mariano Del Olmo who had been postdocs at the CRM. He also had close ties with Mexico, with Sacha Turbiner in particular. Furthermore, when it became possible both Jiří and Pavel reconnected with their roots and put in place collaborative links between Prague and Montreal. This brief overview of the outreach activities of Jiří and Pavel is grossly incomplete but hopefully illustrates how they have both made Montreal an international hub of mathematical physics.

6 Moving in different scientific directions: Relentless creativity

As already indicated, Pavel took a sabbatical year in 1978-79 while for Jiří this happened in 1983-84. At the time, together, they had more or less completed the large undertakings described before and this had monopolized them fully. Without turning their backs on these programs, they then wished to explore new directions and used the occasion of their leaves to do so. As a result, while they kept writing joint papers until 1999 on the classification of maximal Abelian subalgebras [12] and graded contractions [13] in particular, the intensity of their collaborative production diminished as each one of them independently opened up new domains. In France, while pursuing his never-ending interest in the nucleon-nucleon scattering phenomenology [14], Pavel decided to focus his attention on the field of non-linear integrable systems whose study with emphasis on solitonic waves and the introduction of the inverse scattering method had been generating great advances. As for Jiří, his collaboration with Bob Moody had already kicked off with papers on weight multiplicities [15] and on characters of elements of finite order [16] . This would launch four decades of pioneering research by the two of them of which I will only give a succinct overview.[1]

From the late 80's, with several collaborators Jiří developed the profound theory of Lie gradings that he had initiated with Zassenhaus [17]. Joris Van der Jeugt who had collaborated with Bob Sharp held a NSERC Visiting Researcher position at the CRM in that period and got involved in those studies. Then, together with Moody, Jiří made fundamental advances toward the mathematical understanding of quasicrystals viewed as cut and project point sets [11, 18]. This led them, while Jiří was holding a Killam scholarship, to study Voronoi domains [19] and non-crystallographic root systems [20]. Jiří also pursued the applications of these results in cryptography [21]. One additional broad topic that Jiří has much shaped with signal processing in mind, is that of orbit functions. He wrote a foundational paper with Anatolyi Klimyk [22] and much collaborated on this with Jiří Hrivnák [23] who was a postdoc at the CRM and is now on Faculty at the Czech Technical University in Prague. As part of this program, a generalization of the known properties of the Chebyshev polynomials of the second kind in one variable to polynomials of many variables based on the root lattices of compact simple Lie groups of any type and any rank was provided [24]. Another fascinating application that Jiří has explored is the connection that non-crystallographic Coxeter groups have with fullerene and nanotube structures [25].

[1] I shall cite below a number of papers for illustration's sake. This is somewhat arbitrary of course, and I apologize to important collaborators that are not mentioned.

Pavel's first contributions to integrable models had to do with Bäcklund transformations. Renewing with Bob Anderson an acquaintance from the time in Trieste, together with John Harnad, he determined the nonlinear superposition properties of matrix Riccati equations [26]. Subsequently, he launched a broad program aimed at obtaining solutions to these integrable nonlinear partial differential equations through symmetry reduction. This involved finding first the symmetry algebra of the system, a task that he computerized with colleagues [27] and second, imposing invariance under subalgebras with the help of his expertise at classifying those. This was applied fruitfully to many systems and in particular to the KP one [28] in collaboration with Daniel David, a Ph.D. student of Pavel, Niky Kamran a postdoc at the time and now a distinguished Faculty at McGill and Decio Levi with whom Pavel wrote the largest number of papers. Pavel and Decio further introduced the notion of conditional symmetry to treat analogously the Boussinesq equation for example [29]. Michel Grundland who came to the CRM from Poland in the early 80s also participated in these studies. Meanwhile, Pavel pursued his maximal Abelian subalgebras program (see for example [30]) as well as the one aimed at characterizing Lie algebras [31], an undertaking that Libor Šnobl joined as a postdoc [32] to eventually bring it all together in a book [33] co-authored with Pavel. Another major accomplishment of Pavel has been to develop the Lie theory of difference equations [34,35]; the large body of results he has obtained in this area has been collected in a book [36] written with Levi and Yamilov that will be posthumously published. One cannot write about Pavel's scientific production without mentioning his work on superintegrable models which is rooted in his seminal paper [37] of 1966. Throughout his career, he kept returning with numerous co-workers to this fertile topic that he championed connecting it for instance to separation of variables and more lately to Painlevé transcendents. Pavel's discovery in 2009 with his student Tremblay and Turbiner of the so-called TTW model [38] exhibiting constants of motion of arbitrary degrees had the effect of a bomb and gave an enormous impetus to the field. His former student Ian Marquette and former postdocs Sarah Post and Adrian Escobar-Ruiz among others worked actively with Pavel on this topic in more recent times.

Without adequately summarizing their abundant research outputs, I trust this short overview nevertheless gives a sense of the diversity, richness, and importance of their work.

7 Conclusion: Passion for research — Legacy and memories

Jiří and Pavel both had an unquenchable passion for research and science which they followed with talent throughout their professional lives for more than 60 years. They had the good fortune to never lose their vivacity and their curiosity remained high and sharp. Unfettered by fashions, they pursued their interests to achieve bodies of work of great depth and originality. Even afflicted by blindness in his later years, Jiří admirably carried on serenely, supervising in this period many students and postdocs who came from the Czech Republic.

At the Université de Montréal the title of Emeritus Professor is a high distinction and is awarded parsimoniously; there is a yearly quota for these appointments and nominations across the university are carefully assessed. To be eligible, you need of course to announce your retirement. The title of Emeritus Professor was bestowed upon Pavel in June 2020. That this recognition made him very proud reflects how much his belonging to the CRM and the Université de Montréal mattered to him. As for Jiří, he did not find the time to retire!

I had the privilege to be the Director of the CRM from 1993 to 1999 and from 2013 to 2021. This means that I was at the helm in 1996 and 2016 when Jiří and Pavel turned 60 and 80 respectively. For highly distinguished colleagues to whom much is owed, it is a nice and appropriate tradition to organize celebratory events to express esteem and gratitude. Jiří and Pavel deserved such an homage and we made sure not to miss out on this occasion in

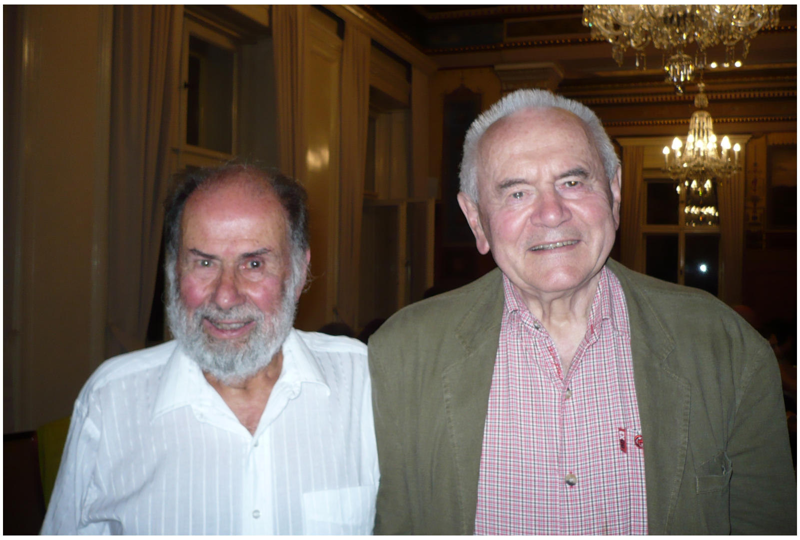


Figure 2: 80th Birthdays of Jiří and Pavel held at Prague in 2016.

1996 and so Yvan Saint-Aubin and I organized a conference entitled Algebraic Methods in Physics: A Symposium for the 60th Birthdays of Jiří Patera and Pavel Winternitz [39]. For obvious reasons, this was one of those instances where they were lauded together. Present at this event and no longer with us were the Wigner medal recipients Louis Michel, Marcos Moshinsky, and Lochlain O'Raifeartaigh as well as David Rowe and Dick Slansky. Jean-Pierre Gazeau, Basil Grammaticos, Ronald King, Frank Lemire, George Pogosyan, Peter Olver, Alfred Ramani, Guy Rideau, Keti Tennenblat and Jiří Tolar whom I have not mentioned before were among the participants.

In 2016, both 80 then, Jiří and Pavel were active as ever and many new generations of colleagues had profited from their interactions with them. Another birthday party was thus in order. This one was held in Prague and put together by the Doppler Institute in collaboration with the CRM. Many Czech colleagues obviously attended, among them Pavel Exner and Igor Jex the Dean of the Faculty of Nuclear and Physical Engineering at the Czech Technical University that was hosting the meeting nicely organized by Libor Šnobl. Let me also add the names of more friends of Jiří and Pavel who spoke on this occasion and who had not appeared in these lines yet: Vladimir Dorodnitsyn, Hubert De Guise, Luigi Martina, Anatoly Nikitin, Alexei Penskoi, Marzena Szajewska, Piergiulio Tempesta, Mark Walton.[2] It was great to have these occasions to express to Jiří and Pavel during their lifetime our deep appreciation for their science, their friendship and the bridge they built between Prague and Montreal.

Alas, they are no longer with us but their legacy lives on. On the scientific front, they have written papers that will keep being touchstones for major areas of mathematics and theoretical physics as well as springboards for many discoveries to come. To all the people they have trained, inspired and befriended, sharing their knowledge, intelligence and culture, they have offered something of themselves that will be transmitted through generations. And, on the human side, they left us with the memory of kind and free men, of proud Czechs and Canadians who were citizens of the World, of fellows who enjoyed life and were caring, of extraordinarily hospitable and welcoming individuals who instilled in the CRM warmth and excellence and taught it to always aim higher. They have set the stage for the members of their Laboratory to carry on and for other researchers from the world over to join the CRM, walk in their footsteps and like them have a global impact. Jiří and Pavel, here is to you.

[2] I have made the perilous choice to identify for memory many individuals whose paths crossed those of Jiří and Pavel one way or the other. To the many who have unfortunately been left out I apologize trusting they will have understood the intent and will not hold grudges.

Acknowledgements

First, I wish to thank Jiří Patera and Pavel Winternitz for all they have done for me. I also want to acknowledge the precious help provided by Josée Savard, Sacha Patera and Michael and Peter Winternitz in the preparation of this presentation.

Funding information My graduate studies in the mathematical physics group of the CRM have been made financially possible thanks to a scholarship from the Natural Sciences, and Engineering Council (NSERC) of Canada. Without this bursary my journey with Jiří and Pavel might not have happened. Like me, both Jiří, and Pavel have continuously held NSERC discovery grants. These allow to provide stipends to deserving students or postdocs, and to carry on the training tradition that Jiří and Pavel initiated. I am sure they would join me in acknowledging this unwavering support. We have also benefited from grants provided by the Fonds de Recherche du Québec (FRQ). Throughout the years, NSERC and FRQ have also kept funding this marvelous institute for research in the mathematical sciences that the CRM is, that Jiří and Pavel contributed so much to and that I was honoured to lead for 14 years. In the name of Jiří and Pavel, I wish to thank all the Canadian taxpayers who, through the various funding agencies, are making scientific treks like theirs possible with all the benefits that these generate.

References

[1] S. F. E. Oliviero, L. Leone, A. Hamma and S. Lloyd, *Measuring magic on a quantum processor*, npj Quantum Inf. **8**, 148 (2022), doi:10.1038/s41534-022-00666-5.

[2] V. Barashenkov, D. Blokhintsev, E. Mihul, J. Patera and G. Semashko, *Pole theory of λ-hyperon production in π–p interactions at high energies*, Nucl. Phys. **48**, 33 (1963).

[3] J. Patera and D. Sankoff, *Tables of branching rules for representations of simple Lie algebras*, Université de Montréal, Montréal, Canada, ISBN 9780840502285 (1973).

[4] W. McKay and J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Marcel Dekker, New York, USA, ISBN 9780824712273 (1981).

[5] M. R. Bremner, R. V. Moody and J. Patera, *Tables of dominant weight multiplicities for representations of simple Lie algebras*, Marcel Dekker, New York, USA, ISBN 9780824772703 (1985).

[6] W. G. McKay, J. Patera and D. W. Rand, *Tables of representations of simple Lie algebras: Exceptional simple Lie algebras*, Centre de Recherches Mathématiques, Montreal, Canada, ISBN 9782921120067 (1990).

[7] S. Kass, R. V. Moody, J. Patera and R. Slansky, *Affine Lie algebras, weight multiplicities, and branching rules*, University of California Press, Oakland, USA, ISBN 9780520067684 (1990).

[8] J. Patera and P. Winternitz, *A new basis for the representations of the rotation group. Lamé and Heun polynomials*, J. Math. Phys. **14**, 1130 (1973), doi:10.1063/1.1666449.

[9] J. Patera, P. Winternitz and H. Zassenhaus, *Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group*, J. Math. Phys. **16**, 1597 (1975), doi:10.1063/1.522729.

[10] J. Beckers, J. Harnad, M. Perroud and P. Winternitz, *Tensor fields invariant under subgroups of the conformal group of space-time*, J. Math. Phys. **19**, 2126 (1978), doi:10.1063/1.523571.

[11] J. Patera, *Quasicrystals and discrete geometry*, American Mathematical Society, Providence, USA, ISBN 9780821806821 (1998).

[12] J. Patera, P. Winternitz and H. Zassenhaus, *Maximal Abelian subalgebras of real and complex symplectic Lie algebras*, J. Math. Phys. **24**, 1973 (1983), doi:10.1063/1.525955.

[13] M. Couture, J. Patera, R. T. Sharp and P. Winternitz, *Graded contractions of sl(3,C)*, J. Math. Phys. **32**, 2310 (1991), doi:10.1063/1.529154.

[14] J. Bystricky, F. Lehar and P. Winternitz, *Formalism of nucleon-nucleon elastic scattering experiments*, J. Phys. France **39**, 1 (1978), doi:10.1051/jphys:019780039010100.

[15] R. V. Moody and J. Patera, *Fast recursion formula for weight multiplicities*, Bull. Amer. Math. Soc. **7**, 237 (1982), doi:10.1090/S0273-0979-1982-15021-2.

[16] R. V. Moody and J. Patera, *Characters of elements of finite order in Lie groups*, SIAM. J. Algebr. Discrete Methods **5**, 359 (1984), doi:10.1137/0605037.

[17] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112**, 87 (1989), doi:10.1016/0024-3795(89)90591-0.

[18] R. V. Moody and J. Patera, *Quasicrystals and icosians*, J. Phys. A: Math. Gen. **26**, 2829 (1993), doi:10.1088/0305-4470/26/12/022.

[19] R. V. Moody and J. Patera, *Voronoi domains and dual cells in the generalized kaleidoscope with applications to root and weight lattices*, Can. J. Math. **47**, 573 (1995), doi:10.4153/CJM-1995-031-2.

[20] L. Chen, R. V. Moody and J. Patera, *Non-crystallographic root systems*, in *Quasicrystals and discrete geometry*, American Mathematical Society, Providence, USA, ISBN 9780821806821 (1998).

[21] J. Patera and M. Nesterenko, *Quasicrystals in cryptography*, in *Aspects of network and information security*, IOS Press, Amsterdam, Netherlands, ISBN 978-1-58603-856-4 (2008).

[22] A. Klimyk and J. Patera, *Orbit functions*, Symmetry Integr. Geom.: Methods Appl. **2**, 006 (2006), doi:10.3842/SIGMA.2006.006.

[23] J. Hrivnák, L. Motlochová and J. Patera, *On discretization of tori of compact simple Lie groups: II*, J. Phys. A: Math. Theor. **45**, 255201 (2012), doi:10.1088/1751-8113/45/25/255201.

[24] R. V. Moody and J. Patera, *Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups*, Adv. Appl. Math. **47**, 509 (2011), doi:10.1016/j.aam.2010.11.005.

[25] M. Bodner, J. Patera and M. Szajewska, *C_{70}, C_{80}, C_{90} and carbon nanotubes by breaking of the icosahedral symmetry of C_{60}*, Acta Cryst Sect A **69**, 583 (2013), doi:10.1107/S0108767313021375.

[26] J. Harnad, P. Winternitz and R. L. Anderson, *Superposition principles for matrix Riccati equations*, J. Math. Phys. **24**, 1062 (1983), doi:10.1063/1.525831.

[27] B. Champagne, W. Hereman and P. Winternitz, *The computer calculation of Lie point symmetries of large systems of differential equations*, Comput. Phys. Commun. **66**, 319 (1991), doi:10.1016/0010-4655(91)90080-5.

[28] D. David, N. Kamran, D. Levi and P. Winternitz, *Symmetry reduction for the Kadomtsev-Petviashvili equation using a loop algebra*, J. Math. Phys. **27**, 1225 (1986), doi:10.1063/1.527129.

[29] D. Levi and P. Winternitz, *Non-classical symmetry reduction: Example of the Boussinesq equation*, J. Phys. A: Math. Gen. **22**, 2915 (1989), doi:10.1088/0305-4470/22/15/010.

[30] V. Hussin, P. Winternitz and H. Zassenhaus, *Maximal Abelian subalgebras of complex orthogonal Lie algebras*, Linear Algebra Appl. **141**, 183 (1990), doi:10.1016/0024-3795(90)90317-6.

[31] D. Rand, P. Winternitz and H. Zassenhaus, *On the identification of a Lie algebra given by its structure constants. I. Direct decompositions, Levi decompositions, and nilradicals*, Linear Algebra Appl. **109**, 197 (1988), doi:10.1016/0024-3795(88)90210-8.

[32] L. Šnobl and P. Winternitz, *A class of solvable Lie algebras and their Casimir invariants*, J. Phys. A: Math. Gen. **38**, 2687 (2005), doi:10.1088/0305-4470/38/12/011.

[33] L. Šnobl and P. Winternitz, *Classification and identification of Lie algebras* American Mathematical Society, Providence, USA, ISBN 9780821843550 (2017).

[34] D. Levi and P. Winternitz, *Continuous symmetries of discrete equations*, Phys. Lett. A **152**, 335 (1991), doi:10.1016/0375-9601(91)90733-O.

[35] D. Levi, L. Vinet and P. Winternitz, *Lie group formalism for difference equations*, J. Phys. A: Math. Gen. **30**, 633 (1997), doi:10.1088/0305-4470/30/2/024.

[36] D. Levi, P. Winternitz and R. I. Yamilov, *Continuous symmetries and integrability of discrete equations*, American Mathematical Society, Providence, USA, ISBN 9780821843543 (2023).

[37] J. Friš, Y. A. Smorodinskii, M. Uhlır and P. Winternitz, *Symmetry groups in classical and quantum mechanics*, Sov. J. Nucl. Phys **4**, 444 (1967).

[38] F. Tremblay, A. V. Turbiner and P. Winternitz, *An infinite family of solvable and integrable quantum systems on a plane*, J. Phys. A: Math. Theor. **42**, 242001 (2009), doi:10.1088/1751-8113/42/24/242001.

[39] Y. Saint-Aubin and L. Vinet, *Algebraic methods in physics: A symposium for the 60th birthdays of Jiří Patera and Pavel Winternitz*, Springer, New York, USA, ISBN 9780387951256 (2012), doi:10.1007/978-1-4613-0119-6.

Remembering David J. Rowe

John L. Wood[1] **and Piet Van Isacker**[2⋆]

1 School of Physics, Georgia Institute of Technology, Atlanta, Georgia, 30332, USA
2 GANIL, CEA/DRF-CNRS/IN2P3, Boulevard Henri Becquerel, F-14076 Caen, France

⋆ isacker@ganil.fr

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

David Rowe was a highly respected theoretical physicist who made seminal contributions that improved our understanding of the atomic nucleus, in particular of the collective behaviour of its constituent nucleons — results he often obtained with the use of sophisticated group-theoretical methods. He will also be remembered as the (co-)author of monographs on nuclear physics, written with the scientific rigour that was characteristic of his research.

Received 16-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.013

David Rowe: Totnes, 4/02/1936 — Toronto, 8/05/2020.

David Rowe was born in Totnes, United Kingdom, on February 4th 1936 and went to school in Kingsbridge. He did his undergraduate studies at the Universities of Cambridge and Oxford and graduated at Oxford University with a PhD in nuclear physics.

David's scientific career started with a study in experimental nuclear physics [1] but quickly his attention shifted to theoretical physics, where it stayed for the rest of his life. Three post-doctoral stays turned out to be of crucial importance in the forging of his scientific interests. The first was in 1962/63 when he was a Ford Foundation fellow at the Niels Bohr Institute in Copenhagen, at the forefront of research in nuclear physics at that time. This post-doctoral stay no doubt must have laid the foundation of his life-long interest in the collective behaviour of nucleons in the nucleus. From 1963 to 1966 David held a post-doctoral position at the Atomic Energy Research Establishment at Harwell in England. There also he thrived in a stimulating intellectual environment where important advances in theoretical physics were made by John Bell, Phill Elliott, Brian (later Lord) Flowers, Tony Skyrme and others; during that period he interacted in particular with Tony Lane. His third post-doctoral stay took place at the University of Rochester in the USA, where he benefitted from the presence of Bruce French who was, among other things, an expert in the application of group theory in physics.

From 1968 onwards he held a permanent position, first as an associate and later as a full professor, at the University of Toronto, where he remained for the rest of his career except for two sabbatical leaves at the University of São Paulo (Brazil) and the University of Canterbury (UK). He was Associate Dean of the School of Graduate Studies in Physical Sciences from 1984 to 1987. For his contributions to theoretical physics he received the Rutherford Memorial Medal and Prize of the Royal Society of Canada in 1983, the CAP/CRM Medal and Prize for Theoretical and Mathematical Physics in 1999 and was elected a Fellow of the Royal Society of Canada in 1986. In 1998 he retired and became emeritus professor at the University of Toronto. Freed from teaching and administrative duties, he could devote more time to research and continued to develop new ideas in theoretical physics until the final days of his life.

A central aim of David's scientific activity at the beginning of his career was to arrive at a microscopic understanding of the collective model of the atomic nucleus. This model, proposed in the 1950s by Aage Bohr and Ben Mottelson, describes nuclear states in terms of vibrations and rotations of a quantised droplet of dense nuclear matter [2]. While this interpretation met with a certain success when confronted with spectroscopic data known at that time, a microscopic understanding of the approach was lacking. That is, its connection with the nuclear shell model, which describes the nucleus in terms of its constituent neutrons and protons, was not well understood. At the time when David began pondering this question (mid-1960s), one important breakthrough had been made by Phil Elliott [3,4], who had shown that rotational states can be realised in the spherical shell model on the basis of an SU(3) (dynamical) symmetry of the nuclear Hamiltonian. Nevertheless, an embedding of the collective model into the shell model, *i.e.*, the formulation of the collective model as a submodel of the shell model, had not yet been achieved. Inspired by Elliott's earlier work, David realised that group theory would play an essential role in establishing this connection since both the shell model and the collective model have an algebraic structure. He also realised that earlier attempts, where the observables are shape coordinates of the nuclear surface and their associated momenta, cannot lead to a microscopic theory and should be replaced by the monopole and quadrupole moments of the nuclear density. The combination of these two features—the algebraic structure of the shell model and the formulation of collective observables in terms of moments—led in a natural way to the symplectic model based on the algebra $\mathrm{Sp}(3,R)$, as proposed by David and his (then) graduate student George Rosensteel in 1977 [5]. Not only is $\mathrm{Sp}(3,R)$ a subalgebra of the full Lie algebra of shell-model observables (which is infinite dimensional) but it contains itself SU(3) and CM(3) (the algebra of the collective model) as subalgebras. The symplectic model therefore provided the first microscopic understanding of the origins of the rotational dynamics of nuclei, including rigid as well as irrotational flows. It continues to inspire present-day nuclear structure. Recently, Sp$(3,R)$ was shown to be a symmetry emerging from *ab initio* large-scale shell-model calculations [6].

Throughout his life David remained interested in nuclear collective models, steadily improving our understanding of them as well as enlarging their applicability. An example of the latter is his proposal of a computationally tractable version of the collective model [7], which made it much more versatile than in the original numerical implementation. As was so often the case, David's formulation was based on an elegant piece of mathematics, namely the correspondence between SO(5), the rotation algebra in five dimensions, and SU(1,1), the algebra of scale transformations in the radial coordinate. He then exploited the presence of the continuous series of SU(1,1) representations to obtain concrete results in terms of greatly improved numerical convergence properties. In subsequent work, David and collaborators showed that the dual pairing of symmetry and dynamical algebras is a feature common to many physical systems [8], the significance of which therefore largely surpasses that of its application to the collective model.

Since most models can be assigned an algebraic structure, it became important to construct unitary representations of Lie algebras in a systematic way. With this goal in mind, David invented a new mathematical structure, namely vector coherent states [9]. VCS theory can be considered as a physically intuitive version of the mathematical theory of induced representations and can be used in the construction of not-so-simple irreducible representations of a Lie algebra, starting from known irreducible representations of one of its subalgebras. VCS theory provides a powerful technique to derive many concrete results, for example, explicit expressions for vector coupling coefficients.

A common thread in all research activities of David was the use of symmetries, which arise if the Hamiltonian of a quantum-mechanical system commutes with a set of transformations that form a Lie algebra. The concept of symmetry can be further generalised to that of a *dynamical* symmetry when the Hamiltonian leaves invariant the subspaces of the total Hilbert space that carry the irreducible representations of a subalgebra of the dynamical algebra. In fact, the algebraic properties of several 'classical' nuclear physics models, such as Wigner's SU(4) supermultiplet scheme, Racah's seniority model of pairing, Elliott's SU(3) description of rotations and the solvable limits of the interacting boson model (IBM) of Arima and Iachello, can all be understood as arising from a dynamical symmetry. Often a single model may display several incompatible dynamical symmetries. This is well known for the IBM, which has three competing dynamical symmetries (or limits): U(5), SU(3) and SO(6). Competing dynamical symmetries also occur in the nuclear shell model, where the short-range pair-coupling interaction among the nucleons keeps the nucleus spherical and induces an SU(2)-type dynamical symmetry while the long-range quadrupole interaction favours a deformed equilibrium shape, corresponding to an SU(3) limit. The properties of systems with competing symmetries can be elucidated with the notion of quasi-dynamical symmetry: the mixing of different representations of a dynamical symmetry caused by a competing symmetry frequently occurs in a highly coherent manner, creating the illusion that the symmetry is preserved. While this concept can be given a precise formulation in terms of embedded representations [10], the intuitive interpretation is that the dominant symmetry is distorted but not broken. As the competing symmetry increases in strength this distortion becomes more important until it reaches breaking point and the system enters a transition phase from where a quasi-dynamical symmetry of the competing phase may emerge. Over the years David and collaborators investigated several models with competing symmetries [11–13] the properties of which can be in terms of quasi-dynamical symmetries.

We close with some heartfelt memories of David as a friend and colleague, which one of us (JLW) enjoyed for 46 years. David was a private and modest person who loved to think, share stories with friends, walk, and travel. He was a master of bird photography. He was an accomplished pianist. He had an infectious sense of humour. Physics discussion could be very intense, his demand was for logical clarity, often with the sense that only the shadows of his thinking were accessible to lesser souls. He was, at least for us, one of the giants of mathematical physics in the latter part of the twentieth century. But one had to listen very carefully: "when the giants walk by, they do so very silently". Walking by his side was a singular experience and a privilege.

David has left a legacy of ideas that we term the Rowe Legacy. To the limits of our ability, we will see this legacy shared in our role as authors and editors.

References

[1] D. J. Rowe, A. B. Clegg, G. L. Salmon and P. S. Fisher, *The $^{16}O(p,p')$ reaction at 150 MeV*, Proc. Phys. Soc. **80**, 1205 (1962), doi:10.1088/0370-1328/80/6/301.

[2] A. Bohr and B. R. Mottelson, *Nuclear structure: II Nuclear deformations*, World Scientific, Singapore, ISBN 9789813105126 (1998), doi:10.1142/3530.

[3] J. P. Elliott, *Collective motion in the nuclear shell model. I. Classification schemes for states of mixed configurations*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **245**, 128 (1958), doi:10.1098/rspa.1958.0072.

[4] J. P. Elliott, *Collective motion in the nuclear shell model II. The introduction of intrinsic wave-functions*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **245**, 562 (1958), doi:10.1098/rspa.1958.0101.

[5] G. Rosensteel and D. J. Rowe, *Nuclear Sp(3,R) model*, Phys. Rev. Lett. **38**, 10 (1977), doi:10.1103/PhysRevLett.38.10.

[6] T. Dytrych, K. D. Launey, J. P. Draayer, D. J. Rowe, J. L. Wood, G. Rosensteel, C. Bahri, D. Langr and R. B. Baker, *Physics of nuclei: Key role of an emergent symmetry*, Phys. Rev. Lett. **124**, 042501 (2020), doi:10.1103/PhysRevLett.124.042501.

[7] D. J. Rowe, *A computationally tractable version of the collective model*, Nucl. Phys. A **735**, 372 (2004), doi:10.1016/j.nuclphysa.2004.02.018.

[8] D. J. Rowe, M. J. Carvalho and J. Repka, *Dual pairing of symmetry and dynamical groups in physics*, Rev. Mod. Phys. **84**, 711 (2012), doi:10.1103/RevModPhys.84.711.

[9] D. J. Rowe, *Coherent state theory of the noncompact symplectic group*, J. Math. Phys. **25**, 2662 (1984), doi:10.1063/1.526497.

[10] D. J. Rowe, P. Rochford and J. Repka, *Dynamic structure and embedded representation in physics: The group theory of the adiabatic approximation*, J. Math. Phys. **29**, 572 (1988), doi:10.1063/1.528049.

[11] P. Rochford and D. J. Rowe, *The survival of rotor and SU(3) bands under strong spin-orbit symmetry mixing*, Phys. Lett. B **210**, 5 (1988), doi:10.1016/0370-2693(88)90337-1.

[12] D. J. Rowe, C. Bahri and W. Wijesundera, *Exactly solvable model of a superconducting to rotational phase transition*, Phys. Rev. Lett. **80**, 4394 (1998), doi:10.1103/PhysRevLett.80.4394.

[13] D. J. Rowe, *Quasidynamical symmetry in an interacting boson model phase transition*, Phys. Rev. Lett. **93**, 122502 (2004), doi:10.1103/PhysRevLett.93.122502.

SciPost Phys. Proc. 14, 014 (2023)

Comments on the negative grade KdV hierarchy

Y. F. Adans, Jose F. Gomes*, G. V. Lobo and A. H. Zimerman

Instituto de Física Teórica, IFT-Unesp, Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II, CEP 01140-070, São Paulo - SP, Brasil

* francisco.gomes@unesp.br

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The construction of negative grade KdV hierarchy is proposed in terms of a Miura-gauge transformation. Such gauge transformation is employed within the zero curvature representation and maps the Lax operator of the mKdV into its couterpart within the KdV setting. Each odd negative KdV flow is obtained from an odd and its subsequent even negative mKdV flows. The negative KdV flows are shown to inherit the two different vacua structure that characterizes the associated mKdV flows.

Received 17-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.014

1 Introduction

Integrable models have been focus of considerable attention in the past few years. These are very peculiar two dimensional field theories admitting an infinite number of conservation laws and soliton solutions. The algebraic construction of integrable models has provided a series of important achievements allowing their construction and classification in terms of the decomposition of the affine algebra into graded subspaces. Structural connection and the derivation of many properties such as the construction of conservation laws and soliton solutions, can be set from the zero curvature representation [1], [2]. In particular the mKdV hierarchy, based on the affine $sl(2)$ algebra, provides the simplest example of systematic construction of a series of evolution equations associated to a universal object called Lax operator. For the mKdV case the relevant decomposition occurs according to the principal gradation. Explicit constructions for positive and negative graded sub-hierarchies have been obtained. The positive flows are known to be labelled by odd numbers whilst there are no restriction for the negative flows [3].

An interesting relation between the KdV and mKdV hierarchies can be realised by the Miura transformation which maps one hierarchy into the other. In ref. [4], [5] we have related the two hierarchies by a gauge transformation that maps one Lax operator into the other. Such *Miura-gauge transformation* acting upon the zero curvature maps the flows from one hierarchy into the other. For the positive sub-hierarchy the mapping is one to one, i.e., each flow equation of mKdV is mapped into its counterpart within the KdV hierarchy. However this is not true for the negative KdV sub-hierarchy. In sec. 3 we argue that only odd flows are consistent for the KdV hierarchy and since there are even and odd flows within the negative mKdV side, there should be a mapping of a pair of mKdV flows into a single KdV flow. This is indeed true, in sect. 4 we construct these mappings and show that an odd and its subsequent even mKdV flows can be mapped into a single KdV flow. An interesting point to mention is that odd mKdV flows admit only *zero vacuum* whilst the even admit strictly *non-zero vacuum* solutions and the associated KdV flow ends up inheriting both types of structure.

2 mKdV negative hierarchy

In this section let us review the construction of mKdV hierarchy within the algebraic formalism. Consider the affine $\mathcal{G} = \hat{sl}(2)$ centerless Kac-Moody algebra generated by

$$h^{(m)} = \lambda^m h^{(0)}, \qquad E_{\pm\alpha}^{(m)} = \lambda^m E_{\pm\alpha}^{(0)}, \quad \text{with} \quad \lambda \in \mathrm{C}, \quad \text{and} \quad n \in \mathrm{Z}, \tag{1}$$

satisfying the following algebra

$$\left[h^{(m)}, E_{\pm\alpha}^{(n)}\right] = \pm 2E_{\pm\alpha}^{(m+n)}, \quad \left[E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}\right] = h^{(m+n)}. \tag{2}$$

Introduce the principal grading operator

$$Q_p = 2\lambda \frac{d}{d\lambda} + \frac{1}{2}h, \tag{3}$$

that decomposes the affine algebra into graded subspaces, i.e., $\mathcal{G} = \bigoplus_i \mathcal{G}_i$ with

$$\left[Q_p, \mathcal{G}_a\right] = a\mathcal{G}_a, \qquad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b}, \quad a, b \in Z, \tag{4}$$

where, for $\mathcal{G} = \hat{sl}(2)$,

$$\mathcal{G}_{2n} = \left\{h^{(n)} = \lambda^n h\right\}, \qquad \mathcal{G}_{2n+1} = \{\lambda^n (E_\alpha + \lambda E_{-\alpha}), \lambda^n (E_\alpha - \lambda E_{-\alpha})\}. \tag{5}$$

A second important ingredient is the choice of a constant grade one element $E^{(1)} \in \mathcal{G}_1$

$$E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}, \tag{6}$$

such that it decomposes the affine algebra as $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$, where $\mathcal{K}$ is the *Kernel* of $E^{(1)}$:

$$\mathcal{K}_E = \left\{y \in \mathcal{K}, [y, E^{(1)}] = 0\right\} = \left\{E^{(2n+1)} \equiv E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}\right\} \in \mathcal{G}_{2n+1}, \tag{7}$$

and $\mathcal{M}$ is its complement subspace. We now define the spatial Lax operator to be an universal algebraic object within the whole hierarchy to be

$$A_x^{\mathrm{mKdV}}(\phi) \;=\; E^{(1)} + A^{(0)}(\phi) = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + \partial_x \phi\, h^{(0)} = \begin{pmatrix} \partial_x \phi & 1 \\ \lambda & -\partial_x \phi \end{pmatrix}, \tag{8}$$

where $v(x, t_{-N}) = \partial_x \phi$ is the field of the theory. We are interested in the negative time flows generated by the temporal Lax operator component of the form [3]

$$A_{t_{-N}}^{\mathrm{mKdV}} = D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}, \qquad N = 1, 2, \cdots, \tag{9}$$

where $D^{(i)} \in \mathcal{G}_i$. Thus, for a given integer N, the zero curvature equation

$$\left[\partial_x + E^{(1)} + A^{(0)},\, \partial_{t_{-N}} + D^{(-N)} + D^{(-N+1)} + \cdots + D^{(-1)}\right] = 0, \tag{10}$$

decomposes according to the grading structure, i.e.,

$$\left[A^{(0)}, D^{(-N)}\right] + \partial_x D^{(-N)} = 0, \tag{11}$$

$$\left[A^{(0)}, D^{(-N+1)}\right] + \left[E^{(1)}, D^{(-N)}\right] + \partial_x D^{(-N+1)} = 0, \tag{12}$$

$$\vdots \qquad \vdots$$

$$\left[E^{(1)}, D^{(-1)}\right] - \partial_{t_N} A^{(0)} = 0. \tag{13}$$

These eqns. can be solved grade by grade in order to determine $D^{(i)}$ and the evolution equation for $A^{(0)}(\phi)$ according to time t_{-N} is given by (13).

The simplest case is found by taking $N = 1$, leading to

$$A_{t_{-1}}^{\mathrm{mKdV}} = e^{-2\phi} E_{\alpha}^{(-1)} + e^{2\phi} E_{-\alpha}^{(0)} = \begin{pmatrix} 0 & \lambda^{-1} e^{-2\phi} \\ e^{2\phi} & 0 \end{pmatrix}, \tag{14}$$

associated with the well known sinh-Gordon equation,

$$\phi_{x,t_{-1}} = e^{2\phi} - e^{-2\phi}. \tag{15}$$

Notice that $v = \partial_x \phi = v_0 = const.$ is the vacuum solution of (15) only if $v_0 = 0 \to \phi = 0$. It therefore follows that the sinh-Gordon equation only admits zero vacuum solution.

Considering now $N = 2$, we find

$$A_{t_{-2}}^{\mathrm{mKdV}} = h^{(-1)} + \left(2e^{-2\phi} d^{-1}(e^{2\phi})\right) E_{\alpha}^{(-1)} - 2e^{2\phi} d^{-1}(e^{-2\phi}) E_{-\alpha}^{(0)}$$

$$= \begin{pmatrix} \lambda^{-1} & \lambda^{-1}\left(2e^{-2\phi} d^{-1}(e^{2\phi})\right) \\ -2e^{2\phi} d^{-1}(e^{-2\phi}) & -\lambda^{-1} \end{pmatrix}, \tag{16}$$

where we have denoted $d^{-1} f = \int_0^x f\, dx'$. It leads to the following nonlocal equation of motion

$$\phi_{x,t_{-2}} = -2\left(e^{-2\phi} d^{-1}(e^{2\phi}) + e^{2\phi} d^{-1}(e^{-2\phi})\right). \tag{17}$$

Notice that for $\phi = \phi_0 = v_0 x$ the following identity

$$e^{-2v_0 x} d^{-1}(e^{2v_0 x}) + e^{2v_0 x} d^{-1}(e^{-2v_0 x}) = 0, \tag{18}$$

holds only for $v_0 \neq 0$ and $v = v_0$ is the vacuum solution of (17), only if $v_0 \neq 0$. In fact, it can be shown that all models associated to negative even values of N only admit non-zero vacuum solutions [3]. Let us consider the zero curvature equation in the vacuum regime, i.e.,

$$\left[A_x^{vac} = E^{(1)} + v_0 h^{(0)},\, A_{t_{-N}}^{vac} = D_{vac}^{(-N)} + D_{vac}^{(-N+1)} + \cdots + D_{vac}^{(-1)}\right] = 0. \tag{19}$$

The lowest grade equation is

$$\left[v_0 h^{(0)},\, D_{vac}^{(-N)}\right] = 0. \tag{20}$$

Thus, if $v_0 \neq 0$ $D_{vac}^{(-N)}$ must commute with $h^{(0)}$ and therefore $D_{vac}^{(-N)} \in \mathcal{G}_{-2n}$ and $N = 2n$. Conversely if $v_0 = 0$ the lowest grade eqn. becomes

$$\left[E^{(1)},\, D_{vac}^{(-N)}\right] = 0, \tag{21}$$

implying $D_{vac}^{(-N)} \in \mathcal{K}_E$ and N is odd. Thus, the negative mKdV hierarchy splits in two sub-hierarchies: one even admitting strictly non-zero vacuum ($v_0 \neq 0$) and one odd admiting, only zero vacuum ($v_0 = 0$) solutions. The systematic construction of soliton solutions for the negative mKdV hierarchies was previously studied and can be written as follows (see [3]). For the odd sub-hierarchy the one soliton solution was constructed from dressing the zero vacuum solution ($A_x^{vac} = E^{(1)}$) leading to

$$v(x,t_{-2n+1}) = \partial_x \ln\left(\frac{1-\beta e^{2kx+\omega_{-2n+1}t_{-2n+1}}}{1+\beta e^{2kx+\omega_{-2n+1}t_{-2n+1}}}\right), \quad \text{with} \quad \omega_{-2n+1} = 2k^{-2n+1}. \tag{22}$$

For the even sub-hierarchy the constant value of the vacuum, v_0 introduces a deformation in the Lax operator, $A_x^{vac} = E^{(1)} + v_0 h^{(0)}$ and hence upon the dressing method. In [3] the solutions were constructed employing deformed vertex operators yielding for the one soliton solution,

$$v(x,t_{-2n}) = v_0 + \partial_x \ln\left(\frac{1+\beta(v_0-k)e^{2kx+\omega_{-2n}t_{-2n}}}{1+\beta(v_0+k)e^{2kx+\omega_{-2n}t_{-2n}}}\right), \quad \text{with} \quad \omega_{-2n} = \frac{2k}{v_0(k^2-v_0^2)^n}, \tag{23}$$

where in both cases, β is a free parameter.

3 KdV negative hierarchy

For the KdV hierarchy we employ the same algebraic structure of section 3, i.e., principal gradation, Q_p (3) and the constant grade one element $E^{(1)}$ (6). We propose the following Lax operator,

$$A_x^{\mathrm{KdV}}(J) = E^{(1)} + A^{(-1)} = E_\alpha^{(0)} + E_{-\alpha}^{(1)} + J\, E_{-\alpha}^{(0)} = \begin{pmatrix} 0 & 1 \\ \lambda + J & 0 \end{pmatrix}, \tag{24}$$

where $A^{(-1)} = J\, E_{-\alpha}^{(0)} \in \mathcal{G}_{-1}$ and $J = J(x,\tau_N)$ is the field of KdV hierarchy. For the sub-hierarchy that leads to negative time-flow τ_{-N}, the temporal-part Lax operator is given by

$$A_{\tau_{-N}}^{\mathrm{KdV}}(J) = \mathcal{D}^{(-N-2)} + \mathcal{D}^{(-N-1)} + \cdots + \mathcal{D}^{(-1)}, \tag{25}$$

where $\mathcal{D}^{(i)} \in \mathcal{G}_i$. The zero curvature decomposes according to the graded structure as

$$\begin{aligned}
\left[A^{(-1)}, \mathcal{D}^{(-N-2)}\right] &= 0, & (26)\\
\partial_x \mathcal{D}^{(-N-2)} + \left[A^{(-1)}, \mathcal{D}^{(-N-1)}\right] &= 0, & (27)\\
\partial_x \mathcal{D}^{(-N-1)} + \left[E^{(1)}, \mathcal{D}^{(-N-2)}\right] + \left[A^{(-1)}, \mathcal{D}^{(-N)}\right] &= 0, & (28)\\
&\vdots \\
\partial_x \mathcal{D}^{(-1)} + \left[E^{(1)}, D^{(-2)}\right] - \partial_{\tau_{-N}} A^{(-1)} &= 0, & (29)\\
\left[E^{(1)}, \mathcal{D}^{(-1)}\right] &= 0, & (30)
\end{aligned}$$

which allows solving for all $\mathcal{D}^{(i)}$ and determines the equation of motion (29) according to τ_{-N}. Notice that the lowest grade equation (26) implies that $\mathcal{D}^{(-N-2)}$ is proportional to $E_{-\alpha}^{(-m)}$ and therefore $N = 2m - 1$. For this reason *all equations of motion for the KdV hierarchy are associated with odd temporal flows*, in contrast to the mKdV case, where there are equations of motion associated to both, even and odd flows.

The equations of motion for KdV hierarchy are more conveniently expresed in terms of non-local field $J(x,\tau_N) = \partial_x \eta(x,\tau_N)$. The first negative flow is obtained from zero curvature with $N = 1$, leads to the following temporal Lax operator,

$$\begin{aligned}
A_{\tau_{-1}}^{\mathrm{KdV}} &= \frac{\eta_{\tau_{-1}}}{2}\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + \frac{\eta_{x,\tau_{-1}}}{4}\, h^{(-1)} + \frac{2\eta_x \eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4}\, E_{-\alpha}^{(-1)} \\
&= \begin{pmatrix} \frac{\eta_{x,\tau_{-1}}}{4\lambda} & \frac{\eta_{\tau_{-1}}}{2\lambda} \\ \frac{2\eta_x\eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4\lambda} + \frac{\eta_{\tau_{-1}}}{2} & -\frac{\eta_{x,\tau_{-1}}}{4\lambda} \end{pmatrix},
\end{aligned} \tag{31}$$

and equation of motion

$$4\eta_x \eta_{x,\tau_{-1}} + 2\eta_{2x}\eta_{\tau_{-1}} - \eta_{3x,\tau_{-1}} = 0\,. \tag{32}$$

This equation was first proposed in [6] using the inverse of recursion operator. Later in [7], its Hamiltonian and soliton solutions were discussed.

If we now take $N = 3$ in (25) and find for the associated temporal Lax operator,

$$A^{\mathrm{KdV}}_{\tau_{-3}} = \frac{\eta_{\tau_{-3}}}{2}\left(E^{(-1)}_{\alpha} + E^{(0)}_{-\alpha}\right) + \frac{\eta_{x,\tau_{-3}}}{4}\, h^{(-1)} - \frac{\mathcal{B}}{8}\left(E^{(-2)}_{\alpha} + E^{(-1)}_{-\alpha}\right)$$

$$+ \frac{2\eta_{\tau_{-3}}\eta_x - \eta_{2x,\tau_{-3}}}{8}\, E^{(-1)}_{-\alpha} - \frac{\mathcal{B}_x}{16}\, h^{(-2)} + \frac{\mathcal{B}_{2x} - \eta_x\,\mathcal{B}}{8}\, E^{(-2)}_{-\alpha}$$

$$= \begin{pmatrix} \frac{\eta_{x,\tau_{-3}}}{4\lambda} - \frac{\mathcal{B}_x}{16\lambda^2} & \frac{\eta_{\tau_{-3}}}{2\lambda} - \frac{\mathcal{B}}{8\lambda^2} \\ \frac{1}{2}\eta_{\tau_{-3}} + \frac{2\eta_{\tau_{-3}}\,\eta_x - \eta_{2x,\tau_{-3}} - \mathcal{B}}{8\lambda} + \frac{\mathcal{B}_{2x} - \eta_x\,\mathcal{B}}{8\lambda^2} & -\frac{\eta_{x,\tau_{-3}}}{4\lambda} + \frac{\mathcal{B}_x}{16\lambda^2} \end{pmatrix}, \tag{33}$$

where

$$\mathcal{B} = d^{-1}(4\eta_x \eta_{x,\tau_{-3}} + 2\eta_{2x}\eta_{\tau_{-3}} - \eta_{3x,\tau_{-3}})\,. \tag{34}$$

The corresponding equation of motion is given by

$$\begin{aligned} &-\frac{1}{2}\eta_{5x,\tau_{-3}} + 4\eta_x\left(-2\eta_{x,\tau_{-3}}\eta_x + \eta_{3x,\tau_{-3}} - \eta_{2x}\eta_{\tau_{-3}}\right) + 5\eta_{2x}\eta_{2x,\tau_{-3}} \\ &+ 4\eta_{x,\tau_{-3}}\eta_{3x} + \eta_{4x}\eta_{\tau_{-3}} + \eta_{2x}\, d^{-1}\left(4\eta_x\eta_{x,\tau_{-3}} + 2\eta_{2x}\eta_{\tau_{-3}} - \eta_{3x,\tau_{-3}}\right) = 0\,. \end{aligned} \tag{35}$$

Notice that vacuum solution $\eta = \eta_0 =$ constant, either zero or non-zero, satisfy both equations of motion (32) and (35). Such behavior differs from the mKdV hierarchy where the equations of motion associated with odd-time flows are satisfied with zero vacuum and the even-time flows with non-zero vacuum (constant). This coalescence in vacuum solution presented in KdV hierarchy can be explained more generally from zero curvature projected around vacuum, i.e,

$$\left[A^{\mathrm{KdV}}_x\big|_{\mathrm{vac}}, A^{\mathrm{KdV}}_{\tau_{-N}}\big|_{\mathrm{vac}}\right] = \left[E^{(1)} + \eta_0\, E^{(0)}_{-\alpha}, \mathcal{D}^{(-N-2)}_{\mathrm{vac}} + \mathcal{D}^{(-N-1)}_{\mathrm{vac}} + \cdots + \mathcal{D}^{(-1)}_{\mathrm{vac}}\right] = 0\,. \tag{36}$$

Its lowest grade component leads to

$$\left[\eta_0\, E^{(0)}_{-\alpha}, \mathcal{D}^{(-N-2)}_{\mathrm{vac}}\right] = \left[\eta_0\, E^{(0)}_{-\alpha},\, a_{-N-2}\, E^{(-1/2(N+1))}_{-\alpha}\right] = 0\,, \tag{37}$$

which is automatically satisfied no matter whether η_0 is zero or non-zero if $N = 2n-1$. It therefore follows that the negative KdV hierarchy are associated to odd flows, $\tau_{-N} = \tau_{-2n-1}$ and admit both, zero and non-zero vacuum solutions.

4 Miura transformation and soliton solutions

In order to map the mKdV and KdV hierarchies let us consider the *Miura-gauge transformation* generated by (see [4], [5])

$$S_1 = e^{\phi_x E^{(0)}_{-\alpha}} = \begin{pmatrix} 1 & 0 \\ \phi_x & 1 \end{pmatrix}, \tag{38}$$

which maps the two Lax operators, A^{mKdV}_x into A^{KdV}_x of eqns. (8) and (24) respectively, i.e.,

$$A^{\mathrm{KdV}}_x = S_1 A^{\mathrm{mKdV}}_x S_1^{-1} + S_1\partial_x S_1^{-1} = E^{(0)}_{\alpha} + E^{(1)}_{-\alpha} + J E^{(0)}_{-\alpha}\,, \tag{39}$$

where

$$J(x,t) = \partial_x \eta(x,t) = (\phi_x)^2 - \phi_{2x}\,. \tag{40}$$

We now analyse how S_1 acts as a local gauge transformation upon A_t^{mKdV}. Let us consider first its action on an even grade element $D^{(-2n)} = c_{-n}h^{(-n)}$:

$$\begin{aligned} D^{(-2\mathrm{n})} \rightarrow & e^{\phi_x E_{-\alpha}^{(0)}}\left(c_{-n}h^{(-n)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} + e^{\phi_x E_{-\alpha}^{(0)}}\partial_t\left(e^{-\phi_x E_{-\alpha}^{(0)}}\right) \\ &= \underbrace{c_{-n}h^{(-n)}}_{\mathcal{G}_{-2n}} + \underbrace{2c_{-n}\phi_x E_{-\alpha}^{(-n)}}_{\mathcal{G}_{-2n-1}} - \underbrace{\partial_t\phi_x E_{-\alpha}^{(0)}}_{\mathcal{G}_{-1}}\,. \end{aligned} \tag{41}$$

On the other hand, if we consider $D^{(-2n+1)} = a_{-n}E_{\alpha}^{(-n)} + b_{-n}E_{-\alpha}^{(-n+1)}$ under the local gauge generated by (38) we find

$$\begin{aligned} D^{(-2\mathrm{n}+1)} \rightarrow\ & e^{\phi_x E_{-\alpha}^{(0)}}\left(a_n E_{\alpha}^{(-n)} + b_n E_{-\alpha}^{(-n+1)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} + e^{\phi_x E_{-\alpha}^{(0)}}\partial_t\left(e^{-\phi_x E_{-\alpha}^{(0)}}\right) \\ = & -\underbrace{a_n(\phi_x)^2 E_{-\alpha}^{(-n)}}_{\mathcal{G}_{-2n-1}} - \underbrace{a_n\phi_x h_1^{(-n)}}_{\mathcal{G}_{-2n}} + \underbrace{a_n E_{\alpha}^{(-n)} + b_n E_{-\alpha}^{(-n+1)}}_{\mathcal{G}_{-2n+1}} - \underbrace{\partial_t\phi_x E_{-\alpha}^{(0)}}_{\mathcal{G}_{-1}}\,. \end{aligned} \tag{42}$$

Thus, any even negative mKdV time flow of the form $A_{t_{-2n}}^{\mathrm{mKdV}} = D^{(-2n)} + D^{(-2n+1)} + \cdots + D^{(-1)}$is mapped into its KdV counterpart with the following graded structure,

$$\begin{aligned} A_{\tau_{-2n+1}}^{\mathrm{KdV}} &= e^{\phi_x E_{-\alpha}^{(0)}}\left(D^{(-2n)} + D^{(-2n+1)} + \cdots + D^{(-1)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-2n}}E_{-\alpha}^{(0)} \\ &= \mathcal{D}^{(-2n-1)} + \mathcal{D}^{(-2n)} + \cdots + \mathcal{D}^{(-1)}\,. \end{aligned} \tag{43}$$

For odd negative mKdV time flow of the form $A_{t_{-2n+1}}^{\mathrm{mKdV}} = D^{(-2n+1)} + D^{(-2n+1)} + \cdots + D^{(-1)}$ will be mapped into

$$\begin{aligned} A_{\tau_{-2n+1}}^{\mathrm{KdV}} &= e^{\phi_x E_{-\alpha}^{(0)}}\left(D^{(-2n+1)} + D^{(-2n+2)} + \cdots + D^{(-1)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-2n+1}}E_{-\alpha}^{(0)} \\ &= \mathcal{D}^{(-2n-1)} + \mathcal{D}^{(-2n)} + \mathcal{D}^{(-2n+1)} + \cdots + \mathcal{D}^{(-1)}\,. \end{aligned} \tag{44}$$

Since A_x^{KdV} is universal for both, even and odd KdV flows, the zero curvature representation (26) - (30) implies that $A_{t_{-2n+1}}^{\mathrm{mKdV}}$ and $A_{t_{-2n}}^{\mathrm{mKdV}}$ are transformed by the Miura-gauge transformation (38), into a single graded KdV structure $A_{\tau_{-2n+1}}^{\mathrm{KdV}}$ (43)-(44) (associated to flow τ_{-2n+1}). We therefore conclude that both *negative even and negative odd mKdV flows collapse within the same KdV odd flow*, i.e.,

$$t_{-2n+1}^{mKdV}, t_{-2n}^{mKdV} \quad \underset{S_1}{\Longrightarrow} \quad \tau_{-2n+1}^{KdV}\,. \tag{45}$$

Notice that this explains why each KdV negative flow admits both zero and non-zero vacuum solutions. They inherit the zero and the non-zero vacuum information from mKdV negative odd and its subsequent negative even flows respectively. Let us illustrate explicitly for the first two negative mKdV flows, namely, t_{-1} and t_{-2}.

For t_{-1}^{mKdV} the field $\phi = \phi(x, t_{-1})$ satisfies the sinh-Gordon eqn (15). We then have

$$\begin{aligned} A_{\tau_{-1}}^{\mathrm{KdV}} &= S_1 A_{t_{-1}}^{\mathrm{mKdV}} S_1^{-1} + S_1\partial_{t_{-1}}S_1^{-1} \\ &= e^{\phi_x E_{-\alpha}^{(0)}}\left(e^{-2\phi}E_{\alpha}^{(-1)} + e^{2\phi}E_{-\alpha}^{(0)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-1}}E_{-\alpha}^{(0)}\,, \end{aligned} \tag{46}$$

leading to

$$A_{\tau_{-1}}^{\mathrm{KdV}} = e^{-2\phi}\left(E_{\alpha}^{(-1)} + E_{-\alpha}^{(0)}\right) + \frac{\eta_{x,t_{-1}}}{4}h^{(-1)} - (\phi_x)^2 e^{-2\phi}E_{-\alpha}^{(-1)}\,, \tag{47}$$

where we used the sinh-Gordon equation of motion, $\phi_{x,t_{-1}} = e^{2\phi} - e^{-2\phi}$ and the *Miura* transformation, $\eta_x = (\phi_x)^2 - \phi_{2x}$ to simplify some terms. Note that in terms of zero curvature, we had already constructed $A^{\mathrm{KdV}}_{\tau_{-1}}$ given in (31),

$$A^{\mathrm{KdV}}_{\tau_{-1}} = \frac{\eta_{\tau_{-1}}}{2}\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + \frac{\eta_{x,\tau_{-1}}}{4}\, h^{(-1)} + \frac{2\eta_x\eta_{\tau_{-1}} - \eta_{2x,\tau_{-1}}}{4}\, E_{-\alpha}^{(-1)}\,. \tag{48}$$

From the condition for eqns (47) and (48) to agree we find

$$\eta_{\tau_{-1}} = 2\cdot e^{-2\phi(x,t_{-1})}\,. \tag{49}$$

On the other hand, if we now consider t_{-2}^{mKdV} with $\phi = \phi(x,t_{-2})$ satisfiyng (17), we get from the Miura gauge transformation $A^{\mathrm{KdV}}_{\tau_{-1}} = S_1 A^{\mathrm{mKdV}}_{t_{-2}} S_1^{-1} + S_1\partial_{t_{-2}} S_1^{-1}$,

$$A^{\mathrm{KdV}}_{\tau_{-1}} = e^{\phi_x E_{-\alpha}^{(0)}}\left(h^{(-1)} + 2e^{-2\phi}d^{-1}(e^{2\phi})E_\alpha^{(-1)} - 2e^{2\phi}d^{-1}(e^{-2\phi})E_{-\alpha}^{(0)}\right)e^{-\phi_x E_{-\alpha}^{(0)}} - \phi_{x,t_{-2}}E_{-\alpha}^{(0)}\,,$$

leading to

$$A^{\mathrm{KdV}}_{\tau_{-1}} = 2e^{-2\phi}d^{-1}(e^{2\phi})\left(E_\alpha^{(-1)} + E_{-\alpha}^{(0)}\right) + \frac{\eta_{x,t_{-2}}}{4}h^{(-1)} + 8(\phi_x - \phi_x^2 e^{-2\phi}d_x^{-1}e^{2\phi})E_{-\alpha}^{(-1)}\,, \tag{50}$$

where we used the equation of motion for t_{-2}^{mKdV} (17) and *Miura transformation*. Thus, (50) only agrees with (48) provided

$$\eta_{\tau_{-1}} = 2\cdot 2e^{-2\phi(x,t_{-2})}d^{-1}(e^{2\phi(x,t_{-2})})\,. \tag{51}$$

Notice that the same $A^{\mathrm{KdV}}_{\tau_{-1}}$ is written in two different ways, one in terms of the sinh-Gordon field $\phi(x,t_{-1})$ given by (47)-(49) and another, in terms of solution of eqn. (17) namely $\phi(x,t_{-2})$ in (50)-(51) . This can be checked explicitly with solutions given in (22) and (23) for $n=1$.

5 Conclusion

We have therefore concluded from the above simple example that solutions of the KdV equation associated to the time flow τ_{-1} inherit different vacuum structures from a pair of mKdV solutions (via Miura transformation) . The first associated to mKdV flow t_{-1}, eqn. (15) (with zero vacuum) satisfying (49) and the second associated to mKdV flow t_{-2}, eqn. (17) (with non-zero vacuum) satisfying (51). The argument can be easily generalized for higher flows, and each KdV flow admits both, zero and non-zero vaccum solutions. They are constructed from pairs of subsequent of mKdV flows each of them admiting different vacuum structures. We expect to report in a future publication the generalization of our construction to the A_r - KdV hierarchy employing the gauge-Miura transformation proposed in [5]. We also expect to discuss the systematic construction of soliton (multisoliton) solutions and their vacuum structure in terms of vertex operators and its deformations along the lines of refs. [3], [4].

Acknowledgements

Funding information JFG and AHZ thank CNPq and Fapesp for support. YFA thanks São Paulo Research Foundation (FAPESP) for financial support under grant #2021/00623−4 and GVL is supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

References

[1] D. Olive and N. Turok, *Local conserved densities and zero-curvature conditions for Toda lattice field theories*, Nucl. Phys. B **257**, 277 (1985), doi:10.1016/0550-3213(85)90347-5.

[2] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge, UK, ISBN 9780511535024 (2009), doi:10.1017/CBO9780511535024.

[3] J. F. Gomes, G. S. França, G. R. de Melo and A. H. Zimerman, *Negative even grade mKdV hierarchy and its soliton solutions*, J. Phys. A: Math. Theor. **42**, 445204 (2009), doi:10.1088/1751-8113/42/44/445204.

[4] J. F. Gomes, A. L. Retore and A. H. Zimerman, *Miura and generalized Bäcklund transformation for KdV hierarchy*, J. Phys. A: Math. Theor. **49**, 504003 (2016), doi:10.1088/1751-8113/49/50/504003.

[5] J. M. de Carvalho Ferreira, J. F. Gomes, G. V. Lobo and A. H. Zimerman, *Gauge Miura and Bäcklund transformations for generalized A n-KdV hierarchies*, J. Phys. A: Math. Theor. **54**, 435201 (2021), doi:10.1088/1751-8121/ac2718.

[6] J. M. Verosky, *Negative powers of Olver recursion operators*, J. Math. Phys. **32**, 1733 (1991), doi:10.1063/1.529234.

[7] Z. Qiao and E. Fan, *Negative-order Korteweg-de Vries equations*, Phys. Rev. E **86**, 016601 (2012), doi:10.1103/PhysRevE.86.016601.

The impact of the diffusion parameter on the passage time of the folding process

Marcelo Tozo Araujo[1], Jorge Chahine[2], Elso Drigo Filho[2] and Regina Maria Ricotta[3⋆]

1 União das Faculdades dos Grandes Lagos, UNILAGO - São José do Rio Preto-SP
2 Instituto de Biociências, Letras e Ciências Exatas, IBILCE-UNESP
3 Faculdade de Tecnologia de São Paulo - Fatec-SP - CEETEPS

⋆ regina@fatecsp.br

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Recently, a mathematical method to solve the Fokker Plank equation (FPE) enabled the analysis of the protein folding kinetics, through the construction of the temporal evolution of the probability density. A symmetric tri-stable potential function was used to describe the unfolded and folded states of the protein as well as an intermediate state of the protein. In this paper, the main points of the methodology are reviewed, based on the algebraic Supersymmetric Quantum Mechanics (SQM) formalism, and new results on the kinetics of the evolution of the system characterized in terms of the diffusion parameter are presented.

Received 18-12-2022
Accepted 21-07-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.015

1 Introduction

Proteins are structures made up of chains of amino acids. In the unfolded state the protein is in a linear configuration of amino acids and is synthesized in a folded three-dimensional structure to perform specific functions in the organism. To reach this final structure, this linear sequence can pass through intermediate conformations, which are transition states in which the protein may not reach its three-dimensional structure. The importance of studying folding lies in understanding how an unfolded linear structure reaches a three-dimensional and functional, folded structure. The subject has been extensively studied within the last decades under the concept of folding funnel, where the energetic scenario has the shape of a funnel, [1–3].

In a previous work, [4], a consistent mathematical model to physically describe the biological process of protein folding was introduced. The approach considers the process as a diffusion model inspired by the concept of a folding funnel, aiming to analyse its dynamic behavior. The protein folding process is described by the Fokker-Planck equation, FPE, associated to a free energy described by a tri-stable symmetric potential function V(x). In turn, the

FPE can be mapped to a Schrödinger-type equation, SE, [5, 6], i. e., both equations share the same spectrum. At this point the methodology of Supersymmetric Quantum Mechanics, SQM, associated with the variational method, [7, 8], is used to obtain the approximate spectrum of energy and eigenfunctions of SE and to evaluate the time-dependent probability function $P(x, x_0, t)$, FPE solution, where x is the reaction coordinate, [9], and the coordinate x_0 is associated to the protein in the unfolded state.

The free energy, given by the tri-stable potential function V(x), is a symmetric function that has lateral minima with the same depth (symmetric wells) that can be interpreted, respectively, as the folded and unfolded protein states; the central minimum is related to an intermediate protein conformation. The kinects of the diffusion process was characterized by the calculation of the particle population of the right well (folded state). The time required for the evolution of the population of the system from its initial state to the well on the right is used as the characteristic passage time of the system to the folding state of the protein. The results in [4] are consistent with those expected in similar diffusion problems, [10].

In this work a short review of the methodology is presented (Section 2) showing the connection of the FPE with the SE, given in terms of the free energy V(x). The model is illustrated by a specific free energy function, a study case different from the one in [4]. Section 3 contains new results, a mapping of the diffusion dependence and its influence on the symmetric free energy profile performed aiming to analyse the way the increase of the diffusion impacts the passage time to the protein folded state. Section 4 contains the conclusions.

2 Methodology: FPE and SE formalism

The probability distribution, FPE's solution, is found by a mapping on an SE, whose solutions are obtained by the variational method associated with the SQM. The free energy to be used is described by tri-stable potentials. Because it is time-dependent, probability distribution describes a characteristic time for the dynamics of protein folding through an estimate of the passage time as a function of the reaction coordinate. The behavior of the population towards the third well is verified, which characterizes the folded state, as a function of time, as an exponential decay characteristic of diffusive processes with a directional force.

The FPE, which describes the time evolution of the probability distribution P(x,t) in diffusion systems is given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[f(x).P(x,t)\right] + Q\frac{\partial^2}{\partial x^2}P(x,t)\,, \tag{1}$$

where x is the characteristic variable of the system, the reaction coordinate (number of native contacts); t is the time variable; Q is the diffusion coefficient and $f(x)$ represents an external force (driving force) acting on the medium, it is associated with the free energy of the medium, the tri-stable potential $V(x)$,

$$f(x) = -\frac{d}{dx}V(x)\,. \tag{2}$$

Writing the probability P(x,t) as a product of a function of x and a function of t,

$$P(x,t) = \Psi(x)e^{-\lambda t}\,, \tag{3}$$

it can be shown that the FPE solutions are solutions of a time-independent Schrödinger-type equation, SE, given by

$$\frac{d^2}{dx^2}\Psi(x) - \frac{1}{2Q}\left(\frac{f(x)^2}{2Q} + \frac{df(x)}{dx}\right)\Psi(x) = \frac{\lambda}{Q}\Psi(x)\,, \tag{4}$$

where λ is proportional to the energy. Expanding $\Psi(x)$ on an orthonormal basis, we obtain the probability distribution given by

$$P(x,t|x_0,t_0) = \frac{\Psi_0(x)}{\Psi_0(x_0)} \sum_{n=0}^{\infty} \Psi_n(x).\,\Psi_n(x_0)\,.\,e^{-\lambda_n(t-t_0)}\,, \tag{5}$$

where $\Psi_0(x)$ is the ground state wave function and x_0 is the starting position. The SE used by SQM is expressed, in reduced units, in general as

$$-\frac{d^2}{dx^2}\Psi(x) + \underbrace{\left(W_1(x)^2 - \frac{dW_1(x)}{dx} + E_0^{(1)}\right)}_{V_{SE}(x)}\Psi(x) = E\,\Psi(x)\,, \tag{6}$$

where $V_{SE}(x)$ is the Schrödinger potential function defined in terms of the superpotential function $W_1(x)$, [7]. Comparing the equations (4) and (6) and considering the relationship of $f(x)$ with $V(x)$ given by Eq. (2), we obtain

$$W_1(x) = \frac{1}{2Q}\frac{dV(x)}{dx}\,, \tag{7}$$

that is, the FPE potential $V(x)$ is related to the superpotential $W_1(x)$ of SQM. Also the energy E is related to the parameter λ as

$$E = \frac{\lambda}{Q}\,. \tag{8}$$

Thus, the SQM methodology associated with the variational method can be used to determine the spectrum, [8]. At this point it is important to remark the relationship between potential function of the SE (6) with the diffusion parameter Q, through the superpotential W_1 in (7),

$$V_{SE}(x) = W_1(x)^2 - \frac{dW_1(x)}{dx} + E_0^{(1)}\,. \tag{9}$$

In other words, when using the SQM methodology the spectrum is explicitly dependent on the value of the diffusion constant Q.

2.1 The tri-stable potential and the spectrum

The tri-stable potentials used are of the type

$$V(x) = ax^6 - 8.93851x^4 + 5.42373x^2\,, \tag{10}$$

illustrated in Figure 1 for various values of the constant a. The lateral minima (V_{min}) have the same depth (symmetrical wells) and are interpreted, respectively, as the unfolded (left well) and folded (right well) states of the protein, and the central minimum is related to a set of intermediate protein conformations, with $\Delta V = V(0) - V_{min}$. The choice of parameters was made in order to have several symmetrical potentials with V(0)=0 and different lateral depths of the wells, as in [10]. Thus, only the variation of the parameter a in each tri-stable potential is enough to deal with the depth of the lateral minima.

2.2 Study case for fixed diffusion parameter Q

To illustrate the model, we choose the potential $V(x)$ with $a = 3.90456$ and fixed diffusion constant, $Q = 0.5$, as in Figure 2. The value of Q is arbitrary but it has to be fixed in order to apply the SQM methodology to obtain the spectrum, as it can be seen from the superpotential W_1 in equation (7). In Section 3 we vary the diffusion constant and evaluate its impact on the

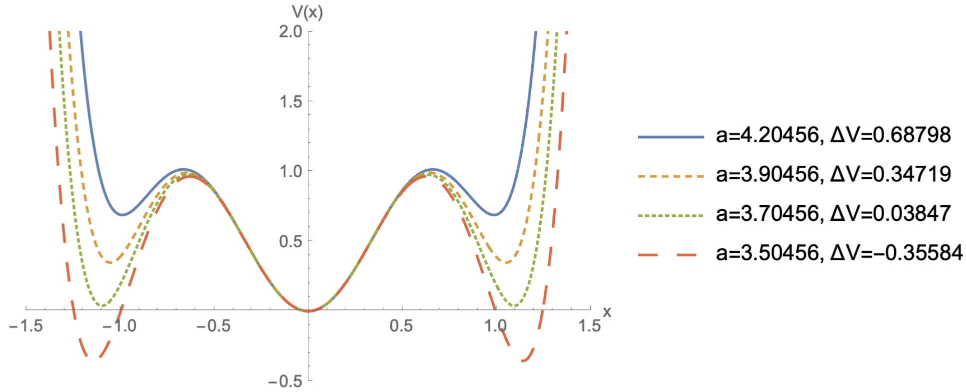


Figure 1: Representation of tri-stable potentials, Eq.(10) for different values of the constant a with the respective values of $\Delta V = V(0) - V_{min}$, [4].

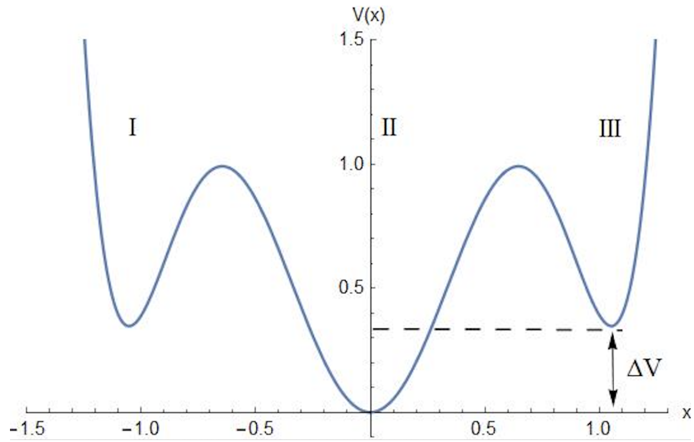


Figure 2: Representation of tri-stable potential function, $V(x) = 3.90456x^6 - 8.93851x^4 + 5.42373x^2$ with the respective value of $\Delta V = 0.34719$, [4].

folding kinetics. It should be mentioned that in other works, [10], the quantities are given in units $\frac{1}{Q}$.

Once defined the free energy given by the potential $V(x)$ we return to the construction of the equivalent SE spectrum, solution of Eq. (6). Then using the SQM methodology, [7]- [8], the approximate spectrum of energies and eigenfunctions is shown by Table 1 and Table 2. It should be stressed that as we are dealing with an approximative method, the number of terms in the probability expansion, Eq. (5), was fixed to six terms in the series, $(n = 0, \ldots, 5)$, since that the contribution of the next exponential term is several orders of magnitude smaller than the previous term and thus can be neglected.

From the SE spectrum, the probability density, given by Eq. (5), can be calculated for different starting points x_0. The diffusion process is then characterized by calculating the population defined by

$$\mathcal{N}(t) = \int_{x_i}^{x_f} P(x,t)dx\,, \tag{11}$$

where the limits of integration x_i and x_f refer to the investigation region of the particle population, regions I, II and III, as denoted in Figure 2.

Table 1: Values of the energy spectrum of the SE when the potential function (free energy) is $V(x) = ax^6 + bx^4 + cx^2$ with the values $a = 3.90456$, $b = -8.93851$, $c = 5.42373$.

n	0	1	2	3	4	5
λ_n	0	1.0361	1.9797	9.2517	17.3589	27.4762

Table 2: Wave functions spectrum of the SE, when the potential function (free energy) is $V(x) = ax^6 + bx^4 + cx^2$ with the values $a = 3.90456$, $b = -8.93851$, $c = 5.42373$.

$\Psi_0^{(1)}(x) = 1.021e^{(-5.42373x^2+8.93851x^4-3.90456x^6)}$
$\Psi_1^{(1)}(x) = e^{(2.73824x^2-3.05175x^4-0.777117x^6)}x(1.99656-8.75317x^2+10.4419x^4)$
$\Psi_2^{(1)}(x) = e^{(-3.68104x^2+1.08564x^4-2.13171x^6)}$ $(-0.904686+5.15415x^2-9.43439x^4+33.953x^6-199.101x^8+303.278x^{10})$
$\Psi_3^{(1)}(x) = e^{(-3.50991x^2-1.40729x^4-0.64738x^6)}(-5.23999x+1.07696x^3+20.5988x^5-$ $74.5237x^7+66.0048x^9-36.7167x^{11}+73.6961x^{13}+69.4599x^{15})$
$\Psi_4^{(1)}(x) = e^{(-4.601x^2-1.49668x^4-0.559156x^6)}(0.928068-17.7878x^2+1.05918x^4+5.10868x^6$ $-84.9985x^8-0.86348x^{10}-24.6105x^{12}+118.086x^{14}+180.684x^{16}+126.316x^{18}$ $+44.1023x^{20})$
$\Psi_5^{(1)}(x) = e^{(-5.52086x^2-1.54808x^4-0.53713x^6)}(4.7712x+0.962939x^3-63.0876x^5-155.71x^7-$ $181.732x^9+13.8384x^{11}+423.621x^{13}+738.097x^{15}+712.503x^{17}+$ $438.794x^{19}+175.5x^{21}+42.7624x^{23}+5.15394x^{25})$

Figure 3 illustrates the results of the numerical calculation of the population of the left well, region I, $\mathcal{N}_I(t)$ as a function of time t and the population of the right well, region III, $\mathcal{N}_{III}(t)$ for the initial value $x_0 = x_{min} = -1.05282$. The initial population $\mathcal{N}_I(t)$ decreases in time until it reaches equilibrium while the population $\mathcal{N}_{III}(t)$ increases in time until reaching the same equilibrium, revealing the diffusion behavior of the process since the wells are symmetrical.

Figure 4 illustrates the best numerical fit of population versus time t of Region I, $\mathcal{N}_I(t)$, as a function of time t and the population of the well on the right, Region III, $\mathcal{N}_{III}(t)$, for the initial value $x_0 = x_{min} = -1.05282$, numeric data from Figure 3. The fit is given by a decreasing exponential (dotted line) and an increasing exponential (dashed line) with characteristic times: $\tau' = 0.787511$ and $\tau = 1.46159$, respectively. The characteristic time is interpreted as the transition time from region I to region III.

2.3 Results for the passage time for different potentials

Figure 5 illustrates the passage time τ versus the initial position x_0 for different potentials $V(x)$ illustrated in Figure 1, revealing a decrease in the value of τ as a function of the initial position x_0, in addition to a decrease in the value of τ with an increase of ΔV, of the depth of the symmetrical wells.

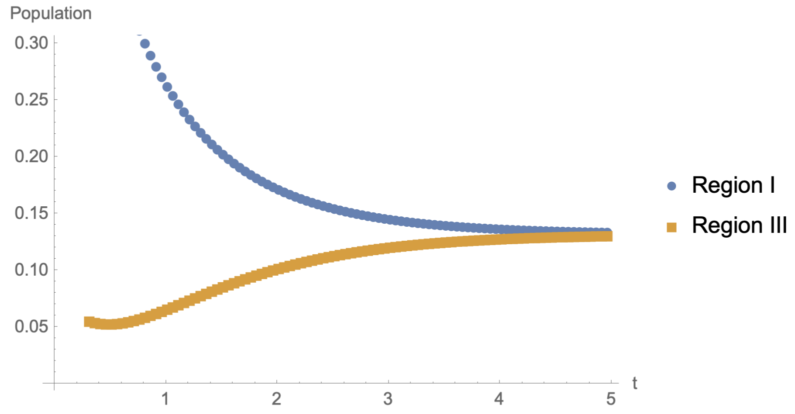


Figure 3: Graph of the population of region I (circle) as a function of time, $\mathcal{N}_I(t)$, and of the population of region III (square) as a function of time, $\mathcal{N}_{III}(t)$, calculated numerically, [4].

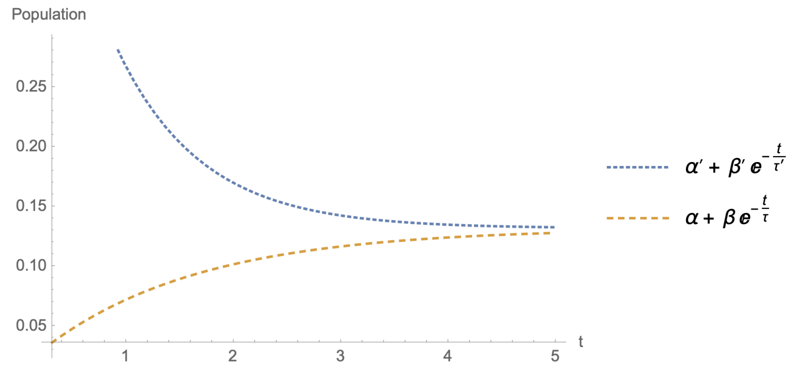


Figure 4: Best numerical fit of population versus region t time of region I, $\mathcal{N}_I(t)$, (dotted line) and the population versus time t of the region III, $\mathcal{N}_{III}(t)$, (dashed line), [4].

3 Diffusion

Using the methodology developed in [4] for the protein folding process, the characteristic passage time τ was evaluated for different values of the diffusion parameter Q for the free energy of Figure 2, $V(x) = 3.90456x^6 - 8.93851x^4 + 5.42373x^2$. For each fixed value of Q in the interval $0.4 < Q < 20$, the passage time τ for the evolution of the population to the right well was calculated, starting from the initial position $x_0 = x_{min} = -1.05282$, as shown in Figure 6 (dotted line). Figure 6 also shows the best fitting for the results (solid line), given as as function of 1/Q.

Figure 6 shows that the passage time τ decreases as the diffusion increases, as expected. The general behavior of the curve of τ *versus* Q is a function proportional to 1/Q which is compatible with that obtained by another method, that uses the stationary state approximation, [10].

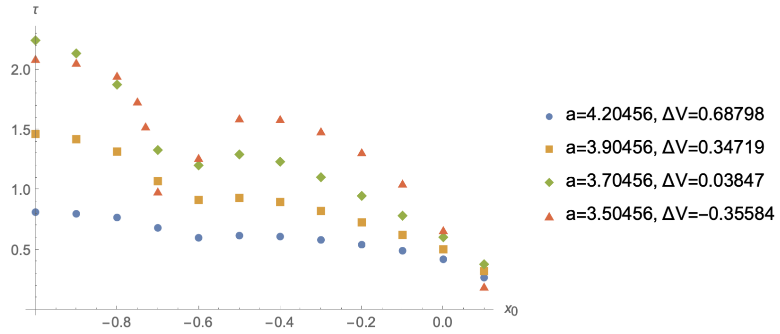


Figure 5: Passage time τ versus the initial position x_0 for the different potentials $V(x)$, as in Figure 1, with $Q = 0.5$, [4].

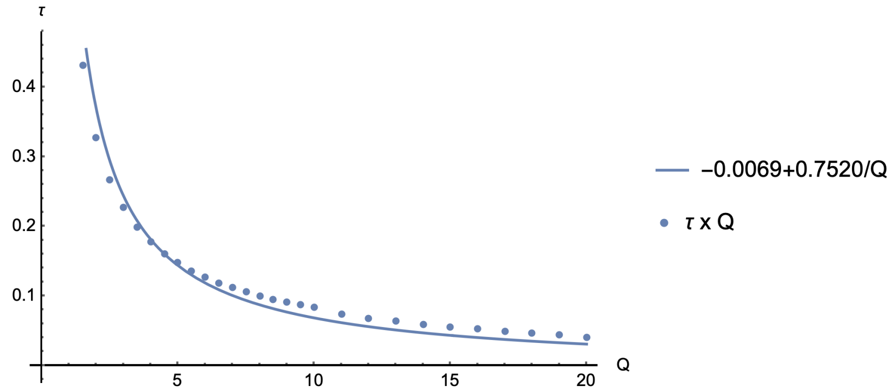


Figure 6: Passage time τ versus the diffusion constant Q for $V(x) = 3.90456x^6 - 8.93851x^4 + 5.42373x^2$ (Figure 2) for analytical results (dotted line) and the best fitting (solid line).

4 Conclusion

The main point of this paper was to determine explicitly the passage time (τ) dependence on the diffusion parameter Q. The general τ *versus* Q curve obtained (Figure 5) is a function proportional to 1/Q which is compatible with that obtained by another method, [8].

The results obtained reinforce the application of the SQM mathematical method proposed for protein folding problems, mainly in the determination of P(x,t) by solving the FPE through its relation with the SE. The passage time of the unfolding-folding process is an important ingredient for the reaction kinetics; the results are consistent with those obtained in [8]. In this reference only the SE ground state is used which makes the probability density depend only on x and not explicitly on t, i.e., their method only use the stationary state. Our approach allows the use of more terms in the expansion of Eq. (5) which makes the dependence of t on $P(x,t)$ appear explicitly.

References

[1] P. G. Wolynes, J. N. Onuchic and D. Thirumalai, *Navigating the folding routes*, Science **267**, 1619 (1995), doi:10.1126/science.7886447.

[2] R. B. Best and G. Hummer, *Diffusion models of protein folding*, Phys. Chem. Chem. Phys. **13**, 16902 (2011), doi:10.1039/c1cp21541h.

[3] S. V. Krivov, *The free energy landscape analysis of protein (FIP35) folding dynamics*, J. Phys. Chem. B **115**, 12315 (2011), doi:10.1021/jp208585r.

[4] M. T. Araujo, J. Chahine, E. Drigo Filho, R. M. Ricotta, *Probability distribution to obtain the characteristic passage time for different tri-stable potentials*, Phys. A **606**, 128121 (2022), doi:10.1016/j.physa.2022.128121.

[5] H. Risken, *The Fokker-Planck equation*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642968075 (1996), doi:10.1007/978-3-642-96807-5.

[6] T. Tomé and M. J. de Oliveira, *Stochastic dynamics and irreversibility*, Springer, Cham, Switzerland, ISBN 9783319117706 (2015), doi:10.1007/978-3-319-11770-6.

[7] E. Drigo Filho and R. M. Ricotta, *Morse potential energy spectra through the variational method and supersymmetry*, Phys. Lett. A **269**, 269 (2000), doi:10.1016/S0375-9601(00)00267-X.

[8] G. R. P. Borges, E. Drigo Filho and R. M. Ricotta, *Variational supersymmetric approach to evaluate Fokker-Planck probability*, Phys. A: Stat. Mech. Appl. **389**, 3892 (2010), doi:10.1016/j.physa.2010.05.027.

[9] F. Polotto, E. Drigo Filho, J. Chahine and R. J. de Oliveira, *Supersymmetric quantum mechanics method for the Fokker-Planck equation with applications to protein folding dynamics*, Phys. A: Stat. Mech. Appl. **493**, 286 (2018), doi:10.1016/j.physa.2017.10.021.

[10] C. Wagner and T. Kiefhaber, *Intermediates can accelerate protein folding*, Proc. Natl. Acad. Sci. USA **96**, 6716 (1999), doi:10.1073/pnas.96.12.6716.

SciPost Phys. Proc. 14, 016 (2023)

Irreducible representations of $\mathbb{Z}_2^2$-graded supersymmetry algebra and their applications

Naruhiko Aizawa⋆

Department of Physics, Osaka Metropolitan University, Sakai, Osaka 599-8531, Japan

⋆ aizawa@omu.ac.jp

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We give a brief review on recent developments of $\mathbb{Z}_2^n$-graded symmetry in physics in which hidden $\mathbb{Z}_2^n$-graded symmetries and $\mathbb{Z}_2^n$-graded extensions of known systems are discussed. This elucidates physical relevance of the $\mathbb{Z}_2^n$-graded algebras. As an example of physically interesting algebra, we take $\mathbb{Z}_2^2$-graded supersymmetry (SUSY) algebras and consider their irreducible representations (irreps). A list of irreps for $\mathcal{N} = 1, 2$ algebras is presented and as an application of the irreps, $\mathbb{Z}_2^2$-graded SUSY classical actions are constructed.

Received 23-11-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.016

1 Introduction

It was more than half a century ago that Ree pointed out that one may generalize Lie algebras by grading with any abelian group [1]. The same object was rediscovered by Rittenberg and Wyler in late 70s [2, 3] (see also [4, 5]). Lie superalgebras are the simplest example of Ree's generalization where the abelian group is taken to be $\mathbb{Z}_2$. Among other possibilities, only $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times) allows us to determine the generalized Lie algebra in terms of commutators and anticommutators so that the $\mathbb{Z}_2^n$-graded algebra is a natural generalization of Lie superalgebras [2, 3]. This implies that the $\mathbb{Z}_2^n$-graded algebras have potential applicability to physical problems, but it is hard to say that such algebra itself is widely recognized in physics community.

One of the purposes of this paper is to emphasize, by providing a brief review on recent developments of $\mathbb{Z}_2^n$-graded symmetry in physical problems, that the $\mathbb{Z}_2^n$-graded algebras are *not* unusual in physics. Rather, they are ubiquitous and would be an important notion to understand nature. This is the contents of §2. The second purpose is to present irreducible representations of $\mathbb{Z}_2^2$-graded version of the supersymmetry algebra [6, 16] which is an algebra of recent particular interest. Here the SUSY-algebra means the super-Poincaré algebra in $(0 + 1)$D spacetime. Knowledge on irreps of an algebra is of fundamental importance for its physical and mathematical applications. In §3.1, irreps of $\mathbb{Z}_2^2$-graded SUSY algebra of $\mathcal{N}=1,2$ are presented. As an application of the irreps, we consider $\mathbb{Z}_2^2$-graded classical mechanics in §3.2. A $\mathbb{Z}_2^2$-graded SUSY transformation is defined by the irrep and classical actions invariant under the transformation and conserved Noether charges are given explicitly.

2 $\mathbb{Z}_2^n$-graded algebras in physics

This section is a very brief review on recently observed relations between $\mathbb{Z}_2^n$-graded algebras and physics. For those discussed in earlier days, readers may refer the references in [7].

2.1 $\mathbb{Z}_2^n$-graded algebras in known systems

(a) symmetries of Lévy-Leblond equation [8, 9].

The Lévy-Leblond equation is a quantum mechanical wave equation describing a spin 1/2 particle in non-relativistic setting. The wave function is a four-component spinor which reproduces, when coupled with electromagnetic field, gyromagnetic ratio two as the Dirac equation does. The equation has Galilean superconformal symmetry, but there exit other symmetry generators which never close in a superalgebra but in $\mathbb{Z}_2^2$-graded algebra. Thus, the symmetry of the equation is given by a $\mathbb{Z}_2^2$-graded algebra.

Similar situation is also observed in the supersymmetric harmonic oscillator discussed in [10]. The system has additonal symmetry generators to the ones in [10] and the whole symmetry generators close in a $\mathbb{Z}_2^2$-graded algebra. This is one of the examples that even simple systems have $\mathbb{Z}_2^2$-graded symmetries.

(b) mixed system of parabosons and parafermions [11].

It is known that parafermion and paraboson algebras are isomorphic to orthogonal algebra and orthosymplectic superalgebra, respectively. There are two possible ways of mixing parabosons and parafermions and form an larger algebra. It has been known that one of them is isomorphic to the superalgebra $osp(2m+1|2n)$. Recently, it was shown that the another one is isomorphic to a $\mathbb{Z}_2^2$-graded extension of orthosymplectic superalgebra. Using this fact, a Fock representation of $\mathbb{Z}_2^2$-graded orthosymplectic superalgebras has been constructed in [12].

(c) Clifford algebras [3, 13, 14].

Clifford algebra $Cl(n,m)$ is regarded as a $\mathbb{Z}_2^n$-graded algebra. This identification is not unique, namely, there are several different ways of assigning $\mathbb{Z}_2^n$-grading to the Clifford algebras. Quaternion and split-quaternion realize $Cl(0,3)$ and $Cl(2,1)$, respectively. Thus, they are also $\mathbb{Z}_2^n$-graded algebra. One may use this fact to realize a $\mathbb{Z}_2^n$-graded algebra in terms of an ordinary superalgebra and a Clifford algebra which leads us to $\mathbb{Z}_2^n$-graded extensions of supersymmetric and superconformal quantum mechanics (see §2.2).

These observations reveal hidden $\mathbb{Z}_2^n$-graded algebraic structure in well-known systems. What is remarkable is that $\mathbb{Z}_2^n$-graded algebras are found in simpler systems compared with the earlier works where SUGRA, string theory were discussed. We believe that this is an illustration of the fact that the $\mathbb{Z}_2^n$-graded algebras are found in many places in physics so that they would play certain roles for deeper understanding of nature.

2.2 $\mathbb{Z}_2^n$-graded extensions of known systems

(a) supersymmetric and superconformal quantum mechanics [15–18].

Supersymmetric quantum mechanics (SQM) can be generalized to $\mathbb{Z}_2^n$-graded setting for arbitrary values of n. It is also possible to have $\mathbb{Z}_2^n$-graded extensions of many models of superconformal mechanics (SCM). This is a consequence of $\mathbb{Z}_2^n$-graded algebraic nature of the

Clifford algebras. One may find an appropriate combination of SQM or SCM and a Clifford algebra which produce its $\mathbb{Z}_2^2$-graded extension. What is remarkable is that these extensions are not unique. That is, for a given model of SQM or SCM, we may have several inequivalent $\mathbb{Z}_2^n$-graded extensions.

(b) supersymmetric classical systems [19–23].

Several $\mathbb{Z}_2^2$-supersymmetric classical actions (field theory, mechanics), which produce $\mathbb{Z}_2^2$-supersymmetric quantum systems upon quantization, have been proposed. Contrast to the quantum mechanics, only $\mathbb{Z}_2^2$-grading is considered in the literature. The actions are constructed by extending D-module presentation and superfield formulation to $\mathbb{Z}_2^2$-setting. Extension of superfield formalism is not straightforward since $\mathbb{Z}_2^2$-graded superspace has an extra bosonic coordinate which is not nilpotent and anticommute with fermionic coordinate so that the superfield is a formal power series in this exotic bosonic coordinate. Furthermore, integration on the $\mathbb{Z}_2^n$-graded superspace is highly non-trivial and only $\mathbb{Z}_2^2$ case is known yet [24].

(c) sine-Gordon equation [25].

It has been shown that a $\mathbb{Z}_2^2$-graded extension of the sine-Gordon equation is solvable. This suggest the existence of a new class of integrable systems which are characterized by $\mathbb{Z}_2^2$-graded symmetry.

(d) detectability of $\mathbb{Z}_2^2$-graded supersymmetry [26, 27].

It is an important question whether the $\mathbb{Z}_2^2$-graded supersymmetry is *physically* different from the ordinary one. The question has been answered affirmatively. Existence of operators which distinguish $\mathbb{Z}_2^2$ and $\mathbb{Z}_2$-graded SQM was shown in multipartite sector of a simple model with harmonic oscillator potential. $\mathbb{Z}_2^2$-graded SUSY describes a kind of para-particle since, by definition of the symmetry algebra, it has commuting fermions and exotic bosons. Recently, it was reported that para-particle oscillators were realized experimentally [28]. This implies the possibility of experimental realization of $\mathbb{Z}_2^2$-graded para-particles.

(e) super division algebras [29].

It is known that the number of inequivalent associative real super division algebras is ten. Three of them are purely even ($\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$) and another seven with odd elements. These ten super division algebras have a deep connection with other objects having ten inequivalent classes such as the periodic table of topological insulators and superconductors, Morita equivalence classes of real and complex Clifford algebras, and classical families of compact symmetric spaces (see [30] for a review).

The superdivision algebra is a $\mathbb{Z}_2$-graded extension of the purely even one. There is no reason to stop at $\mathbb{Z}_2$-grading, one may consider $\mathbb{Z}_2^n$-graded division algebras and would expect surprising connection with other objects. In [29], a classification of $\mathbb{Z}_2^2$-graded division algebras has been done. It was shown that there are 13 $\mathbb{Z}_2^2$-graded division algebras in addition to the $\mathbb{Z}_2$-graded counterpart. Objects having connection with these $\mathbb{Z}_2^2$-graded division algebras are not known yet. Finding them is an exciting problem.

From these observation, one may conclude that $\mathbb{Z}_2^n$-graded symmetries enlarge the concepts of physical importance and open up new fields of research interest.

3 Irreducible representations of $\mathbb{Z}_2^2$-graded SUSY algebras

3.1 Irreps of $\mathcal{N} = 1, 2$ algebras

In this section, we present irreps of $\mathbb{Z}_2^2$-graded SUSY algebras. We deal with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ algebras and consider their representations in a $\mathbb{Z}_2^2$-graded vector space.

Let us first recall the definition of $\mathbb{Z}_2^2$-graded algebra [1–3]. It is a direct sum of four vector spaces each of which is labeled by an element of $\mathbb{Z}_2^2$: $\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,1)}$. The multiplication of two elements $X_{\vec{a}} \in \mathfrak{g}_{\vec{a}}$ and $Y_{\vec{b}} \in \mathfrak{g}_{\vec{b}}$ is defined by generalized Lie bracket $[\![\, , \,]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is a bilinear map and satisfies

$$[\![X_{\vec{a}}, Y_{\vec{b}}]\!] = -(-1)^{\vec{a}\cdot\vec{b}}[\![Y_{\vec{b}}, X_{\vec{a}}]\!] \in \mathfrak{g}_{\vec{a}+\vec{b}} \,, \tag{1}$$

$$[\![X_{\vec{a}}, [\![Y_{\vec{b}}, Z_{\vec{c}}]\!]]\!] = [\![[\![X_{\vec{a}}, Y_{\vec{b}}]\!], Z_{\vec{c}}]\!] + (-1)^{\vec{a}\cdot\vec{b}}[\![Y_{\vec{b}}, [\![X_{\vec{a}}, Z_{\vec{c}}]\!]]\!] \,, \tag{2}$$

where $\vec{a} \cdot \vec{b}$ is the inner product of two-component vectors. The relation (1) implies that the generalized Lie bracket is realized by commutator and anticommutator. The relation (2) is a $\mathbb{Z}_2^2$-graded version of the Jacobi identity.

Now we turn to the $\mathbb{Z}_2^2$-graded SUSY algebra [6,16]. The $\mathcal{N} = 1$ algebra is of four dimension

$$H \in \mathfrak{g}_{(0,0)} \,, \qquad Q_{10} \in \mathfrak{g}_{(1,0)} \,, \qquad Q_{01} \in \mathfrak{g}_{(0,1)} \,, \qquad Z \in \mathfrak{g}_{(1,1)} \,, \tag{3}$$

and the relations is given, in terms of commutator and anticommutator, by

$$\begin{aligned} &\{Q_{01}, Q_{01}\} = \{Q_{10}, Q_{10}\} = 2H \,, && [Q_{01}, Q_{10}] = 2iZ \,, \\ &[H, Q_{01}] = [H, Q_{10}] = 0 \,, && [Z, H] = \{Z, Q_{01}\} = \{Z, Q_{10}\} = 0 \,. \end{aligned} \tag{4}$$

While, the $\mathcal{N} = 2$ algebra is six-dimensional

$$H \in \mathfrak{g}_{(0,0)} \,, \qquad Q_{10}, \; Q_{10}^{\dagger} \in \mathfrak{g}_{(1,0)} \,, \qquad Q_{01}, \; Q_{01}^{\dagger} \in \mathfrak{g}_{(0,1)} \,, \qquad Z \in \mathfrak{g}_{(1,1)} \,. \tag{5}$$

The defining relations are

$$\begin{aligned} &\{Q_a, Q_a^{\dagger}\} = H \,, \qquad [Q_{01}, Q_{10}^{\dagger}] = [Q_{01}^{\dagger}, Q_{10}] = iZ \,, \\ &\{Q_a, Q_a\} = \{Q_a^{\dagger}, Q_a^{\dagger}\} = \{Z, Q_a\} = \{Z, Q_a^{\dagger}\} = [Z, H] = 0 \,, \\ &[Q_a, Q_b] = [Q_a^{\dagger}, Q_b^{\dagger}] = 0 \,, \quad a, \, b = (1,0), \, (0,1) \,. \end{aligned} \tag{6}$$

One may see that H is centeral, while Z is $\mathbb{Z}_2^2$-graded centeral. It follows that H, Z span the Cartan subalgebra, but they are not diagonalizable simultaneously as they belong to the subspaces of different $\mathbb{Z}_2^2$-degree. It is easy to see that H^2 and Z^2 are the second order Casimir which commute with all the elements. Thus, irreps are labeled by their eigenvalues which are denoted by E^2 for H^2 and λ for Z^2. The variables E, λ also labels irreps of the Cartan subalgebra in a $\mathbb{Z}_2^2$-graded representation space [7]:

Lemma 1. *Irreps of the Cartan subalgebra spanned by H, Z are equivalent to one of the followings:*

1. *One dimensional irrep* $v(E, 0) = \text{lin. span} \langle \, v_0 \, \rangle$

$$H v_0 = E v_0 \,, \qquad Z v_0 = 0 \,. \tag{7}$$

2. *Two dimensional irrep* $v(E, \lambda) = \text{lin. span} \langle \, v_0, \; v_1 \, \rangle, \; \lambda \neq 0$

$$H v_0 = E v_0 \,, \qquad v_1 = Z v_0 \,, \qquad Z v_1 = \lambda v_0 \,. \tag{8}$$

Without loss of generality, one may assume the v_0 belongs to the subspace of $\mathbb{Z}_2^2$-degree $(0,0)$.

Representations of the $\mathbb{Z}_2^2$-graded SUSY algebras are induced from $v(E, \lambda)$, but the induced representations are *not* irreducible. For $\mathcal{N} = 1, 2$ cases, due to their simplicity, one may find invariant subspaces explicitly in the induced representation space. For instance, the induced space for $\mathcal{N}=1, \lambda \neq 0$ is eight dimensional whose basis is taken to be

$$\begin{array}{llll} v_0\,, & Q_{10}v_0\,, & Q_{01}v_0\,, & \frac{1}{2}\{Q_{10},Q_{01}\}\,v_0\,, \\ v_1, & Q_{10}v_1\,, & Q_{01}v_1\,, & \frac{1}{2}\{Q_{10},Q_{01}\}\,v_1\,. \end{array} \tag{9}$$

The four dimensional invariant subspace is spanned by the vectors

$$v_{00} := \alpha v_0 + \frac{\beta}{2}\{Q_{10},Q_{01}\}v_1\,, \qquad v_{11} = Z v_{00}\,, \qquad v_{10} = Q_{10}v_{00}\,, \qquad v_{01} = Q_{01}v_{00}\,, \tag{10}$$

provided that α, β satisfy the relations

$$\alpha^2 = \lambda\beta^2(E^2-\lambda)\,, \qquad (Ec+i\lambda)^2 = \lambda(E^2-\lambda)\,, \tag{11}$$

where c is the constant connecting two vectors : $Q_{01}Zv_{00} = cQ_{10}v_{00}$. In the case of $\mathcal{N}=2$ algebra, the induced representation is 16 and 32 dimensional for $\lambda=0$ and $\lambda \neq 0$, respectively. Like the $\mathcal{N}=1$ case, one may find a basis of irreps explicitly. Details are presented in [22,23]. We come to present a list of irreps of the $\mathbb{Z}_2^2$-graded SUSY algebras:

Theorem 2. *The $\mathcal{N}=1$ algebra has a 4D irrep for all possible values of E, λ. While, the $\mathcal{N}=2$ algebra has some inequivalent irreps depending the value of E, λ :*

1. *four inequivalent 4D irreps if $\lambda=0$.*
2. *two inequivalent 4D irreps if $\lambda=E^2$.*
3. *two inequivalent 8D irreps if $\lambda \neq 0$ and $\lambda \neq E^2$.*

Some remarks are in order. If $\lambda=0$, then Z is represented by the zero matrix so that the algebra is almost two copies of the ordinary SUSY algebra. "Almost" means that two supercharges Q_{10} and Q_{01} commute instead of anticommute which is the case of ordinary SUSY. If $\lambda=E^2$, irrep of both $\mathcal{N}=1$ and $\mathcal{N}=2$ algebras is four dimensional and this irrep is peculiar since $H^2=Z^2$ holds for this irrep. In fact, all the physical models discussed in the literature (see §2.2) is limited to this particular irrep. Our theorem shows the existence of wider irreps, the restriction $\lambda=E^2$ is not necessary for both $\mathcal{N}=1$ and $\mathcal{N}=2$ algebras. Therefore, we would expect the existence of physical models in which irreps with $\lambda \neq E^2$ are realized.

3.2 $\mathbb{Z}_2^2$-graded SUSY classical mechanics

As an application of the irreps of §3.1, we construct $\mathbb{Z}_2^2$-graded SUSY actions of classical mechanics. We employ the four dimensional irrep ($\lambda=E^2$) of $\mathcal{N}=2$ algebra. The representation basis is taken to be four complex variables $x(t), z(t), \psi(t), \xi(t)$ which are functions of time t. Their $\mathbb{Z}_2^2$ degree are $(0,0),(1,1),(1,0),(0,1)$, respectively and we assume that they are $\mathbb{Z}_2^2$-commutative: $[\![A,B]\!]=0$. This assignment makes $x(t)$ an ordinary complex number, $\psi(t), \xi(t)$ nilpotent and $z(t)$ anticommute with $\psi(t), \xi(t)$. Thus, $x(t)$ is a bosonic variables, $\psi(t), \xi(t)$ are fermionic and $z(t)$ is an exotic bosonic.

The action of the $\mathcal{N}=2$ algebra on this basis defines a $\mathbb{Z}_2^2$-graded SUSY transformation which reads as follows:

$$\begin{aligned} Q_{10} &: (x,z,\psi,\xi) \to (\psi,\xi,i\dot{x},i\dot{z})\,, & (\bar{x},\bar{z},\bar{\psi},\bar{\xi}) &\to (-\bar{\psi},\bar{\xi},-i\dot{\bar{x}},i\dot{\bar{z}})\,, \\ Q_{01} &: (x,z,\psi,\xi) \to (-i\xi,-i\psi,-\dot{z},-\dot{x})\,, & (\bar{x},\bar{z},\bar{\psi},\bar{\xi}) &\to (-i\bar{\xi},i\bar{\psi},\dot{\bar{z}},-\dot{\bar{x}})\,, \\ Z &: (x,z,\psi,\xi) \to (-\dot{z},-\dot{x},\dot{\xi},\dot{\psi})\,, & (\bar{x},\bar{z},\bar{\psi},\bar{\xi}) &\to (-\dot{\bar{z}},-\dot{\bar{x}},-\dot{\bar{\xi}},-\dot{\bar{\psi}})\,, \end{aligned} \tag{12}$$

where the bar indicates the complex conjugation and H transform all variables to their time derivative.

One may easily write down an action invariant under the transformation (12):

$$L_0 = \dot{\bar{x}}\dot{x} + \dot{\bar{z}}\dot{z} - i(\bar{\psi}\dot{\psi} + \bar{\xi}\dot{\xi}). \tag{13}$$

This is a free theory with four complex dynamical variables and interaction will be introduced by the way similar to [20]. It is also easy to compute the Noether charges. With the notation same as the symmetry generators, they are given by

$$\begin{aligned} H &= \dot{\bar{x}}\dot{x} + \dot{\bar{z}}\dot{z}, \quad Q_{10} = \dot{x}\bar{\psi} + \dot{z}\bar{\xi}, \quad Q_{10}^{\dagger} = \dot{\bar{x}}\psi - \dot{\bar{z}}\xi, \\ Z &= \dot{\bar{x}}\dot{z} + \dot{x}\dot{\bar{z}}, \quad Q_{01} = \dot{x}\bar{\xi} + \dot{z}\bar{\psi}, \quad Q_{01}^{\dagger} = \dot{\bar{x}}\xi - \dot{\bar{z}}\psi. \end{aligned} \tag{14}$$

Like the standard supersymmetry, it is possible to convert dynamical variables to auxiliary ones. We give three examples of such conversion together with the Noether charges.

1) Define $F := \dot{z}$, $\overline{F} := \dot{\bar{z}}$, then all the degree $(1,1)$ variables become auxiliary:

$$L_0 \;\to\; L_1 = \dot{\bar{x}}\dot{x} + |F|^2 - i(\bar{\psi}\dot{\psi} + \bar{\xi}\dot{\xi}), \tag{15}$$

$$H = \dot{\bar{x}}\dot{x}, \quad Z = 0, \quad Q_{10} = \dot{x}\bar{\psi}, \quad Q_{10}^{\dagger} = \dot{\bar{x}}\psi, \quad Q_{01} = \dot{x}\bar{\xi}, \quad Q_{01}^{\dagger} = \dot{\bar{x}}\xi. \tag{16}$$

2) Define $y := \frac{1}{2}(x + \bar{x})$, $A := \frac{i}{2}(\dot{x} - \dot{\bar{x}})$, then one of the degree $(0,0)$ variables becomes auxiliary:

$$L_0 \;\to\; L_2 = \dot{y}^2 + A^2 + \dot{\bar{z}}\dot{z} - i(\bar{\psi}\dot{\psi} + \bar{\xi}\dot{\xi}), \tag{17}$$

$$\begin{aligned} H &= \dot{y}^2 + \dot{\bar{z}}\dot{z}, \quad Q_{10} = \dot{y}\bar{\psi} + \dot{z}\bar{\xi}, \quad Q_{10}^{\dagger} = \dot{y}\psi - \dot{\bar{z}}\xi, \\ Z &= \dot{y}(\dot{z} + \dot{\bar{z}}), \quad Q_{01} = \dot{y}\bar{\xi} + \dot{z}\bar{\psi}, \quad Q_{01}^{\dagger} = \dot{y}\xi - \dot{\bar{z}}\psi. \end{aligned} \tag{18}$$

3) Define $a = \dot{x}$, $\bar{a} = \dot{\bar{x}}$, then all the degree $(0,0)$ variables become auxiliary:

$$L_0 \;\to\; L_3 = |a|^2 + \dot{\bar{z}}\dot{z} - i\left(\bar{\psi}\dot{\psi} + \bar{\xi}\dot{\xi}\right), \tag{19}$$

$$H = \dot{\bar{z}}\dot{z}, \quad Z = 0, \quad Q_{10} = \dot{z}\bar{\xi}, \quad Q_{10}^{\dagger} = \dot{\bar{z}}\xi, \quad Q_{01} = \dot{z}\bar{\psi}, \quad Q_{01}^{\dagger} = \dot{\bar{z}}\psi. \tag{20}$$

An interesting observation is the degree $(1,1)$ charge Z vanishes if all degree $(0,0)$ variables or all degree $(1,1)$ ones are auxiliary. This is understood from the form of Z given in (14). More details of classical actions for $\mathcal{N} = 1,2$ are found in [22,23].

4 Conclusion

Based on many observations of $\mathbb{Z}_2^2$-graded symmetry in physics, we asserted physical relevance of $\mathbb{Z}_2^2$-graded algebras. Supersymmetry is one of the most important symmetries in physics so that its $\mathbb{Z}_2^2$-extensions were considered. We present irreps of $\mathbb{Z}_2^2$-graded SUSY algebras of $\mathcal{N} = 1,2$ on $\mathbb{Z}_2^2$-graded representation space. Even though our investigation was restricted to $(0+1)$D spacetime, due to the $\mathbb{Z}_2^2$-grading of the representation space, the algebras have richer irreps compared with the standard SUSY algebra. We use the four dimensional irrep of $\mathcal{N} = 2$ algebra to construct $\mathbb{Z}_2^2$-graded classical mechanics and discuss its property such as vanishing Noether charge of $\mathbb{Z}_2^2$-degree $(1,1)$.

References

[1] R. Ree, *Generalized Lie elements*, Can. J. Math. **12**, 493 (1960), doi:10.4153/CJM-1960-044-x.

[2] V. Rittenberg, D. Wyler, *Generalized superalgebras*, Nucl. Phys. B **139**, 189 (1978), doi:10.1016/0550-3213(78)90186-4.

[3] V. Rittenberg and D. Wyler, *Sequences of $Z_2 \otimes Z_2$ graded Lie algebras and superalgebras*, J. Math. Phys. **19**, 2193 (1978), doi:10.1063/1.523552.

[4] V. G. Kac, *Lie superalgebras*, Adv. Math. **26**, 8 (1977), doi:10.1016/0001-8708(77)90017-2.

[5] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. **20**, 712 (1979), doi:10.1063/1.524113.

[6] A. Bruce, *On a $\mathbb{Z}_2^n$-graded version of supersymmetry*, Symmetry **11**, 116 (2019), doi:10.3390/sym11010116.

[7] K. Amakawa and N. Aizawa, *A classification of lowest weight irreducible modules over $\mathbb{Z}_2^2$-graded extension of $osp(1|2)$*, J. Math. Phys. **62**, 043502 (2021), doi:10.1063/5.0037493.

[8] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations*, Prog. Theor. Exp. Phys. 123A01 (2016), doi:10.1093/ptep/ptw176.

[9] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *Generalized supersymmetry and the Lévy-Leblond equation*, in *Physical and mathematical aspects of symmetries*, Springer, Cham, Switzerland, ISBN 9783319691633 (2017), doi:10.1007/978-3-319-69164-0_11.

[10] J. Beckers and V. Hussin, *Dynamical supersymmetries of the harmonic oscillator*, Phys. Lett. A **118**, 319 (1986), doi:10.1016/0375-9601(86)90316-6.

[11] V. N. Tolstoy, *Once more on parastatistics*, Phys. Part. Nuclei Lett. **11**, 933 (2014), doi:10.1134/S1547477114070449.

[12] N. I. Stoilova and J. Van der Jeugt, *The $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra $\mathfrak{pso}(2m+1|2n)$ and new parastatistics representations*, J. Phys. A: Math. Theor. **51**, 135201 (2018), doi:10.1088/1751-8121/aaae9a.

[13] H. Albuquerque and S. Majid, *Quasialgebra structure of the octonions*, J. Algebra **220**, 188 (1999), doi:10.1006/jabr.1998.7850.

[14] N. Aizawa, *Generalization of superalgebras to color superalgebras and their representations*, Adv. Appl. Clifford Algebras **28**, 28 (2018), doi:10.1007/s00006-018-0847-x.

[15] A. J. Bruce and S. Duplij, *Double-graded supersymmetric quantum mechanics*, J. Math. Phys. **61**, 063503 (2020), doi:10.1063/1.5118302.

[16] N. Aizawa, K. Amakawa and S. Doi, *$\mathcal{N}$ -extension of double-graded supersymmetric and superconformal quantum mechanics*, J. Phys. A: Math. Theor. **53**, 065205 (2020), doi:10.1088/1751-8121/ab661c.

[17] N. Aizawa, K. Amakawa and S. Doi, *$\mathbb{Z}_2^n$-graded extensions of supersymmetric quantum mechanics via Clifford algebras*, J. Math. Phys. **61**, 052105 (2020), doi:10.1063/1.5144325.

[18] S. Doi and N. Aizawa, *$\mathbb{Z}_2^3$-Graded extensions of Lie superalgebras and superconformal quantum mechanics*, Symmetry Integr. Geom.: Methods Appl. **17**, 071 (2021), doi:10.3842/SIGMA.2021.071.

[19] A. J. Bruce, *$\mathbb{Z}_2\times\mathbb{Z}_2$-graded supersymmetry: 2-d sigma models*, J. Phys. A: Math. Theor. **53**, 455201 (2020), doi:10.1088/1751-8121/abb47f.

[20] N. Aizawa, Z. Kuznetsova and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: The classical theory*, Eur. Phys. J. C **80**, 668 (2020), doi:10.1140/epjc/s10052-020-8242-x.

[21] N. Aizawa, Z. Kuznetsova and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded mechanics: The quantization*, Nucl. Phys. B **967**, 115426 (2021), doi:10.1016/j.nuclphysb.2021.115426.

[22] S. Doi and N. Aizawa, *Comments of $\mathbb{Z}_2^2$-supersymmetry in superfield formalism*, Nucl. Phys. B **974**, 115641 (2022), doi:10.1016/j.nuclphysb.2021.115641.

[23] N. Aizawa and S. Doi, *Irreducible representations of $\mathbb{Z}_2^2$-graded $\mathcal{N} = 2$ supersymmetry algebra and $\mathbb{Z}_2^2$-graded supermechanics*, J. Math. Phys. **63**, 091704 (2022), doi:10.1063/5.0100182.

[24] N. Poncin, *Towards integration on colored supermanifolds*, Banach Center Publ. **110**, 201 (2016), doi:10.4064/bc110-0-14.

[25] A. J. Bruce, *Is the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded sine-Gordon equation integrable?*, Nucl. Phys. B **971**, 115514 (2021), doi:10.1016/j.nuclphysb.2021.115514.

[26] F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parastatistics in multiparticle quantum Hamiltonians*, J. Phys. A: Math. Theor. **54**, 115203 (2021), doi:10.1088/1751-8121/abe2f2.

[27] F. Toppan, *Inequivalent quantizations from gradings and $\mathbb{Z}_2\times\mathbb{Z}_2$ parabosons*, J. Phys. A: Math. Theor. **54**, 355202 (2021), doi:10.1088/1751-8121/ac17a5.

[28] C. H. Alderete, A. M. Green, N. H. Nguyen, Y. Zhu, B. M. Rodríguez-Lara and N. M. Linke, *Experimental realization of para-particle oscillators*, (arXiv preprint) doi:10.48550/arXiv.2108.05471.

[29] Z. Kuznetsova and F. Toppan, *Beyond the 10-fold way: 13 associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superdivision algebras*, Adv. Appl. Clifford Algebras **33**, 24 (2023), doi:10.1007/s00006-023-01263-1.

[30] J. C. Baez, *The tenfold way*, Notices Amer. Math. Soc. **67**, 1599 (2020), https://www.ams.org/journals/notices/202010/rnoti-p1599.pdf?adat=November%202020&trk=2167&cat=column&galt=column.

A general approach to noncommutative spaces from Poisson homogeneous spaces: Applications to (A)dS and Poincaré

Angel Ballesteros[1], Ivan Guitérrez-Sagredo[2★] and Francisco J. Herranz[1†]

1 Departamento de Física, Universidad de Burgos, 09001 Burgos, Spain
2 Departamento de Matemáticas&Computación, Universidad de Burgos, 09001 Burgos, Spain

★ igsagredo@ubu.es , † fjherranz@ubu.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

In this contribution we present a general procedure that allows the construction of noncommutative spaces with quantum group invariance as the quantization of their associated coisotropic Poisson homogeneous spaces coming from a coboundary Lie bialgebra structure. The approach is illustrated by obtaining in an explicit form several noncommutative spaces from (3+1)D (A)dS and Poincaré coisotropic Lie bialgebras. In particular, we review the construction of the κ-Minkowski and κ-(A)dS spacetimes in terms of the cosmological constant Λ. Furthermore, we present all noncommutative Minkowski and (A)dS spacetimes that preserve a quantum Lorentz subgroup. Finally, it is also shown that the same setting can be used to construct the three possible 6D κ-Poincaré spaces of time-like worldlines. Some open problems are also addressed.

Received 22-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.017

Check for updates

Contents

1 Introduction 2

2 Noncommutative spaces from Poisson homogeneous spaces 2

3 κ-Minkowski and κ-(A)dS noncommutative spacetimes 5

4 Noncommutative (A)dS and Minkowski spacetimes with quantum Lorentz subgroups 7

5 κ-Poincaré space of time-like worldlines and beyond 10

6 Concluding remarks and open problems 13

References 13

1 Introduction

The aim of this contribution is twofold. Firstly, we present a systematic "six-step" procedure that allows the construction of different noncommutative spaces with a common underlying homogeneous space G/H where G is a Lie group and H is the isotropy Lie subgroup. The approach requires starting with a coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ such that $\mathfrak{g}$ is the Lie algebra of G and δ is the cocommutator obtained from a classical r-matrix r [1,2]. The main requirement for our development is that δ must satisfy the coisotropic condition $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$ with respect to the isotropy Lie algebra $\mathfrak{h}$ of H [3–5]. Since coboundary Lie bialgebras are the tangent counterpart of Poisson–Lie groups (G, Π) with a Poisson structure Π, the latter just comes from the so-called Sklyanin bracket in this quantum group setting. Therefore, this leads to coisotropic Poisson homogeneous spaces $(G/H, \pi)$ where the Poisson structure π on G/H is obtained via canonical projection of the Poisson–Lie structure Π on the Lie group G. The quantization of $(G/H, \pi)$ gives rise to the corresponding noncommutative space.

Secondly, we illustrate this approach by reviewing, from this general perspective, several very recent noncommutative spaces that could be of interest in a quantum gravity framework [6]. In particular, throughout the paper we will focus on the (3+1)D (Anti-)de Sitter (in short (A)dS)) and Poincaré Lie groups and their associated (3+1)D homogeneous spacetimes together with the 6D Poincaré homogeneous space of time-like geodesics.

The structure of the paper is as follows. In the next section we recall the main necessary mathematical notions and geometric structures. And, as the main result, we present the six-step approach to noncommutative spaces from coisotropic Poisson homogeneous spaces. In Section 3 we apply this procedure in order to recover the well-known κ-Minkowski spacetime [7] as well as the (3+1)D κ-(A)dS spacetimes [8]. In Section 4, we present other noncommutative (3+1)D Minkowski and (A)dS spacetimes, which are quite different from the usual κ-spacetimes ones, by requiring to preserve a quantum Lorentz subalgebra [9].

Now, we stress that in many proposals to quantum gravity theories from quantum groups their cornerstone is usually focused on the (3+1)D noncommutative spacetimes (in general, the κ-Minkowski spacetime), forgetting the role that 6D quantum spaces of geodesics could be played. In fact, in our opinion, any consistent theory should consider, simultaneously, both a (3+1)D noncommutative spacetime and a 6D noncommutative space of worldlines. With this idea and by taking into account the very same six-step procedure of Section 2, we construct the 6D κ-Poincaré quantum space of time-like geodesics [10] in Section 5. Furthermore, there exist two other types of κ-Poincaré deformations beyond the usual "time-like" one; namely, the "space-like" and the "light-like" deformations (see [11,12] and references therein). Thus, we also present in Section 5 these two remaining and very recently obtained 6D noncommutative Poincaré spaces of geodesics [12].

Finally, some remarks and open problems are addressed in the last section.

2 Noncommutative spaces from Poisson homogeneous spaces

In this section, we firstly review the basic mathematical tools necessary for the paper and, secondly, we present a general approach that allows one to construct noncommutative spaces from coisotropic Poisson homogeneous spaces.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ of dimension d. We consider a decomposition of $\mathfrak{g}$, as a vector space, given by the sum of two subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}, \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}. \tag{1}$$

A generic ℓ-dimensional (ℓD) homogeneous space is defined as the left coset space

$$M^\ell = G/H\,, \tag{2}$$

where H is the $(d-\ell)$D isotropy subgroup with Lie algebra $\mathfrak{h}$ (1). Hence we can identify the tangent space at every point $m = gH \in M^\ell$, $g \in G$, with the subspace $\mathfrak{t}$:

$$T_m(M^\ell) = T_{gH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{t} = \operatorname{span}\{T_1,\dots,T_\ell\}\,. \tag{3}$$

The generators of the isotropy subalgebra $\mathfrak{h}$ keep a point on M^ℓ invariant, the origin O, playing the role of rotations around O, while the ℓ generators belonging to $\mathfrak{t}$ move O along ℓ basic directions, thus behaving as translations on M^ℓ. The local coordinates $(t^1,\dots,t^\ell)$ associated with the translation generators of $\mathfrak{t}$ (3) give rise to ℓ coordinates on M^ℓ.

A Poisson–Lie (PL) group is a pair (G,Π) where G is a Lie group and Π is a Poisson structure such that the Lie group multiplication $\mu : G\times G \to G$ is a Poisson map with respect to Π on G and the product Poisson structure $\Pi_{G\times G} = \Pi \oplus \Pi$ on $G\times G$. The relation between the Poisson bivector field and the Poisson bracket is given by

$$(\mathrm{d}f_1 \otimes \mathrm{d}f_2)\Pi = \{f_1, f_2\}_\Pi\,. \tag{4}$$

A Poisson manifold (M,π) is a manifold M endowed with a Poisson structure π on M. A Poisson homogeneous space (PHS) for a PL group (G,Π) is a Poisson manifold (M,π) which is endowed with a transitive group action $\alpha : (G\times M, \Pi\oplus\pi) \to (M,\pi)$ which is a Poisson map. Throughout this paper we shall consider that the manifold is a homogeneous space $M \equiv M^\ell = G/H$ (2). Moreover, we restrict to the case when the Poisson structure π on M^ℓ can be obtained by canonical projection of the PL structure Π on G.

Next, a Lie bialgebra is a pair $(\mathfrak{g},\delta)$ where $\mathfrak{g}$ is a Lie algebra and $\delta : \mathfrak{g} \to \mathfrak{g}\wedge\mathfrak{g}$ is a linear map called the cocommutator satisfying the following two conditions [2]:
(i) δ is a 1-cocycle:

$$\delta\big([X_i,X_j]\big) = [\delta(X_i), X_j\otimes 1 + 1\otimes X_j] + \big[X_i\otimes 1 + 1\otimes X_i\,,\,\delta(X_j)\big]\,,\qquad \forall\, X_i, X_j \in \mathfrak{g}\,. \tag{5}$$

(ii) The dual map $\delta^* : \mathfrak{g}^*\otimes\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on the dual Lie algebra $\mathfrak{g}^*$ of $\mathfrak{g}$.

Coboundary Lie bialgebras [1,2] are those provided by a skewsymmetric classical r-matrix $r \in \mathfrak{g}\wedge\mathfrak{g}$ in the form

$$\delta(X_i) = [X_i\otimes 1 + 1\otimes X_i,\, r]\,,\qquad \forall\, X_i \in \mathfrak{g}\,, \tag{6}$$

such that r must be a solution of the modified classical Yang–Baxter equation (mCYBE)

$$[X_i\otimes 1\otimes 1 + 1\otimes X_i\otimes 1 + 1\otimes 1\otimes X_i, [[r,r]]\,] = 0\,,\qquad \forall\, X_i\in\mathfrak{g}\,, \tag{7}$$

where $[[r,r]]$ is the Schouten bracket defined by

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\,, \tag{8}$$

such that

$$r_{12} = r^{ij} X_i\otimes X_j\otimes 1\,,\qquad r_{13} = r^{ij} X_i\otimes 1\otimes X_j\,,\qquad r_{23} = r^{ij}\, 1\otimes X_i\otimes X_j\,, \tag{9}$$

and hereafter sum over repeated indices will be understood unless otherwise stated. If the Schouten bracket (8) does not vanish the Lie algebra $\mathfrak{g}$ is said to be endowed with a quasitriangular or standard Lie bialgebra structure $(\mathfrak{g},\delta(r))$. The vanishing of the Schouten bracket (8) leads to the classical Yang–Baxter equation (CYBE) $[[r,r]] = 0$ and $(\mathfrak{g},\delta(r))$ is called a triangular or nonstandard Lie bialgebra.

The main point now is that coboundary Lie bialgebras $(\mathfrak{g}, \delta(r))$ are the tangent counterpart of coboundary PL groups (G, Π) [2], where the Poisson structure Π on G is given by the Sklyanin bracket

$$\{f_1, f_2\} = r^{ij}\left(X_i^L f_1 X_j^L f_2 - X_i^R f_1 X_j^R f_2\right), \qquad f_1, f_2 \in \mathcal{C}(G), \tag{10}$$

such that X_i^L and X_i^R are left- and right-invariant vector fields defined by

$$X_i^L f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(g\, \mathrm{e}^{tY_i}\right), \qquad X_i^R f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\mathrm{e}^{tY_i} g\right), \tag{11}$$

where $f \in \mathcal{C}(G)$, $g \in G$ and $Y_i \in \mathfrak{g}$. The quantization (as a Hopf algebra) of a PL group (G, Π) is just the corresponding quantum group.

Given a PHS $(M^\ell = G/H, \pi)$ with an underlying coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ of (G, Π), the Poisson structure π on M^ℓ, coming from canonical projection of the PL structure Π on G, is only ensured to be well-defined whenever the so-called coisotropy condition for the cocommutator δ with respect to the isotropy subalgebra $\mathfrak{h}$ of H is fulfilled [3–5], namely

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \tag{12}$$

This condition means that the commutation relations that define the noncommutative space M_z^ℓ, with underlying classical space M^ℓ (2) and quantum deformation parameter $q = \mathrm{e}^z$, at the first-order in all the quantum coordinates $(\hat{t}^1, \ldots, \hat{t}^\ell)$ close on a Lie subalgebra which is just the annihilator $\mathfrak{h}^\perp$ of $\mathfrak{h}$ on the dual Lie algebra $\mathfrak{g}^*$:

$$\mathfrak{h}^\perp \equiv M_z^\ell. \tag{13}$$

The duality between the generators of $\mathfrak{t}$ (3) and the quantum coordinates $(\hat{t}^1, \ldots, \hat{t}^\ell)$ spanning M_z^ℓ is determined by a canonical pairing given by the bilinear form

$$\langle \hat{t}^j, T_k \rangle = \delta_k^j, \quad \forall\, j, k. \tag{14}$$

Noncommutative spaces can finally be obtained as quantizations of coisotropic PHS in all orders in the quantum coordinates $(\hat{t}^1, \ldots, \hat{t}^\ell)$, so completing the initial quantum space M_z^ℓ (13) which just determines the Lie-algebraic (linear) contribution.

A general approach in order to construct any noncommutative space from any coisotropic PHS $(M^\ell = G/H, \pi)$ with coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$, so fulfilling (12), is summarized in six steps (see [9, 12] and references therein) as follows:

1. Consider a faithful representation ρ of the Lie algebra $\mathfrak{g}$.

2. Compute, by exponentiation, an element of the Lie group G according to the left coset space $M^\ell = G/H$ (2) in the form

$$G_{M^\ell} = \exp\big(t^1 \rho(T_1)\big) \cdots \exp\big(t^\ell \rho(T_\ell)\big) H, \tag{15}$$

   where $(T_1, \ldots, T_\ell)$ are the translation generators on M^ℓ, H is the $(d-\ell)$D isotropy subgroup, and $(t^1, \ldots, t^\ell)$ are local coordinates associated with the above translation generators of $\mathfrak{t}$ (3). Note that these coordinates are independent of the representation chosen in the previous step, provided that it is faithful.

3. Calculate the corresponding left- and right-invariant vector fields (11) from G_{M^ℓ} (15).

4. Consider a classical r-matrix (7) so determining a coboundary Lie bialgebra $(\mathfrak{g}, \delta(r))$ (either of quasitriangular or triangular type), which is the tangent counterpart of the corresponding coboundary PL group (G, Π).

5. Obtain the Poisson brackets among the local translation coordinates $(t^1, \ldots, t^\ell)$ via the Sklyanin bracket (10) from the chosen classical r-matrix. The resulting expressions define the coisotropic PHS.

6. Finally, quantize the PHS thus obtaining the noncommutative space in terms of the quantum coordinates $(\hat{t}^1, \ldots, \hat{t}^\ell)$.

In the next sections we illustrate the above procedure by applying it to several (A)dS and Poincaré quantum deformations giving rise to noncommutative spaces that could be relevant in a quantum gravity framework [6].

3 κ-Minkowski and κ-(A)dS noncommutative spacetimes

Let us consider the (3+1)D Poincaré and (A)dS Lie algebras expressed as a one-parametric family of Lie algebras denoted by $\mathfrak{g}_\Lambda$ depending on the cosmological constant Λ. In a kinematical basis spanned by the generators of time translations P_0, spatial translations $\mathbf{P} = (P_1, P_2, P_3)$, boost transformations $\mathbf{K} = (K_1, K_2, K_3)$ and rotations $\mathbf{J} = (J_1, J_2, J_3)$, the commutation relations of $\mathfrak{g}_\Lambda$ are given by

$$\begin{aligned} &[J_a, J_b] = \epsilon_{abc} J_c\,, && [J_a, P_b] = \epsilon_{abc} P_c\,, && [J_a, K_b] = \epsilon_{abc} K_c\,, \\ &[K_a, P_0] = P_a\,, && [K_a, P_b] = \delta_{ab} P_0\,, && [K_a, K_b] = -\epsilon_{abc} J_c\,, \\ &[P_0, P_a] = -\Lambda K_a, && [P_a, P_b] = \Lambda \epsilon_{abc} J_c\,, && [J_a, P_0] = 0\,. \end{aligned} \tag{16}$$

From now on, Latin indices run as $a, b, c = 1, 2, 3$ while Greek ones run as $\mu = 0, 1, 2, 3$. The Lie algebra $\mathfrak{g}_\Lambda$ comprises the dS algebra $\mathfrak{so}(4,1)$ for $\Lambda > 0$, the AdS algebra $\mathfrak{so}(3,2)$ for $\Lambda < 0$ and the Poincaré one $\mathfrak{iso}(3,1)$ when $\Lambda = 0$.

The first step in our approach is to consider a faithful representation $\rho : \mathfrak{g}_\Lambda \to \mathrm{End}(\mathbb{R}^5)$ for $X \in \mathfrak{g}_\Lambda$, that reads

$$\rho(X) = x^\mu \rho(P_\mu) + \xi^a \rho(K_a) + \theta^a \rho(J_a) = \begin{pmatrix} 0 & \Lambda x^0 & -\Lambda x^1 & -\Lambda x^2 & -\Lambda x^3 \\ x^0 & 0 & \xi^1 & \xi^2 & \xi^3 \\ x^1 & \xi^1 & 0 & -\theta^3 & \theta^2 \\ x^2 & \xi^2 & \theta^3 & 0 & -\theta^1 \\ x^3 & \xi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}. \tag{17}$$

By exponentiation we obtain a one-parametric family of Lie groups, G_Λ, that covers the dS $\mathrm{SO}(4,1)$ for $\Lambda > 0$, the AdS $\mathrm{SO}(3,2)$ for $\Lambda < 0$, and the Poincaré $\mathrm{ISO}(3,1)$ for $\Lambda = 0$. The (3+1)D Minkowski and (A)dS homogeneous spacetimes (2), M^{3+1}_Λ, are defined by

$$M^{3+1}_\Lambda = G_\Lambda / H\,, \qquad H = \mathrm{SO}(3,1) = \langle \mathbf{K}, \mathbf{J} \rangle\,, \tag{18}$$

where the Lie algebra $\mathfrak{h}$ of H is the Lorentz subalgebra and $\mathfrak{t} = \mathrm{span}\,\{P_\mu\}$ (1). Observe that the constant sectional curvature of M^{3+1}_Λ is $\omega = -\Lambda$.

Our aim now is to construct the κ-noncommutative counterpart of M^{3+1}_Λ (18). According to (15) (step 2 in Section 2) we compute G_Λ in terms of local coordinates (x^μ, ξ^a, θ^a) as

$$G_\Lambda = \exp\big(x^0 \rho(P_0)\big) \exp\big(x^1 \rho(P_1)\big) \exp\big(x^2 \rho(P_2)\big) \exp\big(x^3 \rho(P_3)\big) H\,, \tag{19}$$

where the Lorentz subgroup $H = \mathrm{SO}(3,1)$ is parametrized by

$$H = \exp\big(\xi^1\rho(K_1)\big)\exp\big(\xi^2\rho(K_2)\big)\exp\big(\xi^3\rho(K_3)\big)\exp\big(\theta^1\rho(J_1)\big)\exp\big(\theta^2\rho(J_2)\big)\exp\big(\theta^3\rho(J_3)\big). \quad (20)$$

Notice that here the index $\ell = 4$ in (2) and the generic local coordinates (t^1, t^2, t^3, t^4) in (15) corresponds to the spacetime coordinates (x^0, x^1, x^2, x^3).

Following the step 3 in Section 2 we compute the left- and right-invariant vector fields (11) from G_Λ. In the step 4 we have to consider a classical r-matrix and we distinguish two cases between κ-Poincaré with $\Lambda = 0$ and κ-(A)dS with $\Lambda \neq 0$.

The κ-Poincaré classical r-matrix is a solution of the mCYBE (7) and reads [7,13]

$$r_0 = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3), \quad (21)$$

that satisfies the coisotropy condition (12) with respect to $\mathfrak{h} = \mathrm{span}\{\mathbf{K}, \mathbf{J}\}$ and where the quantum deformation parameter $\kappa = 1/z$. The corresponding Sklyanin bracket (10) leads to linear Poisson brackets for the classical coordinates x^μ which determine the κ-Minkowski PHS. This can therefore be quantized directly in terms of the quantum coordinates $\hat{x}^\mu$. Hence we recover well-known κ-Minkowski spacetime [7] (see also [5,11,14,15] and references therein) which is of Lie-algebraic type:

$$[\hat{x}^0, \hat{x}^a] = -\frac{1}{\kappa}\hat{x}^a, \qquad [\hat{x}^a, \hat{x}^b] = 0, \quad (22)$$

completing the final steps 5 and 6 in Section 2.

When $\Lambda \neq 0$ we consider the κ-(A)dS classical r-matrix, which is also a a solution of the mCYBE (7), given by [8,16,17]

$$r_\Lambda = \frac{1}{\kappa}(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \eta J_1 \wedge J_2), \quad (23)$$

such that the parameter η is related to the cosmological constant Λ and the sectional curvature ω of the (A)dS spacetimes (18) by

$$\omega = \eta^2 = -\Lambda. \quad (24)$$

Thus η is real for AdS and a purely imaginary number for dS. The Sklyanin bracket now gives rise to the (nonlinear) κ-(A)dS PHS in the form [8]

$$\begin{aligned}
\{x^0, x^1\} &= -\frac{1}{\kappa}\frac{\tanh(\eta x^1)}{\eta\cosh^2(\eta x^2)\cosh^2(\eta x^3)}, \\
\{x^0, x^2\} &= -\frac{1}{\kappa}\frac{\tanh(\eta x^2)}{\eta\cosh^2(\eta x^3)}, \\
\{x^0, x^3\} &= -\frac{1}{\kappa}\frac{\tanh(\eta x^3)}{\eta},
\end{aligned} \quad (25)$$

$$\begin{aligned}
\{x^1, x^2\} &= -\frac{1}{\kappa}\frac{\cosh(\eta x^1)\tanh^2(\eta x^3)}{\eta}, \\
\{x^1, x^3\} &= \frac{1}{\kappa}\frac{\cosh(\eta x^1)\tanh(\eta x^2)\tanh(\eta x^3)}{\eta}, \\
\{x^2, x^3\} &= -\frac{1}{\kappa}\frac{\sinh(\eta x^1)\tanh(\eta x^3)}{\eta}.
\end{aligned} \quad (26)$$

Consequently, in contrast to the κ-Minkowski spacetime (22) when $\Lambda \neq 0$ the 3-space (26), determined by x^a, is no longer commutative and ordering ambiguities arise in (25) and (26) which precludes a direct quantization. This problem can be circumvented by introducing five ambient coordinates in the (A)dS spacetimes (18) denoted $(s^4, s^\mu) \in \mathbb{R}^5$ such that they fulfil the pseudosphere constraint

$$\Sigma_\Lambda \equiv (s^4)^2 - \Lambda(s^0)^2 + \Lambda\big((s^1)^2 + (s^2)^2 + (s^3)^2\big) = 1\,. \tag{27}$$

These read [8, 9]

$$\begin{aligned}
s^4 &= \cos(\eta x^0)\cosh\left(\eta x^1\right)\cosh\left(\eta x^2\right)\cosh\left(\eta x^3\right),\\
s^0 &= \frac{\sin(\eta x^0)}{\eta}\cosh\left(\eta x^1\right)\cosh(\eta x^2)\cosh\left(\eta x^3\right),\\
s^1 &= \frac{\sinh(\eta x^1)}{\eta}\cosh\left(\eta x^2\right)\cosh\left(\eta x^3\right),\\
s^2 &= \frac{\sinh\left(\eta x^2\right)}{\eta}\cosh\left(\eta x^3\right),\\
s^3 &= \frac{\sinh\left(\eta x^3\right)}{\eta},
\end{aligned} \tag{28}$$

and the spacetime coordinates x^μ are called geodesic parallel coordinates. Notice also that $q^\mu = s^\mu/s^4$ are Beltrami projective coordinates in M_Λ^{3+1} (18) which can be obtained through the projection with pole $(0,0,0,0,0) \in \mathbb{R}^5$ of a point with ambient coordinates (s^4, s^μ) onto the projective hyperplane with $s^4 = +1$ (see [18] for details). Next, if we compute the Poisson brackets among (s^4, s^μ) from (25) and (26), consider the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ along with the ordered monomials $(\hat{s}^0)^k\,(\hat{s}^1)^l\,(\hat{s}^3)^m\,(\hat{s}^2)^n\,(\hat{s}^4)^j$, we finally obtain the κ-(A)dS spacetimes $M_{\Lambda,\kappa}^{3+1}$ expressed as a quadratic algebra [8]

$$\begin{aligned}
&[\hat{s}^0, \hat{s}^a] = -\frac{1}{\kappa}\,\hat{s}^a\hat{s}^4\,, \qquad [\hat{s}^4, \hat{s}^a] = \frac{\eta^2}{\kappa}\,\hat{s}^0\hat{s}^a\,, \qquad [\hat{s}^0, \hat{s}^4] = -\frac{\eta^2}{\kappa}\,\hat{\mathcal{S}}_{\eta/\kappa}\,,\\
&[\hat{s}^1, \hat{s}^2] = -\frac{\eta}{\kappa}(\hat{s}^3)^2\,, \qquad [\hat{s}^1, \hat{s}^3] = \frac{\eta}{\kappa}\,\hat{s}^3\hat{s}^2\,, \qquad [\hat{s}^2, \hat{s}^3] = -\frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^3\,,
\end{aligned} \tag{29}$$

where the quantum 3-space $\hat{\mathcal{S}}_{\eta/\kappa}$ operator is given by

$$\hat{\mathcal{S}}_{\eta/\kappa} = \left(\hat{s}^1\right)^2 + \left(\hat{s}^2\right)^2 + \left(\hat{s}^3\right)^2 + \frac{\eta}{\kappa}\,\hat{s}^1\hat{s}^2\,. \tag{30}$$

Obviously, Jacobi identities are satisfied. We remark that $M_{\Lambda,\kappa}^{3+1}$ (29) has a Casimir operator

$$\hat{\Sigma}_{\Lambda,\kappa} = \left(\hat{s}^4\right)^2 - \Lambda\left(\hat{s}^0\right)^2 + \frac{\Lambda}{\kappa}\,\hat{s}^0\hat{s}^4 + \Lambda\,\hat{\mathcal{S}}_{\eta/\kappa}\,, \tag{31}$$

which is the quantum analogue of the pseudosphere (27) (recall that $\Lambda = -\eta^2$ (24)).

As expected, under the flat limit $\eta \to 0$ (i.e., $\Lambda \to 0$), the ambient coordinates (s^4, s^μ) (28) provide the usual Cartesian ones $(1, x^\mu)$ in the Minkowski spacetime and the κ-(A)dS spacetimes (29) reduce to the κ-Minkowski spacetime (22).

4 Noncommutative (A)dS and Minkowski spacetimes with quantum Lorentz subgroups

In this section we present very recent results concerning (3+1)D noncommutative (A)dS and Minkowski spacetimes that preserve a quantum Lorentz subgroup which were obtained in [9] by following the same six-step procedure described in Section 2. We advance that these are quite different from the κ-Minkowski (22) and κ-(A)dS (29) spacetimes reviewed in the previous section. Hence, we keep the same notation as in Section 3, in particular we shall make use of the expressions (16)–(20), (24), (27) and (28).

We consider the family of the (3+1)D Poincaré and (A)dS Lie algebras $\mathfrak{g}_\Lambda$ (16) and search for classical r-matrices (7) that keep the Lorentz subalgebra $\mathfrak{h} = \text{span}\{\mathbf{K},\mathbf{J}\} = \mathfrak{so}(3,1)$ as a sub-Lie bialgebra, that is,

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}\,, \tag{32}$$

which is a more restrictive version of the coisotropy condition (12). This restriction implies that the corresponding PHS is constructed through the Lorentz isotropy subgroup $H = \mathrm{SO}(3,1)$ such that $(H, \Pi|_H)$ is a PL subgroup of (G_Λ, Π) and it is called a PHS of Poisson subgroup type.

Then we start with the most general element $r \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$. Since the dimension of $\mathfrak{g}_\Lambda$ is $d = 10$, r depends on 45 initial deformation parameters. From it, we directly compute the cocommutator δ (6) such that $(\mathfrak{g}_\Lambda, \delta(r))$ defines a Lie bialgebra if and only if r is a solution of the mCYBE (7). Moreover, we have to impose the condition (32).

The simplest case is to require that $\delta(\mathfrak{h}) = 0$ which means that the Lorentz subgroup remains underformed. The final result is summarized as [9]:

Proposition 1. *The only PL group (G_Λ, Π) such that $\Pi|_H = 0$ is the trivial one.*

Therefore the only PHS $(M^{3+1}_\Lambda = G_\Lambda/H, \pi)$ of Poisson Lorentz subgroup type such that $\Pi|_H = 0$ is the trivial one. In other words, there does not exist any quantum deformation of the (3+1)D Poincaré and (A)dS Lie algebras preserving the Lorentz subalgebra $\mathfrak{h}$ underformed.

Now the main question is whether there exists a quantum deformation of $\mathfrak{g}_\Lambda$ preserving a non-trivial quantum Lorentz subalgebra, that is, $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h} \neq 0$. The answer is positive. By taking into account previous results concerning quantum Poincaré groups [19, 20] and quantum deformations of the Lorentz algebra $\mathfrak{h} = \mathfrak{so}(3,1)$ [21], it can be proven that the classification of the quantum deformations of $\mathfrak{g}_\Lambda$ keeping a quantum Lorentz subalgebra can be casted into three types as follows [9]:

Proposition 2. *There exist three classes of PHS $(M^{3+1}_\Lambda = G_\Lambda/H, \pi)$ for each of the maximally symmetric relativistic spacetimes of constant curvature (Minkowski and (A)dS) (18) such that the isotropy Lorentz subgroup H is a PL subgroup of (G_Λ, Π). All of them are obtained from coboundary PL structures on their respective isometry group G_Λ which are determined, up to $\mathfrak{g}_\Lambda$-isomorphisms, by the classical r-matrices*

$$\begin{aligned} r_{\rm I} &= z\,(K_1\wedge K_2 + K_1\wedge J_3 - K_3\wedge J_1 - J_1\wedge J_2) - z'\,(K_2\wedge K_3 - K_2\wedge J_2 - K_3\wedge J_3 + J_2\wedge J_3)\,,\\ r_{\rm II} &= z\,K_1\wedge J_1\,,\\ r_{\rm III} &= z\,(K_1\wedge K_2 + K_1\wedge J_3)\,, \end{aligned} \tag{33}$$

where z and z' are free quantum deformation parameters. These three classical r-matrices are solutions of the CYBE $[[r,r]] = 0$.

Hence the three classes correspond to triangular or nonstandard deformations. Types II and III would provide one-parametric deformations, while type I would lead to a two-parametric one with arbitrary deformation parameters z and z'. Recall that the κ-$\mathfrak{g}_\Lambda$ deformations described in the previous section have a quasitriangular or standard character.

Next we apply the approach presented in Section 2 in order to construct the corresponding PHS from the above classical r-matrices in terms of the local coordinates x^μ through the Sklyanin bracket (10). However, the resulting expressions are rather cumbersome and strong ordering ambiguities appear, so there is no a direct quantization for any class. In order to solve

Table 1: The three types of (A)dS and Minkowski noncommutive spacetimes with quantum Lorentz subgroups determined by Proposition 2. These are expressed in quantum ambient spacetime coordinates $\hat{s}^\mu$ (28) or in ($\hat{s}^\pm = \hat{s}^0 \pm \hat{s}^1, \hat{s}^2, \hat{s}^3$). The quantum coordinate $\hat{s}^4$ always commutes with $\hat{s}^\mu$.

Type I $r_{\rm I} = z(K_1 \wedge K_2 + K_1 \wedge J_3 - K_3 \wedge J_1 - J_1 \wedge J_2)$
$-z'(K_2 \wedge K_3 - K_2 \wedge J_2 - K_3 \wedge J_3 + J_2 \wedge J_3)$

• Subfamily with $z = 0$

$[\hat{s}^-, \hat{s}^2] = -2z'\hat{s}^+\hat{s}^3 \qquad [\hat{s}^-, \hat{s}^3] = 2z'\hat{s}^+\hat{s}^2 \qquad [\hat{s}^2, \hat{s}^3] = z'(\hat{s}^+)^2 \qquad [\hat{s}^+, \cdot\,] = 0$

• Subfamily with $z' = 0$

$[\hat{s}^-, \hat{s}^+] = 2z\hat{s}^+\hat{s}^2 \qquad [\hat{s}^-, \hat{s}^2] = z\hat{s}^-\hat{s}^+ - 2z(\hat{s}^3)^2 \qquad [\hat{s}^-, \hat{s}^3] = 2z\hat{s}^3\hat{s}^2$
$[\hat{s}^2, \hat{s}^3] = z\hat{s}^+\hat{s}^3 \qquad [\hat{s}^+, \hat{s}^2] = -z(\hat{s}^+)^2 \qquad [\hat{s}^+, \hat{s}^3] = 0$

Type II $r_{\rm II} = zK_1 \wedge J_1$

$[\hat{s}^0, \hat{s}^1] = 0 \qquad [\hat{s}^0, \hat{s}^2] = z\hat{s}^1\hat{s}^3 \qquad [\hat{s}^0, \hat{s}^3] = -z\hat{s}^1\hat{s}^2$
$[\hat{s}^2, \hat{s}^3] = 0 \qquad [\hat{s}^1, \hat{s}^2] = z\hat{s}^0\hat{s}^3 \qquad [\hat{s}^1, \hat{s}^3] = -z\hat{s}^0\hat{s}^2$

Type III $r_{\rm III} = z(K_1 \wedge K_2 + K_1 \wedge J_3)$

$[\hat{s}^2, \hat{s}^+] = z(\hat{s}^+)^2 \qquad [\hat{s}^2, \hat{s}^-] = -z\hat{s}^-\hat{s}^+ \qquad [\hat{s}^-, \hat{s}^+] = 2z\hat{s}^+\hat{s}^2 \qquad [\hat{s}^3, \hat{s}^\mu] = 0$

this problem we proceed similarly to the κ-(A)dS spacetimes (29). We again consider the ambient coordinates (s^4, s^μ) (28) (subjected to the pseudosphere constraint (27)), compute their PL brackets from those initially given in terms of x^μ, and finally obtain the corresponding noncommutive spacetimes by choosing an appropriate order in the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ (so satisfying the Jacobi identities).

As a final result, we display in Table 1 all the (3+1)D Minkowski and (A)dS noncommutive spacetimes that preserve a non-trivial quantum Lorentz subgroup [9].

Now some remarks are in order.

- The ambient quantum coordinate $\hat{s}^4$ is always a central element for all the three types of noncommutive spacetimes, $[\hat{s}^4, \hat{s}^\mu] = 0$, so that these are just defined by the (3+1) quantum variables $\hat{s}^\mu$.

- In this respect, we remark that the corresponding noncommutive Minkowski spacetimes can directly be obtained through the flat limit $\Lambda \to 0$ (or $\eta \to 0$), in such a manner that the quantum coordinates $(\hat{s}^4, \hat{s}^\mu)$ reduce to the usual quantum Cartesian ones $(1, \hat{x}^\mu)$. Since $\hat{s}^4$ is absent in all the expressions presented in Table 1, the noncommutive Minkowski spacetimes adopt the very same formal expressions in the quantum Cartesian coordinates $\hat{x}^\mu$.

- For types I and III it is found that the explicit noncommutive spacetimes are naturally adapted to a null-plane basis [9,22] and for this reason we have considered the quantum coordinates $(\hat{s}^\pm = \hat{s}^0 \pm \hat{s}^1, \hat{s}^2, \hat{s}^3)$ instead of $\hat{s}^\mu$. Thus they lead to $(\hat{x}^\pm = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2, \hat{x}^3)$ for the Minkowski cases.

- In type I we have distinguished two subfamilies with either z or z' equal to zero in order to clarify the presentation of the results. Nevertheless, observe that the general noncommutive spacetimes of type I is just the superposition (the sum) of both subfamilies.

- We remark that the type II noncommutative spacetime has already been obtained for the quadratic Minkowski case in [23] (set $\hat{s}^\mu \equiv \hat{x}^\mu$) by following a different approach from ours; that is, from a twisted quantum Poincaré group and then applying the FRT procedure. Notice that, in fact, the classical r-matrix $r_{\rm II}$ (33) is just a Reshetikhin twist.

- Finally, the type III noncommutative spacetimes can be regarded as (2+1)D quantum spaces since the quantum coordinate $\hat{s}^3$ is a central operator, $[\hat{s}^3, \cdot\,] = 0$. We recall that when this structure is, again, only applied to the Minkowski case ($\hat{x}^\pm = \hat{x}^0 \pm \hat{x}^1, \hat{x}^2$), it was already obtained from a Drinfel'd double structure of the (2+1)D Poincaré group in [24]. In addition, we stress that the corresponding quantum algebra for $\mathfrak{g}_\Lambda$ comes from the lower dimensional Lorentz subalgebra $\mathfrak{so}(2,1)$ spanned by $\{J_3, K_1, K_2\}$ which is just the well-known nonstandard (or Jordanian) quantum deformation of $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1)$ (see [25–28]). For higher-dimensional quantum (A)dS algebras keeping such a nonstandard quantum $\mathfrak{sl}(2,\mathbb{R})$ Hopf subalgebra we refer to [29].

5 κ-Poincaré space of time-like worldlines and beyond

So far we have constructed several (3+1)D Minkowski and (A)dS noncommutive spacetimes by applying the approach given in Section 2. However, we stress that such a procedure is rather general and can be applied to any homogeneous space. Hence in this section we shall consider the 6D homogeneous space of time-like Poincaré geodesics and obtain its κ-noncommutative version [10].

With this aim we consider the following Cartan decomposition of the Poincaré algebra $\mathfrak{g}_\Lambda \equiv \mathfrak{g}$ and $G_\Lambda \equiv G$ with commutation relations (16) with $\Lambda = 0$ (see (1)):

$$\mathfrak{g} = \mathfrak{t}_{\rm tl} \oplus \mathfrak{h}_{\rm tl}\,, \qquad \mathfrak{t}_{\rm tl} = \operatorname{span}\{\mathbf{P}, \mathbf{K}\}\,, \qquad \mathfrak{h}_{\rm tl} = \operatorname{span}\{P_0, \mathbf{J}\} = \mathbb{R} \oplus \mathfrak{so}(3)\,. \tag{34}$$

The homogeneous space of time-like geodesics is of dimension six and is defined by

$$\mathcal{W}_{\rm tl} = G/H_{\rm tl}\,, \tag{35}$$

where the isotropy subgroup $H_{\rm tl} = \mathbb{R} \otimes \mathrm{SO}(3)$ comes from the Lie subalgebra $\mathfrak{h}_{\rm tl}$ (34).

By following the procedure presented in Section 2, we first parametrize the Poincaré Lie group from the 5D matrix representation (17) with $\Lambda = 0$ taking into account the order given in (15), that is,

$$\begin{aligned} G_{\mathcal{W}_{\rm tl}} = {} & \exp\big(\eta^1 \rho(K_1)\big) \exp\big(y^1 \rho(P_1)\big) \exp\big(\eta^2 \rho(K_2)\big) \exp\big(y^2 \rho(P_2)\big) \\ & \times \exp\big(\eta^3 \rho(K_3)\big) \exp\big(y^3 \rho(P_3)\big) H_{\rm tl}\,, \end{aligned} \tag{36}$$

where $H_{\rm tl}$ is the stabilizer of the worldline corresponding to a massive particle at rest at the origin of the (3+1)D Minkowski spacetime, namely

$$H_{\rm tl} = \exp\big(\phi^1 \rho(J_1)\big) \exp\big(\phi^2 \rho(J_2)\big) \exp\big(\phi^3 \rho(J_3)\big) \exp\big(y^0 \rho(P_0)\big)\,. \tag{37}$$

Therefore the classical coordinates $(t^1, \ldots, t^\ell)$ in (15) correspond to (η^a, y^a) in (36) (recall that now $\ell = 6$). Next we consider the κ-Poincaré r-matrix (21) and by projecting the Sklyanin bracket (10) to the homogeneous space (35) we obtain a coisotropic PHS for the classical space of time-like geodesics which can be straightforwardly quantized since no ordering problems appear. In this way, the κ-Poincaré space of time-like geodesics $\mathcal{W}_{\mathrm{tl},\kappa}$ in terms of the six quantum coordinates $(\hat{y}^a, \hat{\eta}^a)$ turns out to be [10]:

$$\begin{aligned}
[\hat{y}^1, \hat{y}^2] &= \frac{1}{\kappa}\left(\hat{y}^2 \sinh \hat{\eta}^1 - \frac{\hat{y}^1 \tanh \hat{\eta}^2}{\cosh \hat{\eta}^3}\right), \\
[\hat{y}^1, \hat{y}^3] &= \frac{1}{\kappa}\big(\hat{y}^3 \sinh \hat{\eta}^1 - \hat{y}^1 \tanh \hat{\eta}^3\big), \\
[\hat{y}^2, \hat{y}^3] &= \frac{1}{\kappa}\big(\hat{y}^3 \cosh \hat{\eta}^1 \sinh \hat{\eta}^2 - \hat{y}^2 \tanh \hat{\eta}^3\big), \\
[\hat{y}^1, \hat{\eta}^1] &= \frac{1}{\kappa}\frac{\big(\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1\big)}{\cosh \hat{\eta}^2 \cosh \hat{\eta}^3}, \\
[\hat{y}^2, \hat{\eta}^2] &= \frac{1}{\kappa}\frac{\big(\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1\big)}{\cosh \hat{\eta}^3}, \\
[\hat{y}^3, \hat{\eta}^3] &= \frac{1}{\kappa}\big(\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1\big),
\end{aligned} \tag{38}$$

together with

$$[\hat{\eta}^a, \hat{\eta}^b] = 0, \quad \forall\, a, b, \qquad [\hat{y}^a, \hat{\eta}^b] = 0, \quad a \neq b. \tag{39}$$

The above commutators can also be written in terms of quantum Darboux operators $(\hat{q}^a, \hat{p}^a)$ on a 6D smooth submanifold $(\eta^1, \eta^2, \eta^3) \neq (0,0,0)$; these are defined by

$$\begin{aligned}
\hat{q}^1 &:= \frac{\cosh \hat{\eta}^2 \cosh \hat{\eta}^3}{\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1}\, \hat{y}^1, \\
\hat{q}^2 &:= \frac{\cosh \hat{\eta}^3}{\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1}\, \hat{y}^2, \\
\hat{q}^3 &:= \frac{1}{\cosh \hat{\eta}^1 \cosh \hat{\eta}^2 \cosh \hat{\eta}^3 - 1}\, \hat{y}^3, \\
\hat{p}^a &:= \hat{\eta}^a,
\end{aligned} \tag{40}$$

where the ordering $(\hat{\eta}^a)^m (\hat{y}^a)^n$ has to be preserved. They lead to the canonical commutation relations

$$\big[\hat{q}^a, \hat{q}^b\big] = \big[\hat{p}^a, \hat{p}^b\big] = 0, \qquad \big[\hat{q}^a, \hat{p}^b\big] = \frac{1}{\kappa}\, \delta_{ab}. \tag{41}$$

From these expressions we find that the noncommutative space $\mathcal{W}_{\mathrm{tl},\kappa}$ can be regarded as three copies of the usual Heisenberg–Weyl algebra of quantum mechanics where the deformation parameter κ^{-1} replaces the Planck constant $\hbar$. We also recall that a first phenomenological analysis for $\mathcal{W}_{\mathrm{tl},\kappa}$, expressed in the form (38) and (39), was performed in [30].

So far we have constructed the noncommutative space $\mathcal{W}_{\mathrm{tl},\kappa}$ from the usual "time-like" κ-Poincaré deformation with classical r-matrix (21). However we remark that there exist two other possible κ-Poincaré deformations provided by "space-like" and "light-like" classical r-matrices [10, 11]. The quantization procedure described in Section 2 can similarly be applied to these remaining cases in order to construct the quantum counterpart of the 6D homogeneous space $\mathcal{W}_{\mathrm{tl}}$ (35). Therefore we shall keep exactly the expressions (36) and (37) together with the associated invariant vector fields and only change the underlying r-matrix. In what follows we summarize the final results which were recently obtained in [12].

We consider the "space-like" r-matrix given by

$$r=\frac{1}{\kappa}\left(K_3\wedge P_0+J_1\wedge P_2-J_2\wedge P_1\right), \tag{42}$$

which is also a solution of the mCYBE (7), so quasitriangular. The corresponding quantum Poincaré algebra was obtained in [31] (c.f. Type 1. (a) with $z=1/\kappa$). When computing the PHS it is found that again there are no ordering problems so that this can be quantized directly leading to the commutation relations defining $\mathcal{W}_{\mathrm{tl},\kappa}$ from the "space-like" κ-Poincaré deformation; these are

$$\begin{aligned}
[\hat{y}^1,\hat{y}^2]&=-\frac{1}{\kappa}\,\hat{y}^1\tanh\hat{\eta}^2\tanh\hat{\eta}^3\,,\\
[\hat{y}^1,\hat{y}^3]&=\frac{1}{\kappa}\,\frac{\hat{y}^1}{\cosh\hat{\eta}^3}\,,\\
[\hat{y}^2,\hat{y}^3]&=\frac{1}{\kappa}\,\frac{\hat{y}^2}{\cosh\hat{\eta}^3}\,,\\
[\hat{y}^1,\hat{\eta}^1]&=-\frac{1}{\kappa}\,\frac{\tanh\hat{\eta}^3}{\cosh\hat{\eta}^2}\,,\\
[\hat{y}^2,\hat{\eta}^2]&=-\frac{1}{\kappa}\tanh\hat{\eta}^3\,,\\
[\hat{y}^3,\hat{\eta}^3]&=-\frac{1}{\kappa}\sinh\hat{\eta}^3\,,
\end{aligned} \tag{43}$$

with the same vanishing brackets given by (39).

Finally, in the kinematical basis (16) with $\Lambda=0$ the "light-like" κ-Poincaré deformation is determined by

$$r=\frac{1}{\kappa}\left(K_3\wedge P_0+K_1\wedge P_1+K_2\wedge P_2+K_3\wedge P_3+J_1\wedge P_2-J_2\wedge P_1\right), \tag{44}$$

which is triangular with vanishing Schouten bracket. This element provides the so-called "null-plane" quantum Poincaré algebra introduced in [32, 33] (where $z=1/\kappa$) in terms of a null-plane basis [22] instead of the kinematical one. Notice that the "light-like" r-matrix (44) is just the sum of the "time-like" r-matrix (21) and the "space-like" one (42). Consequently, as expected, the resulting PHS can directly be quantized giving rise to $\mathcal{W}_{\mathrm{tl},\kappa}$ from the "light-like" κ-Poincaré deformation which turns out to be given by the sum of (38) and (43) (preserving the same vanishing brackets (39)); namely

$$\begin{aligned}
[\hat{y}^1,\hat{y}^2]&=\frac{1}{\kappa}\left(\hat{y}^2\sinh\hat{\eta}^1-\frac{\hat{y}^1\tanh\hat{\eta}^2\big(\sinh\hat{\eta}^3+1\big)}{\cosh\hat{\eta}^3}\right),\\
[\hat{y}^1,\hat{y}^3]&=\frac{1}{\kappa}\left(\hat{y}^3\sinh\hat{\eta}^1-\frac{\hat{y}^1\big(\sinh\hat{\eta}^3-1\big)}{\cosh\hat{\eta}^3}\right),\\
[\hat{y}^2,\hat{y}^3]&=\frac{1}{\kappa}\left(\hat{y}^3\cosh\hat{\eta}^1\sinh\hat{\eta}^2-\frac{\hat{y}^2(\sinh\hat{\eta}^3-1)}{\cosh\hat{\eta}^3}\right),\\
[\hat{y}^1,\hat{\eta}^1]&=\frac{1}{\kappa}\left(\frac{\cosh\hat{\eta}^1\cosh\hat{\eta}^2\cosh\hat{\eta}^3-\sinh\hat{\eta}^3-1}{\cosh\hat{\eta}^2\cosh\hat{\eta}^3}\right),\\
[\hat{y}^2,\hat{\eta}^2]&=\frac{1}{\kappa}\left(\frac{\cosh\hat{\eta}^1\cosh\hat{\eta}^2\cosh\hat{\eta}^3-\sinh\hat{\eta}^3-1}{\cosh\hat{\eta}^3}\right),\\
[\hat{y}^3,\hat{\eta}^3]&=\frac{1}{\kappa}\big(\cosh\hat{\eta}^1\cosh\hat{\eta}^2\cosh\hat{\eta}^3-\sinh\hat{\eta}^3-1\big)\,.
\end{aligned} \tag{45}$$

We remark that quantum Darboux operators $(\hat{q}^a,\hat{p}^a)$ satisfying (41) can also be defined for these latter noncommutative spaces [12].

6 Concluding remarks and open problems

In this "twofold" contribution we have, firstly, presented in Section 2 a general approach to construct noncommutative spaces from coisotropic PHS spaces determined by a coboundary Lie bialgebra structure and, secondly, we have applied it to the physically relevant (3+1)D (A)dS and Poincaré Lie groups. Besides the well-known (3+1)D κ-spacetimes shown in Section 3, we have also presented quite different (i.e. non-equivalent) (3+1)D noncommutative (A)dS and Minkowski spacetimes by requiring to preserve a quantum Lorentz subgroup invariant in Section 4. In addition, we have also considered noncommutative spaces beyond the (3+1)D noncommutative spacetimes, which are the usual models considered in quantum gravity. In this respect, we have presented the only three possible 6D noncommutative spaces of time-like geodesics provided the three types of κ-Poincaré quantum deformations in Section 5. We stress that a classification of all 6D noncommutative spaces of κ-Poincaré geodesics, covering the usual time-like worldlines, already here described, along with the space-like and light-like geodesics can be found in [12].

To conclude, we would like to comment on some open problems. Obviously, the procedure considered here can be applied to any coisotropic PHS space providing new noncommutative spaces. As far as (3+1)D (A)dS and Minkowski noncommutative spacetimes are concerned, we have presented their well-known κ-deformation together with all possible quantum spacetimes preserving a non-trivial quantum Lorentz subgroup. These results constitute the cornerstone of a large number of possibilities for a further development. Nevertheless, we remark that quantum spaces of geodesics have not been considered and studied so deeply. In fact, to the best of our knowledge, only κ-deformations for quantum Poincaré geodesics have been achieved. This fact not only suggests the consideration of other types of quantum Poincaré geodesics but, in our opinion, the relevant open problem is to construct quantum (A)dS spaces of geodescics; there are no results on this problem from a quantum group setting. In fact, for the κ-Poincaré space of time-like worldlines (from the usual κ-Poincaré algebra) its fuzzy properties have been studied in [30] and by following [30, 34] a similar analysis could be faced with the other types of κ-Poincaré geodesics. Consequently, the construction of (A)dS noncommutative spaces of geodesics (covariant under their corresponding (A)dS quantum groups) could be achieved following the same approach here presented, and thus the role of a nonvanishing cosmological constant (or curvature) in this novel noncommutative geometric setting could be further analysed. Work on all these lines is in progress.

Acknowledgements

Funding information This work has been partially supported by Agencia Estatal de Investigación (Spain) under grant PID2019-106802GB-I00/AEI/10.13039/501100011033, by the Regional Government of Castilla y León (Junta de Castilla y León, Spain) and by the Spanish Ministry of Science and Innovation MICIN and the European Union NextGenerationEU(PRTR C17.I1). The authors would like to acknowledge the contribution of the European Cooperation in Science and Technology COST Action CA18108.

References

[1] V. G. Drinfel'd, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, in *Yang-Baxter equation in integrable systems*, World Scientific, Singapore, ISBN 9789810201203 (1990), doi:10.1142/9789812798336_0009.

[2] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, UK, ISBN 9780521558846 (1994).

[3] N. Ciccoli and F. Gavarini, *A quantum duality principle for coisotropic subgroups and Poisson quotients*, Adv. Math. **199**, 104 (2006), doi:10.1016/j.aim.2005.01.009.

[4] A. Ballesteros, I. Gutierrez-Sagredo and F. Mercati, *Coisotropic Lie bialgebras and complementary dual Poisson homogeneous spaces*, J. Phys. A: Math. Theor. **54**, 315203 (2021), doi:10.1088/1751-8121/ac0b8a.

[5] I. Gutierrez-Sagredo and F. J. Herranz, *Cayley-Klein Lie bialgebras: Noncommutative spaces, Drinfel'd doubles and kinematical applications*, Symmetry **13**, 1249 (2021), doi:10.3390/sym13071249.

[6] A. Addazi et al., *Quantum gravity phenomenology at the dawn of the multi-messenger era–A review*, Prog. Part. Nucl. Phys. **125**, 103948 (2022), doi:10.1016/j.ppnp.2022.103948.

[7] P. Maslanka, *The n-dimensional κ-Poincaré algebra and group*, J. Phys. A: Math. Gen. **26**, L1251 (1993), doi:10.1088/0305-4470/26/24/001.

[8] A. Ballesteros, I. Gutierrez-Sagredo and F. J. Herranz, *The κ-(A)dS noncommutative spacetime*, Phys. Lett. B **796**, 93 (2019), doi:10.1016/j.physletb.2019.07.038.

[9] A. Ballesteros, I. Gutierrez-Sagredo and F. J. Herranz, *Noncommutative (A)dS and Minkowski spacetimes from quantum Lorentz subgroups*, Class. Quantum Gravity **39**, 015018 (2021), doi:10.1088/1361-6382/ac3c8d.

[10] A. Ballesteros, I. Gutierrez-Sagredo and F. J. Herranz, *Noncommutative spaces of worldlines*, Phys. Lett. B **792**, 175 (2019), doi:10.1016/j.physletb.2019.03.029.

[11] A. Borowiec, *κ-deformations and extended κ-Minkowski spacetimes*, Symmetry Integr. Geom.: Methods Appl. **10**, 107 (2014), doi:10.3842/SIGMA.2014.107.

[12] A. Ballesteros, I. Gutierrez-Sagredo and F. J. Herranz, *All noncommutative spaces of κ-Poincaré geodesics*, J. Phys. A: Math. Theor. **55**, 435205 (2022), doi:10.1088/1751-8121/ac99af.

[13] S. Zakrzewski, *Quantum Poincare group related to the kappa-Poincare algebra*, J. Phys. A: Math. Gen. **27**, 2075 (1994), doi:10.1088/0305-4470/27/6/030.

[14] J. Lukierski and H. Ruegg, *Quantum in any dimension*, Phys. Lett. B **329**, 189 (1994), doi:10.1016/0370-2693(94)90759-5.

[15] M. Daszkiewicz, *Canonical and Lie-algebraic twist deformations of κ-Poincaré and contractions to κ-Galilei algebras*, Int. J. Mod. Phys. A **23**, 4387 (2008), doi:10.1142/S0217751X08042262.

[16] A. Ballesteros, N. A. Gromov, F. J. Herranz, M. A. del Olmo and M. Santander, *Lie bialgebra contractions and quantum deformations of quasi-orthogonal algebras*, J. Math. Phys. **36**, 5916 (1995), doi:10.1063/1.531368.

[17] Á. Ballesteros, F. J. Herranz, F. Musso and P. Naranjo, *The κ-(A)dS quantum algebra in (3 + 1) dimensions*, Phys. Lett. B **766**, 205 (2017), doi:10.1016/j.physletb.2017.01.020.

[18] A. Ballesteros, G. Gubitosi and F. J. Herranz, *Lorentzian Snyder spacetimes and their Galilei and Carroll limits from projective geometry*, Class. Quantum Gravity **37**, 195021 (2020), doi:10.1088/1361-6382/aba668.

[19] P. Podleś and S. L. Woronowicz, *On the classification of quantum Poincaré groups*, Commun. Math. Phys. **178**, 61 (1996), doi:10.1007/BF02104908.

[20] S. Zakrzewski, *Poisson structures on the Poincaré group*, Commun. Math. Phys. **185**, 285 (1997), doi:10.1007/s002200050091.

[21] S. Zakrzewski, *Poisson structures on the Lorentz group*, Lett. Math. Phys. **32**, 11 (1994), doi:10.1007/BF00761120.

[22] H. Leutwyler and J. Stern, *Relativistic dynamics on a null plane*, Ann. Phys. **112**, 94 (1978), doi:10.1016/0003-4916(78)90082-9.

[23] J. Lukierski and M. Woronowicz, *New Lie-algebraic and quadratic deformations of Minkowski space from twisted Poincaré symmetries*, Phys. Lett. B **633**, 116 (2006), doi:10.1016/j.physletb.2005.11.052.

[24] A. Ballesteros, I. Gutierrez-Sagredo and F. J. Herranz, *The Poincaré group as a Drinfel'd double*, Class. Quantum Gravity **36**, 025003 (2018), doi:10.1088/1361-6382/aaf3c2.

[25] C. Ohn, *A $*$-product on SL(2) and the corresponding nonstandard* $(\mathfrak{sl}(2))$, Lett. Math. Phys. **25**, 85 (1992), doi:10.1007/BF00398304.

[26] O. Ogievetsky, *Hopf structures on the Borel subalgebra of $sl(2)$*, in *Proceedings of the winter school "geometry and physics"*, Circolo Matematico di Palermo, Palermo, Italy (1994), https://dml.cz/handle/10338.dmlcz/701555.

[27] A. Ballesteros and F. J. Herranz, *Universal R-matrix for non-standard quantum*, J. Phys. A: Math. Gen. **29**, L311 (1996), doi:10.1088/0305-4470/29/13/001.

[28] A. Shariati, A. Aghamohammadi and M. Khorrami, *The universal R-matrix for the Jordanian deformation of $sl(2)$, and the contracted forms of $so(4)$*, Mod. Phys. Lett. A **11**, 187 (1996), doi:10.1142/S0217732396000229.

[29] F. J. Herranz, *New quantum conformal algebras and discrete symmetries*, Phys. Lett. B **543**, 89 (2002), doi:10.1016/S0370-2693(02)02452-8.

[30] A. Ballesteros, G. Gubitosi, I. Gutierrez-Sagredo and F. Mercati, *Fuzzy worldlines with κ-Poincaré symmetries*, J. High Energy Phys. **12**, 080 (2021), doi:10.1007/JHEP12(2021)080.

[31] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander, *Four-dimensional quantum affine algebras and space-time q-symmetries*, J. Math. Phys. **35**, 4928 (1994), doi:10.1063/1.530823.

[32] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander, *A new "null-plane" quantum Poincaré algebra*, Phys. Lett. B **351**, 137 (1995), doi:10.1016/0370-2693(95)00386-Y.

[33] A. Ballesteros, F. J. Herranz and C. M. Pereña, *Null-plane quantum universal R-matrix*, Phys. Lett. B **391**, 71 (1997), doi:10.1016/S0370-2693(96)01435-9.

[34] F. Lizzi, M. Manfredonia, F. Mercati and T. Poulain, *Localization and reference frames in κ-Minkowski spacetime*, Phys. Rev. D **99**, 085003 (2019), doi:10.1103/PhysRevD.99.085003.

Computation of entanglement entropy in inhomogeneous free fermions chains by algebraic Bethe ansatz

Pierre-Antoine Bernard[1⋆], Gauvain Carcone[1], Nicolas Crampé[2] and Luc Vinet[1,3]

1 Centre de recherches mathématiques, Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7, Canada
2 Institut Denis-Poisson CNRS/UMR 7013 - Université de Tours - Université d'Orléans, Parc de Grandmont, 37200 Tours, France
3 IVADO, 6666 Rue Saint-Urbain, Montréal (Québec), H2S 3H1, Canada

⋆ bernardpierreantoine@outlook.com

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The computation of the entanglement entropy for inhomogeneous free fermions chains based on q-Racah polynomials is considered. The eigenvalues of the truncated correlation matrix are obtained from the diagonalization of the associated Heun operator via the algebraic Bethe Ansatz. In the special case of chains based on dual q-Hahn polynomials, the eigenvectors and eigenvalues are expressed in terms of symmetric polynomials evaluated on the Bethe roots.

Received 17-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.018

Contents

1 Introduction **2**

2 The model **3**
2.1 Diagonalization of the Hamiltonian 3

3 Entanglement entropy **5**

4 Algebraic Heun operator **6**
4.1 Algebraic Bethe Ansatz 6
4.2 Diagonalization of the truncated correlation matrix 9

5 TQ-relations and thermodynamic limit **10**

6 Conclusion **11**

A Appendix **12**
A.1 q-Racah polynomials 12
A.2 Functions in the algebraic Bethe Ansatz 13

References **13**

1 Introduction

The characterization of entanglement in many-body systems is motivated by its numerous applications in quantum information [1,2] and its role in describing quantum critical points [3]. This endeavour is usually carried out in bipartite situations, where the amount of entanglement between a region and its complement is determined. While many techniques have been developed to perform this task [4], analytical results for entanglement entropy in large systems remain rare.

For spin chains and free fermions models, this problem reduces to diagonalizing a matrix referred to as the truncated correlation matrix [5]. In cases where couplings are homogeneous, for example the XX spin chain, this matrix is Toeplitz or Toeplitz+Hankel and one can use the Fisher–Hartwig conjecture to compute the bipartite entanglement in the thermodynamic limit [6, 7]. For more general couplings and truncated correlation matrices, applying these techniques is not possible and different approaches are required.

Inhomogeneous fermionic chains associated to hypergeometric orthogonal polynomials of the Askey-Wilson scheme [8] are solvable and describe a wide variety of models. It was observed that their truncated correlation matrix admits a commuting tridiagonal matrix, identified as a Heun-Askey-Wilson operator [9–13]. This suggests an interesting connection with the theory of integrable systems. Indeed, these operators arise in the transfer matrices associated to solutions of the reflection equations [14]. They correspond to Hamiltonians of XXZ spin chains with specific boundary fields and have been shown to be diagonalizable via the algebraic Bethe anstaz [15] (other methods have been developed in [16, 17]). They are also examples of the so-called homogeneous case in the context of the modified algebraic Bethe Ansatz, which has been designed to deal with generic Heun operators [14, 18, 19] and diagonalize integrable models with arbitrary boundary conditions (see e.g. [18, 20–24]).

This paper applies the algebraic Bethe Ansatz framework to investigate the spectrum of truncated correlation matrices of models associated to polynomials of the Askey-Wilson scheme. In particular, the eigenvalues of the truncated correlation matrix of free fermionic chains associated to dual q-Hahn polynomials are provided in terms of solutions of a set of Bethe equations. In section 2, we recall the definition of free fermions chains associated to q-Racah polynomials and diagonalize their Hamiltonians. In section 3, we discuss the problem of computing the entanglement entropy and introduce the truncated correlation matrix. In section 4, we exhibit a commuting tridiagonal matrix referred to as the algebraic Heun operator and diagonalize it via the algebraic Bethe Ansatz. This yields a set of relations known as Bethe equations. The eigenvalues of the truncated correlation matrix are then given in terms of roots of these equations. The associated TQ-relation and the thermodynamic limit are briefly discussed in section 5.

2 The model

Let us consider the following free fermions inhomogeneous Hamiltonian with nearest-neighbour interaction J_n and magnetic field μ_n,

$$\widehat{\mathcal{H}} = \sum_{n=0}^{N-1} \left(J_n c_n^\dagger c_{n+1} + J_n c_{n+1}^\dagger c_n \right) - \sum_{n=0}^{N} \mu_n c_n^\dagger c_n \,, \tag{1}$$

where c_n and $c_n^\dagger$ are fermionic annihilation and creation operators satisfying

$$\{c_m^\dagger, c_n^\dagger\} = \{c_m, c_n\} = 0 \,, \qquad \{c_m^\dagger, c_n\} = \delta_{m,n} \,. \tag{2}$$

For convenience, we enumerate the sites of the lattice from 0 to N. This model is equivalent to an inhomogeneous XX spin chain. Indeed, the Jordan-Wigner transformation

$$c_n^\dagger = \sigma_0^z \sigma_1^z \dots \sigma_{n-1}^z \sigma_n^+ \,, \qquad c_n = \sigma_0^z \sigma_1^z \dots \sigma_{n-1}^z \sigma_n^- \,, \tag{3}$$

allows to rewrite the canonical relations of the creation and annihilation operators (2) and the Hamiltonian (1) in terms of spin-1/2 operators,

$$\widehat{\mathcal{H}} = -\frac{1}{2} \sum_{n=0}^{N-1} J_n \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y \right) - \frac{1}{2} \sum_{n=0}^{N} \mu_n \left(1 + \sigma_n^z \right) . \tag{4}$$

We are interested in the case where the coupling parameters J_n and the local magnetic field μ_n are constructed from the recurrence coefficients of the q-Racah polynomials [8]:

$$J_n = \epsilon \sqrt{A_n C_{n+1}} \,, \tag{5}$$

$$\mu_n = A_n + C_n - 1 - \gamma \delta q \,, \tag{6}$$

where $\epsilon = \pm 1$ and A_n, C_n are defined by

$$A_n = \frac{\left(\alpha q^{n+1} - 1\right)\left(\gamma q^{n+1} - 1\right)\left(\alpha\beta q^{n+1} - 1\right)\left(\beta\delta q^{n+1} - 1\right)}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})} \,, \tag{7}$$

$$C_n = \frac{(\beta q^n - 1)(\alpha q^n - \delta)(\alpha\beta q^n - \gamma)(q^{n+1} - q)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})} \,. \tag{8}$$

The choice of such inhomogeneous interactions and magnetic fields yields analytical results for the spectrum, as shown below. It also describes a large class of models thanks to the presence of various parameters. Indeed, the constants A_n, C_n and μ_n depend on the five real parameters q, α, β, γ and δ, restricted only by the requirement that

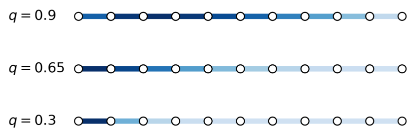
$$A_n C_{n+1} > 0 \,, \qquad J_N = 0 \,. \tag{9}$$

For instance, as shown in figure 1, we can get couplings J_n which are monotone in n or peaking at a certain value. Taking $q < 0$ also gives models with oscillating couplings, reminiscent of alternating spin chains [25].

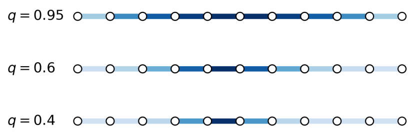
2.1 Diagonalization of the Hamiltonian

In order to diagonalize $\widehat{\mathcal{H}}$, it is convenient to rewrite this operator in matrix form as

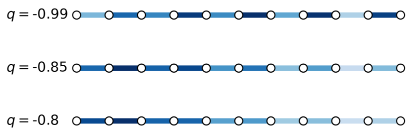
$$\widehat{\mathcal{H}} = \left(c_0^\dagger, \dots, c_N^\dagger\right) A \begin{pmatrix} c_0 \\ \vdots \\ c_N \end{pmatrix} , \tag{10}$$



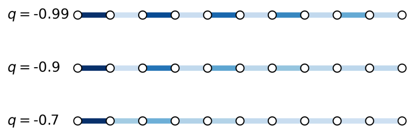
(a) $\alpha = q^{-N-1}$, $\beta = q^{2N}$, $\gamma = q^{-2N}$, $\delta = (q^{-2N} + q^{-N})/2$



(b) $\alpha = q^{-N-1}$, $\beta = -q$, $\gamma = q^2/2$, $\delta = q^2/2$



(c) $\alpha = q^{-N-1}$, $\beta = q^{2N}$, $\gamma = q^{-2N}$, $\delta = q^{-N-1}$



(d) $\alpha = q^{-N-1}$, $\beta = q^{8N}$, $\gamma = q^{-2N}$, $\delta = q^{-8N}$

Figure 1: Inhomogeneous free fermions chains of length $N = 10$, based on q-Racah polynomials, for different parameters $(q, \alpha, \beta, \gamma, \delta)$. The vertices and edges represent respectively the sites and the couplings. The color of the edges indicates the magnitude of J_n, *i.e.* the strength of these couplings. Darker color is associated to stronger couplings.

where $\boldsymbol{A}$ is the hermitian $(N+1) \times (N+1)$ tridiagonal matrix given by

$$\boldsymbol{A} = \sum_{n=0}^{N} \Big(J_n |n\rangle\langle n+1| - \mu_n |n\rangle\langle n| + J_n |n+1\rangle\langle n| \Big), \tag{11}$$

with the convention $J_N = J_{-1} = 0$. The vectors $\{|0\rangle, |1\rangle, \dots, |N\rangle\}$ are naturally associated to sites in the chain and give the canonical orthonormal basis of $\mathbb{C}^{N+1}$. They will be referred to as elements of the position basis. The spectral problem for $\boldsymbol{A}$ reads

$$\boldsymbol{A}|\omega_k\rangle = \omega_k |\omega_k\rangle\,, \tag{12}$$

where

$$|\omega_k\rangle = \sum_{n=0}^{N} \phi_n(\omega_k)|n\rangle\,. \tag{13}$$

Knowing that the entries of $\boldsymbol{A}$ are the recurrence coefficients of the q-Racah polynomials, one deduces (see eq. (A.4)) that its eigenvalues ω_k are

$$\omega_k = q^{-k} + \gamma\delta q^{k+1}\,. \tag{14}$$

The wavefunctions $\phi_n(\omega_k) = \langle \omega_k | n\rangle$ are given in terms of q-Racah polynomials $R_n(\omega_k)$ [8]:

$$\phi_n(\omega_k) = \epsilon^n \sqrt{W_k} \prod_{j=1}^{n} \sqrt{\frac{A_{j-1}}{C_j}} R_n(\omega_k)\,. \tag{15}$$

The definition of $R_n(\omega_k)$ and the normalisation factors W_k are given in appendix A. The latter are chosen such that the wavefunctions $\phi_n(\omega_k)$ are orthonormal *i.e.*

$$\sum_{k=0}^{N} \phi_n(\omega_k)\phi_m(\omega_k) = \delta_{n,m}\,, \quad \text{and} \quad \sum_{n=0}^{N} \phi_n(\omega_k)\phi_n(\omega_{k'}) = \delta_{k,k'}\,. \tag{16}$$

From these wavefunctions, we can define new pairs of fermionic creation and annihilation operators in terms of which the Hamiltonian is diagonal:

$$\widehat{\mathcal{H}} = \sum_{k=0}^{N} \omega_k \tilde{c}_k^\dagger \tilde{c}_k \,, \qquad k \in \{0, 1, \ldots, N\} \,, \tag{17}$$

where

$$\tilde{c}_k = \sum_{n=0}^{N} \phi_n(\omega_k) c_n \,, \qquad \tilde{c}_k^\dagger = \sum_{n=0}^{N} \phi_n(\omega_k) c_n^\dagger \,. \tag{18}$$

Note that the operators $\tilde{c}_k^\dagger$ and c_k are associated to the single particle excitations of the system, with energies given by the spectrum of the matrix $\boldsymbol{A}$. One may further observe that these energies are invariant under arbitrary transformations of α, β and under

$$\delta \to \delta\kappa \,, \qquad \gamma \to \gamma\kappa^{-1} \,, \quad \kappa \in \mathbb{R} \,. \tag{19}$$

This is not true of the coupling parameters J_n and local magnetic field μ_n which depend non-trivially on $(q, \alpha, \beta, \gamma, \delta)$. Important properties characterizing these systems, like the entanglement entropy in the ground state, should thus depend on these parameters.

3 Entanglement entropy

Entanglement in a multipartite system $A \cup B$ is measured by the entanglement entropy S_A, defined as

$$S_A = -\mathrm{tr}_A (\rho_A \ln \rho_A) \,, \tag{20}$$

where A is a subsystem of $A \cup B$ with a reduced density matrix ρ_A given by the trace over the degrees of freedom in B,

$$\rho_A = \mathrm{tr}_B |\Omega\rangle\!\rangle \langle\!\langle \Omega| \,. \tag{21}$$

In the following, we take A to be the first $L+1$ sites of the inhomogeneous free fermionic chain introduced in the previous section. The states considered are obtained by filling up the first $K+1$ single particle states, taken as the Fermi sea,

$$|\Omega\rangle\!\rangle = \prod_{k=0}^{K} \tilde{c}_k^\dagger |0\rangle\!\rangle \,, \tag{22}$$

where $|0\rangle\!\rangle$ is the vacuum state annihilated by all operators $\tilde{c}_k$. For ω_k monotone in k, $|\Omega\rangle\!\rangle$ describes the ground state of Hamiltonians obtained as affine transformations of (10). In other words, it gives the state for which the single particle excitations with negative energy are filled.

As observed in [5], computing the entanglement entropy S_A of free fermions can be done by diagonalizing the truncated correlation matrix. Indeed, it is known that [26]

$$S_A = -\sum_{\ell} c_\ell \ln c_\ell + (1 - c_\ell) \ln(1 - c_\ell) \,, \tag{23}$$

where the coefficients c_ℓ are the eigenvalues of the $(L+1) \times (L+1)$ matrix C with entries C_{nm} given by the 2-point correlation functions,

$$C_{nm} = \langle\!\langle \Omega | c_n^\dagger c_m | \Omega \rangle\!\rangle = \sum_{k=0}^{K} \phi_n(\omega_k) \phi_m(\omega_k) \,, \qquad n, m \in \{0, 1, \ldots L\} \,. \tag{24}$$

This is a submatrix of the complete correlation matrix of the ground state $\widehat{C}$, i.e.

$$C = \pi_A \widehat{C} \pi_A , \qquad \widehat{C} = \sum_{k=0}^{K} |\omega_k\rangle \langle \omega_k| , \tag{25}$$

where π_A is the projector onto the vector space associated to sites of subsystem A,

$$\pi_A = \sum_{n=0}^{L} |n\rangle \langle n| . \tag{26}$$

The computation of the entanglement entropy is thus reduced to determining the eigenvalues c_ℓ. With the help of the algebraic Heun operators, we shall see that the spectral problem for the truncated correlation matrix can be treated in the algebraic Bethe Ansatz framework.

4 Algebraic Heun operator

In this section, we introduce a tridiagonal matrix that commutes with the truncated correlation matrix. To do so, we define an operator $\boldsymbol{A}^*$ which is diagonal in the position basis

$$\boldsymbol{A}^* |n\rangle = \lambda_n |n\rangle , \qquad \lambda_n = q^{-n} + \alpha\beta q^{n+1} . \tag{27}$$

Using the difference relation of the q-Racah polynomials and the expression (15), one finds the tridiagonal action of $\boldsymbol{A}^*$ on the eigenbasis of $\boldsymbol{A}$,

$$\boldsymbol{A}^* |\omega_k\rangle = \bar{J}_k |\omega_{k+1}\rangle - \bar{\mu}_k |\omega_k\rangle + \bar{J}_{k-1} |\omega_{k-1}\rangle . \tag{28}$$

The coefficients $\bar{J}_k$ and $\bar{\mu}_k$ are given in appendix A. The Heun operator T is then defined as

$$T = \{\boldsymbol{A}, \boldsymbol{A}^*\} - (\lambda_L + \lambda_{L+1})\boldsymbol{A} - (\omega_K + \omega_{K+1})\boldsymbol{A}^* , \tag{29}$$

and has the property of commuting with both the projector π_A and the complete correlation matrix $\widehat{C}$,

$$[T, \pi_A] = [T, \widehat{C}] = 0 . \tag{30}$$

This is shown easily by considering the commutators $[T, \pi_A]$ in the position basis and $[T, \widehat{C}]$ in the energy basis. Given relation (25), T also commutes with the truncated correlation C and thus share with it a common set of eigenvectors. This is a crucial observation, in particular because the Heun operator T can be identified in the transfer matrix of integrable models and can hence be diagonalized via the algebraic Bethe Ansatz [14, 27].

4.1 Algebraic Bethe Ansatz

The matrices $\boldsymbol{A}$ and $\boldsymbol{A}^*$ give a representation of the Askey-Wilson algebra [28, 29]:

$$\boldsymbol{A}\boldsymbol{A}\boldsymbol{A}^* - \left(q + \frac{1}{q}\right)\boldsymbol{A}\boldsymbol{A}^*\boldsymbol{A} + \boldsymbol{A}^*\boldsymbol{A}\boldsymbol{A} = \xi\boldsymbol{A} + \chi\boldsymbol{A}^* + \eta\mathcal{I} , \tag{31}$$

$$\boldsymbol{A}^*\boldsymbol{A}^*\boldsymbol{A} - \left(q + \frac{1}{q}\right)\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{A}^* + \boldsymbol{A}\boldsymbol{A}^*\boldsymbol{A}^* = \chi^*\boldsymbol{A} + \xi\boldsymbol{A}^* + \eta^*\mathcal{I} , \tag{32}$$

where $\mathcal{I}$ is the $N+1 \times N+1$ identity matrix and the constants ξ, χ, χ^*, η and η^* can be expressed in terms of the parameters in the Hamiltonian:

$$\chi = -\frac{\gamma\delta\left(q^2-1\right)^2}{q} , \qquad \chi^* = -\frac{\alpha\beta\left(q^2-1\right)^2}{q} , \tag{33}$$

$$\xi = -(q-1)^2(\alpha(\beta\delta+\beta+\gamma+1)+\gamma(\beta\delta+\delta+1)+\beta\delta)\,, \tag{34}$$

$$\eta = (q-1)^2(q+1)(\alpha\gamma(\beta\delta+\delta+1)+\alpha\beta\delta+\gamma\delta(\beta\delta+\beta+\gamma+1))\,, \tag{35}$$

$$\eta^* = (q-1)^2(q+1)\left(\alpha^2\beta+\alpha\left(\beta^2\delta+\beta(\gamma+1)(\delta+1)+\gamma\right)+\beta\gamma\delta\right). \tag{36}$$

The so-called dynamical operators can be defined in terms of the generators of this algebra:

$$\mathcal{A}(u,m) = \frac{q^{-2L}}{(\alpha\beta q^{2m+1}-1)}\left(\frac{q^{m+1}\{\boldsymbol{A},\boldsymbol{A}^*\}}{(q+1)} - \left(\alpha\beta q^{2m+2}+1\right)\boldsymbol{A} - \frac{\left(q^{2m+2}+\gamma\delta u^4\right)}{u^2}\boldsymbol{A}^*\right) + f_1(u,m)\mathcal{I}\,, \tag{37}$$

and

$$\begin{aligned}\mathcal{B}(u,m) = &\frac{\alpha\beta q^{m+2}+q^{-m-1}}{2(q+1)}\{\boldsymbol{A},\boldsymbol{A}^*\} - \frac{q^{-m-1}-\alpha\beta q^{m+2}}{2(1-q)}[\boldsymbol{A},\boldsymbol{A}^*]\\ &-\alpha\beta(q+1)\boldsymbol{A} - \frac{\left(q+\alpha\beta\gamma\delta u^4\right)}{u^2}\boldsymbol{A}^* + f_2(u,m)\mathcal{I}\,.\end{aligned} \tag{38}$$

The functions $f_1(u,m)$ and $f_2(u,m)$ are given in the appendix. These operators verify

$$\mathcal{B}(u,m+1)\mathcal{B}(v,m) = \mathcal{B}(v,m+1)\mathcal{B}(u,m)\,, \tag{39}$$

$$\begin{aligned}\mathcal{A}(u,m+1)\mathcal{B}(v,m) = &f(u,v)\mathcal{B}(v,m)\mathcal{A}(u,m) + g(u,v,m)\mathcal{B}(u,m)\mathcal{A}(v,m)\\ &+ w(u,v,m)\mathcal{B}(u,m)\mathcal{A}(\tau v^{-1},m)\,,\end{aligned} \tag{40}$$

where $\tau = \sqrt{\frac{q}{\alpha\beta\gamma\delta}}$. The functions $f(u,v)$, $g(u,v,m)$ and $w(u,v,m)$ are given in appendix A. Relations (39)-(40) were verified using directly the Askey-Wilson relations (31)-(32). This is similar to the method used in [18, 19] and distinct from the approach based on R and K matrices [14,15]. The Heun operator (29) can be expressed in terms of $\mathcal{A}(u,L)$ and $\mathcal{A}(\tau u^{-1},L)$ as

$$T = r(u)\mathcal{A}(u,L) + r(\tau u^{-1})\mathcal{A}(\tau u^{-1},L) - \left(r(u)f_1(u,L) + r\left(\tau u^{-1}\right)f_1\left(\tau u^{-1},L\right)\right)\mathcal{I}\,, \tag{41}$$

where

$$r(u) = \frac{q^L(q+1)}{\alpha\beta\gamma\delta u^4 - q}\left(\alpha^2\beta^2\gamma\delta u^4 q^{2L} + 1 - \left(\gamma\delta q^{K+1}+q^{-K-1}\right)\alpha\beta u^2 q^L\right). \tag{42}$$

Next, let us consider the vectors $|\bar{u}\rangle$ defined as

$$|\bar{u}\rangle = \boldsymbol{B}(\bar{u},L)\,|0\rangle\,, \quad \bar{u} = \{u_1,u_2,\ldots,u_L\}\,, \tag{43}$$

where

$$\boldsymbol{B}(\bar{u},L) = \mathcal{B}(u_1,L-1)\mathcal{B}(u_2,L-2)\ldots\mathcal{B}(u_L,0)\,. \tag{44}$$

Note that relation (39) implies that $\boldsymbol{B}(\bar{u},L)$ does not depend on the ordering of the variables u_i. Since the vectors $|\bar{u}\rangle$ are obtained by applying L times a tridiagonal matrix on the vector $|0\rangle$, they are contained in the vector space spanned by $\{|0\rangle, |1\rangle, \ldots, |L\rangle\}$. As such, they are eigenvectors of π_A with eigenvalue 1,

$$\pi_A\,|\bar{u}\rangle = \sum_{i=0}^{L}|i\rangle\,\langle i|\bar{u}\rangle = |\bar{u}\rangle\,. \tag{45}$$

The aim is to show that for specific parameters $\bar{u}$, these vectors are also eigenvectors of T. This requires two results. The first is the following relation between the dynamical operator

$\mathcal{A}(u,m)$ and product of dynamical operators $\boldsymbol{B}(\bar{u},L)$:

$$\begin{aligned}\mathcal{A}(u,m)\boldsymbol{B}(\bar{u},L) = &\prod_{i=1}^{L} f(u,u_i)\boldsymbol{B}(\bar{u},L)\mathcal{A}(u,m-L)\\ &+\sum_{i=1}^{L} g(u,u_i,m-1)\prod_{\substack{j=1\\ i\neq j}}^{L} f(u_i,u_j)\boldsymbol{B}(\bar{u}_{\neq i},u,L)\mathcal{A}(u_i,m-L)\\ &+\sum_{i=1}^{L} w(u,u_i,m-1)\prod_{\substack{j=1\\ i\neq j}}^{L} f(\tau u_i^{-1},u_j)\boldsymbol{B}(\bar{u}_{\neq i},u,L)\mathcal{A}(\tau u_i^{-1},m-L),\end{aligned} \tag{46}$$

where

$$\boldsymbol{B}(\bar{u}_{\neq i},u,L) = \mathcal{B}(u_1,L-1)\mathcal{B}(u_2,L-2)\ldots\mathcal{B}(u,m-i)\ldots\mathcal{B}(u_L,0). \tag{47}$$

This relation is obtained by computing the terms $i=1$ and by using the symmetry in the indices u_i induced by relation (39). The second required result is the action of $\mathcal{A}(u,0)$ on the vector $|0\rangle$. From the definition of $\mathcal{A}(u,0)$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^*$, and the action (11)-(27) of these operators on the position basis, it follows that

$$\mathcal{A}(u,0)\,|0\rangle = a(u)\,|0\rangle\ , \tag{48}$$

where

$$a(u) = \left(\frac{q^{-2L}}{(1-\alpha\beta q)}\left(\frac{2\mu_0\lambda_0 q}{(q+1)} - \mu_0(\alpha\beta q^2+1) + \frac{q^2\lambda_0}{u^2} + \gamma\delta u^2\lambda_0\right) + f_1(u,0)\right). \tag{49}$$

In particular, we note that this action is diagonal. This feature shows that the *modified* algebraic Bethe Ansatz is not necessary and that the model we deal with corresponds to the particular case developed in [15]. From this observation and relation (46), it follows that

$$T\,|\bar{u}\rangle = \Lambda(\bar{u})\,|\bar{u}\rangle + \sum_{i=1}^{L} E_i(u,\bar{u})\boldsymbol{B}(\bar{u}_{\neq i},u,L)\,|0\rangle\ , \tag{50}$$

where

$$\begin{aligned}\Lambda(\bar{u}) = r(u)a(u)\prod_{i=1}^{L} f(u,u_i) + r(\tau u^{-1})a\left(\tau u^{-1}\right)\prod_{i=1}^{L} f(\tau u^{-1},u_i)\\ -\left(r(u)f_1(u,L) + r\left(\tau u^{-1}\right)f_1\left(\tau u^{-1},L\right)\right),\end{aligned} \tag{51}$$

and

$$\begin{aligned}E_i(u,\bar{u}) = &\Big(r(u)g(u,u_i,L-1) + r(\tau u^{-1})g(\tau u^{-1},u_i,L-1)\Big)a(u_i)\prod_{\substack{j=1\\ i\neq j}}^{L} f(u_i,u_j)\\ &+\Big(r(\tau u^{-1})w(\tau u^{-1},u_i,L-1) + r(u)w(u,u_i,L-1)\Big)a(\tau u_i^{-1})\prod_{\substack{j=1\\ i\neq j}}^{L} f\left(\tau u_i^{-1},u_j\right).\end{aligned} \tag{52}$$

Thus, for a set of parameters $\bar{u}$ verifying $E_i(u,\bar{u}) = 0$, the vector $|\bar{u}\rangle = \boldsymbol{B}(\bar{u},L)\,|0\rangle$ is an eigenvector of T with eigenvalues $\Lambda(\bar{u})$, *i.e.*

$$T\,|\bar{u}\rangle = \Lambda(\bar{u})\,|\bar{u}\rangle\ . \tag{53}$$

Since the Heun operator T does not depend on the parameter u, the same is true of its eigenvalues $\Lambda(\bar{u})$. In particular, we find by evaluating (51) at $u=0$ that the eigenvalues can be expressed as

$$\Lambda(\bar{u})=-(\omega_K+\omega_{K+1})\lambda_L-\frac{\mu_0(q-1)}{q}\left(\alpha\beta q^2+1+\frac{4\alpha\beta q^2}{\alpha\beta q-1}\right)-\frac{(q^2-1)^2}{q(q+1)}\left(\sum_{i=1}^{L}U_i\right), \tag{54}$$

where $U_i=\frac{q}{u_i^2}+\alpha\beta\gamma\delta u_i^2$. Factorizing terms in the variable u in the conditions $E_i(u,\bar{u})=0$, these reduce to the following conditions on $\bar{u}$, referred to as the *Bethe equations*,

$$\prod_{\substack{j=1\\ i\neq j}}^{L}\frac{q\left(qu_i^2-u_j^2\right)\left(\alpha\beta\gamma\delta u_i^2u_j^2-1\right)}{\left(qu_j^2-u_i^2\right)\left(q^2-\alpha\beta\gamma\delta u_i^2u_j^2\right)}=\frac{\left(q^{K+1}-\alpha\beta u_i^2q^L\right)\left(q^2-\alpha\beta\gamma\delta u_i^4\right)\left(\alpha\beta\gamma\delta u_i^2q^{K+L+1}-1\right)}{\alpha\beta q\left(q^{K+L+2}-u_i^2\right)\left(\alpha\beta\gamma\delta u_i^4-1\right)\left(q^L-\gamma\delta u_i^2q^K\right)}\frac{a(u_i)}{a(\tau u_i^{-1})}. \tag{55}$$

To keep the notation simple, $\bar{u}=\{u_1,u_2,\dots,u_L\}$ will refer from now on to *Bethe roots*, i.e. to solutions of the set of equations (55).

4.2 Diagonalization of the truncated correlation matrix

Since the Heun operator T commutes with the truncated correlation matrix and is non-degenerate, its eigenvectors $|\bar{u}\rangle$ also diagonalize the matrix C,

$$C\,|\bar{u}\rangle=c(\bar{u})\,|\bar{u}\rangle\,,\qquad c(\bar{u})\in\mathbb{R}\,. \tag{56}$$

To obtain an explicit expression for the eigenvalues $c(\bar{u})$ in terms of the parameters $\bar{u}$, we observe that the action of $\mathcal{B}(u,m)$ in the position basis is tridiagonal and given by

$$q^{m+1}\mathcal{B}(u,m)\,|n\rangle=V_{n,m}\,|n+1\rangle+(X_{n,m}+Y_{n,m}U)\,|n\rangle+Z_{n,m}\,|n-1\rangle\,, \tag{57}$$

with $U=\frac{q}{u^2}+\alpha\beta\gamma\delta u^2$. The coefficients $V_{n,m}$, $X_{n,m}$, $Y_{n,m}$ and $Z_{n,m}$ can be computed directly from the definition (38) and the action of $\boldsymbol{A}$ and $\boldsymbol{A}^*$ in the position basis (see appendix A). In the case where $\beta=0$, i.e. the dual q-Hahn special case of the q-Racah polynomials [8], these coefficients simplify greatly,

$$V_{n,m}=\frac{J_n}{q^{n+1}}\,,\quad X_{n,m}=\alpha\gamma q\left(q^{m+1}-q^n\right)\,,\quad Y_{n,m}=1-q^{m+1-n},\quad Z_{n,m}=0\,. \tag{58}$$

In particular, $\mathcal{B}(u,m)$ becomes a raising operator in the sense that $\langle n-1|\,\mathcal{B}(u,m)\,|n\rangle=0$. This allows to compute the wavefunction $\langle n|\bar{u}\rangle$ of Bethe vectors:

$$\langle n|\bar{u}\rangle=q^{-L(L-1)/2}\left(\prod_{\ell=1}^{L-n}1-q^{\ell}\right)\left(\prod_{i=0}^{n-1}\frac{J_i}{q^{i+1}}\right)\sum_{r=0}^{L-n}\left(\frac{\alpha\gamma q(1-q^{n+1})}{q-1}\right)^{L-n-r}S_r(\bar{U})\,, \tag{59}$$

where $\bar{U}=\{U_1,U_2,\dots,U_L\}$ with $U_i=qu_i^{-2}$. The terms $S_r(\bar{U})$ are symmetric polynomials of degree r in the variables U_i defined by

$$S_r(\bar{U})=\sum_{i_1<i_2<\cdots<i_r}U_{i_1}U_{i_2}\dots U_{i_r}\,,\qquad S_0(\bar{U})=1\,. \tag{60}$$

Then, one can use the representation of the truncated correlation matrix in the position basis (24) to obtain a formula for its eigenvalues in terms of Bethe roots. For any $n\in\{0,1,\dots L\}$, we find

$$c(\bar{u})=\frac{\langle n|\,C\,|\bar{u}\rangle}{\langle n|\bar{u}\rangle}=\frac{q^{-L(L-1)/2}}{\langle n|\bar{u}\rangle}\sum_{r=0}^{L}b_{r,n}S_r(\bar{U})\,, \tag{61}$$

where

$$b_{r,n} = \sum_{k=0}^{K} \sum_{n'=0}^{L-r} \phi_n(\omega_k)\phi_{n'}(\omega_k) \left(\prod_{\ell=1}^{L-n'} 1-q^{\ell} \right) \left(\prod_{i=0}^{n'-1} \frac{J_i}{q^{i+1}} \right) \left(\frac{\alpha\gamma q(1-q^{n'+1})}{q-1} \right)^{L-n'-r} . \tag{62}$$

This is valid for parameters $\bar{u}$ which are solutions of the Bethe equations (55). For $\beta = 0$, these equations reduce to

$$\prod_{\substack{j=1 \\ j\neq i}}^{L} \frac{\left(u_i^2 - \frac{u_j^2}{q}\right)}{\left(u_i^2 - qu_j^2\right)} = \frac{q^K \left(q-\alpha u_i^2\right)\left(q-\gamma u_i^2\right)\left(q-\gamma\delta u_i^2\right)}{\left(q^{K+L+2}-u_i^2\right)\left(\alpha\gamma u_i^2-1\right)\left(\gamma\delta u_i^2 q^K - q^L\right)} . \tag{63}$$

5 TQ-relations and thermodynamic limit

Equations (54) and (61) show that the spectra of the Heun operator and of the truncated correlation matrix can be obtained by solving Bethe equations. An alternative approach is given by interpreting the expression (51) for the eigenvalues of T as a q-difference equation. In the case $\beta = 0$, (51) can indeed be rewritten as

$$\begin{aligned} U^2 Q(U)\Lambda(\bar{u}) = &\frac{(q+1)(U-\alpha)(U-\gamma)(U-\gamma\delta)}{q^L} Q(qU) - p(U)Q(U) \\ &+ q^L(q+1)(U-q^{-K-L-1})(U-\alpha\gamma q)(U-\gamma\delta q^{K-L+1})Q(U/q) , \end{aligned} \tag{64}$$

where $p(U)$ is the following polynomial in the variable U,

$$\begin{aligned} p(U) = &-\frac{\alpha\gamma^2\delta(q+1)^2}{q^L} + \frac{\gamma(q+1)U}{q^{K+L}} \left(\alpha\gamma\delta q^{2K+L+2} + q^K(\alpha\delta+\alpha+\gamma\delta+\delta) + \alpha q^L\right) \\ &- 2q(\alpha\gamma+\alpha+\gamma\delta+\gamma)U^2 + 2(q+1)U^3 , \end{aligned} \tag{65}$$

and $Q(U)$ is a polynomial of degree L, the zeros U_i of which are expressed in terms of entries of a Bethe root $\bar{u} = \{u_1, u_2, \dots u_L\}$:

$$Q(U) = \prod_{i=1}^{L} (U - U_i) = \sum_{i=1}^{L} (-1)^{L-i} S_{L-i}(\bar{U}) U^i . \tag{66}$$

Thus, one can use the zeros of polynomial solutions of equation (64) to identify Bethe roots. This equation is referred to as the TQ-relation in the literature.

Let us now further fix[1] $\delta = 0$, $\gamma \in [0,1]$ and $q < 1$. Inserting the r.h.s of (66) in equation (64) yields a three term recurrence relation for the symmetric polynomials $S_n(\bar{U})$:

$$0 = \sigma_{n+1} S_{n+1} + (\rho_n + \Lambda(\bar{u})) S_n + \epsilon_{n-1} S_{n-1} , \tag{67}$$

where

$$\sigma_n = (q+1)q^{-n} + (q+1)q^n - 2(q+1) , \tag{68}$$

$$\rho_n = (q+1)q^{n-L-K} \left(\alpha\gamma q^{K+L+1} + \frac{1}{q} \right) + (q+1)(\alpha+\gamma)q^{-n} - 2q(\alpha\gamma+\alpha+\gamma) , \tag{69}$$

$$\epsilon_n = -\alpha\gamma(q+1)\left(q^{-K} + q^{-L}\right) + \frac{\alpha\gamma(q+1)}{q^{K+L}} q^n + \alpha\gamma(q+1)q^{-n} . \tag{70}$$

[1] The choice of parameters $\delta = 0$, $\beta = 0$ corresponds to the affine q-Krawtchouk limit of the q-Racah polynomials [8].

Table 1: Eigenvalues of the Heun operator ($N = 49$, $L = 9$, $K = 24$, $q = 0.8$, $\alpha = q^{-N-1}$, $\beta = 0$, $\gamma = 0.5$, $\delta = 0$) obtained by three methods. The first column are zeros of $S_{-1}(\bar{U})$ seen as a polynomial of degree $L+1$ in $\Lambda(\bar{u})$. The polynomial was obtained by solving the three term recurrence (67). The second column corresponds to the approximation (72) of the spectrum found in the thermodynamic limit. The third is the result of diagonalizing T using *scipy*'s linear algebra package [30].

Solutions of $S_{-1}(\bar{U}) = 0$	$-\rho_n$ for $n \in \{0, 1, \dots L\}$	Numerical diagonalization of T
-778916	-778741	-778916
-592816	-592623	-592816
-444746	-444544	-444746
-327294	-327099	-327294
-234579	-234418	-234579
-161955	-161865	-161955
-105783	-105813	-105783
-63253.2	-63460.2	-63253.2
-32283.3	-32687.9	-32283.6
-11583.9	-11957.8	-11583.9

In the thermodynamic limit $N \to \infty$, the parameter $\alpha = q^{-N-1}$, with $0 < q < 1$, goes to infinity and (67) becomes effectively a two term recurrence with solution

$$S_n = S_L \prod_{i=n}^{L-1} \frac{\rho_{i+1} + \Lambda(\bar{u})}{\epsilon_i} + O\left(\alpha^{-1}\right) . \tag{71}$$

The condition that $S_{-1}(\bar{U}) = 0$ then requires the eigenvalues $\Lambda(\bar{u})$ of T to take certain values

$$\Lambda(\bar{u}) \in \left\{-\rho_n + O(\alpha^0) \mid n \in \{0, 1 \dots, L\}\right\} . \tag{72}$$

The spectrum of the Heun operator given by this approximation is compared to spectra found using other methods in Table 1. One notes that the values match up to two digits at $N = 49$. This suggests that exact asymptotic results may be obtainable in the thermodynamic limit.

6 Conclusion

Computing bipartite entanglement for free fermionic chains amounts to determining the spectrum of a truncated correlation matrix. For systems associated to q-Racah polynomials, it has been shown how this matrix can be diagonalized via the algebraic Bethe Ansatz. In particular, its eigenvalues and eigenvectors have been given in terms of solutions of Bethe equations. The associated Bethe roots were also found to be related to zeros of polynomial solutions of a q-difference equation, referred to as the TQ-relation. This led to an approximate expression for the eigenvalues of the commuting tridiagonal matrix in the case $\delta = 0$ and $N \to \infty$.

While these results do not provide an explicit formula for the bipartite entanglement, it establishes a clear connection between a central problem in quantum many-body physics and a set of tools coming from the study of integrable models. Future research should thus be directed toward investigating, notably in their thermodynamic limit, the solutions of the Bethe equations and TQ-relation that were found. Derivation of asymptotic expressions for these would provide the groundwork necessary to analyse the interplay between coupling inhomogeneities in free fermions chains and the presence of entanglement in the ground state.

Acknowledgements

We thank Pascal Baseilhac and Rodrigo A. Pimenta for stimulating discussions.

Funding information PAB holds an Alexander-Graham-Bell scholarship from the Natural Sciences and Engineering Research Council of Canada (NSERC). NC is supported by the international research project AAPT of the CNRS and the ANR Project AHA ANR- 18-CE40-0001. The research of LV is founded in part by a Discovery Grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

A Appendix

A.1 q-Racah polynomials

The q-Racah polynomials are defined by [8]

$$R_n(\omega_x) = {}_4\phi_3\left(\begin{matrix} q^{-n} & \alpha\beta q^{n+1} & q^{-x} & \gamma\delta q^{x+1} \\ & \alpha q & \beta\delta q & \gamma q \end{matrix} \middle| q; q \right), \tag{A.1}$$

with

$$\omega_x = q^{-x} + \gamma\delta q^{x+1}. \tag{A.2}$$

The parameters are restricted by the truncation condition $R_{N+1}(x) = 0$. For instance, one can use α and fix

$$\alpha = q^{-N-1}. \tag{A.3}$$

These polynomials also satisfy the following recurrence relation

$$(\omega_x - 1 - \gamma\delta q)R_n(\omega_x) = A_n R_{n+1}(\omega_x) - (A_n + C_n)R_n(\omega_x) + C_n R_{n-1}(\omega_x), \tag{A.4}$$

where

$$A_n = \frac{\left(\alpha q^{n+1} - 1\right)\left(\gamma q^{n+1} - 1\right)\left(\alpha\beta q^{n+1} - 1\right)\left(\beta\delta q^{n+1} - 1\right)}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})}, \tag{A.5}$$

$$C_n = \frac{(\beta q^n - 1)(\alpha q^n - \delta)(\alpha\beta q^n - \gamma)(q^{n+1} - q)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})}. \tag{A.6}$$

The normalisation weight is

$$W_k = \frac{(\beta^{-1}\gamma q, \delta q; q)_N (\gamma\delta q, \alpha q, \beta\delta q, \gamma q; q)_k (1 - \gamma\delta q^{2k+1})}{(\gamma\delta q^2, \beta^{-1}; q)_N (q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_k (\alpha\beta q)^k (1 - \gamma\delta q)}. \tag{A.7}$$

These polynomials also have a difference equation of the form (28), with coefficients given by

$$\begin{aligned} \bar{J}_k = & \sqrt{\frac{(1 - \alpha q^{k+1})(1 - \beta\delta q^{k+1})(1 - \gamma q^{k+1})(1 - \gamma\delta q^{k+1})}{(1 - \gamma\delta q^{2k+1})(1 - \gamma\delta q^{2k+2})}} \\ & \times \sqrt{\frac{(1 - q^{k+1})(1 - \alpha^{-1}\gamma\delta q^{k+1})(1 - \beta^{-1}\gamma q^{k+1})(1 - \delta q^{k+1})(\alpha\beta q)}{(1 - \gamma\delta q^{2k+2})(1 - \gamma\delta q^{2k+3})}}, \end{aligned} \tag{A.8}$$

$$\bar{\mu}_k = \frac{(1-\alpha q^{k+1})(1-\beta\delta q^{k+1})(1-\gamma q^{k+1})(1-\gamma\delta q^{k+1})}{(1-\gamma\delta q^{2k+1})(1-\gamma\delta q^{2k+2})} + \frac{q(1-q^k)(1-\delta q^k)(\beta-\gamma q^k)(\alpha-\gamma\delta q^k)}{(1-\gamma\delta q^{2k})(1-\gamma\delta q^{2k+1})} - 1 - \alpha\beta q\,. \tag{A.9}$$

A.2 Functions in the algebraic Bethe Ansatz

The functions in the definition of the dynamical operators are:

$$f_1(u,m) = \frac{2q^{m+1}\left(q+\alpha\beta\gamma\delta u^4\right)}{u^2\left(\alpha\beta q^{2L+2m+1}-q^{2L}\right)} - \frac{u^2\eta(q+1)q^{-2L+2}}{(q^2-1)^2\left(q^2-\alpha\beta\gamma\delta u^4\right)} + \frac{(q+1)}{(q^2-1)^2\left(q^{2L}-\alpha\beta q^{2L+2m+1}\right)}\left(\frac{\eta^*(\gamma\delta u^4 q^{-2L+1}-q^{2m+4})}{(q^2-\alpha\beta\gamma\delta u^4)} - 2\xi q^{m+2}\right), \tag{A.10}$$

and

$$f_2(u,m) = \frac{\left(\alpha\beta q^{2L+2m+3}+q^{2L}\right)\left(q+\alpha\beta\gamma\delta u^4\right)}{u^2 q^{m+2L+1}} + \frac{\left(\eta^* q^{2L+m+1}+\alpha\beta\xi q^{2L+2m+3}+\xi q^{2L}\right)}{q^{m+2L}(q-1)^2(q+1)}\,. \tag{A.11}$$

The functions in the relation between the dynamical operators are:

$$f(u,v) = \frac{\left(u^2-qv^2\right)\left(\alpha\beta\gamma\delta u^2v^2-q^2\right)}{q\left(u^2-v^2\right)\left(\alpha\beta\gamma\delta u^2v^2-q\right)}, \tag{A.12}$$

$$g(u,v,m) = \frac{(q-1)\left(q^2-\alpha\beta\gamma\delta v^4\right)\left(\alpha\beta v^2 q^{2L+2m+3}-u^2q^{2L}\right)}{q\left(u^2-v^2\right)\left(\alpha\beta q^{2L+2m+3}-q^{2L}\right)\left(q-\alpha\beta\gamma\delta v^4\right)}, \tag{A.13}$$

and

$$w(u,v,m) = \frac{\alpha\beta(q-1)\left(\alpha\beta\gamma\delta v^4-1\right)\left(\gamma\delta u^2v^2q^{2L}-q^{2(L+m+2)}\right)}{\left(q^{2L}-\alpha\beta q^{2L+2m+3}\right)\left(q-\alpha\beta\gamma\delta v^4\right)\left(q-\alpha\beta\gamma\delta u^2v^2\right)}\,. \tag{A.14}$$

The coefficients giving the action of $\mathcal{B}(u,m)$ on vectors in the position basis are:

$$V_{n,m} = J_n\left(q^{-n-1}-\alpha\beta q^{m+2}-\alpha\beta q^{m+1}+\alpha^2\beta^2 q^{2m+n+4}\right), \tag{A.15}$$

$$X_{n,m} = -\frac{\mu_n\lambda_n(\alpha\beta q^{2m+3}+1)}{(q+1)} - \frac{\alpha\beta(q+1)}{q^{-m-1}}\mu_n + \frac{(\eta^* q^{2L+m+1}+\alpha\beta\xi q^{2L+2m+3}+\xi q^{2L})}{(q-1)^2(q+1)q^{2L-1}}, \tag{A.16}$$

$$Y_{n,m} = -q^{m+1}\lambda_n + \frac{\left(\alpha\beta q^{2L+2m+3}+q^{2L}\right)}{q^{2L}}, \tag{A.17}$$

and

$$Z_{n,m} = J_{n-1}\alpha\beta\left(q^{2m+3-n}+q^n-q^{m+2}-q^{m+1}\right). \tag{A.18}$$

References

[1] L. Amico, R. Fazio, A. Osterloh and V. Vedral, *Entanglement in many-body systems*, Rev. Mod. Phys. **80**, 517 (2008), doi:10.1103/RevModPhys.80.517.

[2] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, UK, ISBN 9781107002173 (2012), doi:10.1017/CBO9780511976667.

[3] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, *Entanglement in quantum critical phenomena*, Phys. Rev. Lett. **90**, 227902 (2003), doi:10.1103/PhysRevLett.90.227902.

[4] J. Eisert, M. Cramer and M. B. Plenio, *Colloquium: Area laws for the entanglement entropy*, Rev. Mod. Phys. **82**, 277 (2010), doi:10.1103/RevModPhys.82.277.

[5] I. Peschel, *Calculation of reduced density matrices from correlation functions*, J. Phys. A: Math. Gen. **36**, L205 (2003), doi:10.1088/0305-4470/36/14/101.

[6] B.-Q. Jin and V. E. Korepin, *Quantum spin chain, Toeplitz determinants and the Fisher-Hartwig conjecture*, J. Stat. Phys. **116**, 79 (2004), doi:10.1023/B:JOSS.0000037230.37166.42.

[7] M. Fagotti and P. Calabrese, *Universal parity effects in the entanglement entropy of XX chains with open boundary conditions*, J. Stat. Mech.: Theory Exp. P01017 (2011), doi:10.1088/1742-5468/2011/01/P01017.

[8] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642050138 (2010), doi:10.1007/978-3-642-05014-5.

[9] D. Gioev and I. Klich, *Entanglement entropy of fermions in any dimension and the Widom conjecture*, Phys. Rev. Lett. **96**, 100503 (2006), doi:10.1103/PhysRevLett.96.100503.

[10] V. Eisler and I. Peschel, *Free-fermion entanglement and spheroidal functions*, J. Stat. Mech.: Theory Exp. P04028 (2013), doi:10.1088/1742-5468/2013/04/P04028.

[11] N. Crampé, R. I. Nepomechie and L. Vinet, *Free-Fermion entanglement and orthogonal polynomials*, J. Stat. Mech.: Theory Exp. 093101 (2019), doi:10.1088/1742-5468/ab3787.

[12] N. Crampé, R. I. Nepomechie and L. Vinet, *Entanglement in fermionic chains and bispectrality*, in *Roman Jackiw*, World Scientific, Singapore, ISBN 9789811212284 (2020), doi:10.1142/9789811210679_0013.

[13] P.-A. Bernard, N. Crampé, R. I. Nepomechie, G. Parez, L. P. d'Andecy and L. Vinet, *Entanglement of inhomogeneous free fermions on hyperplane lattices*, Nucl. Phys. B **984**, 115975 (2022), doi:10.1016/j.nuclphysb.2022.115975.

[14] P. Baseilhac and R. A. Pimenta, *Diagonalization of the Heun-Askey-Wilson operator, Leonard pairs and the algebraic Bethe ansatz*, Nucl. Phys. B **949**, 114824 (2019), doi:10.1016/j.nuclphysb.2019.114824.

[15] J. Cao, H.-Q. Lin, K.-J. Shi and Y. Wang, *Exact solution of XXZ spin chain with unparallel boundary fields*, Nucl. Phys. B **663**, 487 (2003), doi:10.1016/S0550-3213(03)00372-9.

[16] R. I. Nepomechie, *Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms*, J. Phys. A: Math. Gen. **37**, 433 (2003), doi:10.1088/0305-4470/37/2/012.

[17] N. Crampé, E. Ragoucy and D. Simon, *Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions*, J. Stat. Mech.: Theory Exp. P11038 (2010), doi:10.1088/1742-5468/2010/11/P11038.

[18] P.-A. Bernard, G. Carcone, N. Crampé and L. Vinet, *Bethe ansatz diagonalization of the Heun-Racah operator*, Lett. Math. Phys. **113**, 8 (2023), doi:10.1007/s11005-023-01633-7.

[19] P.-A. Bernard, N. Crampé, D. S. Kabakibo and L. Vinet, *Heun operator of Lie type and the modified algebraic Bethe ansatz*, J. Math. Phys. **62**, 083501 (2021), doi:10.1063/5.0041097.

[20] J. Avan, S. Belliard, N. Grosjean and R. A. Pimenta, *Modified algebraic Bethe Ansatz for XXZ chain on the segment - III - Proof*, Nucl. Phys. B **899**, 229 (2015), doi:10.1016/j.nuclphysb.2015.08.006.

[21] S. Belliard, *Heisenberg XXX model with general boundaries: Eigenvectors from algebraic Bethe Ansatz*, Symmetry Integr. Geom.: Methods Appl. **9**, 072 (2013), doi:10.3842/SIGMA.2013.072.

[22] S. Belliard et al., *Modified algebraic Bethe Ansatz: Twisted XXX case*, Symmetry Integr. Geom.: Methods Appl. **14**, 054 (2018), doi:10.3842/SIGMA.2018.054.

[23] N. Crampé et al., *Algebraic Bethe Ansatz for the XXZ Gaudin models with generic boundary*, Symmetry Integr. Geom.: Methods Appl. **13**, 094 (2017), doi:10.3842/SIGMA.2017.094.

[24] N. Crampé, *Algebraic Bethe Ansatz for the totally asymmetric simple exclusion process with boundaries*, J. Phys. A: Math. Theor. **48**, 08FT01 (2015), doi:10.1088/1751-8113/48/8/08FT01.

[25] K. Totsuka, *Magnetization processes in bond-alternating quantum spin chains*, Phys. Lett. A **228**, 103 (1997), doi:10.1016/S0375-9601(97)00087-X.

[26] J. A. Carrasco, F. Finkel, A. González-López and P. Tempesta, *A duality principle for the multi-block entanglement entropy of free fermion systems*, Sci. Rep. **7**, 11206 (2017), doi:10.1038/s41598-017-09550-1.

[27] G. Bergeron, N. Crampé, S. Tsujimoto, L. Vinet and A. Zhedanov, *The Heun-Racah and Heun-Bannai-Ito algebras*, J. Math. Phys. **61**, 081701 (2020), doi:10.1063/5.0008372.

[28] N. Crampé, L. Frappat, J. Gaboriaud, L. P. d'Andecy, E. Ragoucy and L. Vinet, *The Askey-Wilson algebra and its avatars*, J. Phys. A: Math. Theor. **54**, 063001 (2021), doi:10.1088/1751-8121/abd783.

[29] A. S. Zhedanov, *"Hidden symmetry" of Askey-Wilson polynomials*, Theor. Math. Phys. **89**, 1146 (1991), doi:10.1007/BF01015906.

[30] P. Virtanen et al, *SciPy 1.0: Fundamental algorithms for scientific computing in Python*, Nat. Methods. **17**, 261 (2020), doi:10.1038/s41592-019-0686-2.

Renormalizable extension of the Abelian Higgs-Kibble model with a dim.6 derivative operator

Daniele Binosi[1] and Andrea Quadri[2⋆]

1 European Centre for Theoretical Studies in Nuclear Physics and Related Areas (ECT*) and Fondazione Bruno Kessler, Villa Tambosi, Strada delle Tabarelle 286, I-38123 Villazzano (TN), Italy
2 INFN, Sez. di Milano, via Celoria 16, I-20133 Milan, Italy

⋆ andrea.quadri@mi.infn.it

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We present a new approach to the consistent subtraction of a non power-counting renormalizable extension of the Abelian Higgs-Kibble (HK) model supplemented by a dim. 6 derivative-dependent operator controlled by the parameter z. A field-theoretic representation of the physical Higgs scalar by a gauge-invariant variable is used in order to formulate the theory by exploiting a novel differential equation, controlling the dependence of the quantized theory on z. These results pave the way to the consistent subtraction by a finite number of physical parameters of some non-power-counting renormalizable models possibly of direct relevance to the study of the Higgs potential at the LHC.

Received 06-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.019

Contents

1 Introduction **2**

2 The z-differential equation **3**
2.1 ST identities in the ℓ-sector 4
2.1.1 Normalization conditions 4

3 Conclusion **5**

A Classical vertex functional in the X-formalism **6**

B Functional identities **7**

References **7**

1 Introduction

In the quest for new physics at the LHC a significant role has been recently played on the theoretical side by the Standard Model (SM) Effective Field Theories [1–3]. Deviations from the SM Lagrangian are described by a set of gauge-invariant higher-dimensional operators suppressed by some large energy scale Λ. The resulting theory is no more power-counting renormalizable and therefore more and more ultraviolet (UV) divergences arise as more loops are included. Their subtraction requires the introduction of more and more higher-dimensional operators compatible with the symmetries of the model.

Power-counting renormalizable theories on the other hand are defined in terms of a finite number of physical parameters in one-to-one correspondence with the finite number of operators required to subtract the UV divergences of the one-particle irreducible (1-PI) amplitudes to all orders in the loop expansion (once linear wave-function renormalization has been taken into account).

This is at variance with the increasing number of higher-dimensional operators required to make effective field theories finite as higher perturbative orders are included. Consequently, effective field theories preserve predictivity only up to the energy scale Λ: below Λ only a finite number of higher dimensional operators are physically relevant and in this sense a finite number of physical parameters control physical observables (up to the relevant energy scale).

The question of the minimal set of independent physical operators required to renormalize an effective field theory is a subtle question. First of all one must take into account redundancies associated with the equations of motion [4], or equivalently by generalized field redefinitions that are in general non-polynomial and prove essential in order to consistently subtract UV divergences by local counter-terms [5].

Moreover, it has been recently advocated [4,6] that additional relations between seemingly independent UV divergent amplitudes are easier to derive within a particular choice of gauge-invariant field coordinates [7–9].

For instance, in the usual formalism of the Abelian Higgs-Kibble model, the complex scalar field $\phi = \frac{1}{\sqrt{2}}(v + \sigma + i\chi)$ is used, v being the vacuum expectation value of ϕ, σ the field describing the physical scalar mode and χ the pseudo-Goldstone field. ϕ transforms in the fundamental representation of the gauge group U(1), $\delta\phi = ie\alpha\phi$ with α the infinitesimal gauge transformation and e the U(1) gauge coupling constant.

One might also consider the alternative choice of using the gauge invariant combination

$$\phi^\dagger\phi - \frac{v^2}{2} \sim vX_2\,,$$

in order to represent the physical scalar mode (this is the so-called X-formalism, based on the set of auxiliary fields X_2 and the Lagrange multiplier X_1).

The resulting theory has been described at length in [4,6,10–12] and its tree-level vertex functional is reported in Eq.(A.1). It is physically equivalent to the Abelian Higgs-Kibble model, after going on-shell with both X_1 and X_2.

At variance with the ordinary formalism, the set of functional identities of the theory in the X-formalism is richer. For instance, 1-PI amplitudes involving at least one X_1 or X_2-fields are uniquely fixed by the $X_{1,2}$-functional equations in Eqs.(B.3,B.4) in terms of amplitudes without.

More importantly, it turns out that the X_2-equation of the Abelian Higgs-Kibble model admits a unique deformation, compatible with all the symmetries of the theory and associated with the addition to the classical action of a bilinear operator in X_2, see the first term in the second line of Eq.(A.1). At $z = 0$ we recover the power-counting renormalizable Abelian Higgs-Kibble model, while at $z \neq 0$ we obtain a non power-counting renormalizabile theory physically equivalent to the one generated by the introduction of the dim.6 operator

$$\frac{z}{2}\partial^{\mu}X_2\partial_{\mu}X_2 \sim \frac{z}{2v^2}\partial^{\mu}(\phi^{\dagger}\phi)\partial_{\mu}(\phi^{\dagger}\phi). \tag{1}$$

A crucial remark is that in the X-formalism the parameter z enters classically only in the quadratic part of the classical action, while in the standard approach it also appears in the interaction vertices. This property allows one to derive in the X-formalism an extremely powerful differential equation.

By solving the latter equation, one obtains a unique prescription for the amplitudes of the non-power-counting renormalizable model at $z \neq 0$ in terms of those at $z = 0$.

If the theory at $z = 0$ is power-counting renormalizable (as in the case we deal with in the present paper, for the sake of definiteness), the model at $z \neq 0$ is defined in terms of the same (finite, to all orders in perturbation theory) number of physical parameters plus z.

If, on the other hand, the model at $z = 0$ is an effective field theory, the results of the present paper show that the addition of the dim.6 interaction in Eq.(1) comes at no cost, since the complete dependence of the amplitudes at $z \neq 0$ is still uniquely determined algebraically by the z-differential equation in terms of the amplitudes at $z = 0$.

The relevant parameters up to the scale energy Λ are those of the effective theory at $z = 0$ plus z. This is a highly non-trivial result that follows from the z-differential equation.

2 The z-differential equation

The starting point is the diagonalization of the quadratic part in the scalar sector, that can be achieved by the field redefintion

$$\sigma = \sigma' + X_1 + X_2 . \tag{2}$$

The propagators read

$$\Delta_{\sigma'\sigma'} = \frac{i}{p^2 - m^2}, \qquad \Delta_{X_1X_1} = -\frac{i}{p^2 - m^2}, \qquad \Delta_{X_2X_2} = \frac{i}{(1+z)p^2 - M^2}. \tag{3}$$

In this basis the dependence on the parameter z only arises via the X_2-propagator. Introducing then the differential operator

$$\mathcal{D}_z^{M^2} = (1+z)\partial_z + M^2\partial_{M^2}, \tag{4}$$

one finds that $\Delta_{X_2X_2}$ is an eigenvector of $\mathcal{D}_z^{M^2}$ with eigenvalue -1:

$$\mathcal{D}_z^{M^2}\Delta_{X_2X_2}(k^2, M^2) = -\Delta_{X_2X_2}(k^2, M^2). \tag{5}$$

The argument generalizes to diagrams with a given number of internal X_2-lines. Let us collectively denote with Φ the set of fields and external sources of the theory, and let us indicate with p_i (with $i = 1, \dots, r$) their external momenta, with $\Phi_i = \Phi(p_i)$ and $p_r = -\sum_1^{r-1} p_i$; in this way a n-loop 1-PI Green's function $\Gamma^{(n)}_{\Phi_1\cdots\Phi_r}$ with r Φ_i insertions can be decomposed as the sum of all 1-PI diagrams with external legs $\Phi_1 \cdots \Phi_r$ with zero, one, two,..., ℓ internal X_2-propagators, *i.e.*,

$$\Gamma^{(n)}_{\Phi_1\cdots\Phi_r} = \sum_{\ell\geq 0}\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r}. \tag{6}$$

Then by applying the differential operator $\mathcal{D}_z^{M^2}$ we find

$$\mathcal{D}_z^{M^2}\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r} = -\ell\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r} \implies \mathcal{D}_z^{M^2}\Gamma^{(n)}_{\Phi_1\cdots\Phi_r} = -\sum_{\ell\geq 0}\ell\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r}. \tag{7}$$

Hence we see that the subdiagrams with a fixed number ℓ of internal X_2-lines are eigenvectors of the $\mathcal{D}_z^{M^2}$ with eigenvalue ℓ. The most general solution to this equation (of the homogeneous Euler's type) reads (indicating explicitly only the dependence on the parameters z and M^2)

$$\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r}(z, M^2) = \frac{1}{(1+z)^\ell}\Gamma^{(n;\ell)}_{\Phi_1\cdots\Phi_r}(0, M^2/(1+z)) . \tag{8}$$

Thus, amplitudes at $z \neq 0$ in each ℓ-sector are obtained from those at $z = 0$ by dividing them by the $(1+z)^\ell$ factor and rescaling by $(1+z)$ the square of the Higgs mass M^2.

Otherwise said, in the X-formalism the existence of the z-differential equation implies that the deformed theory at $z \neq 0$ can be fully characterized once one knows the boundary conditions given by the amplitudes of the power-counting renormalizable theory at $z = 0$.

2.1 ST identities in the ℓ-sector

Another crucial property of the X-formalism is that the ST identities separately hold true in each ℓ-sector. The proof of this statement can be found in [12] and relies on the gauge-invariance of the X_2-field.

At order n in the loop expansion we get a set of ST identities, one for each ℓ:

$$\mathcal{S}_0\left(\Gamma^{(n;\ell)}\right) + \sum_{j=1}^{n-1}\sum_{i=0}^{\ell}\left(\Gamma^{(j;i)}, \Gamma^{(n-j;\ell-i)}\right) = 0 . \tag{9}$$

Such identities encode the conditions required to guarantee physical unitarity of the theory (i.e., the cancellation of the intermediate ghost states). Since X_2 is gauge-invariant, it is physically sensible that it does not participate to such cancellations and therefore that the quartet mechanism [13–15] is at work separately for each sector with a given number ℓ of internal X_2-lines.

2.1.1 Normalization conditions

The normalization conditions that must be imposed in the theory at $z = 0$ can also be consistently decomposed according to the degree induced by the number of internal X_2-lines.

For instance, the on-mass shell normalization condition for the vector meson is obtained by requiring that the position of the pole of the physical components of the vector meson does not shift with respect to the one at tree level and that the residue of the propagator on the pole is one, i.e.

$$\text{Re}\ \Sigma_T(M_A^2) = 0 , \qquad \text{Re}\ \frac{\partial \Sigma_T(p^2)}{\partial p^2}\bigg|_{p^2=M_A^2} = 0 . \tag{10}$$

In the above equation we have denoted by Σ_T the transverse component of the two-point 1-PI gauge function:

$$\Gamma_{A^\mu A^\nu} = g_{\mu\nu}(p^2 - M_A^2) + \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Sigma_T(p^2) + \frac{p^\mu p^\nu}{p^2}\Sigma_L(p^2) . \tag{11}$$

These conditions can be matched by finite renormalizations involving the following ST (and gauge-) invariant operators (we use the notation of Ref. [6]):

$$\lambda_4 \int d^4x\, (D^\mu\phi)^\dagger D_\mu\phi \supset \frac{\lambda_4 v}{2}\int d^4x\, A_\mu^2 , \qquad \frac{\lambda_8}{2}\int d^4x\, F_{\mu\nu}^2 \supset \lambda_8 \int d^4x\, A_\mu(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu . \tag{12}$$

Now, since the ST identities hold true separately at each ℓ-order, we can project the normalization condition Eq.(10) at the relevant ℓ-order and at order n in the loop expansion:

$$\mathrm{Re}\ \Sigma_T^{(1;\ell)}(M_A^2) + v\lambda_4^{(1;\ell)} = 0\,, \qquad \mathrm{Re}\ \frac{\partial \Sigma_T^{(1;\ell)}}{\partial p^2}\Bigg|_{p^2=M_A^2} - 2M_A^2\lambda_8^{(1;\ell)} = 0\,. \tag{13}$$

As can be seen from the above equation, on mass shell renormalization conditions respect the layers in ℓ and consequently the z-differential equation.

Otherwise said, once the appropriate normalization conditions are enforced at order n in the loop expansion at $z = 0$, Eq.(8) fixes the 1-PI amplitudes of the theory at $z \neq 0$ in a unique way.

3 Conclusion

We have obtained a differential equation that controls the deformation of the Abelian Higgs-Kibble model induced by the dim.6 operator

$$\frac{z}{2}\partial^\mu X_2 \partial_\mu X_2 \sim \frac{z}{2}\partial^\mu(\phi^\dagger\phi)\partial_\mu(\phi^\dagger\phi)\,.$$

The solution to the differential equation is uniquely defined in terms of the boundary conditions of the (renormalized) amplitudes of the theory at $z = 0$. This allows one to define the corresponding non-power-counting renormalizable theory in a way that it only depends on the same number of physical parameters of the model at $z = 0$ (either the finite ones, to all orders in perturbation theory, if the model at $z = 0$ is power-counting renormalizable, or those relevant up to the energy scale Λ, if the model at $z = 0$ is an effective field theory), and z.

The results obtained so far for the Abelian gauge group can be generalized to the full electroweak SU(2)×U(1) theory. This is of particular interest, since one could obtain an extension of the SM and of Beyond-the-Standard-Model (BSM) theories by a derivative-dependent dim.6 operator, that still can be defined at the quantum level in a consistent way (to all orders in z). Within this framework, applications to phenomenology should also be studied. In particular one could study the BSM corrections to the SM Higgs potential, that are expected to be explored at the LHC experimental program.

Another interesting problem is whether the present construction can be extended to gauge-invariant fields representing the gauge and fermion degrees of freedom. We hope to report on these issues soon.

Acknowledgements

Funding information A partial financial support by INFN is acknowledged.

A Classical vertex functional in the X-formalism

The classical vertex functional is given by:

$$\begin{aligned}\Gamma^{(0)} = \int \mathrm{d}^4x \Big[& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \frac{M^2-m^2}{2}X_2^2 - \frac{m^2}{2v^2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 \\ & + \frac{z}{2}\partial^\mu X_2\partial_\mu X_2 - \bar{c}(\square + m^2)c + \frac{1}{v}(X_1+X_2)(\square+m^2)\left(\phi^\dagger\phi - \frac{v^2}{2} - vX_2\right) \\ & + \frac{\xi b^2}{2} - b\,(\partial A + \xi e v\chi) + \bar{\omega}\left(\square\omega + \xi e^2 v(\sigma+v)\omega\right) \\ & + \bar{c}^*\left(\phi^\dagger\phi - \frac{v^2}{2} - vX_2\right) + \sigma^*(-e\omega\chi) + \chi^* e\omega(\sigma+v)\Big]. \end{aligned} \tag{A.1}$$

In the above equation D_μ is the covariant derivative

$$D_\mu = \partial_\mu - ieA_\mu\,. \tag{A.2}$$

The first line of Eq.(A.1) is the classical action of the Abelian Higgs-Kibble model. By going on-shell with X_1 and imposing the constraint

$$X_2 = \frac{1}{v}\left(\phi^\dagger\phi - \frac{v^2}{2}\right), \tag{A.3}$$

we recover the usual quartic Higgs potential with coupling $\sim -\frac{M^2}{2v^2}$. Indeed one can prove [6] that the only physical parameter is M, m cancelling out in physical quantities. The first term of the second line contains the deformation proportional to the parameter z. By going on-shell with X_1 we obtain the dimension-six derivative operator $\sim \frac{z}{2v^2}\partial^\mu(\phi^\dagger\phi)\partial_\mu(\phi^\dagger\phi)$, that breaks the power-counting renormalizability of the theory. The second and third terms in the second line of Eq.(A.1) implements off-shell in a BRST-invariant way the constraint in Eq.(A.3) via the Lagrange multiplier X_1. The X_2-dependent term simplifies diagonalization of the quadratic part via the transformation in Eq. (2).

X_1- and σ'- propagators have a relative minus sign responsible for their mutual cancellation inside loops, see Eq.(3), that holds true to all order by virtue of the constraint U(1) BRST symmetry

$$\mathcal{S}X_1 = vc\,, \qquad \mathcal{S}c = 0\,, \qquad \mathcal{S}\bar{c} = \frac{1}{v}\left(\phi^\dagger\phi - \frac{v^2}{2} - vX_2\right), \tag{A.4}$$

all other fields and external sources being invariant under $\mathcal{S}$ and $c, \bar{c}$ being the constraint U(1) ghost and antighost fields.

The third line implements the usual R_ξ-gauge in a BRST-invariant way, $\bar{\omega}, \omega$ being the antighost and ghost fields associated with the gauge group U(1) and b the Nakanishi-Lautrup field. The U(1) BRST symmetry is defined as usual according to

$$sA_\mu = \partial_\mu\omega\,, \quad s\phi = ie\omega\phi\,, \quad s\sigma = -e\omega\chi\,, \quad s\chi = e\omega(\sigma+v)\,, \quad s\bar{\omega} = b\,, \quad sb = 0\,, \tag{A.5}$$

all other fields being invariant. In particular X_2 is BRST-invariant. The cohomological BRST analysis of the physical spectrum of the model is given in [12]. It turns out that the physical modes are the three transverse components of the massive gauge field A_μ and one physical scalar with tree-level mass M.

Finally the last line of Eq.(A.1) contains the external sources required to renormalize the theory. Being coupled to the BRST variation respectively of $\bar{c}, \sigma$ and χ, they are the anti-fields [16] of the BRST differentials $\mathcal{S}$ and s. Invariance of the classical vertex functional under $\mathcal{S}$ and s is translated at the quantum level into the Slavnov-Taylor (ST) identities in Eqs.(B.1) and (B.5).

B Functional identities

The functional identities controlling the theory are listed below:

- The ST identity for the constraint BRST symmetry is

$$\mathcal{S}_c(\Gamma) \equiv \int d^4x \left[vc \frac{\delta\Gamma}{\delta X_1} + \frac{\delta\Gamma}{\delta\bar{c}^*}\frac{\delta\Gamma}{\delta\bar{c}} \right] = \int d^4x \left[vc \frac{\delta\Gamma}{\delta X_1} - (\square + m^2)c \frac{\delta\Gamma}{\delta\bar{c}^*} \right] = 0\,, \tag{B.1}$$

  where in the latter equality we have used the fact that both the ghost c and the antighost $\bar{c}$ are free:

$$\frac{\delta\Gamma}{\delta\bar{c}} = -(\square + m^2)c\,, \qquad \frac{\delta\Gamma}{\delta c} = (\square + m^2)\bar{c}\,. \tag{B.2}$$

- The X_1-equation of motion, that follows from Eq.(B.1) by using the fact that the ghost c is free:

$$\frac{\delta\Gamma}{\delta X_1} = \frac{1}{v}(\square + m^2)\frac{\delta\Gamma}{\delta\bar{c}^*}\,. \tag{B.3}$$

- The X_2-equation of motion:

$$\frac{\delta\Gamma}{\delta X_2} = \frac{1}{v}(\square + m^2)\frac{\delta\Gamma}{\delta\bar{c}^*} - (\square + m^2)X_1 - ((1+z)\square + M^2)X_2 - v\bar{c}^*\,. \tag{B.4}$$

  Notice that the z-term is the only one that affects the right-hand side of the above equation in a linear way (so that no new external source is required to control its renormalization) and that contains at most two derivatives (in order to avoid inconsistencies of higher derivative theories due to the appearance of negative norm states in the physical spectrum).

- The ST identity associated to the gauge group BRST symmetry

$$\mathcal{S}(\Gamma) = \int d^4x \left[\partial_\mu\omega \frac{\delta\Gamma}{\delta A_\mu} + \frac{\delta\Gamma}{\delta\sigma^*}\frac{\delta\Gamma}{\delta\sigma} + \frac{\delta\Gamma}{\delta\chi^*}\frac{\delta\Gamma}{\delta\chi} + b\frac{\delta\Gamma}{\delta\bar{\omega}} \right] = 0\,. \tag{B.5}$$

- The b-equation:

$$\frac{\delta\Gamma}{\delta b} = \xi b - \partial A - \xi e v \chi\,. \tag{B.6}$$

- The antighost equation:

$$\frac{\delta\Gamma}{\delta\bar{\omega}} = \square\omega + \xi e v \frac{\delta\Gamma}{\delta\chi^*}\,. \tag{B.7}$$

References

[1] W. Buchmüller and D. Wyler, *Effective Lagrangian analysis of new interactions and flavour conservation*, Nucl. Phys. B **268**, 621 (1986), doi:10.1016/0550-3213(86)90262-2.

[2] R. Alonso, E. E. Jenkins, A. V. Manohar and M. Trott, *Renormalization group evolution of the Standard Model dimension six operators III: Gauge coupling dependence and phenomenology*, J. High Energy Phys. **04**, 159 (2014), doi:10.1007/JHEP04(2014)159.

[3] I. Brivio and M. Trott, *The Standard Model as an effective field theory*, Phys. Rep. **793**, 1 (2019), doi:10.1016/j.physrep.2018.11.002.

[4] D. Binosi and A. Quadri, *Off-shell renormalization in the presence of dimension 6 derivative operators. Part I. General theory*, J. High Energy Phys. **09**, 032 (2019), doi:10.1007/JHEP09(2019)032.

[5] J. Gomis and S. Weinberg, *Are nonrenormalizable gauge theories renormalizable?*, Nucl. Phys. B **469**, 473 (1996), doi:10.1016/0550-3213(96)00132-0.

[6] D. Binosi and A. Quadri, *Off-shell renormalization in the presence of dimension 6 derivative operators. II. Ultraviolet coefficients*, Eur. Phys. J. C **80**, 807 (2020), doi:10.1140/epjc/s10052-020-8349-0.

[7] J. Fröhlich, G. Morchio and F. Strocchi, *Higgs phenomenon without symmetry breaking order parameter*, Nucl. Phys. B **190**, 553 (1981), doi:10.1016/0550-3213(81)90448-X.

[8] J. Fröhlich, G. Morchio and F. Strocchi, *Higgs phenomenon without a symmetry breaking order parameter*, Phys. Lett. B **97**, 249 (1980), doi:10.1016/0370-2693(80)90594-8.

[9] A. Maas, *Brout-Englert-Higgs physics: From foundations to phenomenology*, Prog. Part. Nucl. Phys. **106**, 132 (2019), doi:10.1016/j.ppnp.2019.02.003.

[10] D. Binosi and A. Quadri, *Off-shell renormalization in Higgs effective field theories*, J. High Energy Phys. **04**, 050 (2018), doi:10.1007/JHEP04(2018)050.

[11] D. Binosi and A. Quadri, *Off-shell renormalization in the presence of dimension 6 derivative operators. Part III. Operator mixing and β functions*, J. High Energy Phys. **05**, 141 (2020), doi:10.1007/JHEP05(2020)141.

[12] D. Binosi and A. Quadri, *Renormalizable extension of the Abelian Higgs-Kibble model with a dimension-six operator*, Phys. Rev. D **106**, 065022 (2022), doi:10.1103/PhysRevD.106.065022.

[13] C. Becchi, A. Rouet and R. Stora, *The Abelian Higgs Kibble model, unitarity of the S-operator*, Phys. Lett. B **52**, 344 (1974), doi:10.1016/0370-2693(74)90058-6.

[14] G. Curci and R. Ferrari, *An alternative approach to the proof of unitarity for gauge theories*, Nuovo Cimento A **35**, 273 (1976), doi:10.1007/BF02730284.

[15] T. Kugo and I. Ojima, *Manifestly covariant canonical formulation of Yang-Mills theories physical state subsidiary conditions and physical S-matrix unitarity*, Phys. Lett. B **73**, 459 (1978), doi:10.1016/0370-2693(78)90765-7.

[16] J. Gomis, J. París and S. Samuel, *Antibracket, antifields and gauge-theory quantization*, Phys. Rep. **259**, 1 (1995), doi:10.1016/0370-1573(94)00112-G.

The tower of Kontsevich deformations for Nambu-Poisson structures on $\mathbb{R}^d$: Dimension-specific micro-graph calculus

Ricardo Buring[1∘] **and Arthemy V. Kiselev**[2⋆§]

1 Institut für Mathematik, Johannes Gutenberg–Universität, Staudingerweg 9, D-55128 Mainz, Germany

2 Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

⋆ A.V.Kiselev@rug.nl

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

In Kontsevich's graph calculus, internal vertices of directed graphs are inhabited by multi-vectors, e.g., Poisson bi-vectors; the Nambu-determinant Poisson brackets are differential-polynomial in the Casimir(s) and density ϱ times Levi-Civita symbol. We resolve the old vertices into subgraphs such that every new internal vertex contains one Casimir or one Levi-Civita symbol $\times\, \varrho$. Using this micro-graph calculus, we show that Kontsevich's tetrahedral γ_3-flow on the space of Nambu-determinant Poisson brackets over $\mathbb{R}^3$ is a Poisson coboundary: we realize the trivializing vector field $\vec{X}$ over $\mathbb{R}^3$ using micro-graphs. This $\vec{X}$ projects to the known trivializing vector field for the γ_3-flow over $\mathbb{R}^2$.

Received 16-12-2022
Accepted 29-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.020

Contents

1 Introduction 2

2 Preliminaries: The tetrahedral flow $\dot{P} = Q_{\gamma_3}(P)$ over twofolds 2

3 Nambu-determinant Poisson brackets: The γ_3-flow over $\mathbb{R}^3$ 5

A (Non)triviality of γ_3-flow for Nambu–Poisson brackets on $\mathbb{R}^4$ 10

References 11

∘ **Address for correspondence**: Team MATHEXP, Centre INRIA de Saclay Île-de-France, Bât. Alan Turing, 1 rue Honoré d'Estienne d'Orves, F-91120 Palaiseau, France

§ **Present address**: Institut des Hautes Études Scientifiques (IHÉS), 35 route de Chartres, Bures-sur-Yvette, F-91440 France

1 Introduction

Kontsevich introduced [5] a universal – for any affine Poisson manifold of dimension d – construction of infinitesimal symmetries for the Jacobi identity: for suitable cocycles γ in the graph complex, one obtains bi-vector flows $\dot{P} = Q_\gamma([P])$ with differential-polynomial right-hand sides (with respect to components $P^{ij}(\boldsymbol{x})$ of Poisson structures $P \in \Gamma(\bigwedge^2 TM^d_{\text{aff}})$). We detect in [4] that for the tetrahedral graph cocycle γ_3 from [5] and for the pentagon-wheel graph cocycle γ_5 (see [3]), the corresponding flows (see [1, 2]) have a well-defined restriction to the subclass of Nambu-determinant Poisson brackets[1] $P(\varrho,[\boldsymbol{a}])$ on $\mathbb{R}^d$ at least in the following three cases: (*i*) γ_3-cocycle flow $\dot{P} = Q_\gamma([P])$ for $P(\varrho,[a])$ over $\mathbb{R}^3$, (*ii*) the same γ_3-cocycle and the flow of $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$, and (*iii*) the next, γ_5-cocycle flow for $P(\varrho,[a])$ over $\mathbb{R}^3$.

To study the (non)triviality of Kontsevich's graph flows in the second Poisson cohomology group for Nambu–determinant brackets $\{\cdot,\cdot\}_{P(\varrho,[\boldsymbol{a}])}$, we consider the coboundary equation,

$$Q_\gamma([P])([\varrho],[\boldsymbol{a}]) = [\![P(\varrho,[\boldsymbol{a}]),\vec{X}([\varrho],[\boldsymbol{a}])]\!], \tag{1}$$

upon vector fields $\vec{X}([\varrho],[\boldsymbol{a}])$ with differential-polynomial coefficients over $\mathbb{R}^d$. We discovered in [4] that the γ_3-flow over $\mathbb{R}^3$ is trivial w.r.t. a unique solution $\vec{X}$ mod $[\![P,H([\varrho],[\boldsymbol{a}])]\!]$. In this text we explain how, for $\gamma=\gamma_3$ and $d=2$, a solution of (1) is constructed using Kontsevich's graphs, and then, for $d=3$, how the trivializing vector field X^{γ_3} is found by using *micro-graphs* that resolve $\varrho(\boldsymbol{x})\cdot$Levi-Civita symbol against the Casimir(s) a_ℓ within copies of Nambu-determinant Poisson brackets $\{\cdot,\cdot\}_{P(\varrho,[\boldsymbol{a}])} = \varrho(\boldsymbol{x})\cdot\sum_{i_1,\ldots,i_d=1}^{d}\varepsilon^{\vec{\imath}}\cdot\partial_{i_1}(a_1)\cdot\ldots\cdot\partial_{i_{d-2}}(a_{d-2})\cdot\partial_{i_{d-1}}\otimes\partial_{i_d}$ in the vertices of Kontsevich's directed graphs for the flow $Q_\gamma([P])$.

2 Preliminaries: The tetrahedral flow $\dot{P} = Q_{\gamma_3}(P)$ over twofolds

Kontsevich's directed graphs are built of $n \geqslant 0$ wedges $\xleftarrow{L}\bullet\xrightarrow{R}$, usually drawn in the upper half-plane $\mathbb{H}^2$, over $m \geqslant 0$ ordered sinks along $\mathbb{R}=\partial\mathbb{H}^2$; tadpoles are allowed. Leibniz graphs are akin: the out-degrees of all but one (or more) vertices equal 2 yet there is (at least) one aerial vertex of out-degree 3 and its outgoing edges are ordered Left $\prec$ Middle $\prec$ Right.[2]

We shall study only those flows $\dot{P} = Q([P])$ on spaces of bi-vectors $P \in \Gamma(\bigwedge^2 TM^{d<\infty}_{\text{aff}})$ which are encoded by Kontsevich's graphs. From [5] (cf. [2]) we know that from suitable *cocycles* γ in the Kontsevich graph complex, one obtains the flows[3] $\dot{P} = Q_\gamma([P])$ which preserve the (sub)set of *Poisson* bi-vectors on M^d_{aff}. The tetrahedron γ_3 and pentagon-wheel cocycle γ_5, see [2], are examples of graph cocycles giving such flows.

[1]The Nambu-determinant Poisson brackets (with $\varrho\not\equiv 1$ and Casimir(s) a_ℓ) of $f,g\in C^1(\mathbb{R}^d)$ are, e.g.,

$$\{f,g\}_{P(\varrho,[a])} = \varrho(x,y,z)\cdot\begin{vmatrix} a_x & f_x & g_x \\ a_y & f_y & g_y \\ a_z & f_z & g_z \end{vmatrix}, \quad \text{on } \mathbb{R}^3 \ni \boldsymbol{x}=(x,y,z),$$

likewise $\{f,g\}_{P(\varrho,[a_1],[a_2])} = \varrho(x^1,x^2,x^3,x^4)\cdot\det\big(\partial(a_1,a_2,f,g)/\partial(x^1,x^2,x^3,x^4)\big)$ on $\mathbb{R}^4$, and so on; all such formulas are tensorial w.r.t. coordinate transformations (as $\varrho(\boldsymbol{x})\cdot\partial_{x^1}\wedge\ldots\wedge\partial_{x^d}$ is a top-degree multivector on $\mathbb{R}^d$).

[2] For example, the tripod is a Leibniz graph; like every Leibniz graph, it expands to a linear combination of Kontsevich graphs, namely to the Jacobiator $\frac{1}{2}[\![P,P]\!]$ for a bi-vector P whose copies are realized by wedges ([1]).

[3] The formula of Kontsevich's graph flow $\dot{P} = Q_\gamma([P])$ can depend on a choice of representative γ for the graph cohomology class $[\gamma]$. Fortunately, the vertex-edge bi-gradings $(4,6)$ for γ_3 and $(6,10)$ for two graphs in γ_5 are not yet big enough to provide room for any nonzero coboundaries (from nonzero graphs on 3 vertices and 5 edges or on 5 vertices and 9 edges, respectively). In other words, the known markers for $[\gamma_3]$ and $[\gamma_5]$ are in fact uniquely defined up to a nonzero multiplicative constant; we prove this by listing all the admissible (non)zero "potentials" and by taking their vertex-expanding differentials in the graph complex. This is why, in our present study of the γ_3- and γ_5-flows on the spaces of Nambu–Poisson brackets, we do not care about a would-be response of trivializing vector fields X^γ in (1) to shifts of the marker cocycle γ within its graph cohomology class $[\gamma]$.

Example 1 ([5] and [1]). The tetrahedral-graph flow $\dot{P} = Q_\gamma(P)$ on the space of Poisson bi-vectors P over any d-dimensional affine Poisson manifold $M^{d\geqslant 2}_{\text{aff}}$ is encoded by the linear combination of three directed graphs,[4]

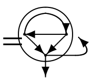
$$Q_{\gamma_3} = 1\cdot\big(0,1;2,4;2,5;2,3\big) - 3\cdot\big(0,3;1,4;2,5;2,3 + 0,3;4,5;1,2;2,4\big), \tag{2}$$

with a copy of the Poisson bi-vector in each internal vertex 2,3,4,5 from which two decorated arrows, Left $\prec$ Right matching the bi-vector indices, are issued to the designated arrowhead vertices. Sink vertices 0 and 1 contain the arguments of bi-vector $Q_{\gamma_3}(P)$; vertices 2,3,4,5 of the tetrahedron γ_3 itself are internal.

The graph construction of the γ_3-flow $\dot{P} = Q_{\gamma_3}(P)$ works in any dimension of the affine Poisson manifold M^d_{aff} at hand. Now if $d = 2$, this infinitesimal deformation of bi-vectors on twofolds is known to be trivial in the second Poisson cohomology, thus amounting to an infinitesimal change of local coordinates (performed along the integral trajectories of the trivializing vector field $\vec{X}$ on M^2_{aff}).

Proposition 1 ([1, App. F]). **1.** In dimension $d = 2$ (where every bi-vector $P = \varrho(x,y)\cdot(\partial_x\otimes\partial_y - \partial_y\otimes\partial_x)$ is Poisson,[5] in absence of nonzero Jacobiator tri-vectors), Kontsevich's tetrahedral flow $\dot{P} = Q_{\gamma_3}(P)$, encoded by three graphs in (2), is Poisson-trivial,

$$Q_{\gamma_3}\big(P(\varrho)\big) - [\![P(\varrho),\vec{X}]\!] = 0 \in \Gamma\Big(\bigwedge^2 TM^2_{\text{aff}}\Big), \qquad \vec{X} = X^{\gamma_3}(P), \quad X^{\gamma_3} =, \tag{3}$$



with respect to the class of 1-vector $\vec{X} = \partial_j\big(\partial_k\partial_m(P^{ij})\cdot\partial_n(P^{k\ell})\cdot\partial_\ell(P^{mn})\big)\,\partial_i$ (modulo Hamiltonian vector fields $[\![P(\varrho),H([\varrho])]\!]$), encoded by the "sunflower" linear combination[6] of Kontsevich graphs $X^{\gamma_3} = \sum_{a=1}^{3}\big(0,a;1,3;1,2\big) = \big(0,1;1,3;1,2\big) + \big(0,2;1,3;1,2\big) + \big(0,3;1,3;1,2\big) = \big(0,1;1,3;1,2\big) + 2\cdot\big(0,2;1,3;1,2\big)$.

2. In dimension $d = 2$, Poisson-coboundary equation (3) is valid as an equality of bi-vectors with differential-polynomial coefficients (w.r.t. ϱ in bi-vector P), but its left-hand side cannot be expressed as a linear combination of zero Kontsevich graphs and Leibniz graphs (that is, differential consequences of the Jacobi identity, see [1]).

Proof. Equality (3) in Part 1 of Proposition 1 is verified in $d = 2$ by straightforward calculation with differential-polynomial coefficient (multi)vectors.

To explore *why* equality (3) is valid in $d = 2$, let us inspect whether it is the standard, working in all dimensions $d \geqslant 2$, Leibniz-graph mechanism that would ensure the vanishing, $Q_{\gamma_3}\big(P(\varrho)\big) - [\![P(\varrho),\vec{X}([\varrho])]\!] \doteq 0$ for any $\varrho(x,y)$, by force of the Jacobi identity realized by Kontsevich graphs. (We claim that it is *not only* this mechanism which does the job.)

For this, we list all connected directed graphs built over two sinks of in-degree 1 from one trident and two wedges, without co-directed double or triple edges, and with none or one tadpole. (The Jacobiator vertex with three outgoing edges will be expanded to the linear

[4] The encoding of each Kontsevich directed graph is an ordered list of ordered pairs of target vertices for the edges issued from the ordered set of arrowtails (here, $\{2,3,4,5\}$), that is of aerial vertices.

[5] For the same reason, the Poisson condition – trivial in $d = 2$ – is preserved by *any* Kontsevich bi-vector graph (not necessarily obtained from a cocycle, on n vertices and $2n-2$ edges, w. r. t. the graph differential [3, 5]). Yet, Proposition 1 condensed to Eq. (3) is not a tautology since the second Lichnerowicz–Poisson cohomology does not vanish *a priori* over $d = 2$; see, e. g., Ph. Monnier *Poisson cohomology in dimension two*, Israel J. Math. **129** (2002) 189–207 (Preprint `arXiv:math.DG/0005261`).

[6] The 'sunflower' 1-vector graph in Eq. (3) expands, by the Leibniz rule, to the linear combination of two Kontsevich graphs, one of them with a 1-cycle (or *tadpole*). This nonzero graph with tadpole in X^{γ_3} survives in the bracket $[\![P,X^{\gamma_3}]\!]$, yet no tadpoles are present in the three graphs of the γ_3-flow Q_{γ_3}. The disappearance of the tadpole from $Q_{\gamma_3}\big(P(\varrho)\big) - [\![P(\varrho),\vec{X}]\!]$ is due to a mechanism which will be explored.

combination of three Kontsevich's graphs, see [1].) There are three admissible Leibniz graphs without tadpoles,

$$1.\quad (3,4;2,4;0,1,2)\,,\qquad 2.\quad (1,3;2,4;0,2,3)\,,\qquad 3.\quad (1,4;2,4;0,2,3)\,,$$

where $0,1$ are the sinks, vertices $2,3$ are wedge tops, and vertex 4 is the top of the trident. Likewise, we have nine admissible Leibniz graphs with one tadpole and sinks of in-degree 1:

$$\begin{array}{llllll} 4. & (2,4;2,4;0,1,2)\,, & 5. & (2,4;2,4;0,1,3)\,, & 6. & (2,3;2,4;0,1,2)\,,\\ 7. & (2,3;2,4;0,1,3)\,, & 8. & (1,2;2,4;0,2,3)\,, & 9. & (1,3;2,3;0,2,3)\,,\\ 10. & (1,4;2,3;0,2,3)\,, & 11. & (1,3;3,4;0,2,3)\,, & 12. & (1,4;3,4;0,2,3) \end{array}$$

(There remain four connected Leibniz graphs with *two* tadpoles but they are irrelevant for our present attempt to balance Eq. (3) using the topologies of Kontsevich graph expansions in the right-hand side).

When the wedge graph P acts on the 'sunflower' 1-vector graph X^{γ_3}, three graph topologies with one tadpole are produced, among others (the tadpole is absent from all the other topologies that appear in $[\![P,X^{\gamma_3}]\!]$):

$$\text{A.}\quad (0,4;1,3;3,5;3,4)\,,\qquad \text{B.}\quad (0,3;1,3;3,5;3,4)\,,\qquad \text{C.}\quad (2,5;2,4;2,3;0,1)\,. \tag{4}$$

In Kontsevich's nonzero graph B, vertex 3 has in-degree four (tadpole counted). But no expansion – of Leibniz graph from the above list of twelve – into Kontsevich graphs results in a graph with vertex of in-degree four. Independently, nonzero bi-vector graph C – with tadpole on vertex 2 at distance two from either sink and with double edge $3 \rightleftarrows 4$ – is not obtained in the Kontsevich graph expansion of any of the relevant bi-vector Leibniz graphs 4–7 from the above list. Thirdly, only the Leibniz graph 8 reproduces nonzero graph A, but the Kontsevich graph expansion of 8 contains five more graphs (with a tadpole at distance one from the sink), none of which appears in the linear combination $Q_{\gamma_3} - [\![P,X^{\gamma_3}]\!]$ of Kontsevich graphs. Therefore, we have that $Q_{\gamma_3} - [\![P,X^{\gamma_3}]\!] \neq \Diamond\big(P,P,\tfrac{1}{2}[\![P,P]\!]\big)$, for any values of coefficients $\in \mathbb{R}$ near Leibniz graphs in the right-hand side. The proof is complete. $\square$

The fact of Poisson trivialization of the tetrahedral-graph flow $\dot{P} = Q_{\gamma_3}(P)$ in dimension two, $P \in \Gamma\big(\bigwedge^2 TM^2_{\text{aff}}\big)$, impossible in this or any higher dimension $d \geqslant 2$ through the mechanism of vanishing by force of the Jacobi identity as the *only* obstruction, implies the existence of other analytic mechanism(s); those can work in combination with the former.

Remark 1. When all the sums over repeated indices, each running from 1 to the finite dimension $d = 2$, are expanded in the left-hand side of the identity $Q_{\gamma_3}\big(P(\varrho)\big) - [\![P(\varrho),\vec{X}\big(P(\varrho)\big)]\!] = 0 \in \Gamma\big(\bigwedge^2 TM^2_{\text{aff}}\big)$, the arithmetic cancellation mechanism works several times: the l.-h.s. splits into sums which cancel out separately from each other. For instance, when the Poisson differential $[\![P(\varrho),\cdot]\!]$ acts on $\vec{X}$ and this 1-vector gets into a sink of the wedge graph of bi-vector, thus forming graphs A and B in particular (see Eq. (4)), the wedge top coefficient ϱ has no derivative(s) falling on it. Yet no such terms occur at $d = 2$ in the graphs of Q_{γ_3} with strictly positive in-degrees of internal vertices. Hence the two parts – with(out) zero-order derivative of ϱ – of the differential-polynomial coefficient in the identity's l.-h.s. vanish simultaneously.

Remark 2 (on $GL(\infty)$-invariants parent to $GL(d)$-invariants). Kontsevich's construction of tensors from Poisson bi-vectors by using graphs is well-behaved under affine coordinate changes on the underlying manifold, thanks to contraction of upper indices of Poisson tensors (P^{ij}) in the arrowtail vertices and of lower indices in derivations at the edge arrowheads. The Jacobians of coordinate changes then belong to the general linear group, while the affine shifts are not felt at all by the index contractions. But, uniform over the dimensions d of affine

Poisson manifolds, the graph technique builds invariants of linear representations for $GL(\infty)$. Linearly independent as graph expressions, such invariants can be linearly tied after projection to a given finite dimension d, where they become tensor-valued $GL(d)$-invariants.[7]

To conclude this section, we note that the original graph language of [5] for construction of tensor-valued invariants of $GL(\infty)$ is no longer enough for the study of Poisson (non)triviality for graph-cocycle flows on spaces of Poisson brackets over affine manifolds of given dimension d. To manage the bound $d < \infty$, one must either take a quotient over the (unknown) new linear relations between Kontsevich's graphs, or work *ab initio* with $GL(d)$-invariants.

We note also that in a given dimension, the problem of Poisson (non)triviality for universal flows (from [5] and subsequent work [2]) itself is meaningful: *a priori* nontrivial deformation can become trivial for a class of Poisson geometries at given d. Let us verify the triviality of the tetrahedral-graph flow $\dot{P}([\varrho],[a]) = Q_{\gamma_3}\big(P(\varrho,[a])\big)$ on the space of Nambu-determinant Poisson bi-vectors $P(\varrho,[a])$ over an affine threefold $\mathbb{R}^3$.

3 Nambu-determinant Poisson brackets: The γ_3-flow over $\mathbb{R}^3$

Definition 1. The *Nambu-determinant Poisson bracket* on $\mathbb{R}^{d\geqslant 3}$ is the derived bi-vector $P(\varrho,[a]) \stackrel{\text{def}}{=} [\![[\![\ldots[\![\varrho\cdot\partial_{x^1}\wedge\ldots\wedge\partial_{x^d}, a_1]\!],\ldots]\!], a_{d-2}]\!]$, where $\varrho(\boldsymbol{x})\cdot\partial_{\boldsymbol{x}}$ is a d-vector field and scalar functions a_ℓ are Casimirs ($1\leqslant \ell\leqslant d-2$). In global (e.g., Cartesian) coordinates $x^1,\ldots,x^d$ on $\mathbb{R}^d$, the Nambu bracket of $f,g\in C^1(\mathbb{R}^d)$ is expressed by the formula

$$\{f,g\}_{P(\varrho,[\boldsymbol{a}])} = \varrho(\boldsymbol{x})\cdot\sum\nolimits_{i_1\ldots,i_d=1}^{d}\varepsilon^{\vec{\imath}}\cdot\partial_{i_1}(a_1)\cdots\partial_{i_{d-2}}(a_{d-2})\cdot\partial_{i_{d-1}}(f)\cdot\partial_{i_d}(g), \tag{5}$$

where $\varepsilon^{\vec{\imath}} = \varepsilon^{i_1,\ldots,i_d}$ is the Levi-Civita symbol on $\mathbb{R}^d$: $\varepsilon^{\sigma(1,\ldots,d)} = (-)^\sigma$ for $\sigma\in S_d$, else zero.[8]

Remark 3. Nambu–Poisson brackets on $\mathbb{R}^{d\geqslant 3}$ can be obtained from Nambu–Poisson brackets on $\mathbb{R}^{d+1}$ by taking $a_{d-1} = \pm x^{d+1}$ on $\mathbb{R}^{d+1}$ and by excluding the last Cartesian coordinate x^{d+1} from the list of arguments for $\varrho(\boldsymbol{x})$ and $a_1,\ldots,a_d(\boldsymbol{x})$. • By doing the above for $d+1=3$, one obtains a generic bi-vector $P=\varrho(x^1,x^2)\,\partial_{x^1}\wedge\partial_{x^2}$, which is Poisson on $\mathbb{R}^2$, and the (Nambu) Poisson bracket $\{f,g\}(x,y) = \varrho(x,y)\cdot(f_x\cdot g_y - f_y\cdot g_x)$.

We recall from [4, §4.1] that, given a suitable graph cocycle γ (e.g., γ_3 which we take here), Kontsevich's γ-flow $\dot{P} = \mathrm{O}\vec{\mathrm{r}}(\gamma)(P^{\otimes\#\mathrm{Vert}(\gamma)})$ restricts to the set of Nambu–Poisson bi-vectors $P(\varrho,[\boldsymbol{a}])$ such that the velocity of a Casimir a_ℓ is still encoded by Formality graphs [4, Proposition 2]: $\dot{a}_\ell = \mathrm{O}\vec{\mathrm{r}}(\gamma)(P\otimes\cdots\otimes P\otimes a_\ell)$, whence the velocity $\dot{\varrho}([\varrho],[\boldsymbol{a}])$ is expressed from the known $\dot{\boldsymbol{a}}$ and $\dot{P}$ (see [4, Corollary 3]). The Leibniz rule, balancing $\dot{P}$ with $\dot{\varrho},\dot{\boldsymbol{a}}$ for P linear in ϱ and the first jets of all a_ℓ, is then a tautology.

Independently, if $\vec{Y}$ is any C^1-vector field on $\mathbb{R}^d$ with Nambu–Poisson bi-vectors $P(\varrho,[\boldsymbol{a}])$, then the evolution $L_{\vec{Y}}(a_\ell) = [\![\vec{Y},a_\ell]\!]$ of scalar functions and $L_{\vec{Y}}(\varrho\cdot\partial_{\boldsymbol{x}}) = [\![\vec{Y},\varrho\cdot\partial_{\boldsymbol{x}}]\!]$ of d-

[7] The main example is given by the linear independence (modulo zero graphs and Leibniz graphs) of three Kontsevich graphs in the γ_3-flow and, on the other hand, of the Poisson-exact bi-vector graphs $[\![P,X^{\gamma_3}]\!]$ with the 'sunflower' 1-vector X^{γ_3}: an insoluble equation for graphs, $Q_{\gamma_3} - [\![P,X^{\gamma_3}]\!] = 0$, turns at $d=2$ into an identity of bi-vector's differential-polynomial coefficients, $Q_{\gamma_3}\big(P(\varrho)\big) - [\![P(\varrho),\vec{X}\big(P(\varrho)\big)]\!] \equiv 0$, for the Poisson structure $P(\varrho) = \varrho(x,y)\cdot\big(\partial_x\otimes\partial_y - \partial_y\otimes\partial_x\big)$ in every chart of the affine twofold M^2_{aff}.

[8] The usual view of Nambu–Poisson bracket (5) on $\mathbb{R}^d$ is that the Jacobian determinant is multiplied by an arbitrary factor $\varrho(x^1,\ldots,x^d)$ which behaves appropriately under coordinate changes $\boldsymbol{x}\rightleftarrows\boldsymbol{x}'$. Our viewpoint is that $\varrho(\boldsymbol{x})\,\partial_{x^1}\wedge\ldots\wedge\partial_{x^d}$ is a top-degree multi-vector on $\mathbb{R}^d$ for any $d\geqslant 2$, so that for all dimensions greater than two, the Nambu-determinant Poisson bi-vector is *derived*: $P(\varrho,[\boldsymbol{a}]) = [\![\ldots[\![\varrho\,\partial_{\boldsymbol{x}}, a_1]\!]\ldots, a_{d-2}]\!]$ with $d-2$ Casimirs $\boldsymbol{a}$. That is, Nambu structures generalize the bi-vector $\varrho(x,y)\cdot(\partial_x\otimes\partial_y - \partial_y\otimes\partial_x)$ from $\mathbb{R}^2$ to $\mathbb{R}^d$ for $d>2$. The Casimirs a_ℓ Poisson-commute with any $f\in C^1(\mathbb{R}^d)$; the symplectic leaves are intersections of level sets $a_\ell = \text{const}(\ell)\in\mathbb{R}$, so that for any $d\geqslant 2$ these leaves are at most two-dimensional.

vectors correlates, by the Leibniz-rule shape of the Jacobi identity for the Schouten bracket $[\![\cdot,\cdot]\!]$, with evolution $L_{\vec{Y}}(P) = [\![\vec{Y}, P]\!]$ of Nambu bi-vector $P = [\![\varrho \cdot \partial_{\boldsymbol{x}}, \cdots \boldsymbol{a} \cdots]\!]$, see [4, §2.1].

Theorem 2 ([4]). *In dimension $d = 3$, the tetrahedral-graph flow $\dot{P} = Q_{\gamma_3}(P)$ on the space of Poisson bi-vectors P has a well-defined restriction to the subspace of Nambu-determinant Poisson bi-vectors $P(\varrho, [a])$ on $\mathbb{R}^3$, and this restriction is Poisson-cohomology trivial: $\dot{P}([\varrho],[a]) = [\![P(\varrho,[a]), \vec{X}^{\gamma_3}([\varrho],[a])]\!]$. The equivalence class $\vec{X}^{\gamma_3}$ mod $[\![P, H(\varrho, a)]\!]$ of trivializing vector field is represented by the vector $\vec{X} = \sum_{\vec{\imath},\vec{\jmath},\vec{k}} \varepsilon^{\vec{\imath}} \varepsilon^{\vec{\jmath}} \varepsilon^{\vec{k}} \cdot X_{\vec{\imath}\vec{\jmath}\vec{k}}$ with*

$$
\begin{aligned}
X_{\vec{\imath}\vec{\jmath}\vec{k}} = {} & 12 \varrho \varrho_{x^{k_2}} \varrho_{x^{i_1} x^{j_1}} a_{x^{k_3}} a_{x^{i_2} x^{j_2}} a_{x^{i_3} x^{j_3}} \cdot \partial/\partial x^{k_1} + 48 \varrho \varrho_{x^{j_3}} \varrho_{x^{i_1} x^{j_1}} a_{x^{k_3}} a_{x^{i_2} x^{j_2}} a_{x^{i_3} x^{k_1}} \cdot \partial/\partial x^{k_2} \\
& + 8 \varrho_{x^{j_2}} \varrho_{x^{i_1} x^{k_1}} \varrho_{x^{i_2} x^{k_2}} a_{x^{i_3}} a_{x^{j_3}} a_{x^{k_3}} \cdot \partial/\partial x^{j_1} - 40 \varrho_{x^{i_3}} \varrho_{x^{j_2}} \varrho_{x^{i_1} x^{k_1}} a_{x^{j_3}} a_{x^{k_3}} a_{x^{i_2} x^{k_2}} \cdot \partial/\partial x^{j_1} \\
& + 8 \varrho_{x^{i_3}} \varrho_{x^{j_2}} \varrho_{x^{k_3}} a_{x^{j_3}} a_{x^{i_1} x^{k_1}} a_{x^{i_2} x^{k_2}} \cdot \partial/\partial x^{j_1} + 24 \varrho_{x^{j_2}} \varrho_{x^{k_3}} \varrho_{x^{i_1} x^{k_1}} a_{x^{i_3}} a_{x^{j_3}} a_{x^{j_1} x^{k_2}} \cdot \partial/\partial x^{i_2} \\
& - 12 \varrho^2 \varrho_{x^{k_2}} a_{x^{i_1} x^{j_1}} a_{x^{i_2} x^{j_2}} a_{x^{i_3} x^{j_3} x^{k_3}} \cdot \partial/\partial x^{k_1} + 24 \varrho \varrho_{x^{j_2}} \varrho_{x^{k_1}} a_{x^{k_2}} a_{x^{i_1} x^{j_1}} a_{x^{i_3} x^{j_3} x^{k_3}} \cdot \partial/\partial x^{i_2} \\
& - 36 \varrho \varrho_{x^{i_2}} \varrho_{x^{j_2}} a_{x^{k_2}} a_{x^{i_1} x^{j_1}} a_{x^{i_3} x^{j_3} x^{k_3}} \cdot \partial/\partial x^{k_1} + 8 \varrho_{x^{i_2}} \varrho_{x^{j_1}} \varrho_{x^{k_1}} a_{x^{j_2}} a_{x^{k_2}} a_{x^{i_3} x^{j_3} x^{k_3}} \cdot \partial/\partial x^{i_1} \\
& - 8 \varrho_{x^{j_1}} \varrho_{x^{k_1}} \varrho_{x^{i_3} x^{j_3} x^{k_3}} a_{x^{i_2}} a_{x^{j_2}} a_{x^{k_2}} \cdot \partial/\partial x^{i_1},
\end{aligned}
$$

where $\vec{\imath} = (i_1, i_2, i_3)$, $\vec{\jmath} = (j_1, j_2, j_3)$, $\vec{k} = (k_1, k_2, k_3)$ and ε^{pqr} is the Levi-Civita symbol on $\mathbb{R}^3$.

Our next finding in [4, Theorem 8] is that for the graph cocycle γ_3 and $d = 3$, the action of vector field $\vec{X}$ (which trivializes the γ_3-flow $\dot{P} = Q_{\gamma_3}([P]) = [\![P, \vec{X}]\!]$ of Nambu brackets P on $\mathbb{R}^3$) upon $P(\varrho, [a])$ factors through the initially known – from γ_3 – velocities of a and ϱ: having solved (1) for $\vec{X}$, we then verified that $\dot{a} = [\![a, \vec{X}]\!]$ and $\dot{\varrho} \cdot \partial_{\boldsymbol{x}} = [\![\varrho \cdot \partial_{\boldsymbol{x}}, \vec{X}]\!]$.

By using this factorization – i.e. the lifting of the sought vector field's action on the elements of $P(\varrho, [a])$ – the other way round, we create an economical scheme to inspect the *existence* of trivializing vector field $\vec{X}$ for larger problems (i.e. for bigger graph cocycles or higher dimension $d \geqslant 3$). When this shortcut works, so that $\vec{X}$ is found, it saves much effort. Otherwise, to establish the (non)existence of $\vec{X}$ one deals with a larger PDE, namely Eq. (1).

Definition 2. Fix the dimension $d \geqslant 2$. A *micro-graph* is a directed graph built over $m \geqslant 0$ sinks, over $n \geqslant 0$ aerial vertices with out-degree d and ordering of outgoing edges, and over n items of $(d-2)$-tuples of aerial vertices with in-degree $\geqslant 1$ and no outgoing edges. • The correspondence between micro-graphs and differential-polynomial expressions in $\varrho, a_1, \ldots, a_{d-2}$ and the content of sink(s) is defined in the same way as the mapping of Kontsevich's graphs to multi-differential operators on $C^\infty(M^d_{\text{aff}})$, see [4, §2.2] or [5]. • Same as for Kontsevich graphs, a micro-graph is *zero* if it admits a sign-reversing automorphism, i.e. a symmetry which acts by parity-odd permutation on the ordered set of edges. But now, in finite dimension d of M^d_{aff}, a micro-graph is *vanishing* if the differential polynomial (in ϱ and a_1, …, a_{d-2}), obtained by expanding all the sums over indices that decorate the edges, vanishes identically.[9]

Example 2. Nambu–Poisson brackets $P(\varrho, [a_1], \ldots, [a_{d-2}])$ on $\mathbb{R}^d$ are realized using micro-graphs, namely by resolving $\varrho(\boldsymbol{x}) \cdot \varepsilon^{i_1,\ldots,i_d}$ in one vertex against $d-2$ vertices with the Casimirs $a_1, \ldots, a_{d-2}$. The out-degree of vertex with $\varrho(\boldsymbol{x}) \cdot \varepsilon^{\vec{\imath}}$ equals d; the in-degree of each vertex with a Casimir equals 1 and its out-degree is zero: the Casimir vertices are terminal (not to be confused with the two sinks, of in-degree 1, for the Poisson bracket arguments). The ordered d-tuple of edges is decorated with summation indices: for the Levi-Civita symbol $\varepsilon^{i_1,\ldots,i_d}$ in their common arrowtail vertex, the range is $1 \leqslant i_\ell \leqslant d$ for $1 \leqslant \ell \leqslant d$.

Remark 4. If the wedge tops contain Nambu–Poisson bi-vectors $P(\varrho, [a])$ on $\mathbb{R}^d$, every Kontsevich graph expands to a linear combination of micro-graphs: the arrow(s) originally in-coming

[9] There are nonzero but still vanishing micro-graphs.

to an aerial vertex with a copy of P, now work(s) by the Leibniz rule over the $d-1$ vertices, with $\varrho \cdot \varepsilon^{\vec{\imath}}$ and with $a_1, \ldots, a_{d-2}$, in the subgraphs $P(\varrho, [a_1], \ldots, [a_{d-2}])$ of the micro-graph.[10]

Example 3. In $d=3$, the micro-graph expansion of $Q_{\gamma_3}(P)$ for $P(\varrho, [a])$ over $\mathbb{R}^3$ consists of directed graphs on 2 sinks for f and g, on four terminal vertices – for copies of the Casimir a – without outgoing arrows, and on four vertices for $\varrho \cdot \varepsilon^{ijk}$ with three ordered outgoing edges. In every micro-graph in bi-vector $Q_{\gamma_3}(P)$ there are 12 edges, with exactly one going towards f and one to g in the sinks. To have a solution $\vec{X}$ of the equation $Q_{\gamma_3}(P(\varrho, [a])) = [\![P, \vec{X}]\!]$ using micro-graphs that encode $\vec{X}([\varrho], [a])$, we thus need micro-graphs on one sink, three terminal vertices with a, and three trident vertices for $\varrho \cdot \varepsilon^{ijk}$. Of the nine edges in each micro-graph, exactly one goes to the sink, so that $\vec{X}$ is a 1-vector.

Example 4. In $d=3$, two Kontsevich's graphs of the 'sunflower' 1-vector (which trivializes the restriction of tetrahedral-graph flow (2) to the space of bi-vectors on an affine twofold) expand to a linear combination of 42 one-vector micro-graphs over one sink, three trident vertices, three terminal vertices, and $3 \times 3 = 9$ edges (one into the sink). A tadpole is met in $10 = 3+3+4$ such micro-graphs, and the other $32 = 8 \times 4$ have none.

Let us illustrate how the shortcut scheme works. We now tune a 1-vector field $\vec{X}(\varrho, [a])$ for the flow $\dot{P} = Q_{\gamma_3}([P])$ of $P(\varrho, [a])$ over $\mathbb{R}^3$ such that $\dot{a} = [\![a, \vec{X}]\!]$ and $\dot{\varrho} \cdot \partial_{\boldsymbol{x}} = [\![\varrho \cdot \partial_{\boldsymbol{x}}, \vec{X}]\!]$, whence we verify that $\dot{P} = [\![P, \vec{X}([\varrho], [a])]\!] \in \operatorname{im} \partial_P$ for the Nambu-determinant class of Poisson brackets on $\mathbb{R}^3$.

We first generate all suitable unlabeled micro-graphs (i.e. without distinguishing which sinks are for Casimirs) without tadpoles and with exactly one tadpole. Next, by deciding on the run which of the four sinks is the argument of 1-vector, we produce 366 1-vector fields with differential-polynomial coefficients in ϱ and a, encoded by micro-graphs. Some of the coefficients are identically zero when the sums over three triples of indices in Levi-Civita symbols are fully expanded; there remain 244 nonvanishing marker micro-graphs in the ansatz for the trivializing vector field $\vec{X}$. Now, we do not attempt to solve the big problem $Q_{\gamma_3}(P) = [\![P, \vec{X}]\!]$ directly with respect to the 244 coefficients of nonvanishing marker micro-graphs. Instead, let us find a vector field $\vec{X}$, realized by 1-vector micro-graphs X^γ, which reproduces the known velocities [4, Eq. (11)] of ϱ and Casimir a, that is, we solve the equations $\dot{a} = -[\![\vec{X}, a]\!]$ and $\dot{\varrho}\, \partial_x \wedge \partial_y \wedge \partial_z = [\![\varrho\, \partial_x \wedge \partial_y \wedge \partial_z, \vec{X}]\!]$ with respect to the coefficients in the micro-graph ansatz for X^γ. To determine exactly the number of equations in either linear algebraic system we keep track of the number of differential monomials appearing when $\vec{X}$ acts on either a or ϱ as above, and we recall also the differential monomials which already appeared in $\dot{a}$ and $\dot{\varrho}$ in the left-hand sides, that is in [4, Eq. (11)]. In this way, we detect that the linear algebraic system for $\dot{a}$ contains 2961 equations and the system for $\dot{\varrho}$ contains 6679 equations. Each equation is a balance of the coefficient of one differential monomial. We now merge these two systems of linear algebraic equations upon the coefficients of micro-graphs in the ansatz for the trivializing vector field $\vec{X}$, and we find a solution. Only 11 coefficients are nonzero. The analytic formula of this vector field is reported in Theorem 2. The three equalities, namely $\dot{a} = -[\![\vec{X}, a]\!]$ and $\dot{\varrho}\, \partial_x \wedge \partial_y \wedge \partial_z = [\![\varrho \partial_x \wedge \partial_y \wedge \partial_z, \vec{X}]\!]$ implying $Q_{\gamma_3}(P) = [\![P, \vec{X}]\!]$, are verified immediately. Here is the encoding[11] of the weighted sum X^γ of these 11 micro-graphs which

[10] But not every micro-graph is obtained from a Kontsevich graph by resolving the old aerial vertices into subgraphs. This is what makes interesting our graphical rephrasing of the Poisson (non)triviality problem for the restriction of Kontsevich graph flows to the class of Nambu-determinant Poisson brackets.

[11] Vertices are labelled and the sink is indicated; for each micro-graph, its directed edges are listed in due ordering: e.g., $(6,6)$ is the tadpole $6 \to 6$ and $(0,4)$, $(0,5)$, $(0,6)$ is a trident Left $\prec$ Middle $\prec$ Right.

realize the trivializing vector field $\vec{X}$ for the tetrahedral flow $Q_{\gamma_3}(P) = [\![P,\vec{X}]\!]$ on the space of Nambu–Poisson structures over $\mathbb{R}^3$:

```
 16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,0), (6,2), (6,3)]      (sink 2),
 24 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,1), (6,2), (6,3)]      (sink 2),
 16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,2), (6,1), (6,3), (6,5)]      (sink 2),
-16 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,2), (6,1), (6,3), (6,5)]      (sink 4),
 12 * [(0,4), (0,5), (0,6), (5,1), (5,2), (5,6), (6,1), (6,2), (6,3)]      (sink 3),
-12 * [(0,4), (0,5), (0,6), (5,1), (5,2), (5,6), (6,1), (6,2), (6,3)]      (sink 4),
 24 * [(4,0), (4,1), (4,6), (5,0), (5,1), (5,2), (6,0), (6,2), (6,3)]      (sink 3),
-24 * [(4,0), (4,1), (4,6), (5,0), (5,2), (5,4), (6,0), (6,1), (6,3)]      (sink 2),
  8 * [(4,0), (4,1), (4,5), (5,0), (5,2), (5,6), (6,0), (6,3), (6,4)]      (sink 1),
 -8 * [(4,0), (4,1), (4,5), (5,2), (5,3), (5,6), (6,0), (6,1), (6,4)]      (sink 2),
  8 * [(0,4), (0,5), (0,6), (5,0), (5,1), (5,6), (6,2), (6,3), (6,6)]      (sink 2).
```

Remark 5. At the level of micro-graphs, the solution X^{γ_3} contains a tadpole, i.e. a 1-cycle, in the last graph. In terms of differential polynomials this means the presence of a deriative ∂_{x^i} acting on the coefficient $\varrho(\boldsymbol{x})$ near the Levi-Civita symbol ε^{ijk} containing the index i of the base coordinate x^i in that derivative; that is, the last term in the vector field $\vec{X}$ contains $\partial_{x^i}(\varrho(\boldsymbol{x}))\cdot\varepsilon^{ijk}$.

Proposition 3. Without tadpoles in the micro-graph ansatz X^{γ_3}, there is no solution $\vec{X}$ to the trivialization problem $Q_{\gamma_3}(P(\varrho,[a])) = [\![P,\vec{X}]\!]$ at the level of differential polynomials.[12]

Remark 6. If $d = 2$ and (Nambu–)Poisson brackets on $\mathbb{R}^2$ are $\{f,g\}(x,y) = \varrho\cdot(f_x\cdot g_y - f_y\cdot g_x)$ as in Remark 3, the only possible Kontsevich 'sunflower' graphs, built from $n = 3$ wedges over one sink (see (3)), tautologically expand to a nontrivial linear combination of nonzero micro-graphs on three aerial vertices with $\varrho(x,y)\cdot\varepsilon^{i^\alpha j^\alpha}$, $1 \leqslant \alpha \leqslant 3$. Independently, the linear combination X^γ of micro-graphs that encode the trivializing vector field $\vec{X}([\varrho],[a])$ for Kontsevich's γ_3-flow for Nambu bi-vectors $P(\varrho,[a])$ on $\mathbb{R}^3 \ni (x,\, y, z)$, see Theorem 2, under the reduction $a := z$ and $\varrho = \varrho(x,y)$ becomes a well-defined vector field on the plane $\mathbb{R}^2 \subset \mathbb{R}^3$: the z-component of $\vec{X}([\varrho(x,y)],[z])$ vanishes. Let us compare the two vector fields on $\mathbb{R}^2$.

Proposition 4. The old 'sunflower' vector field which trivializes the tetrahedral γ_3-flow for all Poisson brackets on $\mathbb{R}^2$ *coincides* with the new vector field $\vec{X}([\varrho(x,y)],[a=z])$ from the trivialization of γ_3-flow for the Nambu brackets $P(\varrho,[a])$ on $\mathbb{R}^3$ (both viewed as 1-vector fields on $\mathbb{R}^2$ with differential coefficients in $[\varrho]$). • Yet the linear combination $X_3^{\gamma_3}$ of micro-graphs over $d = 3$ contains *not only* and *not all* the expansions of Kontsevich's graphs from the 'sunflower' 1-vector into micro-graphs over $d = 3$.

Proof. The micro-graph expansion of the 'sunflower' graph is not enough in $d = 3$ because, in particular, the before-last micro-graph in $X_3^{\gamma_3}$, namely $(-8)\cdot[(4,0),(4,1),(4,5),(5,2),(5,3),$ $(5,6),(6,0),(6,1),(6,4)]$ with trident vertices $4,5,6$, sink 2, and terminal vertices $0,1,3$, does not originate from either graph in the 'sunflower' $X_2^{\gamma_3}$. Indeed, the above micro-graph contains a 3-cycle $4 \to 5 \to 6 \to 4$ but no edge from 6 to either 5 or its Casimir 3 in a would-be expansion of $P(\varrho,[a])$ with 5 in the trident top.[13] □

Let us examine how, by which mechanism(s), Eq. (1) is verified for the tetrahedral cocycle γ_3 and the respective flow of Nambu brackets on $\mathbb{R}^3$.

[12] By setting to zero the coefficients of micro-graphs with one tadpole, that is by excluding all the differential polynomials in ϱ and a which stem from those micro-graphs with a tadpole, we detect that the linear algebraic system for the coefficients of micro-graphs without tadpoles has no solution at all.

[13] Not all of the micro-graphs appearing in the expansion of Kontsevich's graphs $X_{d=2}^{\gamma_3}$ mod $[\![P,H]\!]$ are needed for a solution $X_3^{\gamma_3}$, see Example 4 and the eleven micro-graphs on p. 8.

Lemma 5. If $d=3$ and $P(\varrho,[a])$ is Nambu, the Jacobiator tri-vector graph Jac(P) remains a nontrivial linear combination, Jac$(P)([\varrho],[a])$, of 3 or 6 nonvanishing micro-graphs, each on $m=3$ sinks of in-degree 1, on two trident vertices with $\varrho\cdot\varepsilon^{i_1^\alpha i_2^\alpha i_3^\alpha}$, and on two terminal vertices of in-degrees 1 and 2 (if $\varrho\equiv$ const) or $(1,1)$ and $(1,2)$ if $\varrho\not\equiv$ const.

Proposition 6. The trivialization mechanism, $Q_{\gamma_3}(P(\varrho,[a]))-[\![P(\varrho,[a]),\vec{X}_3^{\gamma_3}([\varrho],[a])]\!]\doteq 0$, for the tetrahedral-graph flow on the space of Nambu-determinant Poisson bi-vectors $P(\varrho,[a])$ over $\mathbb{R}^3$ and for the linear combination of micro-graphs $X_3^{\gamma_3}$, does not amount *only to* the Leibniz (micro)graphs and zero (micro)graphs in the right-hand side of coboundary equation (1).

Proof. Suppose for contradiction that a linear combination $\Diamond$ of Leibniz and zero micro-graphs turns the coboundary equation, $Q_{\gamma_3}-[\![P,X^{\gamma_3}]\!]=\Diamond\big([\varrho],[a],\tfrac{1}{2}[\![P,P]\!]\big)$, into an equality of bi-vector micro-graphs. By setting the Casimir $a:=z$ we fix the values of indices decorating the arrows which run into the terminal vertices a; now for $\varrho(x,y)$ independent of z, the rest of each micro-graph becomes a Kontsevich graph over $d=2$. Zero micro-graphs from dimension 3 remain zero over dimension 2. This dimensional reduction would yield a Leibniz-graph realization of the corresponding r.-h.s. for the two-dimensional problem from Part 2 of Proposition 1. But this is impossible; therefore, at least one new mechanism of differential polynomials' vanishing is at work. □

Conclusion

Kontsevich's symmetries $\dot{P}=Q_\gamma(P)$ of the Jacobi identity $\frac{1}{2}[\![P,P]\!]=0$ are produced from suitable graph cocycles γ as described in [2,5]. We study the (non)triviality of these flows w.r.t. the Poisson differential $\partial_P=[\![P,\cdot]\!]$. The original formalism of [5] yields tensorial $GL(\infty)$-invariants; but in finite dimension d of Poisson manifolds, these tensors, encoded by Kontsevich's graphs, can become linearly dependent, whence the 'incidental' trivializations (e.g., in $d=2$). The graph language can be adapted to the subclass of Nambu-determinant Poisson brackets $P(\varrho,[\boldsymbol{a}])$; Kontsevich's graph-cocycle flows do restrict to the Nambu subclass (see [4]). The new calculus of micro-graphs is good for encoding known flows and vector fields. Still the core task of this research is *finding* these fields $\vec{X}\big([\varrho],[\boldsymbol{a}]\big)$ or proving their non-existence over $\mathbb{R}^d$.

The calculus of micro-graphs is (almost) well-behaved[14] under the dimensional reduction $d\mapsto d-1$ by the loss of one Casimir $a_{d-2}:=x^d$ and last coordinate x^d in ϱ and all other Casimirs a_ℓ, $\ell<d-2$. The forward move, $d\mapsto d+1$, is not well defined. *A priori* there is no guarantee that any solution X_{d+1}^γ exists at all: the dimensions $d_0<d_0+1$ can mark the threshold where the $GL(\infty)$-invariants from Kontsevich's graphs lose their linear dependence in lower dimensions $d\leqslant d_0$, and the flow $\dot{P}\big([\varrho],[a_1],\ldots,[a_{d-1}]\big)=Q_{d+1}^\gamma\big(P(\varrho,[\boldsymbol{a}])\big)$ becomes Poisson-nontrivial is dimension d_0+1 and onwards.

Lemma 7. Suppose there exists a trivializing vector field X_{d+1}^γ for a γ-flow of $P(\varrho,[\boldsymbol{a}])$ over $\mathbb{R}^{d+1}$, and this solution projects – when $a_1:=x^3$, ..., $a_{d-1}:=x^{d+1}$ – onto the known linear combination $X_{d=2}^\gamma$ of Kontsevich's graphs. Then X_{d+1}^γ contains (at least) those micro-graphs from the expansion of $X_{d=2}^\gamma$ over $d+1$ in which all the old edges between Poisson structures $P(\varrho,[\boldsymbol{a}])$ head to arrowtail vertices for ϱ but not to terminal vertices for the Casimirs a_ℓ.

Indeed, it is the differential polynomials from these micro-graphs which, staying nonzero, retract to the bottom-most solution.

[14]The projection – from micro-graphs to differential-polynomial coefficients (in ϱ and a_ℓ) of multi-vectors – has a kernel which contains, as a strict subset, the space of Leibniz and zero (micro)graphs, but does not amount only to it. Can the dimension reduction result in an identically zero vector field $\vec{X}_{d-1}^\gamma([\varrho],[\boldsymbol{a}])\equiv\vec{0}$ on $\mathbb{R}^{d-1}$? That is, can a tower of micro-graph solutions $X_{d\geqslant d_0}^\gamma$ start at bottom dimension $d_0>2$ above the main case of $\mathbb{R}^2$?

Remark 7. Another constraint – upon the derivative order profiles in $X^i([\varrho],[\boldsymbol{a}])$, hence in $[\![P,\vec{X}]\!]$ – comes from the bi-vector $Q_{\gamma_3}(P(\varrho,[\boldsymbol{a}]))$ with known differential-polynomial coefficients. In effect, terms in X^i can contain only those orders of derivatives which, under $[\![P,\cdot]\!]$, reproduce the actually existing profiles of derivatives in $Q_{\gamma_3}([\varrho],[\boldsymbol{a}])$.

Open problem 1. Is there a dimension $d+1<\infty$ at which the tetrahedral-graph flow on the space of Nambu structures over $\mathbb{R}^{d+1}$ becomes nontrivial in the second Poisson cohomology?

Acknowledgements

The authors thank the organizers of GROUP34 colloquium on group theoretical methods in Physics on 18–22 July 2022 in Strasbourg for a very warm atmosphere during the meeting. The authors thank M. Kontsevich and G. Kuperberg for helpful discussions; the authors are grateful to the referee for useful comments and suggestions.

Funding information A part of this research was done while A.K. was visiting at the IHÉS; A.K. thanks the IHÉS for hospitality, and thanks the IHÉS and Nokia Fund for financial support. The travel of A.K. was partially supported by project 135110 at the Bernoulli Institute, University of Groningen. The research of R.B. was supported by project 5020 at the Institute of Mathematics, Johannes Gutenberg–Universität Mainz and by CRC-326 grant GAUS "Geometry and Arithmetic of Uniformized Structures".

A (Non)triviality of γ_3-flow for Nambu–Poisson brackets on $\mathbb{R}^4$

From [4] we know that Kontsevich's tetrahedral γ_3-flow restricts to the space of Nambu-determinant Poisson bi-vectors $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$: the differential-polynomial velocities $\dot{\varrho}$ and $\dot{a}_1,\dot{a}_2$ inducing the graph cocycle evolution $\dot{P}([\varrho],[a_1],[a_2])$ are stored externally.[15] The evolutions $\dot{a}_1,\dot{a}_2(\varrho,[a_1],[a_2])$ are realized by Kontsevich graphs $\mathrm{O}\vec{\mathrm{r}}(\gamma_3)(P\otimes P\otimes P\otimes a_\ell)$, hence they are immediately expanded to micro-graph realizations. The evolution $\dot{\varrho}([\varrho],[a_1],[a_2])$ can then be expressed by using micro-graphs with minimal effort.

The problem of Poisson (non)triviality of the tetrahedral γ_3-flow for Nambu brackets in dimension $d=4$ is open. At the level of micro-graphs and Nambu-determinant Poisson structures $P(\varrho,[a_1],[a_2])$ over $\mathbb{R}^4$, a solution X^γ of (1) for the graph cocycle γ_3 would be realized by micro-graphs possibly with tadpoles, on one sink of in-degree 1, three vertices of out-degree 4, and two triples of terminal vertices for Casimirs a_1,a_1,a_1 and a_2,a_2,a_2.

Proposition 8. • There are 1,079 isomorphism classes of directed graphs on one sink, three vertices of out-degree four, six terminal vertices, and at most one tadpole (of them, 352 are without a tadpole and 727 have one tadpole).
• Taking those graphs containing a vertex of in-degree one (for the sink), and dynamically appointing the Casimirs from the multi-set $\{a_1,a_1,a_1,a_2,a_2,a_2\}$ to the six terminal vertices of the above graphs, we obtain 38,120 micro-graphs.
• Excluding repetitions in the above set of micro-graphs (e.g., if those micro-graphs are isomorphic), still not excluding micro-graphs which equal minus themselves under a symmetry (automorphism of micro-graph with outgoing edge ordering and known location of a_1's and a_2's) we obtain 19,957 micro-graphs in the ansatz for X^γ that would encode the trivializing vector field solutions, if any, of the coboundary equation $Q_{\gamma_3}\big(P(\varrho,[a_1],[a_2])\big)=[\![P,\vec{X}_4^{\gamma_3}([\varrho],[a_1],[a_2])]\!]$.

[15] https://rburing.nl/gcaops/adot_rhodot_g3_4D.txt

• Of these 19,957 micro-graphs, one tadpole is present in 13,653 micro-graphs, and there are no tadpoles in 6,304 micro-graphs.

Construction sketch. The representatives of isomorphism classes of graphs without tadpoles are generated by the nauty command-line call `geng 10 9:12 | directg -e12 | pickg -d0 -m7 -D4 -M3` [7]. Likewise, the graphs with one tadpole are generated by first producing graphs with one edge fewer, using `geng 10 0:12 | directg -e11 | pickg -d0 -m7 -D4`, and adding a tadpole to the lonely vertex of out-degree 3 in each graph. For the appointment of Casimirs to 6 vertices in all *different* ways, one uses an efficient algorithm to generate the 20 permutations of the multi-set $\{a_1, a_1, a_1, a_2, a_2, a_2\}$. □

Proposition 9 (R. Buring, PhD thesis (2022))**.** If $d = 2$, the Poisson cocycles $Q_\gamma(P)$ for graph cocycles $\gamma \in \{\gamma_3, \gamma_5, \gamma_7\}$ are Poisson-trivial: $Q_\gamma(P) = [\![P, \vec{X}^\gamma_2(P)]\!]$. Every such vector field $\vec{X}^\gamma_2(P)$ is Hamiltonian w.r.t. the standard symplectic structure $\omega = dx \wedge dy$ on $\mathbb{R}^2$ and Hamiltonian $H^\gamma(P)$. The differential polynomials $H^\gamma(P)$ are encoded by sums of Kontsevich graphs.

The case of γ_3 was known to Kontsevich [5], and the respective Hamiltonian was found by Bouisaghouane (see `arXiv:1702.06044` [math.DG]). The cases of γ_5 and chosen representative for the graph cocycle γ_7 are new.

Example 5. Let $P = u\partial_x \wedge \partial_y$ be the generic Poisson bi-vector on $\mathbb{R}^2$; we have $H^{\gamma_3} = 8u_y^2 u_{xx} - 16u_x u_y u_{xy} + 8u_x^2 u_{yy}$ and $H^{\gamma_5} = 6u_y^2 u_{xx} u_{xy}^2 - 12u_x u_y u_{xy}^3 - 6u_y^2 u_{xx}^2 u_{yy} + 12u_x u_y u_{xx} u_{xy} u_{yy} + 6u_x^2 u_{xy}^2 u_{yy} - 6u_x^2 u_{xx} u_{yy}^2 - 2u_y^3 u_{xy} u_{xxx} + 2u_x u_y^2 u_{yy} u_{xxx} + 2u_y^3 u_{xx} u_{xxy} + 2u_x u_y^2 u_{xy} u_{xxy} - 4u_x^2 u_y u_{yy} u_{xxy} - 4u_x u_y^2 u_{xx} u_{xyy} + 2u_x^2 u_y u_{xy} u_{xyy} + 2u_x^3 u_{yy} u_{xyy} + 2u_x^2 u_y u_{xx} u_{yyy} - 2u_x^3 u_{xy} u_{yyy} - 2u_y^4 u_{xxxx} + 8u_x u_y^3 u_{xxxy} - 12u_x^2 u_y^2 u_{xxyy} + 8u_x^3 u_y u_{xyyy} - 2u_x^4 u_{yyyy}$.

References

[1] A. Bouisaghouane, R. Buring and A. Kiselev, *The Kontsevich tetrahedral flow revisited*, J. Geom. Phys. **119**, 272 (2017), doi:10.1016/j.geomphys.2017.04.014.

[2] R. Buring and A. V. Kiselev, *The orientation morphism: From graph cocycles to deformations of Poisson structures*, J. Phys.: Conf. Ser. **1194**, 012017 (2019), doi:10.1088/1742-6596/1194/1/012017.

[3] R. Buring, A. V. Kiselev and N. J. Rutten, *The heptagon-wheel cocycle in the Kontsevich graph complex*, J. Nonlin. Math. Phys. **24**, 157 (2021), doi:10.1080/14029251.2017.1418060.

[4] R. Buring, D. Lipper and A. V. Kiselev, *The hidden symmetry of Kontsevich's graph flows on the spaces of Nambu-determinant Poisson brackets*, J. Nonlin. Math. Phys. **2**, 186 (2022), doi:10.46298/ocnmp.8844.

[5] M. Kontsevich, *Formality conjecture*, in *Deformation theory and symplectic geometry*, Springer, Dordrecht, Netherlands, ISBN 9780792345251 (1997).

[6] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. D **7**, 2405 (1973), doi:10.1103/PhysRevD.7.2405.

[7] B. D. McKay and A. Piperno, *Practical graph isomorphism. II*, J. Symb. Comput. **60**, 94 (2014), doi:10.1016/j.jsc.2013.09.003.

Hilbert space structure and classical limit of the low energy sector of U(N) quantum Hall ferromagnets

Manuel Calixto[1,2⋆], Alberto Mayorgas[1] and Julio Guerrero[2,3]

1 Department of Applied Mathematics, University of Granada,
Fuentenueva s/n, 18071 Granada, Spain
2 Institute Carlos I for Theoretical and Computational Physics, University of Granada,
Fuentenueva s/n, 18071 Granada, Spain
3 Department of Mathematics, University of Jaen,
Campus Las Lagunillas s/n, 23071 Jaen, Spain

⋆ calixto@ugr.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Using the Lieb–Mattis ordering theorem of electronic energy levels, we identify and construct the Hilbert space of the low energy sector of U(N) quantum Hall/Heisenberg ferromagnets at filling factor M for L Landau/lattice sites. The carrier Hilbert space of irreducible representations of U(N) is described by rectangular Young tableaux of M rows and L columns, and associated with Grassmannian phase spaces U(N)/U(M)×U($N-M$). Replacing U(N)-spin operators by their expectation values in a Grassmannian coherent state allows for a semi-classical treatment of the low energy U(N)-spin-wave coherent excitations (skyrmions) of U(N) quantum Hall ferromagnets in terms of Grasmannian nonlinear sigma models.

Received 15-12-2022
Accepted 11-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.021

1 Introduction

The magnetic interaction between adjacent $\langle\alpha,\beta\rangle$ dipoles is described by the U(2) (two-component electrons) Quantum Heisenberg Model Hamiltonian

$$H=-\frac{1}{2}\sum_{\langle\alpha,\beta\rangle}\mathcal{J}_x\sigma_x(\alpha)\sigma_x(\beta)+\mathcal{J}_y\sigma_y(\alpha)\sigma_y(\beta)+\mathcal{J}_z\sigma_z(\alpha)\sigma_z(\beta),\tag{1}$$

with $\sigma_{x,y,z}(\alpha)$ Pauli matrices at site α and $\mathcal{J}_{x,y,z}$ coupling constants. For positive $\mathcal{J}$, the dominant coupling between two dipoles may cause nearest-neighbors $\langle\alpha,\beta\rangle$ to have lowest energy when they are aligned (ferromagnetic case). The generalization of this model to N-component electrons arises in, for example, the two-body exchange interaction for N-component planar electrons in a perpendicular magnetic field [1], which adopts the form of a U(N) Quantum Hall Ferromagnet (QHF) Hamiltonian on a square lattice

$$H = -\mathcal{J} \sum_{\langle\alpha,\beta\rangle} \sum_{i,j=1}^{N} S_{ij}(\alpha) S_{ji}(\beta), \tag{2}$$

written in terms of U(N)-spin operators

$$S_{ij}(\alpha) = c_i^\dagger(\alpha) c_j(\alpha), \qquad \left[S_{ij}(\alpha), S_{kl}(\beta)\right] = \delta_{\alpha\beta}\left(\delta_{jk} S_{il}(\beta) - \delta_{il} S_{kj}(\beta)\right), \tag{3}$$

realized in terms of creation $c_i^\dagger(\alpha)$ and annihilation $c_i(\alpha)$ operators of an electron with component $i, j \in \{1, \ldots, N\}$ in a given Landau/lattice site $\alpha \in \{1, \ldots, L\}$ of a given Landau level (namely, the lowest one). The sum over $\langle\alpha,\beta\rangle$ extends over all near-neighbor Landau/lattice sites, and $\mathcal{J}$ is the exchange coupling constant (the spin stiffness for the XY model).

In particular, the electrons become multicomponent when, for example, in addition to the usual two spin components $\uparrow$ and $\downarrow$, they acquire extra "pseudospin" internal components associated with: (a) layer (for multilayer arrangements), (b) valley (like in graphene and other 2D Dirac materials), (c) sub-lattice, etc. In the case of a bilayer quantum Hall system in the lowest Landau level, one Landau site can accommodate $N = 4$ internal states/components $|i\rangle, i = 1, 2, 3, 4$ ("flavors")

$$|1\rangle = |\uparrow t\rangle, \quad |2\rangle = |\uparrow b\rangle, \quad |3\rangle = |\downarrow t\rangle, \quad |4\rangle = |\downarrow b\rangle, \tag{4}$$

where t and b make reference to the "top" and "bottom" layers, respectively. Since the electron field has $N = 4$ degenerate components, the bilayer system possesses an underlying U(4) symmetry. Likewise, the ℓ-layer case carries a U(2ℓ) symmetry.

For N-component electrons, the Pauli exclusion principle allows $M \leq N$ electrons per Landau/lattice site (the filling factor). Selecting a ground state ($|0\rangle_\mathrm{F}$ denotes de Fock vacuum)

$$|\Phi_0\rangle = \Pi_{\alpha=1}^{L} \Pi_{i=1}^{M} c_i^\dagger(\alpha) |0\rangle_\mathrm{F}, \tag{5}$$

which fills all L lattice sites with the first M internal levels $i = 1, \ldots, M \leq N$, spontaneously breaks the U(N) symmetry (SSB) since a general unitary transformation mixes the first M "spontaneously chosen" occupied internal levels with the $N - M$ unoccupied ones. The ground state $|\Phi_0\rangle$ is still invariant under the stability subgroup U(M) × U($N - M$) of transformations among the M occupied levels and the $N - M$ unoccupied levels, respectively. Therefore, the transformations that do not leave $|\Phi_0\rangle$ invariant are parametrized by the Grassmannian coset $\mathbb{G}_M^N = \mathrm{U}(N)/\mathrm{U}(M) \times \mathrm{U}(N-M)$, which reduces to the well known Bloch sphere $\mathbb{S}^2 = \mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$ for $N = 2$ spin components and $M = 1$ electron per Landau site ("symmetric multi-qubits" [2]).

In this article, we aim to describe the carrier Hilbert space associated with these U(N) representations, their coherent states [3], and the classical limit. The structure of the Hilbert space for a U(N) QHF with L Landau/lattice sites and filling factor M is sketched in Section 2. U(N) irreducible representations (IRs) are classified with Young diagrams. Lieb-Mattis ordering of electronic energy levels (based on the pouring principle for Young diagrams) identifies rectangular Young diagrams of L columns and M rows as the carrier Hilbert space of the lower energy sector. We also provide a Fock (boson and fermion) representation of basis states alternative to the Young tableau representation.

In the classical/continuum limit $L \to \infty$ (large U(N)-spin representations or large number of lattice sites), the U(N)-spin operators S_{ij} become c-numbers, and the low energy U(N)-spin-wave coherent excitations are named "skyrmions" [4–6]. These coherent excitations turn out to be governed by a ferromagnetic order parameter associated with this SSB and labeled by $(N-M)\times M$ complex matrices Z parametrizing the complex Grassmannian manifold $\mathbb{G}_M^N$ in Section 3. In fact, Grassmannian nonlinear sigma models (NLσM) describe the classical dynamics associated with these SU(N) quantum spin chains [7–12], generalizing the SU(2) NLσM for the continuum dynamics of Heisenberg (anti)ferromagnets [13–15]. In references such as [9, 10], N represents the number of fermion "flavors", whereas L is referred to as the number of "colours" n_c.

2 Lieb-Mattis theorem and low energy U(N) ferromagnetism

Given the Fourier transform

$$\mathcal{S}_{ij}(q)=\sum_{\alpha=1}^{L} e^{\mathrm{i}q\alpha} S_{ij}(\alpha)\,, \tag{6}$$

the long-wavelength (low momentum $q \simeq 0$) ground state excitations of QHFs are described by the collective operators

$$\mathcal{S}_{ij}(0)=\sum_{\alpha=1}^{L} S_{ij}(\alpha)\,, \tag{7}$$

which are invariant under site permutations $\alpha \leftrightarrow \alpha'$. The kind of IRs of U($N$) related to translation invariance are those described by rectangular Young diagrams of M rows and L columns

$$[L^M]=M\left\{\overbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \vdots & \vdots & \vdots \\ \hline & \cdots & \\ \hline\end{array}}^{L}\right.\,. \tag{8}$$

This means that physical states are symmetric (bosonic) under permutations of the L lattice sites and antisymmetric (fermionic) under permutation of the M electrons (the filling factor) at each lattice site. This reasoning gives an introductory and heuristic proof of the main Proposition 2.

As an interesting comment, in the quantum Hall effect approach, each electron occupies on average a surface area of $2\pi\ell_B^2$ (a Landau site, with ℓ_B the magnetic length) that is pierced by one magnetic flux quantum $\phi_0 = 2\pi\hbar/e$. This image allows for a dual bosonic Schwinger realization of collective U(N)-spin operators

$$\mathcal{S}_{ij}=\sum_{\mu=1}^{M} a_{i\mu}^{\dagger}a_{j\mu}\,, \qquad i,j=1,\dots,N\,, \tag{9}$$

this time in terms of creation $a_{i\mu}^{\dagger}$ and annihilation $a_{j\mu}$ *boson* operators of magnetic flux quanta attached to the electron $\mu=1,\dots,M$ with component $i=1,\dots,N$. From the usual bosonic commutation relations $[a_{i\mu},a_{j\nu}^{\dagger}] = \delta_{ij}\delta_{\mu\nu}$ we recover the U(N)-spin commutation relations (3). We shall not further pursue this bosonic picture here. For more information, we address the reader to the Reference [16].

The Hilbert space of a U(N) QHF with L Landau/lattice sites at integer filling factor M is the $\binom{N}{M}^L$-dimensional L-fold tensor product space $\mathcal{H}_N^{\otimes L}[1^M]=\bigotimes_{\alpha=1}^{L}\mathcal{H}_N^{\alpha}[1^M]$. In Young diagram notation

$$M\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right. \otimes \overset{L\text{ times}}{\cdots} \otimes \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array} \quad\longleftrightarrow\quad [1^M]^{\otimes L}=[1^M]\otimes \overset{L}{\cdots}\otimes[1^M]\,. \tag{10}$$

Basis vectors of $\mathcal{H}_N^\alpha[1^M]$ are the M-particle Slater determinants (for $M=1$ we have "quNits", as a N-ary quantum-digit generalization of qubits) written in Fock and Young tableau notation as

$$\Pi_{\mu=1}^M c^\dagger_{i_\mu}(\alpha)|0\rangle_\mathrm{F} = \begin{ytableau} i_1 \\ \vdots \\ i_M \end{ytableau}\,, \tag{11}$$

obtained by filling out columns of the corresponding Young diagram with components $i_\mu \in \{1,\dots,N\}$ in strictly increasing order $i_1 < \cdots < i_M$. One can see that there are exactly $\binom{N}{M}$ different arrangements of this kind (the dimension of $\mathcal{H}_N^\alpha[1^M]$). This tensor product representation of U(N) is reducible. For example, the Clebsch-Gordan decomposition of a tensor product of $L=2$ IRs of U(N) of shape $[1^M]$, with filling factor $M=2$ and $N\geq 4$ components, is represented by the following Young diagrams

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = {\color{red}\ydiagram{2,2}} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} \quad \longleftrightarrow \quad [1^2]\otimes[1^2] = {\color{red}[2^2]} \oplus [2,1^2] \oplus [1^4]\,, \tag{12}$$

where we have highlighted in red the rectangular case $[2^2]$ for later discussion. The $P(=ML)$-particle ground state (5) can be written in Young tableau notation

$$|\Phi_0\rangle = \Pi_{\alpha=1}^L \Pi_{i=1}^M c_i^\dagger(\alpha)|0\rangle_\mathrm{F} = \begin{ytableau} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ M & \dots & M \end{ytableau}\,, \tag{13}$$

and then it belongs to the carrier Hilbert space $\mathcal{H}_N[L^M]$ of the rectangular IR $[L^M]$ with dimension

$$D[L^M] = \frac{\prod_{i=N-M+1}^N \binom{i+L-1}{i-1}}{\prod_{i=2}^M \binom{i+L-1}{i-1}} \overset{M=1}{\longrightarrow} \binom{L+N-1}{L} \overset{N=2}{\longrightarrow} L+1\,. \tag{14}$$

Note that $\mathcal{H}_2[L^1]$ is just the usual $(2j+1)$-dimensional Hilbert space for the angular momentum $j=L/2$ representation of SU(2). We denote Young diagrams of $P=ML$ boxes/particles by (a partition of P)

$$h = [h_1,\dots,h_N] = \overbrace{\begin{ytableau} \none[] & \dots & \dots & \dots & \dots & \dots & \none[] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \none[] & \dots & \none[] \end{ytableau}}^{h_1} \qquad \begin{array}{l} h_1 \geq \cdots \geq h_N\,, \\ h_1+\cdots+h_N = P\,. \end{array} \tag{15}$$

The shorthand $[h,\overset{M}{\dots},h,0,\dots,0] = [h^M]$ is often used. Before presenting the central proposition of this work, we should define the concept of "dominance order $\succeq$" of Young diagrams of P particles as: h dominates h' (h is "more symmetric" than h') if

$$[h_1,\dots,h_N] \succeq [h'_1,\dots,h'_N] \Leftrightarrow h_1+\cdots+h_k \geq h'_1+\cdots+h'_k \quad \forall\, k\,. \tag{16}$$

Lieb-Mattis' theorem [17] states that, under general conditions on the symmetric Hamiltonian of the system, if $h \succeq h'$ then $E(h) < E(h')$, with $E(h)$ the ground state energy inside each IR h of U(N). Then we can establish the following

Proposition: The rectangular Young diagram of shape $[L^M]$ dominates all Young diagrams arising in the Clebsch-Gordan direct sum decomposition of the L-fold tensor product (10).

Therefore, the ground state will always belong to the rectangular $[L^M]$ sector. For instance, the rectangular sector ${\color{red}[2^2]} \succeq [2,1^2] \succeq [1^4]$ dominates in the Clebsch-Gordan decomposition (12). Intuitively, dominance means that one can go from h to h' by moving a certain number of boxes from upper rows to lower rows, so that h is "more symmetric". Therefore, we shall concentrate on the low-energy carrier Hilbert space $\mathcal{H}_N[L^M]$ of the rectangular IR $[L^M]$ to which the ground state $|\Phi_0\rangle$ in (5) belongs. In particular, we shall construct *coherent* (Skyrmion) ground state excitations. For the role of other mixed permutation symmetry sectors we address the reader to [18].

3 Grassmannian coherent states and nonlinear sigma models

Grassmannian (fermionic) coherent states can be seen as U(N) rotations/excitations over the ground state $|\Phi_0\rangle$

$$|Z\rangle^L = \frac{\exp\left[\sum_{1\leq j\leq M, M+1\leq i\leq N+M} Z_{ij}\mathcal{S}_{ij}\right]|\Phi_0\rangle}{\sqrt{\det(\mathbb{1}_M + Z^\dagger Z)}}\,, \tag{17}$$

created by appying U(N)-spin collective $\mathcal{S}_{ij}, i > j$, ladder operators. These Grassmannian coherent states are then labeled by $(N-M)\times M$ complex matrices Z. For $N = 2$ spin components, $\uparrow$ and $\downarrow$, and $M = 1$ we recover spin $j = L/2$ (atomic) coherent states

$$|z\rangle^L = \frac{e^{zS_{21}}|\Phi_0\rangle}{\sqrt{1+|z|^2}} = (1+|z|^2)^{-j}\sum_{m=-j}^{j}\sqrt{\binom{2j}{j-m}}z^{j-m}|j,m\rangle\,, \tag{18}$$

where we have spanned in terms of the usual angular momentum (Dicke) states $\{|j,m\rangle, m = -j,\ldots,j\}$, with $|\Phi_0\rangle = |j,-j\rangle$ and $z = \tan(\theta/2)e^{i\phi}$ is the stereographic projection of the Bloch sphere $\mathbb{S}^2$ onto the complex plane $\mathbb{C}$. Actually, atomic coherent states can also be written as a tensor product of qubits

$$|z\rangle^L = \big[\underbrace{\cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)e^{i\phi}|\downarrow\rangle}_{|z\rangle}\big]^{\otimes L} = |z\rangle^{\otimes L}\,. \tag{19}$$

For $L = 2 \Rightarrow j = L/2 = 1$, we identify the spin triplet $|j,m\rangle$ states

$$|1,1\rangle = |\uparrow\uparrow\rangle\,,\quad |1,0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\,,\quad |1,-1\rangle = |\downarrow\downarrow\rangle\,. \tag{20}$$

For $N = 4$ and filling factor $M = 1$ we have

$$|Z\rangle^L = \frac{[|1\rangle + z_2|2\rangle + z_3|3\rangle + z_4|4\rangle]^{\otimes L}}{(1+|z_2|^2+|z_3|^2+|z_4|^2)^{L/2}}\,, \tag{21}$$

where $Z = (1, z_2, z_3, z_4)^t$ denotes a point on the complex projective space $\mathbb{C}P^3 = \mathrm{U}(4)/\mathrm{U}(1)\times\mathrm{U}(3)$ or the Grassmannian $\mathbb{G}_1^4$.

In order to study the semi-classical/thermodynamical limit $L\to\infty$ of $U(N)$ QHF, one has to replace U(N)-spin operators S_{ij} by their coherent state expectation values $\langle Z|S_{ij}|Z\rangle$, which play the role of a matrix order parameter

$$\mathcal{S}(Z) \equiv \frac{2}{L}\langle Z|\left(S - \frac{L}{2}\mathbb{1}_N\right)|Z\rangle^L = Q(Z)^\dagger E_M Q(Z)\,, \tag{22}$$

$$E_M = \mathrm{diag}(1,\overset{M}{\ldots},1,-1,\overset{N-M}{\ldots},-1)\,, \tag{23}$$

with

$$Q(Z) = \left(\begin{array}{c|c} \Delta_1 & -Z^\dagger\Delta_2 \\ \hline Z\Delta_1 & \Delta_2 \end{array}\right)\,, \tag{24}$$

$$\Delta_1 = (\mathbb{1}_M + Z^\dagger Z)^{-1/2}, \qquad \Delta_2 = (\mathbb{1}_{N-M} + ZZ^\dagger)^{-1/2}. \tag{25}$$

The low energy physics of the U(N) QHF [when considering only nearest-neighbor interactions $\mathcal{J}_{\alpha\beta} = \mathcal{J}\delta_{\alpha,\beta\pm1}$ in the exchange Hamiltonian (1)] is described by a NLσM field theory with action in the continuum limit ($L \to \infty$ and lattice constant $\ell \to 0$)

$$A[Z] = \int dx_0 dx_1 dx_2 \left[\mathrm{tr}(E_M Q^\dagger \partial_{x_0} Q) + \mathcal{J}\,\mathrm{tr}(\vec{\nabla}\mathcal{S} \cdot \vec{\nabla}\mathcal{S})\right], \tag{26}$$

where $\partial_{x_0} \equiv \partial_0$ means partial derivative with respect to time $t = x_0$, $\vec{\nabla} = (\partial_{x_1}, \partial_{x_2}) \equiv (\partial_1, \partial_2)$ is the gradient and $\vec{\nabla}\mathcal{S}\cdot\vec{\nabla}\mathcal{S}$ is the scalar product. The first (kinetic) term of the action is the Berry term, provided by the coherent state representation of the path integral quantization. The second term describes the energy cost when the order parameter $\mathcal{S}$ is not uniform (see [7–12] and [16] for more information). The topological current

$$J^\mu = \frac{\mathrm{i}}{16\pi}\varepsilon^{\mu\nu\lambda}\mathrm{tr}(\mathcal{S}\partial_\nu\mathcal{S}\partial_\lambda\mathcal{S}) \tag{27}$$

(ε is the Levi-Civita antisymmetric symbol in 1+2 dimensions), leads to the topological (Pontryagin) charge or Skyrmion number

$$\mathcal{C} = \int dx_1 dx_2 J^0. \tag{28}$$

See e.g. Ref. [12] for more information.

4 Conclusion

We have presented several group-theoretical tools to study interacting N-component fermions on a lattice, like U(N) quantum Hall ferromagnets arising from two-body exchange interactions of N-component fermions. In particular, we have restricted ourselves to the lower energy permutation symmetry sector (according to the Lieb-Mattis theorem) corresponding to fermion mixtures described by rectangular Young diagrams with M rows (the filling factor) and L columns (Landau/lattice sites).

The "spontaneously chosen" ground state $|\Phi_0\rangle$ breaks the original U(N) symmetry and the associated U(N) ferromagnetic order parameter $\mathcal{S}$ [the expectation value of collective U(N)-spin operators S in a Grassmannian coherent state $|Z\rangle$] describes coherent state excitations ("Skyrmions") in the semi-classical $L \to \infty$ limit, whose dynamics is governed by a Grassmannian nonlinear sigma model.

The subject of SU(N) fermions and SU(N) magnetism has been recently further fueled in condensed matter physics with exciting advances in cooling, trapping and manipulating fermionic alkaline-earth atoms trapped in optical lattices (see e.g. [19, 20] for a realization of a SU(N) generalization of the Hubbard model). Multilayer quantum Hall arrangements, bearing larger U(N) symmetries, also display interesting new physics (see [21] for the bilayer case); Such is the case of superconducting properties of twisted bilayer (and trilayer) graphene predicted by [22] and observed by [23]. Furthermore, magnetic Skyrmion materials display a robust topological magnetic structure, being a candidate for the next generation of spintronic memory devices.

Acknowledgements

We thank the organizers of the 34th edition of the ICGTMP for their hospitality and the preparation of the congress. We are also glad to have shared interesting discussions with J.P. Gazeau, M.A. del Olmo, M.A. Lledó, P. Van Isacker, D. Schuch, and F.J. Herranz during the sessions.

Author contributions Conceptualization, M.C.; methodology, M.C.; software, A.M.; validation, M.C., A.M. and J.G.; formal analysis, M.C.; investigation, M.C., A.M. and J.G.; resources, M.C.; data curation, M.C. and A.M.; writing—original draft preparation, M.C.; writing—review and editing, M.C., A.M. and J.G.; visualization, A.M.; supervision, M.C.; project administration, M.C.; funding acquisition, M.C. and J.G.

Funding information We thank the support of the Spanish MICINN through the project PGC2018-097831-B-I00 and Junta de Andalucía through the projects FEDER/UJA-1381026, UHU-1262561 and FQM-381. AM thanks the Spanish MIU for the FPU19/06376 predoctoral fellowship.

References

[1] Z. F. Ezawa, *Quantum Hall effects*, World Scientific, Singapore, ISBN 9789814360753 (2011), doi:10.1142/8210.

[2] M. Calixto, A. Mayorgas and J. Guerrero, *Entanglement and U(D)-spin squeezing in symmetric multi-qudit systems and applications to quantum phase transitions in Lipkin-Meshkov-Glick D-level atom models*, Quantum Inf. Process. **20**, 304 (2021), doi:10.1007/s11128-021-03218-6.

[3] J. Gazeau, *Coherent states in quantum physics*, Wiley, Hoboken, USA, ISBN 9783527407095 (2009), doi:10.1002/9783527628285.

[4] S. Seki and M. Mochizuki, *Skyrmions in magnetic materials*, Springer, Cham, Switzerland, ISBN 9783319246499 (2016), doi:10.1007/978-3-319-24651-2.

[5] J. H. Han, *Skyrmions in condensed matter*, Springer, Cham, Switzerland, ISBN 9783319692449 (2017), doi:10.1007/978-3-319-69246-3.

[6] S. Zhang, *Chiral and topological nature of magnetic skyrmions*, Springer, Cham, Switzerland, ISBN 9783319982519 (2018), doi:10.1007/978-3-319-98252-6.

[7] I. Affleck, *The quantum Hall effects, σ-models at $\theta = \pi$ and quantum spin chains*, Nucl. Phys. B **257**, 397 (1985), doi:10.1016/0550-3213(85)90353-0.

[8] I. Affleck, *Exact critical exponents for quantum spin chains, non-linear σ-models at $\theta = \pi$ and the quantum Hall effect*, Nucl. Phys. B **265**, 409 (1986), doi:10.1016/0550-3213(86)90167-7.

[9] I. Affleck, *Critical behaviour of SU(n) quantum chains and topological non-linear σ-models*, Nucl. Phys. B **305**, 582 (1988), doi:10.1016/0550-3213(88)90117-4.

[10] N. Read and S. Sachdev, *Some features of the phase diagram of the square lattice SU(N) antiferromagnet*, Nucl. Phys. B **316**, 609 (1989), doi:10.1016/0550-3213(89)90061-8.

[11] N. Read and S. Sachdev, *Spin-Peierls, valence-bond solid, and Néel ground states of low-dimensional quantum antiferromagnets*, Phys. Rev. B **42**, 4568 (1990), doi:10.1103/PhysRevB.42.4568.

[12] D. P. Arovas, A. Karlhede and D. Lilliehöök, SU(n) *quantum Hall skyrmions*, Phys. Rev. B **59**, 13147 (1999), doi:10.1103/PhysRevB.59.13147.

[13] F. D. M. Haldane, *Continuum dynamics of the 1-d Heisenberg antiferromagnet: Identification with the O(3) nonlinear sigma model*, Phys. Lett. A **93**, 464 (1983), doi:10.1016/0375-9601(83)90631-X.

[14] F. D. M. Haldane, *Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state*, Phys. Rev. Lett. **50**, 1153 (1983), doi:10.1103/PhysRevLett.50.1153.

[15] F. D. M. Haldane, *O(3) nonlinear σ model and the topological distinction between integer- and half-integer-spin antiferromagnets in two dimensions*, Phys. Rev. Lett. **61**, 1029 (1988), doi:10.1103/PhysRevLett.61.1029.

[16] M. Calixto, A. Mayorgas and J. Guerrero, *Hilbert space structure of the low energy sector of U(N) quantum Hall ferromagnets and their classical limit*, Symmetry **14**, 872 (2022), doi:10.3390/sym14050872.

[17] E. Lieb and D. Mattis, *Theory of ferromagnetism and the ordering of electronic energy levels*, Phys. Rev. **125**, 164 (1962), doi:10.1103/PhysRev.125.164.

[18] M. Calixto, A. Mayorgas and J. Guerrero, *Role of mixed permutation symmetry sectors in the thermodynamic limit of critical three-level Lipkin-Meshkov-Glick atom models*, Phys. Rev. E **103**, 012116 (2021), doi:10.1103/PhysRevE.103.012116.

[19] C. Honerkamp and W. Hofstetter, *Ultracold fermions and the* SU(N) *Hubbard model*, Phys. Rev. Lett. **92**, 170403 (2004), doi:10.1103/PhysRevLett.92.170403.

[20] M. A. Cazalilla, A. F. Ho and M. Ueda, *Ultracold gases of ytterbium: Ferromagnetism and Mott states in an SU(6) Fermi system*, New J. Phys. **11**, 103033 (2009), doi:10.1088/1367-2630/11/10/103033.

[21] M. Calixto, C. Peón-Nieto and E. Pérez-Romero, *Hilbert space and ground-state structure of bilayer quantum Hall systems at* $\nu = 2/\lambda$, Phys. Rev. B **95**, 235302 (2017), doi:10.1103/PhysRevB.95.235302.

[22] R. Bistritzer and A. H. MacDonald, *Moiré bands in twisted double-layer graphene*, Proc. Natl. Acad. Sci. **108**, 12233 (2011), doi:10.1073/pnas.1108174108.

[23] Y. Cao et al., *Correlated insulator behaviour at half-filling in magic-angle graphene superlattices*, Nature **556**, 80 (2018), doi:10.1038/nature26154.

Generalisation of affine Lie algebras on compact real manifolds

Rutwig Campoamor-Stursberg[1], Marc de Montigny[2] and Michel Rausch de Traubenberg[3⋆]

1 Instituto de Matemática Interdisciplinar and Dpto. Geometría y Topología, UCM, E-28040 Madrid, Spain
2 Faculté Saint-Jean, University of Alberta, 8406 91 Street, Edmonton, Alberta T6B 0M9, Canada
3 Université de Strasbourg, CNRS, IPHC UMR7178, F-67037 Strasbourg Cedex, France

⋆ Michel.Rausch@iphc.cnrs.fr

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We report on recent work concerning a new type of generalised Kac-Moody algebras based on the spaces of differentiable mappings from compact manifolds or homogeneous spaces onto compact Lie groups.

Received 14-12-2022
Accepted 11-08-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.022

1 Introduction

Among the infinite-dimensional groups and algebras motivated by physical problems, the Virasoro, Kac-Moody, current and W-algebras and their representations are the most relevant representatives, and constitute a fundamental tool in several theories, such as Conformal Field Theory, gauge and string theories or SUGRA models (see [1–3] and references therein). It turns out that Kac-Moody algebras, as well as the associated Virasoro algebras, provide a natural framework for the unification of symmetry and locality properties [4]. Basing on different physical assumptions, several generalisations of these algebraic structures have been proposed, usually from an analytic point of view, rather than on the axiomatic construction of these entities [5]. In this context, the quasisimple Lie algebras [6], generalised Kac-Moody algebras based on geometrical properties of closed surfaces [7] as well as several hierarchies of centrally extended algebras are worthy to be mentioned [8–11].

In most of these constructions, the one-dimensional sphere $\mathbb{S}^1$ plays a relevant role, a fact that suggests that, for other physical models involving more than one degree of freedom and related to some basis manifold, a similar procedure can be proposed, provided that the manifold is either compact or presents some peculiar properties that guarantee convergence of integrals. This situation was the starting point for the general procedure initiated in [12], where a systematic construction of generalised Kac-Moody algebras based on compact manifolds $\mathcal{M}$ related to either a Lie group or an appropriate homogeneous space was proposed. Under these assumptions, harmonic functions on the manifold can be described in terms of the representation theory of the corresponding Lie group, allowing us, in particular, to identify a complete set of Hermitean labelling operators. An important difference of this generalisation with respect to the well-known class of usual Kac-Moody algebras and other generalisations resides in the fact that our construction, based on the Fourier expansion on compact manifolds, does not imply in general the existence of simple roots, even if a root structure can always be identified.

Besides the interest of these generalised Kac-Moody algebras from the mathematical point of view of, as this kind of algebras is naturally related related to higher-dimensional compact manifolds, the question of their relevance in theories involving higher dimensional space-times such as Kaluza-Klein theories, supergravity, *etc* is of certainly of physical interest.

2 The algorithmic construction of generalised Kac-Moody algebras

The construction of generalised Kac-Moody algebras proposed in [12] for the case of manifolds associated to either a compact Lie group G_c or a coset space G_c/H (via the exponential map, see [12] for details) starts with a simple compact[1] Lie algebra $\mathfrak{g}$, a given basis $\{T_a, a = 1, \cdots, \dim \mathfrak{g}\}$ with structure tensor

$$\left[T_a, T_b\right] = \mathrm{i}\, f_{ab}{}^c T_c\,,$$

and Killing form

$$\left\langle T_a, T_b\right\rangle_0 = g_{ab} \equiv \mathrm{Tr}\Big(\mathrm{ad}(T_a)\mathrm{ad}(T_b)\Big). \tag{1}$$

Denoting by V the volume elements of the associated compact $n = (p+q)$—dimensional manifold $\mathcal{M}$ (with $\mathcal{M} \simeq G_c$ or $\mathcal{M} \simeq G_c/H$, we consider a local coordinate frame $y^A = (\varphi^i, u^r)$ with $1 \le i \le p$, $1 \le r \le q$, such that the condition

$$\int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M}) = \frac{1}{V}\int_{\mathcal{M}} \mathrm{d}^p\varphi\; d^q u = 1$$

holds. On $\mathcal{M}$ we consider the set of square integrable functions periodic in all φ—directions, but non-periodic in the u—directions. The space $L^2(\mathcal{M})$ admits a complete orthonormal Hilbert basis

$$\mathcal{B} = \Big\{\rho_I(\varphi, u)\,,\;\; I \in \mathcal{I}\Big\}, \tag{2}$$

with respect to the Hermitean scalar product on $L^2(\mathcal{M})$, where $\mathcal{I}$ denotes a minimal (finite) set of labels required to identify the states unambiguously [13]. In these conditions, we define a space of smooth mappings from $\mathcal{M}$ into $\mathfrak{g}$ as

$$\mathfrak{g}(\mathcal{M}) = \Big\{T_{aI} = T_a\rho_I(\varphi, u)\,, a = 1, \ldots, \dim\mathfrak{g}\,, I \in \mathcal{I}\Big\}.$$

On this space, that inherits the structure of a Lie algebra, the Lie brackets are well defined and adopt the generic form

$$\left[T_{aI}, T_{bJ}\right] = \mathrm{i}\, f_{ab}{}^c c_{IJ}{}^K T_{cK}\,, \tag{3}$$

[1] This analysis can of course be extended to any simple (real or complex) Lie algebra. However, only in the case of compact Lie algebras, the representation theory has been analysed (see below).

where the coefficients ${c_{IJ}}^K$ are those of the Fourier expansion of products of elements in the basis $\mathcal{B}$. For the case where the manifold $\mathcal{M}$ is related to a compact Lie groups G_c, these can be associated to the Clebsch-Gordan coefficients of G_c. In particular, the Killing form in $\mathfrak{g}(\mathcal{M})$ is given by

$$\left\langle X,Y\right\rangle_1 = \int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M})\left\langle X,Y\right\rangle_0, \quad X,Y \in \mathfrak{g}(\mathcal{M}) \tag{4}$$

(with $\left\langle X,Y\right\rangle_0$ being the Killing form in (1)), from which the relations

$$\rho_I(\varphi,u) = \eta_{IJ}\overline{\rho}^J(\varphi,u), \quad \left\langle T_{aI},T_{bJ}\right\rangle_1 = g_{ab}\eta_{IJ}\,,$$

follow at once. The first relation simply means that $\overline{\rho}^J \in L^2(\mathcal{M})$, and thus extends in the basis $\mathcal{B}$ given by (2).

In a second step, the existence of central extensions for the preceding algebras is analyzed. Following a general approach based on cohomological methods, the central extension is obtained through the 2-cocycle

$$\omega_C(X,Y) = \int_{\mathcal{M}} \left\langle X, \partial_i Y\ \mathrm{d}\varphi^i + \partial_s Y\ \mathrm{d}u^s\right\rangle_0 \wedge \gamma\,, \tag{5}$$

with γ being a closed $(n-1)$-current associated to a closed loop C. In this context, it should be taken into account that central extensions are associated to compact one-dimensional submanifolds of $\mathcal{M}$, *i.e.* curves, and that the procedure cannot be extrapolated to maps from higher-dimensional manifolds onto $\mathcal{M}$ [14]. Specifically, we consider

$$\gamma_{(A)} = (-1)^A k_A \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{A-1} \wedge \mathrm{d}y^{A+1} \wedge \cdots \mathrm{d}y^n\,, \quad A = 1,\cdots,n\,, \quad k_A \in \mathbb{R}\,.$$

This leads to the identity

$$\omega_{(A)}(T_{aI},T_{bJ}) = k_A g_{ab} \int_{\mathcal{M}} \mathrm{d}\mu(\mathcal{M})\ \rho_I(\varphi,u)\partial_A\rho_J(\varphi,u) = k_A g_{ab} d_{AIJ}\,, \tag{6}$$

hence for the centrally extended algebra $\mathfrak{g}(\mathcal{M})$ we get the commutator

$$\left[T_{aI},T_{bJ}\right] = \mathrm{i}\, {f_{ab}}^c {c_{IJ}}^K T_{cK} + g_{ab}\sum_{A=1}^{n} k_A d_{AIJ}\,. \tag{7}$$

It is not casual that this algebra has a deep similitude with the current algebra defined through

$$\left[T_a(y),T_{a'}(y')\right] = \mathrm{i}\, {f_{aa'}}^b T_b(y)\delta^n(y-y') - \mathrm{i}\sum_{A=1}^{n} k_A\partial_A\delta^n\left(y-y'\right), \tag{8}$$

and possessing Schwinger terms. Actually, centrally extended extensions of the generalised Kac-Moody algebras associated to the compact manifolds $\mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^1$ were determined in [15] by means of current algebras, showing the validity of the procedure.

In a third step, derivations ∂_A of the generalised Kac-Moody algebra $\mathfrak{g}(\mathcal{M})$ are considered. This is a technically delicate step, as the variables φ are periodic, whereas the variables u do not exhibit periodicity properties. In other words, the operators $d_j = -\mathrm{i}\partial_{\varphi^j}$ are (commuting) Hermitean, while the operators $d_s = -\mathrm{i}\partial_{u^s}$ are not Hermitean. In order to obtain a complete set of commuting Hermitean operators, we use the identification of the manifold $\mathcal{M}$ with a compact Lie group (coset space). To this extent, an embedding $\mathfrak{g}_c \subseteq \mathfrak{g}_m$ of $\mathfrak{g}_c$ into a higher-rank Lie algebra $\mathfrak{g}_m$ is used, with $\mathfrak{g}_c$ the Lie algebra of G_c, and such that the basis functions of (2) belong to an irreducible unitary representation of $\mathfrak{g}_m$. The generators of the latter can be realised as Hermitean differential operators acting naturally on the manifold; in particular, the elements $h_1, \cdots, h_k$ of the Cartan subalgebra of $\mathfrak{g}_m$ (where k is the rank of $\mathfrak{g}_m$), are realised as the Hermitean operators

$$h_j = -\mathrm{i} f_j^A(y)\partial_A \,, \quad 1 \le j \le k \,.$$

A particularity of these operators is that the boundary term associated to all u—directions vanishes. Among the operators $\{d_1, \cdots, d_p, h_1, \cdots, h_k\}$ we determine a maximal set of commuting operators

$$D_j = -\mathrm{i} f_j^A(y)\partial_A \,, \quad j = 1, \cdots, r \,,$$

that satisfy the constraints

$$\partial_A f_j^A(y) = 0 \,, \quad \text{and} \quad f_j^r| = 0 \,, \; j = 1, \cdots, r \,, \tag{9}$$

related to Hermiticity. In these expressions, $f_j^r|$ represent the boundary terms associated to all u—directions that must vanish. Note further that when $\mathcal{M} = \mathbb{T}^n$, as all directions are periodic, we have $r = n$, but for a generic n—dimensional manifold $\mathcal{M}$ we have $r < n$. It can be easily shown that the D_i act diagonally on the functions ρ_I, leading to an eigenvalue problem

$$D_j(\rho_I(y)) = I(j)\rho_I(y) \,,$$

with $I(j)$ the corresponding eigenvalue.

The Hermitean operators D_j and central extensions of the generalised Kac-Moody algebra are deeply related through the closed $(n-1)$—forms $(j = 1, \cdots, r)$

$$\gamma_j = k_j \sum_{A=1}^{n} (-1)^A f_j^A(y)\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{A-1} \wedge \mathrm{d}y^{A+1} \wedge \cdots \wedge \mathrm{d}y^n \,, \quad j = 1, \ldots, r \,, \quad k_j \in \mathbb{R} \,, \tag{10}$$

with corresponding 2-cocycles given by (see equation (6))

$$\omega_k(T_{aI}, T_{bJ}) = k_k J(k) g_{ab} \eta_{IJ} \,. \tag{11}$$

Summarising, the generalised Kac-Moody algebra $\widehat{\mathfrak{g}}(\mathcal{M})$ associated to the compact Lie algebra $\mathfrak{g}$ and the compact manifold $\mathcal{M}$ is determined by the following data

1. Generators T_{aI} belonging to $\mathfrak{g}(\mathcal{M})$;
2. Commuting Hermitean operators $D_1, \cdots, D_r$;
3. Central charges $k_1, \cdots, k_r$ associated to the Hermitean operators.

If $I(j)$ denotes the eigenvalue of D_j (see (10)), the non-vanishing brackets of the generalised Kac-Moody algebra associated to $\mathcal{M}$ are

$$\begin{aligned} \left[T_{aI}, T_{bJ}\right] &= \mathrm{i}\, f_{ab}{}^c c_{IJ}{}^K T_{cK} + g_{ab}\eta_{IJ} \sum_{j=1}^{r} k_j I(j) \,, \\ \left[D_j, T_{aI}\right] &= I(j) T_{aI} \,, \end{aligned} \tag{12}$$

where $I(j)$ is the eigenvalue of D_j. Recall again that the central charges and the Hermitian operators are both associated to the closed $(n-1)$—form given by (10).

As shown in [12], the choice of $G_c = U(1)^n$ leads to a generalised Kac-Moody algebra that structurally coincides with the generalised algebras based on the torus $\mathbb{T}^n$ studied and analyzed in [6], and that actually correspond to specific cases of the wide class of so-called 'quasi-simple Lie algebras'.

3 Identification of roots in $\widehat{\mathfrak{g}}(\mathcal{M})$

The fourth step is devoted to the identification of a root structure based on equation (12) associated to generalised Kac-Moody algebras. Supposed that the initial simple Lie algebra $\mathfrak{g}$ has rank ℓ and let Σ denotes the root system with respect to the Cartan subalgebra $H^i, i = 1, \cdots, \ell$, we consider the root operators $E_\alpha, \alpha \in \Sigma$ in the Cartan-Weyl basis. Defining

$$\hat{\mathfrak{g}}(\mathcal{M}) = \text{Span}\Big\{T_{aI}, D_j, k_j, a = 1, \cdots, \dim\mathfrak{g}, I \in \mathcal{I}, j = 1, \cdots, r\Big\}, \tag{13}$$

the Cartan subalgebra of the latter is spanned by H^i, D_j and k_j $(i = 1, \ldots, \ell,\ j = 1, \cdots, r)$. Taking the Cartan-Weyl basis H^i_I, $E_{\alpha I}$ and the Killing form as defined in (4), application of the procedure described in [16] shows that the Killing form of $\hat{\mathfrak{g}}(\mathcal{M})$ satisfies

$$\begin{aligned} \Big\langle T_{aI}, T_{bJ}\Big\rangle &= \eta_{IJ}\, g_{ab}\,, \quad \Big\langle D_j, T_{aI}\Big\rangle = \Big\langle k_j, T_{aI}\Big\rangle \ = \ 0\,, \\ \Big\langle k_i, k_j\Big\rangle &= \Big\langle D_i, D_j\Big\rangle \ = \ 0\,, \quad \Big\langle D_i, k_j\Big\rangle = \delta^i_j\,. \end{aligned} \tag{14}$$

From this we get the (infinite-dimensional) root spaces (where $\mathbf{n} = (n_1, \cdots, n_r)$)

$$\begin{aligned} \mathfrak{g}_{(\alpha,\mathbf{n})} &= \Big\{E_{\alpha I,}\ \text{ with } I(1) = n_1, \cdots, I(r) = n_r\Big\}, \quad \alpha \in \Sigma, \quad n_1, \cdots, n_r \in \mathbb{Z}, \\ \mathfrak{g}_{(0,\mathbf{n})} &= \Big\{H^i_I,\ \text{ with } I(1) = n_1, \cdots, I(r) = n_r\Big\},\ n_1, \cdots, n_r \in \mathbb{Z}, \end{aligned} \tag{15}$$

with commutation relations

$$\begin{aligned} \Big[\mathfrak{g}_{(0,\mathbf{n})}, \mathfrak{g}_{(\alpha,\mathbf{m})}\Big] &\subset \mathfrak{g}_{(\alpha,\mathbf{m}+\mathbf{n})}\,, \\ \Big[\mathfrak{g}_{(\alpha,\mathbf{m})}, \mathfrak{g}_{(\beta,\mathbf{n})}\Big] &\subset \mathfrak{g}_{(\alpha+\beta,\mathbf{m}+\mathbf{n})}\,,\ \alpha+\beta \in \Sigma\,. \end{aligned}$$

An important difference with respect to the usual Kac-Moody algebras is that, in this case, the commutator between elements depends also on the representation theory of G_c, specifically in connection with the Clebsch-Gordan coefficients $c_{IJ}{}^K$. This shows that the construction goes beyond the traditional root theory, as it also involves the so-called labelling problem for embedded algebras [13].

Explicit construction of these generalised structures was obtained in [12] for the case of manifolds isomorphic to the spheres $\mathbb{S}^n$, specifically for the values $n = 2$ with $SU(2)/U(1)$, $n = 3$ for $SU(2)$ and $SO(4)/SO(3)$, $n = 5$ for $SU(3)/SU(2)$ and $n = 6$ for $G_2/SU(3)$.

Concerning the representation theory of generalised Kac-Moody algebras, the case of the n-dimensional torus $\mathbb{T}^n$ has been inspected in some detail in [12], corresponding to the Lie group $U(1)^n$. An extrapolation to other more complicated manifolds is a delicate task, the technical difficulties of which have not yet been solved satisfactorily. However, for the two-dimensional case and the manifolds $\mathbb{T}^2$ and $\mathbb{S}^2$, an alternative ansatz has been proposed in [17] and [18], based on the observation that the Kac-Moody and the corresponding Virasoro algebras associated to these manifolds can be constructed naturally from the usual Kac-Moody and Virasoro algebras. More specifically, in this case we have assumed that the Laurent modes of the usual Kac-Moody and Virasoro algebras can themselves be (Fourier) developed in an adapted manner on the two-sphere and the two-torus, respectively [17]. This assumption enabled us to reproduce the generalised Kac-Moody algebras associated to $\mathbb{S}^2$ or $\mathbb{T}^2$, in a semi-direct product with a subalgebra of vector fields of the two-torus and the two-sphere

$$\text{Vir}(\mathcal{M}) \ltimes \widehat{g}(\mathcal{M}), \quad \text{with} \quad \mathcal{M} = \mathbb{S}^1, \quad \text{or} \quad \mathbb{T}^2\,. \tag{16}$$

The algebras Vir($\mathcal{M}$) can been seen as extensions of the Virasoro algebras in these cases. The interesting observation of this construction is that it leads naturally to central extensions. The fermions [17] and boson [18] realisations subsequently obtained lead automatically to a Fock space construction and thus to a unitary representation bounded from below. In the case of the bosonic construction, we have introduced vertex operators along the lines of the vertex operator in string theory [19, 20].

4 Concluding remarks and future prospects

We have reported on recent work concerning on the construction of generalised Kac-Moody algebras for the class of compact Lie groups and certain coset spaces determined by a closed subgroup, and the analysis of some of its main features that may be of interest in physical applications, such as the existence of a root system and central extensions. The procedure can formally be developed for any compact manifold or homogeneous space of the specified type, with the main difficulties being of computational nature. Whether this class of extensions fits naturally in the description of physical phenomenology, is a problem that has still to be explored in more detail.

The next natural step, besides specific applications, consists in proposing an analogous construction for the case where the basis manifold $\mathcal{M}$ is no more compact. Some results in this direction actually exist, such as the work [9], but a general approach has not been formulated yet. Among the obstructions observed in this general frame, the acute divergence problems that arise in the integration theory of non-compact manifolds, as well as the technical difficulties emerging in the cohomological formulation of central extensions (see equation (5)), are the most relevant. Inspection of several examples suggest that additional techniques have to be considered to cover this case appropriately, in order to obtain a description the validity of which is not restricted to very particular manifolds. A successful approach in this sense could possibly be of interest in the context of $M-$theory or supergravity models, studying whether there is a connection between the central extensions of the generalised Kac-Moody algebra and super-membrane solutions in extended SUGRA.

The fermionic [17] and bosonic [18] construction obtained in the case of $\mathbb{S}^2$ and $\mathbb{T}^2$ can be easily extended to the $n-$tori $\mathbb{T}^n$. This extension leads to a hierarchy of algebras (with the notations of (16))

$$\mathrm{Vir}(\mathbb{T}^n) \ltimes \widehat{g}(\mathbb{T}^n) \subset \mathrm{Vir}(\mathbb{T}^{n-1}) \ltimes \widehat{g}(\mathbb{T}^{n-1}) \subset \cdots \subset \mathrm{Vir} \ltimes \widehat{g}\,, \tag{17}$$

where at the last stage we have the usual Virasoro and Kac-Moody algebras. This series of embeddings could play a role in toroidal compactifications, a very important notion in higher dimensional supergravity.

Acknowledgements

Funding information RCS acknowledges financial support by the research grants MTM2016-79422-P (AEI/FEDER, EU) and PID2019-106802GB-I00/AEI/10.13039/501100011033 (AEI/FEDER, UE). MdeM is grateful to the Natural Sciences and Engineering Research Council (NSERC) of Canada for partial financial support (grant number RGPIN-2016-04309).

References

[1] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras. A reprint volume for physicists*, World Scientific, Singapore, ISBN 9789814507707 (1988), doi:10.1142/0485.

[2] P. West, *Introduction to strings and branes*, Cambridge University Press, Cambridge, UK, ISBN 9781139045926 (2012), doi:10.1017/CBO9781139045926.

[3] S. Ferrara, A. Marrani and M. Trigiante, *Super-Ehlers in any dimension*, J. High Energy Phys. **11**, 068 (2012), doi:10.1007/JHEP11(2012)068.

[4] L. Dolan, *Kac-Moody algebras and exact solvability in hadronic physics*, Phys. Rep. **109**, 1 (1984), doi:10.1016/0370-1573(84)90134-0.

[5] I. G. Macdonald, *Kac–Moody-algebras*, in *Lie algebras and related topics*, American Mathematical Society, Providence, USA, ISBN 9780821860090 (1986).

[6] R. Høegh-Krohn and B. Torresani, *Classification and construction of quasisimple Lie algebras*, J. Funct. Anal. **89**, 106 (1990), doi:10.1016/0022-1236(90)90006-7.

[7] L. Frappat, E. Ragoucy, P. Sorba, F. Thuillier and H. Høgåsen, *Generalized Kac-Moody algebras and the diffeomorphism group of a closed surface*, Nucl. Phys. B **334**, 250 (1990), doi:10.1016/0550-3213(90)90663-X.

[8] R. E. Borcherds, *Central extensions of generalized Kac-Moody algebras*, J. Algebra **140**, 330 (1991), doi:10.1016/0021-8693(91)90158-5.

[9] E. Ragoucy and P. Sorba, *Extended Kac-Moody algebras and applications*, Int. J. Mod. Phys. A **07**, 2883 (1992), doi:10.1142/S0217751X92001307.

[10] E. Ragoucy and P. Sorba, *An attempt to relate area-preserving diffeomorphisms to Kac-Moody algebras*, Lett. Math. Phys. **21**, 329 (1991), doi:10.1007/BF00398331.

[11] K. Harada, P.-M. Ho, Y. Matsuo and A. Watanabe, *Dimensional oxidization on coset space*, J. High Energy Phys. **10**, 198 (2020), doi:10.1007/JHEP10(2020)198.

[12] R. Campoamor-Stursberg, M. de Montigny and M. Rausch de Traubenberg, *An overview of generalised Kac-Moody algebras on compact real manifolds*, J. Geom. Phys. **180**, 104624 (2022), doi:10.1016/j.geomphys.2022.104624.

[13] R. Campoamor-Stursberg, *The external labelling problem and Clebsch-Gordan series of semisimple Lie algebras*, J. Phys. A: Math. Theor. **52**, 125201 (2019), doi:10.1088/1751-8121/ab0485.

[14] A. Pressley and G. Segal, *Loop groups*, Oxford University Press, Oxford, UK, ISBN 9780198535614 (1986).

[15] I. Bars, *Local charge algebras in quantum chiral models and gauge theories*, in *Vertex operators in mathematics and physics*, Springer, Berlin, Heidelberg, Germany, ISBN 9781461395508 (1985), doi:10.1007/978-1-4613-9550-8_18.

[16] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. A **01**, 303 (1986), doi:10.1142/S0217751X86000149.

[17] R. Campoamor-Stursberg and M. Rausch de Traubenberg, *Fermion realisations of generalised Kac–Moody and Virasoro algebras associated to the two-sphere and the two-torus* Mod. Phys. Lett. A **37**, 2250240 (2022), doi:10.1142/S0217732322502406.

[18] R. Campoamor-Stursberg and M. Rausch de Traubenberg, *Vertex operator for generalised Kac–Moody algebras associated to the two-sphere and the two-torus* Mod. Phys. Lett. A **37**, 2250239 (2022), doi:10.1142/S021773232250239X.

[19] I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62**, 23 (1980), doi:10.1007/BF01391662.

[20] G. Segal, *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. **80**, 301 (1981), doi:10.1007/BF01208274.

Generalized Heisenberg-Weyl groups and Hermite functions

Enrico Celeghini[1,2], Manuel Gadella[2] and Mariano A. del Olmo[2⋆]

1 Dipartimento di Fisica, Università di Firenze and INFN-Sezione di Firenze,
150019 Sesto Fiorentino, Firenze, Italy
2 Departamento de Física Teórica, Atómica y Optica and IMUVA,
Universidad de Valladolid, 47011 Valladolid, Spain

⋆ marianoantonio.olmo@uva.es

Group ICGTMP
34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

A generalisation of Euclidean and pseudo-Euclidean groups is presented, where the Weyl-Heisenberg groups, well known in quantum mechanics, are involved. A new family of groups is obtained including all the above-mentioned groups as subgroups. Symmetries, like self-similarity and invariance with respect to the orientation of the axes, are properly included in the structure of this new family of groups. Generalized Hermite functions on multidimensional spaces, which serve as orthogonal bases of Hilbert spaces supporting unitary irreducible representations of these new groups, are introduced.

Received 06-12-2022
Accepted 11-08-2023
Published 23-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.023

---

1 Introduction

It is well-known the interest of the Heisenberg-Weyl (HW) group in physics, mainly in Quantum Mechanics (QM). The indetermination principle, fundamental in QM, is closely linked to this group and the Fourier transform (FT) [1, 2]. It is also related with the Gabor formalism [3] on the theory of wavelets, where an uncertainty principle for time-frequency operators appears [4]. On the other hand, (the affine spaces) Euclidean, $\mathbb{R}^n$, or pseudo-Euclidean spaces, $\mathbb{R}^{p,q}$ $(p+q=n)$, are the arena of the physical events, where their invariance properties are described by the Euclidean type groups $E_n = \mathbb{R}^n \odot SO(n)$ or $E_{p,q} = \mathbb{R}^{p,q} \odot SO(p,q)$, respectively. The HW and Euclidean groups are involved in relevant invariance properties used in the study of the physical systems. Thus, we can mention, first of all, the pairs of sets of conjugate variables, connected through the HW group, that allows us to get equivalent physical descriptions either in the position or in the momentum representations. The freedom of the choice of the origin in each coordinate system (either position or momenta) that it is know as "homogeneity" and it is related to both kind of groups. The freedom to choose the unit of length or "self-similarity", that can be implemented via dilations. And finally the freedom to select the orientation of the unit vectors for the orthogonal bases of the physical space ("invariance from orientation"). In these last two cases the Euclidean groups are involved in. However, all these invariances are not completely independent because the FT, which matches coordinate and momentum representations [5], does not allow to fix independently self-similarity and orientation. Both family of groups are independent although some times they appear together in the implementation of the invariances above mentioned, that we consider as a whole.

Recently in [6] we have studied the case related with $\mathbb{R}$, It has been the point of departure for a generalization of our analysis to $\mathbb{R}^n$ and $\mathbb{R}^{p,q}$ realized in [7]. Here the Euclidean-like groups E_n and $E_{p,q}$ and the HW groups H_n and $H_{p,q}$ (where $\mathbb{R}^n \subset H_n$ has been replaced by $\mathbb{R}^{p,q}$) have been enlarged to the groups K_n and $K_{p,q}$ that contain the Euclidean groups and the HW groups as subgroups. Tentatives in this direction has been done but with different motivations and only considering the cases with positively defined metric [8–10].

Here, we present the lower dimensional cases (1D and 2D) in Sections 2 and 3, respectively. The representations of the groups here studied are supported by square integrable functions. The fact that the Hermite functions (HF) constitute a (discrete) basis of $L^2(\mathbb{R})$ (see Subsection 2.2) allows us to introduce in Subsection 3.4 a generalization of the HF in order to describe the above mentioned invariance in 2D. We end with a Section 4 devoted to conclusions.

2 Heisenberg-Weyl groups in the real line $\mathbb{R}$

2.1 The Heisenberg-Weyl group H_1

The HW group in 1D can be realized on the coordinate space $\mathbb{R}$ providing the basic commutation relations of QM as $[x,p] \equiv [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar$. A matrix representation of H_1 in terms of real 3×3 upper triangular matrices of the group $M_3(\mathbb{R})$ [11] is given by

$$H_1[a,b,c] = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R}. \tag{1}$$

Self-similarity and orientation are included by extending H_1 to a new group K_1 realized as

$$K_1[a,b,c,k] = \begin{bmatrix} 1 & a & c \\ 0 & k & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R}, \;\; k \in \mathbb{R}^*. \tag{2}$$

Obviously, the group laws in both cases are obtained through matrix multiplication.

The group K_1 has two connected components: the connected component of the identity (K_1^o) characterized by $k > 0$; and a 2nd component with $k < 0$ (K_1^1).

The parameters a,b,c of H_1 (and K_1) are in correspondence to the three generators X,P,I of the Lie algebra of H_1 (and K_1), Lie$[H_1]$ (Lie$[K_1]$), respectively; and the generator D associated to k only belongs to Lie$[K_1]$. The explicit form of these generators in (1) and (2) is

$$\begin{aligned} X &= \left.\frac{\partial K_1[\ldots]}{\partial a}\right|_{Id} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P = \left.\frac{\partial K_1[\ldots]}{\partial b}\right|_{Id} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ I &= \left.\frac{\partial K_1[\ldots]}{\partial c}\right|_{Id} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \left.\frac{\partial K_1[\ldots]}{\partial k}\right|_{Id} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{3}$$

with Id the identity element. The commutation relations for both Lie algebras are

$$[X,P] = I\,, \quad [D,X] = -X\,, \quad [D,P] = P\,, \quad [I,\bullet] = 0\,. \tag{4}$$

The real line $\mathbb{R}$ is a metric space that supports two continuous conjugate (in the sense of position-momentum conjugation) bases for $L^2(\mathbb{R})$: $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ obtained by means of the generalized eigenvectors of the operators X and P, i.e., $X\,|x\rangle = x\,|x\rangle\,, P\,|p\rangle = p\,|p\rangle\,.$ The basis elements of $\{|x\rangle\}_{x\in\mathbb{R}}$ satisfy (and similarly for $\{|p\rangle\}_{p\in\mathbb{R}}$)

$$\langle x|x'\rangle = \sqrt{2\pi}\,\delta(x-x')\,, \qquad \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,|x\rangle\langle x| = \mathbb{I}\,. \tag{5}$$

As we mention before these generalized bases are well defined on certain extensions of the Hilbert space (the Gel'fand triplets or the rigged Hilbert spaces) [12].

As is well known the Fourier transform (FT) and its inverse (IFT) connect both bases [5]

$$FT\,[|x\rangle,x,p] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,e^{ipx}|x\rangle\, = |p\rangle\,, \quad IFT\,[|p\rangle,p,x] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\,e^{-ipx}|p\rangle\, = |x\rangle\,. \tag{6}$$

There exists a representation of H_1 by unbounded operators on $L^2(\mathbb{R})$, where P and X may be represented by

$$[Pf](x) = -i\,\frac{d}{dx}f(x)\,, \qquad [Xf](x) = x\,f(x)\,, \qquad f(x)\in L^2(\mathbb{R})\,, \tag{7}$$

satisfying $[X,P] = I$. We also may choose another representation of P and X on an abstract infinite dimensional separable Hilbert space $\mathcal{H}$. Since there is always a unitary map $U:\mathcal{H}\to L^2(\mathbb{R})$, the commutation relation between P and X on $L^2(\mathbb{R})$ is translated to $\mathcal{H}$. In order to simplify the notation we also denote the operators on $\mathcal{H}$ by P and X.

The relationship between the elements $|f\rangle\in\mathcal{H}$ and $f(x)\in L^2(\mathbb{R})$ is given by [5]

$$|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,f(x)\,|x\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\,\hat{f}(-p)\,|p\rangle\,, \tag{8}$$

with $f(x) = \langle x|f\rangle$, $\hat{f}(p) = FT[f(x);x,p]$ and $\hat{f}(-p) = \langle p|f\rangle$. Remember that only the vectors $|f\rangle$ belonging to a dense space in $\mathcal{H}$ (i.e., the space of test vectors) can be written as (8).

The action of the group elements e^{-iPb} and e^{-iXa} on the continuous bases is given by

$$e^{-iPb}\,|x\rangle = |x+b\rangle\,, \qquad e^{-iXa}\,|p\rangle = |p-a\rangle\,, \qquad \forall a,b\in\mathbb{R}\,. \tag{9}$$

From these relations we conclude that $\{|x\rangle\}$ ($\{|p\rangle\}$) is equivalent to $\{|x+b\rangle\}$ ($\{|p-a\rangle\}$).

These bases support each an infinite dimensional unitary irreducible representation (UIR) of H_1, $U_h(g)$, $h\in\mathbb{R}^*$ [6,13],

$$U_h(g)\equiv U_h(c,a,b) := e^{ihcI}\,e^{ih(aX-bP)} = e^{ih(c-ab/2)I}\,e^{ihaX}\,e^{-ihbP}\,. \tag{10}$$

For instance, in the cases of $\{|x\rangle\}$ as well as $L^2(\mathbb{R})$ the action is given by

$$U_h(g)\,|x\rangle = e^{ihc}\,e^{iha(x+b/2)}\,|x+b\rangle\,, \qquad (\mathcal{U}_h(g)f)(x) = e^{ihc}\,e^{iha(x-b/2)}\,f(x-b)\,. \tag{11}$$

We mentioned before that H_1 does not exhaust the invariances of the real line if we add the hypothesis of self-similarity and orientation and we have to considerer K_1. Since $\{|x\rangle\}$ ($\{|p\rangle\}$) is equivalent to $\{|k\,x\rangle\}$ $\big(\{|k'\,p\rangle\}\big)$ and from (6) we find that $k' = k^{-1}\in\mathbb{R}^*$. In other words, $\mathbb{R}$

supports a UIR, $U_{h,\mathcal{C}}$, of K_1. For the connected component K_1^o of K_1 and for the dilations we use the formula (53) of [6] obtaining that $e^{idD}\,|x\rangle = e^{d/2}\,|e^d\,x\rangle$. Therefore,

$$U_{h,\mathcal{C}}(\tilde{g})\,|x\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x+b/2)}\,|e^d\,x+b\rangle\,, \quad \tilde{g}=(a,b,c,d)\in K_1^o\,, \tag{12}$$

where $\mathcal{C}\in\mathbb{R}$ denotes the eigenvalues of the quadratic Casimir of K_1^o, $\mathcal{C}=X\,P-I\,D$. When we consider also the dilations with $k<0$ we introduce the (unitary) parity operator $\mathcal{P}$ $(x\to -x)$ and we obtain in a unified manner that

$$\mathcal{U}_{h,\mathcal{C}}(\tilde{g},\alpha)\,|x\rangle = \mathcal{U}_{h,\mathcal{C}}(\tilde{g})\,|x^\alpha\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x^a+b/2)}\,|e^d\,x^\alpha+b\rangle\,, \tag{13}$$

where α stands either for the identity ($x^{\mathcal{I}}=x$) and $(\tilde{g},\mathcal{I})\in K_1^o$ or the parity ($x^{\mathcal{P}}=-x$) and $(\tilde{g},\mathcal{P})\in K_1^1$. We can rewrite (13) in terms of $k\in R^*$ with $|k|=e^d$ and $d\in\mathbb{R}$ as

$$\mathcal{U}_{h,\mathcal{C}}(c,a,b,k)\,|x\rangle = \sqrt{|k|}\,e^{ih(c+\mathcal{C})}\,e^{iha(k\,x+b/2)}\,|k\,x+b\rangle\,. \tag{14}$$

The corresponding action on the functions of $L^2(\mathbb{R})$ is given by

$$\big(\mathcal{U}_{h,C}(\tilde{g},\alpha)f\big)(x) = \frac{1}{\sqrt{|k|}}\,e^{ih(c+C)}\,e^{i\,h\,a(x-b/2)}\,f\left(k^{-1}\,(x-b)\right)\,. \tag{15}$$

2.2 The Hermite functions appear on the scene

It is well known that the FT of the Hermite Functions $\{\psi_m(x)\}_{m\in\mathbb{N}}$ are also HF, i.e.

$$FT[\psi_m(x),x,p] = i^m\,\psi_m(p)\,, \quad IFT[\psi_m(p),p,x] = (-i)^m\,\psi_m(x)\,. \tag{16}$$

Hence, both are complete orthonormal bases in $L^2(\mathbb{R})$ [14].

Invariance properties of K_1 are implemented to a generalization of the HF obtained using the UIR's of K_1 (13) in position coordinates x (and similarly for p) as follows

$$\chi_m(x,a,b,k) := |k|^{1/2}\;e^{-ia(k\,x+b/2)}\,\psi_m(k\,x+b)\,, \qquad a,b\in\mathbb{R}\,,\; k\in\mathbb{R}^*\,. \tag{17}$$

In this way we obtain two families of functions depending on 3 real parameters (a,b,k)

$$\{\chi_m(x,a,b,k)\}\,, \quad \{\chi_m(p,a,b,k)\}\,, \quad \forall\; k\neq 0, a, b\in\mathbb{R}\,. \tag{18}$$

Orthonormal and completeness relations of the HF induce similar relations for these families of generalized HF, so they are also orthonormal bases in $L^2(\mathbb{R})$. However, these generalized HF are not eigenfunctions of the FT and its inverse, contrarily to the ordinary HF (16)

$$\begin{aligned} FT[\chi_m(x,a,b,k),x,p] &= i^m\;\chi_m(p,b,-a,k^{-1})\,,\\ IFT[\chi_m(p,a,b,k),p,x] &= (-i)^m\;\chi_m(x,-b,a,k^{-1})\,. \end{aligned} \tag{19}$$

3 Euclidean and pseudo-Euclidean plane cases

In this Section we will consider the 2D configuration spaces: the Euclidean plane ($\mathbb{R}^2$) and the pseudo-Euclidean plane ($\mathbb{R}^{1,1}$) with metrics of signature $(+,+)$ and $(+,-)$, respectively.

3.1 The groups H_2 and K_2 on the plane

The HW group on 2D, H_2, admits a finite representation by real 4×4 matrices as

$$H_2[\mathbf{a},\mathbf{b},c]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1,a_2,b_1,b_2,c\in\mathbb{R}. \tag{20}$$

This group can be enlarged by adding the group of proper rotations $SO(2)$ and the dilations on the plane, $\mathbb{R}^*$, so as to obtain the group K_2

$$K_2[\mathbf{a},\mathbf{b},c,k,R(\theta)]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & kR(\theta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad R(\theta)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\in SO(2), \tag{21}$$

with $\theta\in[0,2\pi)$ and $k\in\mathbb{R}^*$. The group law is obtained by matrix multiplication, as usual,

$$K_2[\mathbf{a},\mathbf{b},c,k,R]\cdot K_2[\mathbf{a}',\mathbf{b}',c',k',R']=K_2[\mathbf{a}'+k'R'^{\,T}\mathbf{a},\ \mathbf{b}+kR\,\mathbf{b}',\ c+c'+\mathbf{a}\cdot\mathbf{b}',\ kk',\ RR'], \tag{22}$$

where $RR'\equiv R(\theta)R(\theta')=R(\theta+\theta')$.

3.2 The groups $H_{1,1}$ and $K_{1,1}$ on the pseudo-plane

A new generalization of H_2, $H_{1,1}$, can be obtain by replacing $\mathbb{R}^2$ by $\mathbb{R}^{(1,1)}$. It formally is like H_2 (20) but replacing $SO(2)$ by $SO_0(1,1)$), the connected component of the identity of $SO(1,1)$. The group $K_{1,1}$ comes from $H_{1,1}$ by adding $\mathbb{R}^*$,

$$K_{1,1}[\mathbf{a},\mathbf{b},c,k,\Lambda(\eta)]=\begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\Lambda(\eta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad \Lambda(\eta)=\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\in SO_0(1,1), \tag{23}$$

with $\eta\in\mathbb{R}$ and $k\in\mathbb{R}^*$. The group law for $K_{1,1}$ is similar to that of K_2 (22), provided R is replaced by Λ. Note that K_2 has only a connected component while $K_{1,1}$ has two.

3.3 The Lie algebras of K_2 and $K_{1,1}$

Both algebras are 7D with infinitesimal generators X_1, X_2, P_1, P_2, I, D and, moreover, J for Lie$[K_2]$ and K for Lie$[K_{1,1}]$. A 4×4 matrix realization of the generators is

$$\begin{aligned} X_\alpha &= \frac{\partial K_-}{\partial a^\alpha}\Big|_{Id}=\begin{pmatrix} 0 & \boldsymbol{\alpha}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad P_\alpha=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & \boldsymbol{\alpha} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \\ I &= \frac{\partial K_-}{\partial c}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & c \\ \mathbf{0} & \mathbb{O}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad D=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \end{aligned} \tag{24}$$

where $\boldsymbol{\alpha}$ is either the column vector $(1,0)^T$ for $\alpha=1$ or $(0,1)^T$ for $\alpha=2$ and $\mathbb{O}_2$ is the 2×2 zero matrix. The generators J and K are represented as

$$J=\frac{\partial K_-}{\partial \theta}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & -i\sigma_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad K=\frac{\partial K_-}{\partial \eta}\Big|_{Id}=\begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \sigma_1 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \tag{25}$$

where σ_i are Pauli matrices. The non-vanishing commutation relations are

$$[X_\alpha, P_\beta] = \delta_{\alpha\beta} I\,, \quad [D, X_\alpha] = -X_\alpha\,, \quad [D, P_\alpha] = +P_\alpha\,, \tag{26}$$

together with these for Lie$[K(2)]$

$$[J, X_\alpha] = \epsilon_{\alpha\beta} X_\beta\,, \quad [J, P_\alpha] = \epsilon_{\alpha\beta} P_\beta\,, \tag{27}$$

where $\epsilon_{\alpha\beta}$ is the skew-symmetric tensor, and these ones for Lie$[K(1,1)]$

$$[K, X_\alpha] = (-1)^\alpha \epsilon_{\alpha\beta} X_\beta\,, \quad [K, P_\alpha] = (-1)^{\alpha+1} \epsilon_{\alpha\beta} P_\beta\,. \tag{28}$$

3.4 Bases on the plane and the hyperplane

Now we will consider together the 2D real affine space $\mathbb{X}$ associated to either the vector space $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ and the Hilbert space $L^2(\mathbb{X})$ on which we define the position operator $\mathbf{X} \equiv (X_1, X_2)$ and their conjugate momentum operator $\mathbf{P} \equiv (P_1, P_2)$. These operators act on the eigenvectors $|\mathbf{x}\rangle\, (\equiv |x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle)$ and $|\mathbf{p}\rangle$, respectively, as $X_\alpha\,|\mathbf{x}\rangle = x_\alpha |\mathbf{x}\rangle$ and $P_\alpha\,|\mathbf{p}\rangle = p_\alpha |\mathbf{p}\rangle\,, \alpha = 1, 2\,.$ These eigenvectors are transformed into each other by means of Fourier type transformations (6) but in 2D

$$|\mathbf{p}\rangle = \frac{1}{2\pi} \int_{\mathbb{X}} d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}\, |\mathbf{x}\rangle\,, \qquad |\mathbf{x}\rangle = \frac{1}{2\pi} \int_{\mathbb{X}} d\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, |\mathbf{p}\rangle\,. \tag{29}$$

As for the 1D case (9) we have similar relations: $e^{-i\mathbf{b}\cdot\mathbf{P}}\,|\mathbf{x}\rangle = |\mathbf{x}+\mathbf{b}\rangle$ and $e^{-i\mathbf{a}\cdot\mathbf{X}}\,|\mathbf{p}\rangle = |\mathbf{p}-\mathbf{a}\rangle$ $(\mathbf{a}, \mathbf{b} \in \mathbb{X})$. Hence the basis $\{|\mathbf{x}\rangle\}$ is equivalent to $\{|\mathbf{x}+\mathbf{b}\rangle\}$ and the same for $\{|\mathbf{p}\rangle\}$ and $\{|\mathbf{p}-\mathbf{a}\rangle\}$.

The use of the 2D FT serves us to realize that the five operators given by $\mathbf{X}$, $\mathbf{P}$ and I determine a UIR representation of H_2 or $H_{1,1}$ by exponentiation.

Let $\mathcal{H}$ be an abstract infinite-D separable Hilbert space and $S : \mathcal{H} \to L^2(\mathbb{X})$ a unitary map. If $|f\rangle \in \mathcal{H}$ and $S\,|f\rangle = f(x)$ we have the following relation in a suitable dense subspace of $\mathcal{H}$

$$|f\rangle = \int_{\mathbb{X}} d\mathbf{x}\, f(\mathbf{x})\, |\mathbf{x}\rangle\,, \qquad f(\mathbf{x}) = \langle \mathbf{x}|f\rangle\,. \tag{30}$$

The action of an element of K_2 (or $K_{1,1}$) on $\mathbb{X}$ implies that $|\mathbf{x}\rangle$ transforms as

$$|\mathbf{x}\rangle \to |\mathbf{x}'\rangle = |k|\, e^{ih(c+\mathcal{C}+\mathbf{a}\cdot\mathbf{b}/2)}\, e^{ih\mathbf{a}\cdot(k\Lambda\mathbf{x}+\mathbf{b})}\, |k\,\Lambda\mathbf{x}+\mathbf{b}\rangle\,, \tag{31}$$

see (10), (13) and (14). This action allows to calculate the action of a UIR on $L^2(\mathbb{X})$

$$(U(g)f)(\mathbf{x}) = |k|^{-1}\, e^{ic}\, e^{-ik^{-1}\mathbf{a}\cdot\Lambda^{-1}(\mathbf{x}-\mathbf{b})}\, f(k^{-1}\Lambda^{-1}(\mathbf{x}-\mathbf{b}))\,. \tag{32}$$

The interested reader can easily compute similar expressions for $|\mathbf{p}\rangle$ and $f(\mathbf{p})$.

3.5 Bases on $L^2(\mathbb{X})$

The HFs $\psi_\alpha(x_\alpha)$ determine an orthonormal basis on $L^2(\mathbb{R})$ (Subsection 2.2). So the functions

$$\Psi_{\mathbf{m}}(\mathbf{x}) := \psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \qquad \mathbf{m} = (m_1, m_2) \in \mathbb{N}^2\,, \tag{33}$$

constitute an orthonormal basis on $L^2(\mathbb{X})$, i.e. for any $f(x) \in L^2(\mathbb{X})$ we have that

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{N}^2}^{\infty} c^{\mathbf{m}}\, \Psi_{\mathbf{m}}(\mathbf{x}) \equiv \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c^{m_1,m_2}\; \psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \quad c^{m_1,m_2} \in \mathbb{C}\,. \tag{34}$$

The double HF or the 2D HF functions $\Psi_{\mathbf{m}}(\mathbf{x})$ verify the following relations

$$\begin{aligned}\int_{\mathbb{R}^2} d\mathbf{x}\,[\Psi_{\mathbf{m}'}(\mathbf{x})]^*\,\Psi_{\mathbf{m}}(\mathbf{x}) &= \delta_{\mathbf{m},\mathbf{m}'} \equiv \delta_{m_1,m_1'}\,\delta_{m_2,m_2'}\,,\\ \sum_{\mathbf{m}\in\mathbb{N}^2}[\Psi_{\mathbf{m}}(\mathbf{x})]^*\,\Psi_{\mathbf{m}}(\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}) \equiv \delta(x^1-y^1)\,\delta(x^2-y^2)\,.\end{aligned} \tag{35}$$

They are real functions and eigenfunctions of the FT and of its inverse, i.e.,

$$FT\,[\Psi_{\mathbf{m}}(\mathbf{x});\mathbf{x};\mathbf{p}] = i^{\tilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{p})\,,\qquad IFT\,[\Psi_{\mathbf{m}}(\mathbf{p});\mathbf{p};\mathbf{x}] = (-i)^{\tilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{x})\,,\quad \tilde{\mathbf{m}} := \sum_\alpha m_\alpha\,. \tag{36}$$

In this 2D case we can profit from the invariance properties of 2D HF to construct a representation of the groups K_2 (or $K_{1,1}$) supported on a set of *generalized* HF. We start by defining

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) := |k|\,e^{-i\,\mathbf{a}(k\Lambda\mathbf{x}+\mathbf{b}/2)}\,\Psi_{\mathbf{m}}(k\Lambda\mathbf{x}+\mathbf{b})\,. \tag{37}$$

Now we are able to obtain an explicit form of the 2D generalized HF in terms of the 1D generalized HF, (17) and (18), as

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) = \chi_{m_1}((\Lambda\mathbf{x})^1,a^1,b^1,k)\,\chi_{m_2}((\Lambda\mathbf{x})^2,a^2,b^2,k)\,, \tag{38}$$

where $(\Lambda\mathbf{x})^\alpha$ denotes la α-th contravariant component of the vector $\Lambda\mathbf{x}$.

The 2D GHF determine an orthonormal basis on $L^2(\mathbb{X})$ since

$$\begin{aligned}\int_{\mathbb{R}^2} d\mathbf{x}\,\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}'}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)]^* &= \delta_{\mathbf{m},\mathbf{m}'}\,,\\ \sum_{\mathbf{m}\in\mathbb{N}^2}\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{y},\mathbf{a},\mathbf{b},k,\Lambda)]^* &= \delta(\mathbf{x}-\mathbf{y})\,.\end{aligned} \tag{39}$$

In addition, for the FT in 2D and its inverse we have the following relations:

$$\begin{aligned}FT\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,;\,\mathbf{x},\mathbf{p}] &= i^{\tilde{\mathbf{m}}}\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,,\\ IFT\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{a},\mathbf{b},k,\Lambda)\,;\mathbf{p};\mathbf{x}] &= (-i)^{\tilde{\mathbf{m}}}\,\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,.\end{aligned} \tag{40}$$

4 Conclusion

We present a revision of some generalizations of the Euclidean groups [8–10] by considering as an ensemble the equivalence of conjugate variables, and the properties of homogeneity, self-similarity and invariance from orientation that are present in the description of physical systems. The group extensions of the Euclidean-like groups by the HW group give rise to new groups that amalgamate the symmetries associated to both groups together with the invariances that we have just mentioned above. Moreover these groups $K_{p,q}$ (with $q+p=n$) admit a representation in terms of $(n+2)\times(n+2)$ matrices. In particular, we have displayed here the low dimensional cases (1D and 2D). The nD case can be easily implemented from the 2D case [7]. Thus, the elements of the n-D Heisenberg-Weyl group are given (see expression (20)) by

$$H_{p,q}[\mathbf{a},\mathbf{b},c] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c\\ \mathbf{0} & \mathbb{I}_n & \mathbf{b}\\ 0 & \mathbf{0}^T & 1\end{pmatrix}, \qquad \mathbf{a},\mathbf{b}\in\mathbb{R}^{(p,q)}\,,\;\; c\in\mathbb{R}\,. \tag{41}$$

Now according to (21) and (23) we can write the matrix elements of the new group $K_{p,q}$ as

$$K_{p,q}[\mathbf{a},\mathbf{b},c,k,\Lambda] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\,\Lambda & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \qquad \mathbf{a},\mathbf{b} \in \mathbb{R}^{(p,q)},\ k \in \mathbb{R}^*,\ \Lambda \in SO(p,q)\,. \tag{42}$$

Since the HF are an orthogonal basis of $L^2(\mathbb{R}^1)$ a basis on $L^2(\mathbb{R}^{p,q})$ is obtained in terms of nD HF, which can be easily obtained taking into account formula (33). The function spaces $L^2(\mathbb{R}^{p,q})$ support a UIR of the group $K_{p,q}$, that allows us to define a new set of orthonormal functions, the nD generalized Hermite functions following the expressions (37) and (38).

The existence of both discrete and continuous bases supporting representations of $K_{p,q}$ lead us to introduce a generalization of the Hilbert spaces: the rigged Hilbert spaces (or Gel'fand triplets) [12]. Then the infinitesimal generators of $K_{p,q}$ realized by self-adjoint operators on $L^2(\mathbb{R}^{p,q})$ are, generally, unbounded become bounded (continuous) operators (on two different locally convex topologies) using these rigged Hilbert spaces [7].

The nD Hermite functions appear in many quantum systems with quadratic Hamiltonians [15, 16], hence our results could be of interest, for instance, in Quantum Optics (photon distribution on multimodes mixed states [17]), in multidimensional signals analysis (decomposition of signals in terms of wavelets involves Fourier transform or Gabor transform [3, 18, 19]) and in vision studies [20–22].

Acknowledgements

Funding information We acknowledge partial financial support by the Ministerio de Ciencia e Innovación (MCIN) of Spain with funding from the European Union NextGenerationEU (PRTRC17.I1) and MCIN project PID2020-113406GB-I0.

References

[1] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, USA, ISBN 9780691085289 (1989).

[2] R. B. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Am. Soc. **3**, 821 (1980), doi:10.1090/S0273-0979-1980-14825-9.

[3] D. Gabor, *Theory of communication*, J. Inst. Electr. Eng. - III: Radio Commun. Eng. **93**, 429 (1946), doi:10.1049/ji-3-2.1946.0074.

[4] W. Czaja and J. Zienkiewicz, *Uncertainty principle for Gabor systems and the Zak transform*, J. Math. Phys. **47**, 123507 (2006), doi:10.1063/1.2393146.

[5] E. Celeghini, M. Gadella and M. A. del Olmo, *Applications of rigged Hilbert spaces in quantum mechanics and signal processing*, J. Math. Phys. **57**, 072105 (2016), doi:10.1063/1.4958725.

[6] E. Celeghini, M. Gadella and M. A. del Olmo, *Heisenberg-Weyl groups and generalized Hermite functions*, Symmetry **13**, 1060 (2021), doi:10.3390/sym13061060.

[7] E. Celeghini, M. Gadella and M. A. del Olmo, *Symmetry groups, quantum mechanics and generalized Hermite functions*, Mathematics **10**, 1448 (2022), doi:10.3390/math10091448.

[8] J. A. Hogan and J. D. Lakey, *Extensions of the Heisenberg group by dilations and frames*, Appl. Comput. Harmon. Anal. **2**, 174 (1995), doi:10.1006/acha.1995.1013.

[9] B. Torresani, *Time-frequency representations: Wavelet packets and optimal decomposition*, Ann. I. H. P. Phys. Théor. **56**, 215 (1992), http://www.numdam.org/item/AIHPA_1992__56_2_215_0/.

[10] C. Kalisa and B. Torresani, *N-dimensional affine Weyl-Heisenberg wavelets*, Ann. I. H. P. Phys. Théor. **59**, 201 (1993), http://www.numdam.org/item/AIHPA_1993__59_2_201_0/.

[11] B. C. Hall, *Lie groups, Lie algebras, and representations*, Springer, Berlin, Heidelberg, Germany, ISBN 9783319374338 (2015), doi:10.1007/978-3-319-13467-3.

[12] A. Bohm, *Boulder lectures in theoretical physics*, Gordon and Breach, New York, USA (1967).

[13] S. T. Ali, J. P. Antoine and J. P. Gazeau, *Coherent states, wavelets and their generalizations*, Springer, New York, USA, ISBN 9780387989082 (1999), doi:10.1007/978-1-4614-8535-3.

[14] G. B. Folland, *Fourier analysis and its applications*, Wadsworth, Pacific Grove, USA, ISBN 9780821847909 (1992).

[15] V. V. Dodonov, V. I. Man'ko and V. V. Semjonov, *The density matrix of the canonically transformed multidimensional Hamiltonian in the Fock basis*, Nuovo Cimento B **83**, 145 (1984), doi:10.1007/BF02721587.

[16] V. V. Dodonov and V. I. Man'ko, *New relations for two-dimensional Hermite polynomials*, J. Math. Phys. **35**, 4277 (1994), doi:10.1063/1.530853.

[17] V. V. Dodonov, O. V. Man'ko and V. I. Man'ko, *Multidimensional Hermite polynomials and photon distribution for polymode mixed light*, Phys. Rev. A **50**, 813 (1994), doi:10.1103/PhysRevA.50.813.

[18] H. G. Feichtinger and G. Zimmermann, *A Banach space of test functions for Gabor analysis*, in *Gabor analysis and algorithms*, Birkhäuser, Boston, USA, ISBN 9781461273820 (1998), doi:10.1007/978-1-4612-2016-9_4.

[19] K. Gröchenig and Y. Lyubarskii, *Gabor (super)frames with Hermite functions*, Math. Ann. **345**, 267 (2009), doi:10.1007/s00208-009-0350-8.

[20] J. Yang and A. Reeves, *Bottom-up visual image processing probed with weighted Hermite polynomials*, Neural Netw. **8**, 669 (1995), doi:10.1016/0893-6080(95)00023-S.

[21] A. Reeves and J. Yang, *Visual pattern encoding with weighted Hermite polynomials*, Spatial Vis. **14**, 391 (2001), doi:10.1163/156856801753253609.

[22] J. D. Victor, F. Mechler, M. A. Repucci, K. P. Purpura and T. Sharpee, *Responses of V1 neurons to two-dimensional Hermite functions*, J. Neurophysiol. **95**, 379 (2006), doi:10.1152/jn.00498.2005.

SciPost Phys. Proc. 14, 024 (2023)

Newton mechanics, Galilean relativity, and special relativity in α-deformed binary operation setting

Won S. Chung[1] and Mahouton N. Hounkonnou[2⋆]

1 Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea
2 International Chair in Mathematical Physics and Applications, ICMPA-UNESCO Chair, University of Abomey-Calavi, 072 BP 50, Cotonou, Rep. of Benin

⋆ norbert.hounkonnou@cipma.uac.bj

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We define new velocity and acceleration having dimension of $(Length)^{\alpha}/(Time)$ and $(Length)^{\alpha}/(Time)^2$, respectively, based on the fractional addition rule. We discuss the formulation of fractional Newton mechanics, Galilean relativity and special relativity in the same setting. We show the conservation of the fractional energy, characterize the Lorentz transformation and group, and derive the expressions of the energy and momentum. The two body decay is discussed as a concrete illustration.

Received 14-12-2022
Accepted 21-07-2023
Published 23-11-2023
doi:10.21468/SciPostPhysProc.14.024

1 Introduction

This work is based on pseudo analysis (see [6] and [7], and references therein for a review). It is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$. It has different applications in mathematics and physics, e.g. in modeling nonlinearity, uncertainty in optimization problems, nonlinear partial differential equations, nonlinear difference equations, optimal control, fuzzy systems, decision making, game theory, etc. It also gives solutions in the forms, which are not achieved by other approaches, e.g., Bellman difference equation, Hamilton Jacobi equation with non-smooth Hamiltonians.

Definition 1.1. The pseudo binary operations are defined by the help of a monotonous bijective map f, called their generator, as:

$$x \oplus_f y = f^{-1}(f(x) + f(y)), \qquad x \ominus_f y = f^{-1}(f(x) - f(y)), \tag{1}$$

$$x \otimes_f y = f^{-1}(f(x) f(y)), \quad \text{and} \quad x \oslash_f y = f^{-1}(f(x)/f(y)), \tag{2}$$

$$x^n \oplus_f y^m = x^n + y^m, \qquad x^n \ominus_f y^m = x^n - y^m, \tag{3}$$

$$x^n \otimes_f y^m = x^n y^m, \quad \text{and} \quad x^n \oslash_f y^m = x^n / y^m, \tag{4}$$

$$\left(x \oplus_f y\right)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad \left(x \ominus_f y\right)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k y^{n-k}, \tag{5}$$

$$\left(x \otimes_f y\right)^n = (xy)^n, \qquad \text{and} \qquad \left(x \oslash_f y\right)^n = (x/y)^n. \tag{6}$$

It can be easily checked that the operation $\oplus_f$ and $\otimes_f$ satisfy the commutativity and associativity properties. Through the map f, we can perform many deformed binary operations [2,8]. The velocity addition formula can be recovered by using the pseudo-analysis, (see Chung *et al.* [4] about some applications of the pseudo-binary operations). The pseudo binary operations were also used to construct the q-additive entropy theory [1].

Recently, in 2019, Chung and Hassanabadi [4] considered a special choice of f,

$$f(x) = |x|^{\alpha-1} x, \qquad \alpha > 0, \tag{7}$$

so that the deformed multiplication and deformed division may be the same as the ordinary ones. Using this, these authors studied the anomalous diffusion process by using the α-deformed mechanics which possesses the α-translation in space $x \to x \oplus \delta x$. The α-deformed binary operations, i. e. α-addition, α-subtraction, α-multiplication and α-division take the form:

$$a \oplus_\alpha b = |a|a|^{\alpha-1} + b|b|^{\alpha-1}|^{1/\alpha-1} \left(a|a|^{\alpha-1} + b|b|^{\alpha-1}\right), \tag{8}$$
$$a \ominus_\alpha b = |a|a|^{\alpha-1} - b|b|^{\alpha-1}|^{1/\alpha-1} \left(a|a|^{\alpha-1} - b|b|^{\alpha-1}\right), \tag{9}$$
$$a \otimes_\alpha b = ab, \quad a \oslash_\alpha b = \frac{a}{b}. \tag{10}$$

Interestingly, the multiplication and division are invariant under α-deformation.

In this same spirit, in 2022, Hounkonnou *et al* proved that a Minkowski phase space endowed with a bracket relatively to a conformable (α-deformed) differential realizes a conformable Poisson algebra, confering a bi-Hamiltonian structure to the resulting manifold. They deduced that the related α-Hamiltonian vector field for a free particle is an infinitesimal Noether symmetry and computed the corresponding $\alpha-$deformed recursion operator [5].

The present paper is organized as follows. In Section 2, we derive the Newton law of $\alpha-$deformed Newton mechanics. Section 3 is devoted to the characterization of α-deformed Galilean relativity. The α-deformed Galilei group is described, and energy conservation law is deduced. In Section 4, we study the special relativity with α-translation symmetry. Section 5 deals with an analysis of two body decay.

2 α-deformed Newton mechanics

In an ordinary Newtonian mechanics in one dimension, the Newton velocity is defined as

$$v = \frac{dx}{dt}. \tag{11}$$

The infinitesimal displacement is invariant under spatial translation $x \to x + \delta x$ and the infinitesimal time interval is invariant under temporal translation $t \to t + \delta t$. If we impose new translation symmetry based on α-addition rule, we need to change the definition of velocity so that it may possess this new symmetry. Here we impose two translation symmetries: the α-translation in position, $x \to x \oplus_\alpha \delta x$, and α-translation in time, $t \to t \oplus_\alpha \delta t$.

Note in 2019, Chung and Hassanabadi [4] defined the deformed velocity, which is invariant under α-translation in position and ordinary translation in time. Their average velocity is given by

$$v_{ave} = \frac{f_\alpha(x' \ominus_\alpha x)}{t' - t} = \frac{\Delta_\alpha x}{\Delta t} = \frac{|x'|^{\alpha-1}x' - |x|^{\alpha-1}x}{t' - t}. \tag{12}$$

Taking $t' \to t$, we obtain the velocity:

$$v = \frac{d_\alpha x}{dt} = \alpha |x|^{\alpha-1}\frac{dx}{dt}. \tag{13}$$

If we impose the α-translation in both time and position, we have to change the definition of the velocity. In this case, the average α-velocity is furnished by the expression

$$v_{\alpha,ave} = \frac{f_\alpha(x' \ominus_\alpha x)}{f(t' \ominus_\alpha t)} = \frac{\Delta_\alpha x}{\Delta_\alpha t} = \frac{|x'|^{\alpha-1}x - |x|^{\alpha-1}x}{|t'|^{\alpha-1}t' - |t|^{\alpha-1}t}. \tag{14}$$

Taking $t' \to t$ leads to the α-velocity:

$$v_\alpha = \frac{d_\alpha x}{d_\alpha t} = t^{1-\alpha}|x|^{\alpha-1}\frac{dx}{dt}. \tag{15}$$

Because v_α is α-translation invariant, the α-acceleration is defined as

$$a_\alpha = \frac{dv_\alpha}{d_\alpha t} = \frac{1}{\alpha}t^{1-\alpha}\frac{dv_\alpha}{dt}. \tag{16}$$

Since the α-velocity and α-acceleration have dimension $[Length]^\alpha/[Time]^\alpha$ and dimension $[Length]^\alpha/[Time]^{2\alpha}$, respectively, the Newton equation is obtained by the relation

$$|F|^{\alpha-1}F = m^\alpha a_\alpha, \quad \text{or equivalently}, \quad F = m|a_\alpha|^{\frac{1}{\alpha}-1}a_\alpha. \tag{17}$$

In mechanics with α-translation symmetry, the α-velocity and α-acceleration have the fractional dimensions which are different from the ordinary case $\alpha = 1$. But, for the force, we assumed that it has the same dimension as in the $\alpha = 1$—mechanics.

3 α-deformed Galilean relativity

Based on the new definition of α-velocity and α-acceleration, we define the α-inertial frames of reference possessing the property that a body with zero net force acting upon these frames does not α-accelerate; that is, such a body is at rest or moving at a constant α-velocity. Here we assume *the physical laws must be the same in all α-inertial frames of reference.* Now let us consider two inertial frames $S(t,x)$ and $S'(t',x')$ moving at a relative constant α-velocity u_α with x-axes. The Newton equation is invariant under the transformations

$$v'_\alpha = v_\alpha - u_\alpha, \quad v'_\alpha = \frac{d_\alpha x'}{d_\alpha t}, \quad x' = x \ominus_\alpha |u_\alpha|^{\frac{1}{\alpha}-1}u_\alpha t, \quad t' = t. \tag{18}$$

3.1 α-deformed Galilei group

Based on the α-operations for matrices, we can rewrite the coordinate transformations of the Newton equation as:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = T_\alpha(u_\alpha) \otimes_\alpha \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & -|u_\alpha|^{\frac{1}{\alpha}-1} u_\alpha \\ 0 & 1 \end{pmatrix} \otimes_\alpha \begin{pmatrix} x \\ t \end{pmatrix}. \tag{19}$$

The transformation matrix $T_\alpha(u_\alpha)$ forms a Lie group with the α-multiplication. The following properties are indeed satisfied:

- $T_\alpha(u_\alpha) \otimes_\alpha T_\alpha(v_\alpha) = T_\alpha(u_\alpha + v_\alpha)$.
- The α-multiplication is associative.
- The identity is $T_\alpha(0)$.
- The inverse is $T_\alpha(-u_\alpha)$.

3.2 Energy conservation

Because dx is not invariant under the α-translation, we use α-translational invariant infinitesimal displacement to get $d_\alpha x = \alpha |x|^{\alpha-1} dx$ and define the work,

$$|W|^{\alpha-1} W = -\int d_\alpha x |F|^{\alpha-1} F\,, \tag{20}$$

having the same dimension as in the $\alpha = 1$—mechanics. We define the potential energy through the conservative force,

$$|F|^{\alpha-1} F = -\frac{d_\alpha U}{d_\alpha x} = -|x|^{1-\alpha} |U|^{\alpha-1} \frac{dU}{dx}\,. \tag{21}$$

Thus, for the conservative force, we have

$$|W_{1\to 2}|^{\alpha-1} W_{1\to 2} = -\left(|U_2|^{\alpha-1} U_2 - |U_1|^{\alpha-1} U_1\right). \tag{22}$$

Inserting the Newton equation obtained previously (see (17)) into (20), we get

$$|W_{1\to 2}|^{\alpha-1} W_{1\to 2} = K_2 - K_1\,, \tag{23}$$

where the kinetic energy is given by

$$K = \frac{1}{2} m^\alpha v_\alpha^2\,. \tag{24}$$

Considering the dimension, the conservation of energy is provided by

$$|E|^{\alpha-1} E = K + |U|^{\alpha-1} U = \frac{1}{2} m^\alpha v_\alpha^2 + |U|^{\alpha-1} U = \frac{p_\alpha^2}{2m^\alpha} + |U|^{\alpha-1} U\,, \tag{25}$$

where the linear momentum is expressed as $p_\alpha = m^\alpha v_\alpha$. The energy has the same dimension as in the $\alpha = 1$—mechanics, while the linear momentum has fractional dimension.

4 Special relativity with α-translation symmetry

The 3-position in non-relativistic mechanics is changed into 4-position (or event) in the relativistic mechanics. Let us consider the event $P(ct,x,y,z)$, where c is the Newton speed of light, (i. e. speed with $\alpha=1$). Based on the definition of α-translation invariant infinitesimal displacement and α-translation invariant infinitesimal time interval, the α-translation invariant distance (α-distance) of infinitesimally close space-time events denoted by ds_α is given by

$$d_\alpha s^2 = c^{2\alpha} d_\alpha t^2 - d_\alpha x^2 - d_\alpha y^2 - d_\alpha z^2 \,. \tag{26}$$

The α-deformed proper time τ_α is

$$d_\alpha \tau^2 = \frac{d_\alpha s^2}{c^{2\alpha}} \,. \tag{27}$$

4.1 α-Lorentz transformations

The α-Lorentz transformations making invariant the space-time interval

$$(\Delta_\alpha s)^2 = \left(c^\alpha (|t|^{\alpha-1} t)\right)^2 - \left((|x|^{\alpha-1} x)\right)^2 \,, \tag{28}$$

are given by

$$|x|^{\alpha-1} x = c^\alpha |t'|^{\alpha-1} t' sh_\alpha(\psi) + |x'|^{\alpha-1} x' ch_\alpha(\psi) \,, \tag{29}$$

$$c^\alpha |t|^{\alpha-1} t = c^\alpha |t'|^{\alpha-1} t' ch_\alpha(\psi) + |x'|^{\alpha-1} x' sh_\alpha(\psi) \,, \tag{30}$$

where the little α-deformed hyperbolic functions are defined by

$$sh_\alpha(\psi) := \frac{1}{2}(e_\alpha(\psi) - e_\alpha(-\psi)) = \sinh(|\psi|^{\alpha-1}\psi) \,, \tag{31}$$

$$ch_\alpha(\psi) := \frac{1}{2}(e_\alpha(\psi) + e_\alpha(-\psi)) = \cosh(|\psi|^{\alpha-1}\psi) \,, \tag{32}$$

$$th_\alpha(\psi) := \frac{sh_\alpha(\psi)}{ch_\alpha(\psi)} = \tanh(|\psi|^{\alpha-1}\psi) \,, \quad e_\alpha(x) := e^{|x|^{\alpha-1}x} \,. \tag{33}$$

The little α-deformed hyperbolic functions obey the relations

$$ch_\alpha^2(\psi) - sh_\alpha^2(\psi) = 1 \,. \tag{34}$$

In terms of the α-deformed binary operations, we get

$$x = ct' Sh_\alpha(\psi) \oplus x' Ch_\alpha(\psi) \,, \qquad ct = ct' Ch_\alpha(\psi) \oplus x' Sh_\alpha(\psi) \,, \tag{35}$$

where the big α-deformed hyperbolic functions are

$$Ch_\alpha(\psi) := |ch_\alpha(\psi)|^{\frac{1}{\alpha}-1} ch_\alpha(\psi) \,, \quad Sh_\alpha(\psi) := |sh_\alpha(\psi)|^{\frac{1}{\alpha}-1} sh_\alpha(\psi) \,, \tag{36}$$

$$Th_\alpha(\psi) := \frac{Sh_\alpha(\psi)}{Ch_\alpha(\psi)} \,, \tag{37}$$

obeying $|Ch_\alpha(\psi)|^2 \ominus |Sh_\alpha(\psi)|^2 = 1$. Consider in the coordinate system (ct,x) the origin of the coordinate system (ct',x'). Then, $x'=0$, and

$$x = ct' Sh_\alpha(\psi) \,, \quad ct = ct' Ch_\alpha(\psi) \,. \tag{38}$$

Dividing the two equations gives

$$\frac{x}{ct} = Th_\alpha(\psi) \,, \quad \text{or} \quad \frac{|x|^{\alpha-1}x}{c^\alpha |t|^{\alpha-1} t} = th_\alpha(\psi) \,. \tag{39}$$

Since $\frac{|x|^{\alpha-1}x}{|t|^{\alpha-1}t} = v_\alpha$ is the relative uniform α-velocity of the two systems, we identify the physical meaning of the imaginary "rotation angle ψ" as

$$th_\alpha(\psi) = \frac{v_\alpha}{c^\alpha} = \beta_\alpha\,. \tag{40}$$

Using the following identities

$$ch_\alpha(\psi) = \gamma_\alpha\,, \quad sh_\alpha(\psi) = \gamma_\alpha\beta_\alpha\,, \quad \text{where} \quad \gamma_\alpha = \frac{1}{\sqrt{1-\beta_\alpha^2}}\,, \tag{41}$$

we obtain the α-deformed Lorentz transformation of the form

$$|x|^{\alpha-1}x = \gamma_\alpha\left(|x'|^{\alpha-1}x' + v_\alpha|t'|^{\alpha-1}t'\right), \quad |t|^{\alpha-1}t = \gamma_\alpha\left(|t'|^{\alpha-1}t' + \frac{v_\alpha}{c^{2\alpha}}|x'|^{\alpha-1}x'\right). \tag{42}$$

Expressing the spatial and temporal coordinates in terms of the α-deformed binary operations, we get

$$x = \Gamma_\alpha(x' \oplus v_\alpha^{1/\alpha}t')\,, \quad t = \Gamma_\alpha\left(t' \oplus \frac{v_\alpha^{1/\alpha}}{c^2}x'\right), \quad \text{where} \quad \Gamma_\alpha = \gamma_\alpha^{1/\alpha} = (1-\beta_\alpha)^{-\frac{1}{2\alpha}}\,. \tag{43}$$

If we set

$$u_\alpha = \left(\frac{|x|^{\alpha-1}}{|t|^{\alpha-1}}\right)\frac{dx}{dt}\,, \qquad u'_\alpha = \left(\frac{|x'|^{\alpha-1}}{|t'|^{\alpha-1}}\right)\frac{dx'}{dt'}\,, \tag{44}$$

the addition of α-velocity becomes

$$u_\alpha = \frac{u'_\alpha + v_\alpha}{1 + \frac{v_\alpha u_\alpha}{c^{2\alpha}}}\,. \tag{45}$$

If we regard the α-speed of light as c^α, the eq.(44) shows that the α-speed of light remains invariant, and, hence, the speed of light also remains invariant under the α-deformed Lorentz transformation.

4.2 α-Lorentz group

Now, let us introduce the four α-velocity. For that, we change the notation as:

$$ct = x^0\,, \qquad x = x^1\,, \qquad y = x^2\,, \qquad z = x^3\,. \tag{46}$$

Then, the four α-velocity is given by

$$u_\alpha^a = \frac{|x^a|^{\alpha-1}dx^a}{(\tilde{d}\tau)_\alpha}\,, \quad \text{or explicitly}\,, \quad u_\alpha^0 = c^\alpha\gamma_\alpha\,, \quad u_\alpha^i = v_\alpha^i\gamma_\alpha\,, \quad i = 1,2,3\,. \tag{47}$$

Therefore, we have

$$\eta_{ab}u_\alpha^a u_\alpha^b = c^{2\alpha}. \tag{48}$$

4.3 Energy and α-momentum

The four α-momentum is defined as

$$p_\alpha^a = m^\alpha u_\alpha^a\,, \tag{49}$$

explicitly giving

$$p_\alpha^0 = m^\alpha c^\alpha\gamma_\alpha\,, \quad p_\alpha^i = m^\alpha v_\alpha^i\gamma_\alpha\,, \quad \text{and thus} \quad \eta_{ab}p_\alpha^a p_\alpha^b = m^{2\alpha}c^{2\alpha}\,. \tag{50}$$

Here, we have $p_\alpha^a \neq (E/c, \vec{p}_\alpha)$ because the energy in α-deformed mechanics has the same unit as in the undeformed case. Therefore, we set

$$p_\alpha^a = \left(\left(\frac{E}{c}\right)^\alpha, \vec{p}_\alpha\right). \tag{51}$$

Thus, the eq.(50) gives

$$E^{2\alpha} = c^{2\alpha}|\vec{p}_\alpha|^2 + m^{2\alpha}c^{4\alpha}, \quad \text{and when} \quad |\vec{v}_\alpha| \ll c^\alpha, \quad E^\alpha \approx \frac{|\vec{p}_\alpha|^2}{2m^\alpha}, \tag{52}$$

which is the same as the non-relativistic case.

5 Two body decay

The simplest particle reaction is the two-body decay of unstable particles. A well known example from nuclear physics is the alpha decay of heavy nuclei. In particle physics, one observes, for instance, decays of charged pions or kaons into muons and neutrinos, or decays of neutral kaons into pairs of pions, etc. Consider the decay of a particle of mass M which is initially at rest. Its four α-momentum is $P = (M^\alpha, \vec{0})$, where we set $c = 1$.

This reference frame is called the centre-of mass frame (CMS). Denote the four α-momenta of the two daughter particles by $p_1 = (E_1^\alpha, \vec{p}_{\alpha,1}), p_2 = (E_2^\alpha, \vec{p}_{\alpha,2})$. From the momentum conservation, we get

$$\vec{p}_{\alpha,1} + \vec{p}_{\alpha,2} = 0. \tag{53}$$

The energy conservation is

$$M^\alpha = \sqrt{|\vec{p}_{\alpha,1}|^2 + m_1^{2\alpha}} + \sqrt{|\vec{p}_{\alpha,2}|^2 + m_2^{2\alpha}}. \tag{54}$$

If we set

$$p = |\vec{p}_{\alpha,1}| = |\vec{p}_{\alpha,2}|, \tag{55}$$

we have

$$p = \frac{1}{2M^\alpha}\sqrt{\left(M^{2\alpha} - \left(m_1^\alpha - m_2^\alpha\right)^2\right)\left(M^{2\alpha} - \left(m_1^\alpha + m_2^\alpha\right)^2\right)}. \tag{56}$$

Thus, we have

$$M \geq m_1 \oplus_\alpha m_2. \tag{57}$$

6 Conclusion

This work has focused on the formulation of Newton mechanics, Galilean relativity, and special relativity in α-deformed binary operation setting. The Galilei and Lorentz groups have been explored, but there still remain other interesting aspects and finer questions relating to the analysis of their corresponding Lie algebras and interrelationships, which cannot be discussed within the framework of this short article without leaving the reader hungry. Such a study is currently under consideration and will be covered in detail in the forthcoming paper.

Acknowledgements

The authors thank the anonymous referees for their useful comments which helped to improve the paper.

Funding information M.N.H. is supported by the ICMPA-UNESCO Chair, which is in partnership with the Daniel Iagolnitzer Foundation (DIF), France, and the Association pour la Promotion Scientifique de l'Afrique (APSA), supporting the development of mathematical physics in Africa.

References

[1] E. P. Borges, *A possible deformed algebra and calculus inspired in nonextensive thermostatistics*, Phys. A: Stat. Mech. Appl. **340**, 95 (2004), doi:10.1016/j.physa.2004.03.082.

[2] W. S. Chung and H. Hassanabadi, *The f-deformation I: f-deformed classical mechanics*, Rep. Math. Phys. **85**, 1 (2020), doi:10.1016/S0034-4877(20)30007-0.

[3] W. S. Chung and H. Hassanabadi, *The f-deformation II: f-deformed quantum mechanics in one dimension*, Rep. Math. Phys. **85**, 305 (2020), doi:10.1016/S0034-4877(20)30038-0.

[4] W. S. Chung and H. Hassanabadi, *Deformed classical mechanics with α-deformed translation symmetry and anomalous diffusion*, Mod. Phys. Lett. B **33**, 1950368 (2019), doi:10.1142/S0217984919503688.

[5] M. N. Hounkonnou, M. J. Landalidji and M. Mitrović, *Einstein field equation, recursion operators, Noether and master symmetries in conformable Poisson manifolds*, Universe **8**, 247 (2022), doi:10.3390/universe8040247.

[6] E. Pap, *Generalized real analysis and its applications*, Internat. J. Approx. Reason. **47**, 368 (2008), doi:10.1016/j.ijar.2007.05.015.

[7] E. Pap, *g-calculus*, Novi Sad J. Math. **23**, 145 (1993).

[8] N. Ralević, *Pseudo-analysis and applications on solution nonlinear equations*, PhD thesis, PMF Novi Sad, Novi Sad, Serbia (1997).

On old relations of Lie theory, classical geometry and gauge theory

Rolf Dahm⋆

Beratung für Informationssysteme und Systemintegration, D-55116 Mainz, Germany

⋆ dahm@bf-is.de

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Having been led by hadron interactions and low-energy photoproduction to SU(4) and non-compact SU∗(4) symmetry, the general background turned out to be projective geometry (PG) of P^3, or when considering line and Complex geometry to include gauge theory, aspects of P^5. Point calculus and its dual completion by planes introduced quaternary (quadratic) 'invariants' $x_\mu x^\mu = 0$ and $p_\mu p^\mu = 0$, and put focus on the intermediary form (xu) and its treatment. Here, the major result is the identification of the symmetric $\underline{20}$ of SU(4) comprising nucleon and Delta states as related to the quaternary cubic forms discussed by Hilbert in his work on full invariant systems. So PG determines *geometrically* the scene by representations (reps) and invariant theory without having to force affine restrictions and additional (spinorial or gauge) rep theory.

Received 18-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.025

1 Introduction

When analyzing low-energy hadronic experiments and their degrees of freedom in the context of effective chiral theories, it turned out that with respect to the pion-nucleon-delta system SU(4) linear states were able to describe the fermionic N- and Δ-states and their properties linearly. Explicitly, by starting over from current algebra and spectral descriptions (i.e. Goldberger-Treiman, PCAC,... [2]), Sudarshan [24] proposed to saturate the Adler-Weisberger sum rule [1], [26] of the (axial) charge commutators by quasi-particle calculations based on usual spin-isospin states. Thus, by requiring that the quasi-particle ansatz

$$N'_{\text{dyn}} = \lambda N_{\text{stat}} + \sqrt{1-\lambda^2} \int \pi(x) N(x) d^4x \,, \tag{1}$$

describes the axial coupling g_A^2, we've showed that the 'dynamic' states N'_{dyn} fit perfectly to the members of the (linear) threefold symmetric rep $\underline{\mathbf{20}}$ of SU(4) [6], [7], [8], built symmetrically out of three fundamental reps $\underline{\mathbf{4}}$. The light non-strange mesons π, ω, and ρ fit into the linear SU(4) rep $\underline{\mathbf{15}}$, and due to SU(4) $\cong A_3$ it is evident from the Dynkin diagram that we can 'embed' (or identify) two commuting (chiral) SU(2) groups, $A_1 \otimes A_1$. But because SU(4) still yields well-defined 3-projections of spin and isospin, we cannot only treat the various actions of SU(2)×SU(2) or the 'spontaneously broken symmetry' SU(2)×SU(2)/SU(2) of the (nonlinear) chiral approaches – this rep theory can also handle partially conserved axial currents ('PCAC') in terms of the SU(2) pion field(s). Moreover, in contrary to somewhat tedious and 'higher order' calculations in effective chiral approaches, the threshold production amplitude of π^0 on the nucleon (the interactions of $\underline{\mathbf{20}}$ with $\underline{\mathbf{15}}$ when reduced to the observed 'spin-isospin states') yields strong suppression in first order by superselection rules. Last not least, pion scattering on the nucleon when treated by SU(4) reps yields small charge dependence ('isospin breaking'). With respect to usual quark descriptions, it is noteworthy that $\underline{\mathbf{20}}$ in the spinorial rep yields three symmetric constituents while $\underline{\mathbf{15}}$ comprises the fundamental rep $\underline{\mathbf{4}}$ and its conjugate $\underline{\mathbf{4}}^*$.

So the manifest question 'Why this?' lead us to a series of papers (see refs in [13]) to discuss SU(4) and the non-compact group SU∗(4), their common maximal compact subgroup USp(4) and certain aspects of the associated Riemannian spaces AII∼SU∗(4)/USp(4) and CI∼USp(4)/SU(2)×U(1). Here, however, it is time to step back from the transformation groups and related mathematical constructions, and to recall some old relations of Lie theory with physical and geometrical aspects.

An essential 'in-between' has been achieved by considering line geometry which allowed to associate reps of gauge bosons and reps of line Complexe [11], and to identify Lorentz transformations in Special Relativity as a special transformations of the Plücker-Klein quadric M_2^4 onto itself [13]. This emphasized the importance of treating line Complexe in P^5 and their 'reduction' via the Plücker-Klein quadric to line sets in P^3 so that in terms of (line) geometry of P^3 the symplectic transformations reflect mappings of Complexe onto each other, and the Lorentz (point) transformations of Special Relativity ensure 'invariance of line geometry' by restricting the Complexe to transformations of the Plücker-Klein quadric ([13], IV.C).

This concept on the one hand paved the path to identify the photon rep with a special line Complex, and it pointed to a possible geometrical/physical background of the 5-dim coset space SU∗(4)/USp(4), a rank-1 irreducible globally symmetric Riemannian space AII [9], and the occurrence of symplectic symmetries. On the other hand, it pointed to the necessary treatment of line Complexe, line sets, Congruences or ray systems, and associated reps from scratch. The 10-dim rank-2 CI-space can be represented once more by SU(2)×U(1) symmetric cosets which yield a simple background when restricting PG to affine geometry.

Needless to say, that besides the abstraction of 'a point x' and the associated evangelism of (Lagrangean) point motion, we have to consider its dual – the plane u – in P^3 as well, or – as a substitute of both – quadratic line geometry (lines being dual to lines in P^3) and using Hamilton's approach. So in all cases, reps of 'non-local' or 'extended' objects like lines or planes enter rep theory, although in PG of P^3 we can still found on their linearity. More generally, all such identifications require a priori a stricter treatment of the reps by (Lie) transformation theory and of their geometry, and a common treatment of lines versus points and planes (which compels a thorough discussion of conjugation, or duality, too!). In other words, as long as we treat linear reps and symmetries, we should apply projective geometry (PG) from scratch in order to *derive* and treat two aspects consistently: the breakdown to affine geometry by fixing an 'absolute plane' in order to connect geometrically to Weyl's concepts and gauge symmetries used throughout field and quantum theories, and the metric aspects from the viewpoint of Cayley and Klein with respect to a given (invariant) polar system ('absolute quadric') and the respective transformation groups.

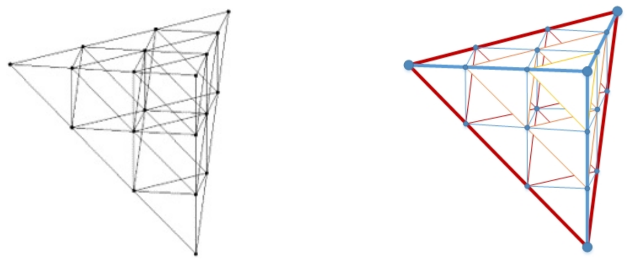


Figure 1: Symmetric $\underline{\mathbf{20}}$; left: construction by $\underline{\mathbf{4}}$ of SU(4) [8], F.6; right: subdivision by Hilbert [16], §19.

2 The results

Now, instead of taking the long approach to answer 'why' SU(4) obviously works with respect to hadron reps and symmetries, we argue 'top down' using both our construction scheme of SU(4) reps, and especially of $\underline{\mathbf{20}}$ (see details in [8], app. F.6), as well as Hilbert's (almost forgotten) foundations on invariant systems [16].

Whereas due to the rank-3 group A_3 the root system can be 'rotated' to P^3 and serve as a coordinate system by identifying $\underline{\mathbf{4}}$ with the fundamental tetrahedron, the construction of $\underline{\mathbf{20}}$ yields a 'threefold' tetrahedron subdivided by '$\underline{\mathbf{4}}$'s, see fig. 1, left. The individual states of $\underline{\mathbf{20}}$ are given and discussed e.g. in [6], [7], and [8].

But using the symmetry group SU(4) requires a treatment of its invariant theory, especially of the full invariant system. As such, being concerned to construct the full invariant system with respect to (quaternary) linear transformations, we can use Hilbert's approach [16] and the related forms. The approach to rep theory via forms is suitable because the transformation determinant is 1, so all occurring determinant powers throughout invariant theory are 1, too, i.e. the respective forms are not altered by additional determinant factors.

Citing Hilbert's construction scheme ([16], §19), we find:[1] *'(...) For example, to construct the quaternary forms of the 3^{rd} order, we construct a regular tetrahedron in 3-space with edge length 3, then divide each edge into three equal pieces and draw through the partial points two parallel planes to each of the four side faces; these planes cut the tetrahedron into regular tetrahedra with edge length 1. Each corner point (n_1, n_2, n_3, n_4) of these tetrahedra corresponds to a member of the quaternary cubic form. (...)'.*

So while we have constructed the rep $\underline{\mathbf{20}}$ (fig. 1, left) in a bottom-up approach by means of roots and the fundamental tetrahedron $\underline{\mathbf{4}}$ [8], Hilbert subdivides 'top-down' the 'large' tetrahedron (fig. 1, right) and identifies each of the 20 intersection/corner points (n_1,n_2,n_3,n_4) with a member of the quaternary cubic form, i.e. with one member of $\underline{\mathbf{20}}$, or what we denoted initially by a 'Chiron' [6], [7]. Based on our construction scheme, besides the bridge to well-established classical invariant theory, we thus have a symbolism at hand to treat the geometry of P^3 in terms of quaternary forms. From the physical point of view, when recalling the historic and ongoing quest for hadronic states and equations of motions (see e.g. [22] and references where one tries to separate spin content), we have identified the irrep $\underline{\mathbf{20}}$ yielding physical as well as geometrical background while additionally subordinating into the algebraic framework of invariant theory.[2] Whereas *synthetic* geometry proposes additional rich background and strategy (see e.g. [20] or [21] §2ff.), the *analytical* frameworks and tools beyond just linear algebra, affine geometry and gauge theory still have to be established consistently. Please note also with respect to physics and affine geometry, that by means of the structures above,

[1] ibd. p. 366, translated from German...

[2] In other words, the symmetric threefold 'spinorial' structure of $\underline{\mathbf{20}}$ is based on nothing but the very origins of invariant theory of transformation groups without the need to introduce additional 'gauge glue'.

it is straightforward to introduce quaternary barycentric coordinates. As such, masses and mass relations are linked to geometrical properties, especially with respect to the interior of the convex hull like in the case of $\underline{\mathbf{20}}$, and – appropriately normalized – the four 'coordinates' sum up to 1. Here, the major result is the *geometrical* identification of $\underline{\mathbf{20}}$ in SU(4) with reps of cubics, which is a dead giveaway with respect to PG and higher order representations [21].

3 The context

Carrying forward the results of the last section, we can test the symbolism with respect to P^3 if we identify the four members of $\underline{\mathbf{4}}$ with points, i.e. the rep $\underline{\mathbf{4}}$ with the quaternary point coordinates x_α. So with respect to quaternary forms when multiplying two (a priori different) point reps $\underline{\mathbf{4}} \sim \square$, we expect a bilinear (symmetric) form (or the polarized form of a quadric) with dim 10 for the symmetric part, and a line rep of dim 6 for the antisymmetric part. The symbolism yields $\square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix} = \underline{\mathbf{10}} \oplus \underline{\mathbf{6}}$, and from both approaches it is evident that the limit $x \to y$ 'destroys' the antisymmetric part, and the quadric 'survives'. Formally, we can introduce a bilinear form $B(x,y)$ so that *the set of points* $\{y|B(x,y)=0\}$ defines 'the polar' of x, or an associated linear map $m_B : V \to V^*$ by $B(x,y) = \langle m_b x, y\rangle$, and by symmetry $\langle m_b x, y\rangle = \langle m_b y, x\rangle$. So even analytically, we have tools to treat linear mappings as well as quadrics in V and V^* (by the adjoint map m_b^* and the induced quadratic form). To grasp the physical notation, we can use the 'old geometrical' notion of points x and planes u, and their 'products' $(a_\alpha x_\alpha)^n \equiv (ax)^n$, $(bu)^n$, as well as $x \cdot u \equiv (xu)$, higher orders thereof and appropriate forms.[3] So from the symbolism above, $\underline{\mathbf{10}}$ relates to the (symmetric) quaternary quadric whereas $\underline{\mathbf{6}}$ relates to (antisymmetric) line reps (which by appropriate complexification of the Plücker coordinates or using Klein's linear Complexe [12] can serve as $\square$ of SO(6)).

If we look for the conjugate of $\underline{\mathbf{4}}$, SU(4) requires the conjugate rep $\square^*$ to transform according to the threefold antisymmetric rep. Written in terms of determinants, it is easy to see that $\square^*$ has to represent quaternary plane coordinates of 3-space. SU(4) yields $\square^* \otimes \square = \underline{\mathbf{15}} \oplus [0]$ where the $[0]$ represents a vanishing 4×4-determinant (or a point-plane incidence $(ux) = 0$). On the same footing, using $\square^*$ as rep for three linear independent points ('a plane'), non-vanishing $\underline{\mathbf{15}}$ associates a 4^{th} point a_α to the plane $u_\alpha = \epsilon_{\alpha\beta\gamma\delta} x_\beta y_\gamma z_\delta$, and $a_\alpha u_\alpha = \epsilon_{\alpha\beta\gamma\delta} a_\alpha x_\beta y_\gamma z_\delta$ represents a determinant, or geometrically a tetrahedron ('volume'). This requires a thorough discussion of $(ux) \equiv u_x$ (or $u_\alpha x_\beta$, or $(ax)(bu) \equiv a_x b_u$) in quaternary invariant theory.[4]

Note, however, that this symbolism works by means of the initial analytic rep of linearly transforming point and plane coordinates in $\mathbb{R}^3$, or P^3, and their respective analytic reps by forms, not as a feature of space geometry itself.

4 The background

Now please recall, that given an irreducible polynomial $f \in K[x_\alpha]$ and a related hypersurface V, in order to define a tangential plane (and the tangential space) of V at a regular point p, we can invoke the hyperplane definition $\left\{v \in K^n | \sum v_\alpha \frac{\partial f}{\partial x_\alpha}(p) = 0\right\}$ describing a plane normal to $\nabla f(p)$. The same mechanism in (finite) geometry can be achieved by considering null

[3]While we have discussed $(bu)^2$ in relation to Dirac's linearization of $p_\mu p^\mu$ (see e.g. [12]), $(bu)^n$ in general relates to moments of order n and the tetrahedral Complex; due to the 8-page limit here, we postpone this discussion.

[4]To pursue the combinatorial aspects in contemporary considerations, one can follow Rota (see e.g. [14], [18], [15], however, according to his foreword in the reprint of Study's marvellous work [23] it is worth considering the classical path, too, as well as Study's geometrical concepts [23].

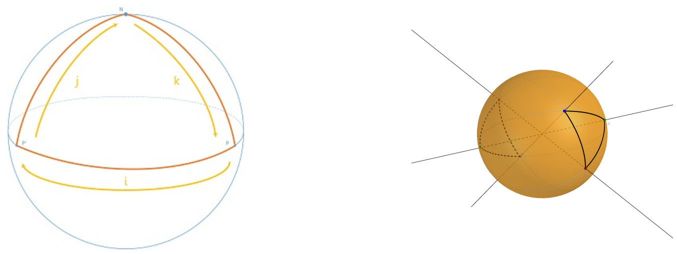


Figure 2: Left: Quaternionic action induced by rays and lines; right: line vs. ray intersections with quadric.

systems [13]. So we have to treat two 'competing' descriptions, 'moving' the tangential plane with respect to the quadric and the ('orthogonal') null plane with respect to motions along the normal and their so(4) Lie algebra [13]. Whereas for the tangential discussion and two point x, y of the quadric, we can use the incidence relations $(ux) = (u'y) = 0$, a (null) plane $u_\alpha = A_{\alpha\beta} x_\beta$ in general will be mapped to the (null) plane of a different linear Complex, $u'_\alpha = B_{\alpha\beta} y_\beta$. So relating A and B requires symplectic transformation groups.

Formally, if in the special incident/tangential case we represent the plane by $u = \frac{1}{2}(u^p + u^N)$ where u^p denotes the polar/tangential part and u^N the null plane, we can express the 'parts' of the plane according to $u_\mu \to \partial'_\mu \sim \frac{1}{2}(\partial_\mu + A_{\mu\nu} x_\nu)$, A describes the 6-dim antisymmetric rep of the null system, or the rep of a (general) linear Complex. Now expressing A via its dual/conjugate A^c, i.e. $A_{\mu\nu} \sim \epsilon_{\mu\nu\alpha\beta} A^c_{\alpha\beta}$, we find the plane rep $u^N_\mu \sim \epsilon_{\mu\nu\alpha\beta} A^c_{\alpha\beta} x_\nu$. Here, the ϵ-'tensor' formally ensures the threefold antisymmetry of the coordinate expression, and moreover, it ensures the point-plane incidence $(xu) = 0$. For electromagnetism and the electromagnetic field, an (affine) replacement $A_{\mu\nu} \to F_{\mu\nu}$ has been discussed in [25] by 3-vectors $\vec{E}$, $\vec{B}$, $\vec{M}$ and $\vec{H}$ to derive Maxwell's equations. So the difference in the tangent planes can be seen as a necessary rotation (or readjustment) of the null lines (i.e. of moments), and symbolically as $\partial'_\mu = \partial_\mu + \tilde{u}^N_\mu$, i.e. by correcting the plane appropriately. The Jacobian J benefits from the polar decomposition, i.e. for $x'_\alpha = f_\alpha(x_\beta) = A_{\alpha\beta} x_\beta$ and $S^2 = x_\alpha f_\alpha(x_\beta)$, we find $\frac{\partial f_\alpha}{\partial x_\beta} = \frac{\partial x'_\alpha}{\partial x_\beta} = A_{\alpha\beta} \cong J_{\alpha\beta}$.

So using a sphere to represent the quadric above (and to connect to what Weyl[5] and Wigner understood as features of quantum theories), we want to emphasize the underlying line geometric picture. By considering the sphere as a hull with center common to the center of a ray or line bundle (see fig. 2, right), this introduces immediately two well-known algebraic reps. In the first approach, we can define operators *on the sphere* S^2 to shift the point p quarterwise along the great circles while inherently respecting the quadric constraint of the sphere. It is easy to see that these quarterwise transformation operators fulfill the quaternion algebra (see fig. 2, left), where $-kP = N = jiP \longleftrightarrow ij = k, ijk = -\mathbb{1}$. The negative squares map the points to their 'antipodes' on the sphere, i.e. using this quadratic algebra, we have an operator system at hand which respects the (invariant) geometry of the surface by means of appropriate transformations of points. In general, we can use the quadratic algebras (or especially Clifford algebras or hypercomplex number systems) to represent the three possible signatures of the various real cases of quadrics when the base elements square to $q_N^2 = \pm 1, 0$, i.e. also in the hyperbolic and parabolic cases. For rays or oriented lines this approach yields a 2π-periodicity, i.e. $q^4 = \mathbb{1}$ or reps in terms of sin() or cos(), whereas lines yield π-periodicity and tan(). So already this simple classical picture analytically introduces 'quantum notion', and if e.g. instead of three rays we use three lines, the spherical triangle has a 'mirror image' on the opposite side of the sphere.[6] So instead of a single quaternionic rep (or SU(2)), we can discuss a twofold

[5]Recall e.g. [27], III § 16: the system space of quantum mechanics is a ray space, no vector space.

[6]In PG, we can also perform the shift of the center from 0 to ∞ easily (or an appropriate change of points in

quaternionic rep transformed by SL(1,$\mathbb{H}$)×SL(1,$\mathbb{H}$) (or SU(2)×SU(2)), SL(1,$\mathbb{H}$)×$\overline{\mathrm{SL}(1,\mathbb{H})}$, or coverings like SL(2,$\mathbb{H}$) or S(SL(1,$\mathbb{H}$)×SL(1,$\mathbb{H}$)).

The second associated algebraic rep, Study's kinematical mapping,[7] uses a similar reasoning to treat SU(2)×SU(2), where in addition special emphasis is given to projections onto the conic in the equatorial plane. Like in the first picture by using the equatorial great circle, we can thus switch to an alternative, rational parametrization of the (planar) conic, $\Phi : \mathbb{R} \to \mathbb{R}^2$, by means of $t \to \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$ which recovers the 'spinor' introduction (see e.g. [10], III.C and III.F). By recalling the projective generation of a conic by two line pencils, we can introduce the respective pencil coordinates so that the theory of binary forms applies and Clebsch-Gordan decompositions enter naturally. So these examples reveal an obvious mismatch on the interpretations of algebraic reps vs. physical notion, and more important and induced by the focus on point calculus, between the number of different physical processes and the amount of allegedly independent algebraical descriptions. Here, based on the additional geometrical hint by Hilbert and invariant theory above, in the next section we follow Plücker, Klein and Lie and introduce another linear geometrical object, 'the plane', to keep in touch with PG and classical (quaternary) invariant theory of P^3. Evidently, this well-defined geometrical notion is able to treat certain tensorial notions common in contemporary ('quantum') rep theory.

5 Some consequences

Thus, relying on point reps x_α, the geometrical rep theory of P^3 has to be completed by dual plane reps u_α, also in order to complete invariant theory.[8] We have discussed above few combinatorial aspects of a symbolism, so before entering algebraical details on invariant systems comprising x^2, u^2, $x \cdot u$, etc., it is worth to scrutinize plane reps $u \sim \square^*$ and their use.

Now, while from classical geometry we know various forms of plane reps in $\mathbb{R}^3$, in the usual (metric) interpretation when given e.g. in the Hesse form $n_i x_i - d = 0$ (and which relates to our tangential definition above), $\vec{n}$ describes the normal vector to the plane and d the distance of the plane with respect to the origin. If we *formally* introduce homogeneous point coordinates, $\vec{x}_i \to \frac{x_i}{x_0}$, and rewrite the form by $n_i x_i - d x_0 = 0 = u_\alpha x_\alpha = (ux)$, we thus have metric interpretations of the formal 'plane' coordinates u_α. Exhausting (or even overexciting) this formalism, we can think of the plane as a tangential plane to a sphere at distance d, and if we assume propagating spherical waves with velocity c and time t, then $d = ct$ corresponds formally to the 'energy component' u_0.[9]

Here, as a further aspect with respect to the use of exponentials, their partial differentiation and 'plane waves' in physics, we want to use 'intermediary forms'[10] (ux). From above it is obvious, that this 'distance' measurement of points with respect to planes can be used to define the point coordinates, however, one has to work out the dependence from the usual metric terminology e.g. in the definition of the (Euclidean) coordinates or the distances. To approach this problem, we can go back to the Cayley-Klein approach, and define the distance d of two points q_1 and q_2 by $\mathrm{dist}(q_1, q_2) \sim -2i \log DV(q_1, q_2, R, S)$ with intersections R, S on the absolute quadric; DV denotes the anharmonic ratio. If we rewrite the distance d of a point x to a plane p by $d = \mathrm{dist}(q_1, q_2)$, then $id = ix \cdot p \sim \log DV(q_1, q_2, R, S)$. Exponentiation

the anharmonic ratio), and discuss the associated orientation(s) of the second 'projected triangle'.

[7] See e.g. [17], Abb. 87, see also Study's transfer principle and dual numbers [5], §103.

[8] Here, we exclude line and Complex reps and their use in force systems, kinematics and gauge theory.

[9] So based on this identification, care has to be taken when discussing linear roots of quadrics in order to not produce 'anti-particles' with opposite sign in the component u_0.

[10] german: 'Zwischenformen'; invariants using both variables and their duals when set to zero represent projective relations.

results in $\exp(id) = \exp(i\,x \cdot p) \sim DV(q_1, q_2, R, S)$ which yields some insight into the 'plane wave'-approaches, 'second quantization', 'exponential forms', etc. in the affine setup $x_0 = 1$. Whereas the rhs, $DV(q_1, q_2, R, S)$, is a priori well-defined in PG of P^3 for general projective generalization and accessible by von Staudt's 'Würfe' or derived concepts like binary forms or Hesse transfer, a naive 'relativistic' expansion $x_i \to x_\alpha$, $p_i \to p_\alpha$ has to be treated carefully. It is obvious that these well-defined, projective properties 'scatter' to individual analytical expansion of exponentials on the lhs, or parts of such series, a fortiori with respect to non-commuting parameters, higher orders or necessary applications of Baker-Campbell-Hausdorff formulæ. As such, it is easier to start from an intermediary form (xp) and apply quaternary invariant theory from scratch; its exponentiation forces the use of expansions, power series, ordering and grouping schemes, and their respective comparison(s), i.e. it discharges into series of $(ux)^n$. The lhs thus complies with the operator and expansion methodology of usual invariant theory.

Now, whereas in the Cayley-Klein approach we can rely on the logarithm to produce metric and *additive* quantities from the DV and from PG, the rhs here 'lives' in pure and strict PG. So we can use the (partial) differential operators to restore the linearity and the 'vector' properties of the originally linear vectors x and p if we use 'new forms' $\exp(i(xp))$. Then, the action of ∂_μ reproduces 'contravariant' linear elements, $\partial_\mu \exp(i(xp)) \sim ip_\mu \exp(i(xp))$, so that effectively $p_\mu \sim -i\partial_\mu$ provides a 'quantization'. To a certain extent, differential operations in this rep theory thus replace linear operators or vector space behaviour from general PG. To get rid of the 'new forms', however, people have had to introduce additional rules and frameworks, e.g. the necessary $\mathbb{1}$-operator in terms of 'delta functions', 'integration' over homogeneous point variables x_μ, etc. (see e.g. [19], [3], [4], ...). Because such reasoning leads us back to enrich the intermediary form and its exponential by additional invariants and possible exponentials, we are faced with the original problem (see above or [16]) of determining the full invariant system either analytically in terms of points *and* planes, or with respect to (quadratic) line geometry and its relations to Complex geometry in P^5.

6 Conclusion

By identifying the physical rep $\underline{\mathbf{20}}$ geometrically as cubic by means of Hilbert's construction [16], we have strengthened the foundations of our SU(4) Ansatz within PG. Although it is hard to recover physically relevant concepts from today's jungle of physical and algebraical phenomenology and empiricism, starting over from quaternary invariant theory provides a reliable basis and stable guidelines. By means of point and plane reps of P^3, we can treat invariants (especially covariants) where already the linear and quadratic orders have enormous physical relevance. Using PG and invariant theory to start over again seems to establish correct descriptions and an ordering scheme, the more as P^5 provides subtle and profound Complex background as well as important transformation theory and relevant mappings to P^3.

Acknowledgements

Funding information This work has not been externally funded.

References

[1] S. L. Adler, *Calculation of the axial–vector coupling constant renormalization in β decay*, Phys. Rev. Lett. **14**, 1051 (1965), doi:10.1103/PhysRevLett.14.1051.

[2] V. de Alfaro, S. Fubini, G. Furlan, C. Rossetti, *Currents in hadron physics*, Elsevier, Amsterdam, Nethelands, ISBN 9780720402124 (1973).

[3] J. D. Bjorken, S. D. Drell, *Relativistic quantum mechanics*, McGraw-Hill, New York, USA, ISBN 9780070054936 (1964).

[4] J. D. Bjorken, S. D. Drell, *Relativistic quantum fields*, McGraw-Hill, New York, USA, ISBN 9780070054943 (1965).

[5] W. Blaschke, *Vorlesungen über Differentialgeometrie I*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642988004 (1945), doi:10.1007/978-3-642-99615-3.

[6] R. Dahm and M. Kirchbach, *Linear wave equations and effective Lagrangians for Wigner supermultiplets*, Int. J. Mod. Phys. A **10**, 4225 (1995), doi:10.1142/S0217751X95001960.

[7] R. Dahm, *Relativistic SU(4) and quaternions*, Adv. Applied Clifford Algebra **7(S)** 337 (1995).

[8] R. Dahm, *Spin-Flavour-Symmetrien und das $\pi N\Delta$-System*, Shaker Verlag, Düren, Germany, ISBN 9783826517822 (1996).

[9] R. Dahm, *A symmetry reduction scheme of the Dirac algebra without dimensional defects*, Phys. Atom. Nuclei **73**, 276 (2010), doi:10.1134/S1063778810020122.

[10] R. Dahm, *On a microscopic representation of space-time VII – On spin*, J. Phys.: Conf. Ser. **965**, 012012 (2018), doi:10.1088/1742-6596/965/1/012012.

[11] R. Dahm, *On a microscopic representation of space-time VIII – On relativity*, Phys. Atom. Nuclei **81**, 819 (2018), doi:10.1134/S1063778818060108.

[12] R. Dahm, *On a microscopic representation of space-time III*, Adv. Appl. Clifford Algebras **29**, 20 (2019), doi:10.1007/s00006-018-0936-x.

[13] R. Dahm, *On some geometrical aspects of space-time description and relativity*, in *The future of relativity, gravitation and cosmology*, Nova Science Publishers, Hauppauge, USA, ISBN 9798886974553 (2023), doi:10.13140/RG.2.2.23119.02729.

[14] P. Doubilet and G.-C. Rota, *Skew-symmetric invariant theory*, Adv. Math. **21**, 196 (1976), doi:10.1016/0001-8708(76)90075-X.

[15] R. Ehrenborg and G.-C. Rota, *Apolarity and canonical forms for homogeneous polynomials*, Eur. J. Comb. **14**, 157 (1993), doi:10.1006/eujc.1993.1022.

[16] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42**, 313 (1893), doi:10.1007/BF01444162.

[17] F. Klein, *Vorlesungen über höhere Geometrie*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642498480 (1926), doi:10.1007/978-3-642-49848-0.

[18] J. P. S. Kung and G.-C. Rota, *The invariant theory of binary forms*, Bull. Amer. Math. Soc. **10**, 27 (1984), doi:10.1090/S0273-0979-1984-15188-7.

[19] D. Lurie, *Particles and fields*, Wiley, Hoboken, USA, ISBN 9780470556429 (1968).

[20] H. Müller, *Zur Geometrie auf den Flächen zweiter Ordnung*, Math. Ann. **1**, 407 (1869), doi:10.1007/BF01445866.

[21] T. Reye, *Collineare Gundgebilde und ihre Erzeugnisse.*, J. Reine Angew. Math. **1872**, 1 (1872), doi:10.1515/crll.1872.74.1.

[22] V. M. Simulik and I. I. Vyikon, *On the choice of relativistic wave equation for the particle having spin $s = 3/2$*, J. Phys. Commun. **6**, 075008 (2022), doi:10.1088/2399-6528/ac7eae.

[23] E. Study, *Methoden zur Theorie der ternären Formen*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642675331 (1982).

[24] E. C. G. Sudarshan, *Theory of approximate symmetries*, in *High energy physics and elementary particles*, International Atomic Energy Agency, Vienna, Austria (1965).

[25] M. von Laue, *Zur Minkowskischen Elektrodynamik der bewegten Körper*, Z. Phys. **128**, 387 (1950), doi:10.1007/BF01339439.

[26] W. I. Weisberger, *Renormalization of the weak axial-vector coupling constant*, Phys. Rev. Lett. **14**, 1047 (1965), doi:10.1103/PhysRevLett.14.1047.

[27] H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, Germany (1931).

Linking ladder operators for the Rosen-Morse and Pöschl-Teller systems

Simon Garneau-Desroches[1⋆] and Véronique Hussin[2,3]

1 Département de Physique, Université de Montréal, QC, H3C 3J7, Canada
2 Département de Mathématiques et de Statistique,
Université de Montréal, QC, H3C 3J7, Canada
3 Centre de Recherches Mathématiques, Université de Montréal, QC, H3C 3J7, Canada

⋆ simon.garneau-desroches@umontreal.ca

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

An analysis of the realizations of the ladder operators for the Rosen-Morse and Pöschl-Teller quantum systems is carried out. The failure of the *algebraic method* of construction in the general Rosen-Morse case is exposed and explained. We present the reduction of a recently obtained set of $(2n \pm 1)$-th-order Rosen-Morse ladder operators to the usual first-order realization for the Pöschl-Teller case known in the literature.

Received 06-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.026

Contents

1 Introduction 2
2 The Rosen-Morse and Pöschl-Teller systems 2
3 Ladder operators and the algebraic method 3
4 Ladder operators for the Rosen-Morse system 4
5 Reduction of Pöschl-Teller ladder operators 5
5.1 Reduction of $A^{+}_{s,0}(n)$ 6
5.2 Reduction of $A^{-}_{s,0}(n)$ 7
6 Conclusion 7
References 8

1 Introduction

Ladder operators are objects of fundamental importance in the context of exactly solvable quantum systems. While they connect adjacent eigenspaces of the Hamiltonian, they appear as key operators in the definition of coherent and squeezed states extensively studied for their properties in quantum optics, for instance [1]. Ladder operators also participate in describing the underlying structure of the system through the spectrum generating algebra (SGA) [2]. For the most common one-dimensional (1D) exactly solvable systems (harmonic oscillator, infinite square well, Morse, etc.), a systematic method has been developed to obtain a realization of the ladder operators as first-order differential operators [2]. However, more elaborated one-dimensional exactly solvable systems like the Rosen-Morse system fall outside the range of application of this algebraic method. This system, originally introduced as a model to study vibrations of polyatomic molecules, has been studied in different contexts recently [3–6]. Indeed, two different ladder operators realizations have been proposed in the literature, both as higher-order differential operators [4,5]. The first of which was motivated by an analogy with classical mechanics [6]; while the second arises purely from quantum mechanics through the concept of shape invariance in supersymmetric quantum mechanics (SUSYQM) [7].

In this paper, we investigate why realization as first-order differential operators of Rosen-Morse ladder operators cannot be achieved with the standard algebraic method. We find that the rational dependence of the bounded eigenstates parameters on the excitation number is crucial in providing an explanation. Moreover, the Rosen-Morse system is a generalization of the Pöschl-Teller system for which first-order ladder operators are known [8]. We then address the natural question of relating these sets of ladder operators. Starting from the most recent set of higher-order Rosen-Morse ladder operators, we explicitly show how they reduce to the known first-order realization in the Pöschl-Teller limit.

The present work is linked to the study of ladder operators in the context of SUSYQM and of exactly solvable systems of the Pöschl-Teller type on a larger scale. Indeed, ladder operators have been constructed and studied for rational extensions of the harmonic oscillator [9, 10] and of the Rosen-Morse I and II systems [4], among others. Besides, reflectionless cases of the Pöschl-Teller systems investigated in the following paper intervene in the context of soliton physics and have shown to exhibit non-linear supersymmetries [11, 12] similar to that of the rational extensions.

The plan of the paper is as follows. In Section 2, we review the Rosen-Morse and Pöschl-Teller exactly solvable systems. Then, we introduce ladder operators in Section 3 together with the algebraic method for obtaining realizations as first-order differential operators. The resulting ladder operators are presented for the Pöschl-Teller while the failure of the method is demonstrated for the general Rosen-Morse system. In Section 4, the construction of the most recent realization of higher-order ladder operators for the Rosen-Morse system is exposed. Then, we show explicitly in Section 5 how the higher-order ladder operators for the Rosen-Morse system reduce to their usual first-order realization in the Pöschl-Teller case. We make final conclusions in Section 6.

2 The Rosen-Morse and Pöschl-Teller systems

The (hyperbolic) Rosen-Morse (RM) [13] system is an exactly solvable quantum system with Hamiltonian labelled by the parameters s and λ:

$$H_{s,\lambda} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\lambda \tanh(x) - s(s+1)\,\mathrm{sech}^2(x)\,, \quad x \in \mathbb{R}\,, \quad s > 0\,, \quad 0 \leq \lambda < s^2\,. \tag{1}$$

This system is also named Rosen-Morse II in the literature, as opposed to its trigonometric analogue (Rosen-Morse I) [14]. The particular case $\lambda = 0$ is the Pöschl-Teller (PT) system [2,15]. The normalizable eigenstates solving the time-independent Schrödinger equation $H_{s,\lambda}\psi_{s,\lambda}(n) = E_{s,\lambda}(n)\psi_{s,\lambda}(n)$ are given in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ [19]:

$$\psi_{s,\lambda}(n;x) = M_{s,\lambda}(n)\cosh^{-(s-n)}(x)\mathrm{e}^{-\frac{\lambda x}{s-n}} P_n^{(a_{s,\lambda}(n),b_{s,\lambda}(n))}(\tanh(x)), \tag{2}$$

where the parameters are

$$a_{s,\lambda}(n) = s - n + \frac{\lambda}{s-n}, \qquad b_{s,\lambda}(n) = s - n - \frac{\lambda}{s-n}, \tag{3}$$

and $M_{s,\lambda}(n)$ is a normalization constant. There exists a finite number of bounded eigenstates and the associated energies are rational in the excitation:

$$E_{s,\lambda}(n) = -(s-n)^2 - \frac{\lambda^2}{(s-n)^2}, \qquad n = 0, 1, \ldots, n_{\max} < s - \sqrt{\lambda}. \tag{4}$$

The energies are related to the parameters through $E_{s,\lambda}(n) = -[a_{s,\lambda}^2(n) + b_{s,\lambda}^2(n)]/2$. In the Pöschl-Teller case, $a_{s,0}(n) = b_{s,0}(n)$ are linear in n and the eigenstates can be expressed in terms of the associated Legendre polynomials $P_l^{\mu}(y)$ [19]. Moreover, the energy spectrum becomes quadratic in n.

3 Ladder operators and the algebraic method

In this work we define ladder operators $\left\{A^{\pm}(n)\right\}_{n=0}^{n_{\max}}$ for a given Hamiltonian H by the following action on the bounded eigenstates:

$$A^{\pm}(n)\psi(n;x) \propto \psi(n \pm 1; x), \qquad A^{-}(0)\psi(0;x) = 0. \tag{5}$$

Here, $n_{\max}$ is either finite or infinite depending on H. They connect eigenspaces of adjacent energies: $A^{+}(n)$ is referred to as a raising operator and $A^{-}(n)$ is a lowering operator. This definition allows for different realizations of the ladder operators for a unique given system. Indeed, the proportionality constant can be chosen arbitrarily either to close an algebra or to construct certain types of coherent states, for instance. In the Rosen-Morse case, the bounded spectrum is finite and the action $A^{+}(n_{\max})\psi(n_{\max};x)$ yields an unbounded state; we refer to [4] for more details.

For numerous exactly solvable systems (harmonic oscillator, infinitely deep square-well, Morse potential, etc.), there exists a standard technique to realize the ladder operators as first-order differential operators using the action on the eigenstates. This technique is sometimes referred to as the algebraic method in the literature [16,17]. In particular, it has shown to be efficient for the Pöschl-Teller system. Starting with the assumption that $A^{\pm}(n)$ may be realized as

$$A^{\pm}(n) = g^{\pm}(n;x) + f^{\pm}(n;x)\frac{\mathrm{d}}{\mathrm{d}x}, \tag{6}$$

we act on an eigenstate $\psi(n;x)$ in order to get (5). The result is well-known and detailed in this case (see [2] , for example). Indeed, ladder operators are found to be given as

$$A_{PT}^{\pm}(n) \propto -(s-n)\sinh(x) \pm \cosh(x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{7}$$

Let us now try to apply this technique to the Rosen-Morse case. We will show that it fails to obtain (5) in a straight way.

The idea is first to act with a derivative on the eigenstate, and then to use functional relations among the eigenstates to express the result in terms of the adjacent eigenstates. We take the usual change of variable $z = \tanh(x)$ and act with $\frac{\mathrm{d}}{\mathrm{d}x} = \mathrm{sech}^2(x)\frac{\mathrm{d}}{\mathrm{d}z}$ on an eigenstate of (2). This yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{s,\lambda}(n;x) = &\left[-(s-n)\frac{\sinh x}{\cosh x} - \frac{\lambda}{s-n}\right]\psi_{s,\lambda}(n;x) \\ &+ M_{s,\lambda}(n)\cosh^{-(s-n)}(x)\mathrm{e}^{-\frac{\lambda x}{s-n}}\,\mathrm{sech}^2(x)\frac{\mathrm{d}}{\mathrm{d}z}P_n^{(a_{s,\lambda}(n),b_{s,\lambda}(n))}(z)\,.\end{aligned} \tag{8}$$

Now, one wants to make use of the functional relation [18]

$$\frac{\mathrm{d}}{\mathrm{d}z}P_n^{(a_{s,\lambda}(n),b_{s,\lambda}(n))}(z) = \frac{2s-n+1}{2}\,P_{n-1}^{(a_{s,\lambda}(n)+1,b_{s,\lambda}(n)+1)}(z)\,, \tag{9}$$

to recover $P_{n-1}^{(a_{s,\lambda}(n-1),b_{s,\lambda}(n-1))}(z)$ in order to have $\psi_{s,\lambda}(n-1;x)$ in (8). However, recalling the expression (3) for $a_{s,\lambda}(n)$ and $b_{s,\lambda}(n)$, this is only possible in the Pöschl-Teller case:

$$\left.\begin{aligned}a_{s,\lambda}(n)+1 = a_{s,\lambda}(n-1)\\ b_{s,\lambda}(n)+1 = b_{s,\lambda}(n-1)\end{aligned}\right\} \qquad \Longleftrightarrow \qquad \lambda = 0\,. \tag{10}$$

Therefore, one cannot recover the eigenstate $\psi_{s,\lambda}(n-1;x)$ using this relation in the general Rosen-Morse setting. In fact, the problem comes directly from the rational dependence of the parameters $a_{s,\lambda}(n)$ and $b_{s,\lambda}(n)$ with respect to the excitation number n. Indeed, the functional relations that share the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ only allow integer shifts of the parameters α and β [19]. The same problem occurs when trying to recover $\psi_{s,\lambda}(n+1;x)$ instead. Consequently, the algebraic method fails to provide ladder operators for the Rosen-Morse system. The next section summarizes the most recent alternative way of constructing ladder operators for the Rosen-Morse system [4].

4 Ladder operators for the Rosen-Morse system

To simplify notation, we omit the explicit x-dependence of the eigenstates and use $\psi_{s,\lambda}(n)$ from this point on. We apply first-order supersymmetric (SUSY) transformation (see [7, 20], for example) to the Rosen-Morse Hamiltonian $H_{s,\lambda}$ and the corresponding eigenstates $\psi_{s,\lambda}(n)$. We get the so-called intertwining first-order differential operators

$$B^{\pm}_{s,\lambda} = -s\tanh(x) - \frac{\lambda}{s} \pm \frac{\mathrm{d}}{\mathrm{d}x}\,. \tag{11}$$

The Rosen-Morse system $H_{s,\lambda}$ is known to be shape invariant with SUSY partner $H_{s-1,\lambda}$ with translated parameter $s \to s-1$ [7]. We have the usual eigenstates connections

$$\psi_{s-1,\lambda}(n) = \frac{B^-_{s,\lambda}\psi_{s,\lambda}(n+1)}{\sqrt{E_{s,\lambda}(n+1)-E_{s,\lambda}(0)}}\,, \qquad \psi_{s,\lambda}(n+1) = \frac{B^+_{s,\lambda}\psi_{s-1,\lambda}(n)}{\sqrt{E_{s,\lambda}(n+1)-E_{s,\lambda}(0)}}\,, \tag{12}$$

together with the ground state annihilation $B^-_{s,\lambda}\psi_{s,\lambda}(0) = 0$. The energies are preserved under the application of $B^{\pm}_{s,\lambda}$ as $E_{s-1,\lambda}(n) = E_{s,\lambda}(n+1)$. Successive applications of the SUSY transformation generate a hierarchy of Rosen-Morse Hamiltonians with fixed λ and translating s. Since the system loses its ground state energy at every step of the procedure, the state $\psi_{s,\lambda}(n)$ of the initial system is connected to the ground state of the system $H_{s-n,\lambda}$ and vice versa:

$$\psi_{s-n,\lambda}(0) \propto \left(B^-_{s-n+1,\lambda}B^-_{s-n+2,\lambda}\cdots B^-_{s-1,\lambda}B^-_{s,\lambda}\right)\psi_{s,\lambda}(n)\,, \tag{13}$$

$$\psi_{s,\lambda}(n) \propto \left(B^+_{s,\lambda}B^+_{s-1,\lambda}\cdots B^+_{s-n+2,\lambda}B^+_{s-n+1,\lambda}\right)\psi_{s-n,\lambda}(0)\,. \tag{14}$$

$$
\begin{tikzcd}
H_{s,\lambda} & H_{s-1,\lambda} & \cdots & H_{s-n+1,\lambda} & H_{s-n,\lambda} & H_{s-n-1,\lambda} \\
\psi_{s,\lambda}(n+1) & \psi_{s-1,\lambda}(n) \arrow[l, "B^{+}_{s,\lambda}"'] & \cdots \arrow[l, "B^{+}_{s-1,\lambda}"'] & \psi_{s-n+1,\lambda}(2) \arrow[l, "B^{+}_{s-n+2,\lambda}"'] & \psi_{s-n,\lambda}(1) \arrow[l, "B^{+}_{s-n+1,\lambda}"'] & \psi_{s-n-1,\lambda}(0) \arrow[l, "B^{+}_{s-n,\lambda}"'] \\
\psi_{s,\lambda}(n) \arrow[r, "B^{-}_{s,\lambda}"] & \psi_{s-1,\lambda}(n-1) \arrow[r, "B^{-}_{s-1,\lambda}"] & \cdots \arrow[r, "B^{-}_{s-n+2,\lambda}"] & \psi_{s-n+1,\lambda}(1) \arrow[r, "B^{-}_{s-n+1,\lambda}"] & \psi_{s-n,\lambda}(0) \arrow[ur, bend right, "\gamma_{s-n,\lambda}"'] \arrow[dl, bend left, "\gamma^{-1}_{s-n+1,\lambda}"] & \\
\psi_{s,\lambda}(n-1) & \psi_{s-1,\lambda}(n-2) \arrow[l, "B^{+}_{s,\lambda}"'] & \cdots \arrow[l, "B^{+}_{s-1,\lambda}"'] & \psi_{s-n+1,\lambda}(0) \arrow[l, "B^{+}_{s-n+2,\lambda}"'] & &
\end{tikzcd}
$$

Figure 1: Product decomposition of $A^{\pm}_{s,\lambda}(n)$ acting on $\psi_{s,\lambda}(n)$ to obtain $\psi_{s,\lambda}(n\pm 1)$ through the Rosen-Morse shape invariance hierarchy scheme.

Furthermore, defining

$$\gamma_{s,\lambda} = \cosh(x) e^{-\frac{\lambda x}{s(s-1)}} , \tag{15}$$

we obtain the ground states connections

$$\psi_{s-1,\lambda}(0) \propto \gamma_{s,\lambda}\psi_{s,\lambda}(0) , \qquad \psi_{s+1,\lambda}(0) \propto \gamma^{-1}_{s+1,\lambda}\psi_{s,\lambda}(0) , \tag{16}$$

which respectively raise and lower the value of the energy (4) in the hierarchy.

A ladder operator is constructed by applying successive intertwining operators from (13) on $\psi_{s,\lambda}(n)$ until a ground state is reached, then applying the connection (16), and finally applying successive intertwining operators from (14) to climb back in the hierarchy until $\psi_{s,\lambda}(n\pm 1)$ is reached [4]. The ladder operators $A^{\pm}_{s,\lambda}(n)$ write as the $(2n\pm 1)$-th-order differential operators

$$A^{+}_{s,\lambda}(n) \propto \left(B^{+}_{s,\lambda}B^{+}_{s-1,\lambda}\cdots B^{+}_{s-n+1,\lambda}B^{+}_{s-n,\lambda}\right)\gamma_{s-n,\lambda}\left(B^{-}_{s-n+1,\lambda}B^{-}_{s-n+2,\lambda}\cdots B^{-}_{s-1,\lambda}B^{-}_{s,\lambda}\right) , \tag{17}$$

$$A^{-}_{s,\lambda}(n) \propto \left(B^{+}_{s,\lambda}B^{+}_{s-1,\lambda}\cdots B^{+}_{s-n+3,\lambda}B^{+}_{s-n+2,\lambda}\right)\gamma^{-1}_{s-n+1,\lambda}\left(B^{-}_{s-n+1,\lambda}B^{-}_{s-n+2,\lambda}\cdots B^{-}_{s-1,\lambda}B^{-}_{s,\lambda}\right) . \tag{18}$$

The previous equations are valid with the exception of $A^{-}_{s,\lambda}(0)$, $A^{+}_{s,\lambda}(0)$ and $A^{-}_{s,\lambda}(1)$ for which they do not hold. For the latter two, one of the products should be interpreted as unity:

$$A^{+}_{s,\lambda}(0) \propto B^{+}_{s,\lambda}\gamma_{s,\lambda} , \qquad A^{-}_{s,\lambda}(1) \propto \gamma^{-1}_{s,\lambda}B^{-}_{s,\lambda} . \tag{19}$$

The particular case $A^{-}_{s,\lambda}(0)$ is also of the first order. For consistency with (18)[1] we take

$$A^{-}_{s,\lambda}(0) \propto \gamma^{-1}_{s+1,\lambda}B^{-}_{s,\lambda,} , \tag{20}$$

even though $B^{-}_{s,\lambda}$ already annihilates $\psi_{s,\lambda}(0)$. The ladder operators $A^{\pm}_{s,\lambda}(n)$ satisfy the action (5) and they are illustrated in Figure 1 where their action is decomposed within the hierarchy.

5 Reduction of Pöschl-Teller ladder operators

This section addresses the reduction of the Rosen-Morse ladder operators $A^{\pm}_{s,\lambda}(n)$ to the known Pöschl-Teller first-order realization $A^{\pm}_{PT}(n)$ presented in Section 3. We set $\lambda = 0$ and remove the λ-label so that $A^{\pm}_{s}(n)$, $B^{\pm}_{s}$, γ_s and $\psi_s(n)$ are understood to be that of the Pöschl-Teller system.

[1] As well as for technical reasons in view of Section 5.

5.1 Reduction of $A^+_{s,0}(n)$

We show how the action of $A^+_s(n)$ reduces to that of $A^+_{PT}(n)$. To do so, we use ideas from [8]. Raising the $\psi_s(n)$ state develops as

$$A^+_s(n)\psi_s(n) \propto B^+_s B^+_{s-1}\cdots B^+_{s-n+1}\big[B^+_{s-n}\cosh(x)B^-_{s-n+1}B^-_{s-n+2}\cdots B^-_{s-1}B^-_s\psi_s(n)\big], \tag{21}$$

where the factor in brackets yields the state $\psi_{s-n}(1)$ (see Figure 1). Knowing the expression for $\psi_{s-n}(1)$, we write it in terms of the ground state of the same system in the hierarchy:

$$\psi_{s-n}(1) \propto \sinh(x)\cosh^{-(s-n)}(x) \propto \sinh(x)\psi_{s-n}(0). \tag{22}$$

Substituting back in (21), we arrive at

$$A^+_s(n)\psi_s(n) \propto B^+_s B^+_{s-1}\cdots B^+_{s-n+1}\sinh(x)\psi_{s-n}(0). \tag{23}$$

The $\sinh(x)$ must be commuted to the left of the product of intertwining operators in order to recover $\psi_s(n)$ on the right hand side via the relation (14). This can be done by first writing the product of intertwining operators in the form

$$B^+_s B^+_{s-1}\cdots B^+_{s-n+1} = \cosh^{s+1}(x)\left(\operatorname{sech}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \cosh^{-(s-n+1)}(x), \tag{24}$$

and then by making use of the commutation relation

$$\left[\left(\operatorname{sech}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)^n, \sinh(x)\right] = n\left(\operatorname{sech}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}, \tag{25}$$

among differential operators [8]. We obtain

$$\begin{aligned} A^+_s(n)\psi_s(n) &\propto \sinh(x)B^+_s B^+_{s-1}\cdots B^+_{s-n+1}\psi_{s-n}(0) + n\cosh(x)B^+_{s-1}\cdots B^+_{s-n+1}\psi_{s-n}(0) && (26)\\ &\propto \sinh(x)B^+_s\psi_{s-1}(n-1) + n\cosh(x)\psi_{s-1}(n-1), && (27) \end{aligned}$$

where we again used (14) in the last line. Then, from (12), we use respectively

$$B^+_s\psi_{s-1}(n-1) = \sqrt{n(2s-n)}\,\psi_s(n), \quad \text{and} \qquad \psi_{s-1}(n-1) = \frac{B^-_s\psi_s(n)}{\sqrt{n(2s-n)}}, \tag{28}$$

on the first and second terms of (27) to recover the action of a first-order differential operator on $\psi_s(n)$:

$$A^+_s(n)\psi_s(n) \propto \left[\sqrt{n(2s-n)}\sinh(x) + \frac{n\cosh(x)}{\sqrt{n(2s-n)}}B^-_s\right]\psi_s(n). \tag{29}$$

Simplifying using the expression for B^-_s, the remaining operator is proportional to the usual raising operator for the Pöschl-Teller system:

$$A^+_s(n)\psi_s(n) \propto \left[-(s-n)\sinh(x) + \cosh(x)\frac{\mathrm{d}}{\mathrm{d}x}\right]\psi_s(n) \propto A^+_{PT}(n)\psi_s(n). \tag{30}$$

In the general Rosen-Morse case, the operators $B^\pm_{s,\lambda}$ contain a λ/s term which complicates the generalization of the identity (24). Then, the association (22) contains two terms with exponentials. Put together, the commutation of the $B^\pm_{s,\lambda}$ cannot be performed similarly and prevents the reduction of the ladder operators.

5.2 Reduction of $A^-_{s,0}(n)$

The reduction of $A^-_s(n)$ is similar to that of $A^+_s(n)$. Developing the lowering of the $\psi_s(n)$ state in a similar fashion as done in (21) and (22) yields

$$A^-_s(n)\psi_s(n) \propto B^+_s B^+_{s-1}\cdots B^+_{s-n+2}\,\mathrm{sech}(x)B^-_{s-n+1}\psi_{s-n+1}(1)\,. \tag{31}$$

Note that we have made the association with $\psi_{s-n+1}(1)$ before reaching the step of ground state connexion (see Figure 1). To continue further, we make use the equivalence of the following operators on $\psi_{s-n+1}(1)$:

$$\left.\begin{array}{l}\mathrm{sech}(x)B^-_{s-n+1,\lambda}\\ \cosh(x)B^-_{s-n,\lambda}\end{array}\right\}:\ \psi_{s-n+1,\lambda}(1)\mapsto\psi_{s-n+1,\lambda}(0)\,, \tag{32}$$

to write

$$A^-_s(n)\psi_s(n) \propto B^+_s B^+_{s-1}\cdots B^+_{s-n+2}\cosh(x)B^-_{s-n}\psi_{s-n+1}(1)\,. \tag{33}$$

Noticing $B^-_{s-n} = B^-_{s-n+1} + \tanh(x)$, we get

$$A^-(n)\psi_s(n) \propto B^+_s B^+_{s-1}\cdots B^+_{s-n+2}\big[\cosh(x)B^-_{s-n+1} + \sinh(x)\big]\psi_{s-n+1}(1)\,. \tag{34}$$

We use $B^+_s\cosh(x) = \cosh(x)B^+_{s-1}$ repeatedly on the first term of (34) to commute $\cosh(x)$ to the left. The second term is treated similarly as in the previous section by using the formula (24) and the commutation relation (25) on $\sinh(x)$, yielding two terms. Keeping track of the relative constants between the terms, we obtain

$$\begin{aligned}A^-_s(n)\psi_s(n) \propto\ &\cosh(x)B^+_{s-1}\cdots B^+_{s-n+2}B^+_{s-n+1}B^-_{s-n+1}\psi_{s-n+1}(1)\\ &+\sinh(x)B^+_s B^+_{s-1}\cdots B^+_{s-n+2}\psi_{s-n+1}(1)\\ &+(n-1)\cosh(x)B^+_{s-1}\cdots B^+_{s-n+2}\psi_{s-n+1}(1)\,.\end{aligned} \tag{35}$$

The product $B^+_{s-n+1}B^-_{s-n+1}$ factorizes H_{s-n+1} in the first term and the three terms can then be combined. We act with the product $B^+_{s-1}\cdots B^+_{s-n+2}$ to get $\psi_{s-1}(n-1)$ (see Figure 1). We are left with

$$A^-_s(n)\psi_s(n) \propto \big[(2s-n)\cosh(x) + \sinh(x)B^+_s\big]\psi_{s-1}(n-1)\,. \tag{36}$$

We again make use of (28) and rearrange to recover the usual Pöschl-Teller lowering operator

$$A^-_s(n)\psi_s(n) \propto \left[-(s-n)\sinh(x) - \cosh(x)\frac{\mathrm{d}}{\mathrm{d}x}\right]\psi_s(n) \propto A^-_{PT}(n)\psi_s(n)\,. \tag{37}$$

6 Conclusion

In this paper, we have studied ladder operators for the Rosen-Morse system and the Pöschl-Teller particular case. We exposed how the algebraic method of constructing ladder operators fails for the general Rosen-Morse system, and we found that the rational dependence of the parameters $a_{s,\lambda}(n)$ and $b_{s,\lambda}(n)$ on n is responsible for this failure. Next, we recalled the construction of a set of known $(2n\pm1)$-th-order Rosen-Morse ladder operators. It was expected that the later should reconcile with the well-known first-order realization of the Pöschl-Teller ladder operators obtained from the algebraic method. Indeed, we have explicitly obtained the reduction of the Pöschl-Teller ladder operators from order $2n\pm1$ to order 1. Besides, a point canonical transformation [21] has been used to map the ladder operators presented in Section 4 onto $(2n\pm1)$-th-order analogous ladder operators for the trigonometric Rosen-Morse system [4]. We expect that similar results apply in the trigonometric case.

Acknowledgements

Funding information V. Hussin acknowledges the support of research grant from NSERC of Canada.

References

[1] J. Gazeau, *Coherent states in quantum physics*, Wiley, Weinheim, Germany, ISBN 9783527407095 (2009), doi:10.1002/9783527628285.

[2] S.-H. Dong, *Factorization method in quantum mechanics*, Springer, Dordrecht, Netherlands, ISBN 9781402057953 (2007), doi:10.1007/978-1-4020-5796-0.

[3] D. J. Fernández and R. Reyes, *Spectral manipulation of the trigonometric Rosen-Morse potential through supersymmetry*, Phys. Scr. **96**, 125250 (2021), doi:10.1088/1402-4896/ac33f9.

[4] S. Garneau-Desroches and V. Hussin, *Ladder operators and coherent states for the Rosen-Morse system and its rational extensions*, J. Phys. A: Math. Theor. **54**, 475201 (2021), doi:10.1088/1751-8121/ac2549.

[5] L. Delisle-Doray and V. Hussin, *Ladder operators for the Rosen-Morse system through classical analogy*, J. Phys.: Conf. Ser. **1540**, 012001 (2020), doi:10.1088/1742-6596/1540/1/012001.

[6] L. Delisle-Doray, V. Hussin, Ş. Kuru and J. Negro, *Classical ladder functions for Rosen-Morse and curved Kepler-Coulomb systems*, Ann. Phys. **405**, 69 (2019), doi:10.1016/j.aop.2019.03.004.

[7] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, Phys. Rep. **251**, 267 (1995), doi:10.1016/0370-1573(94)00080-M.

[8] Ş. Kuru and J. Negro, *Dynamical algebras for Pöschl-Teller Hamiltonian hierarchies*, Ann. Phys. **324**, 2548 (2009), doi:10.1016/j.aop.2009.08.004.

[9] J. F. Cariñena and M. S. Plyushchay, *ABC of ladder operators for rationally extended quantum harmonic oscillator systems*, J. Phys. A: Math. Theor. **50**, 275202 (2017), doi:10.1088/1751-8121/aa739b.

[10] J. F. Cariñena, L. Inzunza and M. S. Plyushchay, *Rational deformations of conformal mechanics*, Phys. Rev. D **98**, 026017 (2018), doi:10.1103/PhysRevD.98.026017.

[11] F. Correa and M. S. Plyushchay, *Hidden supersymmetry in quantum bosonic systems*, Ann. Phys. **322**, 2493 (2007), doi:10.1016/j.aop.2006.12.002.

[12] F. Correa, V. Jakubský and M. S. Plyushchay, *Aharonov-Bohm effect on AdS_2 and nonlinear supersymmetry of reflectionless Pöschl-Teller system*, Ann. Phys. **324**, 1078 (2009), doi:10.1016/j.aop.2009.01.009.

[13] N. Rosen and P. M. Morse, *On the vibrations of polyatomic molecules*, Phys. Rev. **42**, 210 (1932), doi:10.1103/PhysRev.42.210.

[14] C. B. Compean and M. Kirchbach, *The trigonometric Rosen-Morse potential in the supersymmetric quantum mechanics and its exact solutions*, J. Phys. A: Math. Gen. **39**, 547 (2005), doi:10.1088/0305-4470/39/3/007.

[15] G. Pöschl and E. Teller, *Bemerkungen zur Quantenmechanik des anharmonischen Oszillators*, Z. Phys. **83**, 143 (1933), doi:10.1007/BF01331132.

[16] S.-H. Dong, R. Lemus and A. Frank, *Ladder operators for the Morse potential*, Int. J. Quantum Chem. **86**, 433 (2002), doi:10.1002/qua.10038.

[17] D. Mikulski, K. Eder and M. Molski, *The algebraic approach for the derivation of ladder operators and coherent states for the Goldman and Krivchenkov oscillator by the use of supersymmetric quantum mechanics*, J. Math. Chem. **52**, 1610 (2014), doi:10.1007/s10910-014-0341-1.

[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Elsevier, Amsterdam, Netherlands, ISBN 9780123849335 (2015), doi:10.1016/C2010-0-64839-5.

[19] H. Bateman, *Higher transcendental functions [Volumes I-III]*, McGraw-Hill, New York, USA (1953).

[20] D. J. Fernández C., *Trends in supersymmetric quantum mechanics*, in *Integrability, supersymmetry and coherent states*, Springer, Cham, Switzerland, ISBN 9783030200862 (2019), doi:10.1007/978-3-030-20087-9_2.

[21] J. V. Mallow, A. Gangopadhyaya, J. Bougie and C. Rasinariu, *Inter-relations between additive shape invariant superpotentials*, Phys. Lett. A **384**, 126129 (2020), doi:10.1016/j.physleta.2019.126129.

New perspectives in gravity-mediated supersymmetry breaking

Robin Ducrocq⋆

Theory Group, IPHC, Strabourg, France

⋆ robin.ducrocq@iphc.cnrs.fr

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

New solutions in supersymmetry breaking through gravity mediation have been recently discovered. Such solutions have interesting properties regarding renormalisation and introduce new contributions in the scalar potential that may help to resolve some issues of the Standard Model. The purpose of this article is to investigate the consequences of these new structures. We construct a model related to these new solutions, the S2MSSM, and present some preliminary results on the effects of these new contributions, especially on the Standard Model's Higgs boson mass.

Received 16-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.027

1 Introduction

The Standard Model of particle physics provides a robust framework to describe the behaviour of particles and fundamental interactions. However, several issues still remain in this model. Some of these problems may be solved by embedding the Standard Model in a more fundamental theory. There exist several possibilities. Two of them are supersymmetry and its local version, supergravity. Such theories are defined within the framework of Lie superalgebras in the line of the Haag–Lopuszański–Sohnius theorem [1]. This theorem strongly constrains the possible symmetries of the spacetime. In the simplest case ($N = 1$ supersymmetry and supergravity), this theorem restricts the symmetry to be:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 , \quad \text{with} \quad \mathfrak{g}_0 = \mathfrak{Iso}(1,3) \times \mathfrak{g}_C , \quad \mathfrak{g}_1 = S_L \oplus S_R ,$$

with $\mathfrak{Iso}(1,3)$ the Poincaré algebra, $\mathfrak{g}_C$ a compact Lie algebra related to internal symmetries and S_L (resp. S_R), the left-(right-)handed spinor representation where $S_L = \{Q_\alpha,\ \alpha = 1,2\}$ ($S_R = \{\bar{Q}^{\dot\alpha},\ \dot\alpha = 1,2\}$) and $(Q_\alpha)^\dagger = \bar{Q}_{\dot\alpha}$. The subspace $\mathfrak{g}_0$ is called even whereas $\mathfrak{g}_1$ is called odd.

However, the spectrum of such theory is incompatible with the actual measurements. Supersymmetry and supergravity must then be broken. Several consistent mechanisms exist in supergravity. We focus on one of them, namely, gravity-mediated supersymmetry breaking. In such scenarios, supergravity is assumed to be broken in a hidden sector. Such a configuration induces, through gravitational effects, supersymmetry breaking in the usual field sector. Solutions to this mechanism were first classified in the 80s [2]. Recently, new solutions have been discovered [3] with new supersymmetry breaking terms and a new field sector with specific properties.

After a general presentation of the gravity-mediated supersymmetry breaking mechanism, we present the new solutions. A particular model, the S2MSSM, is then constructed. The mass matrix of the scalar sector is finally analysed.

2 Supersymmetry breaking through gravitational interactions

To construct a model in supergravity, we choose a gauge group G. In our case, we consider $G = SU(3)_c \times SU(2)_L \times U(1)_Y$. Vector superfields associated to the strong $SU(3)_c$ and the electroweak $SU(2)_L \times U(1)_Y$ interactions are naturally introduced in the adjoint representation of G. We also include a matter sector with chiral superfields subdivided into two subsectors. The first is the visible (or observable) sector $\{\Phi^a,\ a = 1,\dots,n_a\}$ where $\Phi^a = (\phi^a, \chi^a_\phi, F^a_\phi)$,[1] containing the usual fields of the Standard Model with their associated supersymmetric partners. The second is the hidden sector $\{Z^i,\ i = 1,\dots,n_i\}$ with $Z^i = (\zeta^i, \chi^i_\zeta, F^i_\zeta)$. Finally, two gauge invariant functions of the chiral superfields are introduced: a real function called the Kähler potential K which leads to the kinetic term of chiral superfields and a holomorphic function, the superpotential W, which generates the Yukawa couplings of the Standard Model. Supergravity is then assumed to be broken in the hidden sector with $\langle \zeta^i \rangle = \mathcal{O}(m_p)$ where m_p is the Planck mass (we also have $\langle \phi^a \rangle \ll m_p$). The F-term of the scalar potential of supergravity can then be computed:

$$V_F = \exp\Big(K/m_p^2\Big)\Big(\mathcal{D}_A W (K^{-1})^A{}_{B^*} \mathcal{D}^{B^*} \bar{W} - \frac{3}{m_p^2}|W|^2\Big), \tag{1}$$

with:

$$\mathcal{D}_A W = \partial_A W + \frac{W}{m_p^2}\partial_A K\,, \quad (K^{-1})^A{}_{B^*} = \left(\frac{\partial^2 K}{\partial X^A \partial X^\dagger_{B^*}}\right)^{-1},$$

($\{X^A\} = \{Z^i, \Phi^a\}$). Considering the low energy limit (*i.e.*, taking $m_p \to \infty$), we obtain the classical potential of supersymmetry V_{SUSY} with additional terms $V_{\cancel{SUSY}}$ which explicitly break supersymmetry:

$$V_F = V_{SUSY} + V_{\cancel{SUSY}}\,. \tag{2}$$

Note that the form of the superpotential and the Kähler potential must not induce dangerous couplings in Eq. 1. Indeed, at low energy, couplings proportional to the Planck mass m_p generate instabilities in the matter sector. We must therefore impose that the interactions in the visible sector must be proportional in the potential to m_p^n with $n \le 0$.

We are interested in solutions for which the Kähler potential K and the superpotential W can be expanded as power of the Planck mass:

$$K(Z,Z^\dagger,\Phi,\Phi^\dagger) = \sum_{n=0}^{r} K_n(Z,Z^\dagger,\Phi,\Phi^\dagger) m_p^n\,, \quad W(Z,\Phi) = \sum_{n=0}^{s} W_n(Z,\Phi) m_p^n\,, \tag{3}$$

(we thus exclude no-scale solutions). Under these assumptions, one obtain two solutions using a canonical Kähler potential. The first one is the historical solution discovered by Soni &

[1]We denote the components of a chiral superfield X^A by $X^A = (x^A, \chi^A_X, F^A_X)$ with x^A a scalar field, χ^A_X a left-handed Weyl spinor and F^A_X an auxiliary field.

Weldon [2], which is the cornerstone for all the studies involving gravity-mediated supersymmetry breaking up to now. The second is a new structure that will be described in the next section.

3 New solutions in gravity-mediated supersymmetry breaking

We briefly present the new solutions [3] associated to a canonical Kähler potential:

$$K(Z,Z^\dagger,\Phi,\Phi^\dagger)=Z^iZ_i^\dagger+\Phi^a\Phi_a^\dagger\,.$$

Following Eq. 3, two possible forms for the superpotential have been identified. The first corresponds to the known solution developed in [2]. The second has a new structure and introduces a new singlet superfield sector $\{\mathcal{S}^p,\ p=1,\dots,n_p\}$ (with $\mathcal{S}^p=(S^p,\chi_S^p,F_S^p)$):

$$W(Z,\Phi,\mathcal{S})=m_pW_1(Z,\mathcal{S})+W_0(Z,\Phi,\mathcal{S})\,, \tag{4}$$

with:

$$W_1(Z,\mathcal{S})=W_{1,0}(Z)+W_{1,p}(Z)\mu_p^*\mathcal{S}^p\,,\quad W_0(Z,\Phi)=W_{0,p}(Z)\mathcal{S}^p+W_0(Z,\mathcal{U},\Phi)\,, \tag{5}$$

and

$$\mathcal{U}^{pq}=\mu^p\mathcal{S}^q-\mu^q\mathcal{S}^p\,. \tag{6}$$

The functions $W_{1,0}$, $W_{1,p}$, $W_{0,p}$ and W_0 are holomorphic functions of chiral superfields. The form of $W(Z,\Phi,\mathcal{S})$ in Eqs. 4 & 5 and $\mathcal{U}$ in Eq. 6 is dictated to avoids dangerous couplings between the visible and the hidden sector in the low energy limit. The new singlet superfield sector $\{\mathcal{S}^p\}$ is called "hybrid". It involves in $W_1(Z,\mathcal{S})$ a term proportional to the Planck mass m_p (see Eq. 4), but still produces a divergent-free low energy potential. The low energy scalar potential Eq. 7 associated with Eqs. 4 and 5 is:

$$V=V_{SUSY}+\Lambda m_p^2+V_{SOFT}+V_{HARD}\,, \tag{7}$$

where Λ is the cosmological constant. The two remaining terms break supersymmetry explicitly. The terms V_{SOFT} are *soft* supersymmetric breaking terms, *i.e.*, terms that lead to logarithmic divergences through loop corrections. Such contributions are already present in the historical classification. The general form of V_{SOFT} has been classified [4] and takes the form:

$$V_{SOFT}=\Big(C_i\tilde\phi^i+\frac{1}{2}B_{ij}\tilde\phi^i\tilde\phi^j+\frac{1}{6}A_{ijk}\tilde\phi^i\tilde\phi^j\tilde\phi^k+\text{h.c.}\Big)+m^2_{\tilde\phi}\tilde\phi^i\tilde\phi_i^\dagger\,, \tag{8}$$

with $\{\tilde\phi^i\}=\{\phi^a,S^p\}$. The first terms are holomorphic while the last term is real and correspond to the mass term to each chiral fields ϕ^i. Note that the parameters C_i, B_{ij} and A_{ijk} are not arbitrary but are related to the form of the superpotential W_0 in Eq. 5 (which is polynomial of degree three).

The specific structure of the S-sector generates the last term in Eq. 7. Such couplings are *hard* breaking terms, *i.e.*, induce quadratic loop divergences. The general form of the *hard* breaking terms takes the form:

$$\begin{aligned}V_{HARD}=&\Big(\big(D_i^p\phi^i+\frac{1}{2}E_{ij}^p\phi^i\phi^j+\frac{1}{6}F_{ijk}^p\phi^i\phi^j\phi^k\big)S_p^\dagger+G_{ijk}{}^l\phi^i\phi^j\phi^k\phi_l^\dagger+H_{ijp}{}^l\phi^i\phi^jS^p\phi_l^\dagger+\text{h.c.}\Big)\\&+Q_{i,p}{}^q\phi^i\phi_i^\dagger S^pS_q^\dagger+T_{i,p}{}^qS^pS_p^\dagger S^rS_q^\dagger\,.\end{aligned}$$

The presence of *hard* breaking terms in the potential is new. Such terms differ from *soft* breaking terms since couplings between holomorphic and anti-holomorphic superfields are present.

These couplings allow to close S-loops and induce new contributions to the mass of fields ϕ. Since such hard terms are suppressed by an intermediate scale, the quadratic divergences are reduced and may be sizeable to solve some actual issues of the Standard Model. The *hard* parameters D_i^p, E_{ij}^p and F_{ijk}^p are also correlated with the holomorphic *soft* breaking terms through the hybrid fields couplings.

4 Hybrid extension of the MSSM: The S2MSSM

In the previous section, we have presented new solutions obtained from a canonical Kähler potential. It is desirable to extend the analysis to the non-canonical case to get a richer mass spectrum. Thus, along the lines of the results of Brignole, Ibanez & Munoz [5] and Guidicce & Masiero [6], we have considered a solution assuming a non-canonical Kähler metric. This enables us to identify a possible extension of the Minimal Supersymmetric Standard Model (MSSM) [7] involving hybrid fields S^p.

4.1 Definition of the model

We assume a hidden sector containing one superfield $Z = (\zeta, \chi_\zeta, F_\zeta)$ and the observable sector of the MSSM $\{\Phi^a\} = \{\Phi^a\}_{MSSM}$. This model is the simplest supersymmetric extension of the Standard Model. The visible sector contains superfields associated to quarks, leptons and the two $SU(2)$ superfields Higgs doublets H_U and H_D. We also introduce a hybrid sector $\{\mathcal{S}^p\}$ $(p = 1, \ldots, n_p)$. Following the results above, the superfield $\mathcal{U}$ is the only superfield that couples to the observable sector. Among these n_p hybrid superfields, we assume that only two superfields $\mathcal{S}^1$ and $\mathcal{S}^2$ interact with $\{\Phi^a\}$ via $\mathcal{U}$:

$$\mathcal{U} = \mu^1 \mathcal{S}^2 - \mu^2 \mathcal{S}^1 \,.$$

The $n_p - 2$ other fields $\{S^3, \ldots, S^{n_p}\}$ will play an important role as we will see later. The superpotential and the Kähler potential are:

$$\begin{aligned}
W(\Phi, \mathcal{S}, Z) &= m_p\big(W_{1,0}(Z) + \mathcal{S}^p \mu_p^* W_{1,p}(Z)\big) + \mathcal{S}^p W_{0,p}(Z) + W_0(\Phi, \mathcal{U}, Z)\,,\\
K(\Phi, \Phi^\dagger, \mathcal{S}, \mathcal{S}^\dagger, Z, Z^\dagger) &= m_p^2 \hat{K}(Z, Z^\dagger) + \mathcal{S}_p^\dagger S^p + \sum_a \Lambda_a(Z, Z^\dagger)\Phi_a^\dagger \Phi^a\,,
\end{aligned}$$

where:

$$W_0(\Phi, \mathcal{U}, Z) = \lambda(Z)\mathcal{U} H_U \cdot H_D + \frac{1}{6}\kappa(Z)\mathcal{U}^3 + W_{MSSM}|_{\mu=0}\,,$$

with $W_{MSSM}|_{\mu=0}$, the superpotential of the MSSM (not given here) where the quadratic Higgs doublets coupling is not present. Such a superpotential contains then only Yukawa couplings, *i.e.*, cubic terms. The matrix $\Lambda_a(Z, Z^\dagger)$ leads to a non-universality of the breaking terms in the usual matter sector. Since the hybrid superfields $\mathcal{S}^p$ are gauge invariant, quadratic and linear contributions can be added. However, we restrict ourselves to a $\mathbb{Z}_3$-invariant $W_0(\Phi, \mathcal{U}, Z)$ superpotential (only cubic couplings) assuming superconformal invariance.

As seen previously, such solutions generate *soft* and *hard* terms that affect the mass spectrum of particles at the tree level and through loop corrections. We now investigate the mass matrix of such a model.

4.2 A simple case

This theory contains many fields. It is then difficult to determine the set of parameters leading to interesting results. In order to find the optimal configuration, we first analyse a simplified model. We assume that only the scalar field from the hidden sector gets a nonzero vacuum expectation value (or *v.e.v.*) with $\langle\zeta\rangle = \mathcal{O}(m_p)$.[2] We also assume:

$$W_0(\Phi,\mathcal{U},Z) = W_0(\Phi,Z)\,.$$

The S-sector then only contributes through the two components $W_{1,p}$ and $W_{0,p}$ of the superpotential Eq. 5.

We compute the scalar potential Eq. 1 following these hypotheses. Such a potential can be written as Eq. 7. The vanishing of the cosmological constant and the minimisation of the potential are also imposed:

$$\langle V\rangle = 0\,,\quad \left\langle \frac{\partial V}{\partial X^A}\right\rangle = 0\,,\quad \left\langle \frac{\partial V}{\partial X^\dagger_{A^*}}\right\rangle = 0\,, \tag{9}$$

(with $X^A = \{z, \phi^a, S^p\}$, the scalar part of the chiral superfields and we introduce $z = \zeta/m_p$). These relations highly constrain the parameter space.

The mass matrix of this model is a $(n_a + n_p + 1)\times(n_a + n_p + 1)$ matrix mixing the three different sectors. It can be shown that it decouples into two submatrices related to the two sectors $\{\Phi^a\}$ and $\{S^p, z\}$ at first order of $1/m_p^2$, and so can be diagonalised separately. The mass matrix $\mathbb{M}'^2$ in the sector $X'^A = \{S^p, z\}$ reads:

$$\mathbb{M}'^2 = \left\langle \frac{\partial^2 V}{\partial X'^A X'^\dagger_{B^*}}\right\rangle = \begin{pmatrix} \delta_p{}^q m_{3/2} + b\mathcal{I}_p\bar{\mathcal{I}}^q & c\mathcal{I}_p \\ \bar{c}\bar{\mathcal{I}}^q & d\end{pmatrix}\,,\quad \text{with}\quad m_{3/2} = \frac{1}{m_p^2} e^{\langle K\rangle/m_p^2}\langle W\rangle\,, \tag{10}$$

where $\mathcal{I}_p = \left\langle \mu_p^* W_{1,p}(z) m_p + W_{0,p}(z)\right\rangle$, b,c and d are some constants related to the parameters of the superpotential, and $m_{3/2}$ is the gravitino mass. Performing a change of basis, we rewrite the mass matrix $\mathbb{M}'^2$ in the form:

$$\mathbb{M}'^2 = \begin{pmatrix} m_{3/2}\mathbb{I}_{n-1} & 0 & 0 \\ 0 & m_{3/2} + b|\mathcal{I}|^2 & c|\mathcal{I}| \\ 0 & \bar{c}|\mathcal{I}| & d\end{pmatrix}\,,\quad \text{with}\quad \sum_{p=1}^{n_p}\mathcal{I}_p\bar{\mathcal{I}}^p = |\mathcal{I}|^2\,. \tag{11}$$

We then obtain one S-field mixing in a non-trivial way with the hidden field z and the $n_p - 1$ remaining S fields with a mass equal to the gravitino mass $m_{3/2}$. Proving inductively that $\mathrm{Tr}[(\mathbb{M}'^2)^n]$ (with $n \in \mathbb{N}$) is only a function of $|\mathcal{I}|^2$, one can show that the eigenvalues do not depend on $\mathcal{I}_p$ thanks to the vanishing of the cosmological constant:

$$\langle V\rangle = e^{|\langle z\rangle|^2}\Big(|\mathcal{I}|^2 + \sum_a \big|\langle\partial_a W_0\rangle\big|^2\Big) + m_p^2(|m'_{3/2}|^2 - 3|m_{3/2}|^2) = 0 \text{ with } m'_{3/2} = \frac{1}{m_p^2} e^{\langle K\rangle/m_p^2}\langle \mathfrak{d}_z W\rangle\,,$$

where $\mathfrak{d}_z W = \partial_z W + z^\dagger W$.

To understand the effects of the hybrid sector $\{\mathcal{S}^p\}$ on the Standard Model, we have investigated the consequences of this new sector on the Higgs boson mass. Several points can be mentioned:

- Since there are no interactions between the hybrid and the observable sector in W_0, the only contributions of the S-sector on the Higgs boson mass are obtained through the *hard* breaking terms and thus through loop-corrections.

[2] Note that fields from the observable sector can develop a nonzero *v.e.vs* but much smaller than the Planck mass, *i.e.*, $\langle S^p\rangle \ll \langle\zeta\rangle$ and $\langle\phi^a\rangle = M_\phi \ll \langle\zeta\rangle$ (with $M_\phi = M_{EW} \approx 10^2$ GeV or $M_{GUT} \approx 10^{16}$ GeV). The effect of these nonzero *v.e.vs.* are taken in account in Section 4.3.

- The order of magnitude of the one loop-correction is proportional to the energy scale of the visible sector. In order to increase such contributions, we embed the Standard Model in a GUT model such that $\langle \phi^a \rangle \approx M_{GUT}$.

- A quantitative study on the order of magnitude of the one S-loop contribution to the Higgs boson mass has been done. With such hypotheses, we have highlighted several configurations leading to a Higgs boson mass of 125 GeV. Nevertheless, such configurations require a certain level of fine-tuning in the hidden sector.

This study enables us to put in evidence the correct strategy for analysing the S2MSSM without the simplifying assumptions imposed previously.

4.3 Towards the general S2MSSM

We now reintroduce the dependence of $\mathcal{U}$ in the superpotential $W_0(\Phi,\mathcal{U},Z)$ Eq. 5. Non-null vacuum expectation values for the hybrid fields $\langle S^p \rangle \neq 0$ and the visible sector $\langle \phi^a \rangle \neq 0$ are also assumed. The energy scale of the Φ-sector corresponds to the electroweak scale ($M_{EW} \approx 10^2$ GeV) or a GUT scale ($M_{GUT} \approx 10^{16}$ GeV).

The complete form of the scalar potential of the S2MSSM is given in Appendix A. The mass matrix in the sub-sector $\{S^p, z\}$ can be written in the following form:

$$\mathbb{M}'^2 = \begin{pmatrix} \delta_p{}^q a' + e' + b' \mathcal{J}_p \bar{\mathcal{J}}^q & c' \mathcal{J}_p + f'_p \\ \bar{c}' \bar{\mathcal{J}}^q + \bar{f}'^q & d' \end{pmatrix}, \quad \text{with} \quad \mathcal{J}_p = \mathcal{I}_p + \langle \partial_p W_0 \rangle\,, \tag{12}$$

with a', b', c', d', e' and f'_p some constants. Note that e', f' and d' depend on the parameter $\mathcal{J}_p$. Due to these new contributions, the simple structure $b\mathcal{I}_p\bar{\mathcal{I}}^q$ in the $\{S^p\}$-sector (see Eq. 10) is lost. Consequently, the spectrum is not degenerate with a mass equal to $m_{3/2}$.

Mention again that a complete qualitative analysis is tedious due to the number of new contributions in the scalar potential (see Appendix A). A numerical computation of the mass matrix is necessary to find configurations that reduce the mass matrix to a form equivalent to Eq. 11. Such a study is in progress [8]. Such new terms may also help to resolve two actual issues in the Standard Model and in supersymmetry, *i.e.*:

- reduce the fine-tuning on the Higgs boson mass through tree-level and loop corrections and help to naturally obtain a mass near 125 GeV,

- push the squark masses to higher energy which may explain the non-detection of supersymmetry in particle physics experiments.

Note also that this model can have an interesting relationship with a model called NMSSM [9] (extension of the MSSM with one singlet superfield). The relation between these two models is also under investigation [8].

5 Conclusion

New solutions where supersymmetry is broken through gravitational mediation involving *hard* breaking terms have been investigated. The contributions of hard breaking terms have been studied in this paper through the construction of a model related to these new solutions, the S2MSSM. The mass spectrum of this new model has been calculated assuming the vanishing of the cosmological constant, *i.e.*, $\langle V \rangle = 0$. Following some simplifying assumptions, we obtain in the particle spectrum several degenerated states with a mass equal to the gravitino mass. However, such structure is lost when assuming all the contributions in the S2MSSM.

A complete numerical analysis through a spectrum generator may be useful to investigate all the (tree-level and loop-level) contributions of the new field sector $\{\mathcal{S}^p\}$.

Acknowledgements

I'm grateful to the organisers of the 34th International Colloquium on Group Theoretical Methods in Physics to allowed me to present these results in a parallel session.

A Scalar potential of the S2MSSM

The purpose of this appendix is to give the scalar potential of the S2MSSM in the low energy limit. We define $W_0 = M_4^3 \omega_0$ and $\phi^a = M_4 \varphi^a$ with $M_4 = M_{EW}$ or M_{GUT}. We also introduce the notation

$$\Delta f(z, S, \Phi) = f(\langle z\rangle, S + \langle S\rangle, \Phi + \langle \Phi\rangle) - f(\langle z\rangle, \langle S\rangle, \langle \Phi\rangle)\,.$$

The definition of $\mathcal{I}_p$ is given in Eq. 10. The complete form of the scalar potential is:

$$\begin{aligned}
V =& m_p^2 \big|m_{3/2}\big|^2 \Big(\frac{1}{\big|\xi_{3/2}\big|^2} - 3\Big) + e^{\left|\langle z\rangle\right|^2} \Big(\sum_p \big|\mathcal{I}_p + M_4^3 \partial_p \omega_0\big|^2 + M_4^4 \partial_a \omega_0 \partial^{a^*} \bar{\omega}_0 \langle (\Lambda^{-1})^a{}_{a^*}\rangle\Big) \\
&+ \Big((\langle S^p\rangle + S^p)(\langle S_p^\dagger\rangle + S_p^\dagger)\Big)\Big(\big|m_{3/2}\big|^2 T + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \big[\bar{m}_{3/2} S^r T_r + \text{h.c.}\big] + \frac{1}{m_p^4} e^{\left|\langle z\rangle\right|^2} S^r S_t^\dagger T^t{}_s\Big) \\
&+ \frac{1}{m_p^2} e^{\left|\langle z\rangle\right|^2} S^p S_q^\dagger \Big(\mathfrak{d}_z \mathcal{I}_p \mathfrak{d}^z \bar{\mathcal{I}}^q - 3 \mathcal{I}_p \bar{\mathcal{I}}^q\Big) + e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \bar{m}_{3/2} S^p \Big(\frac{1}{\bar{\xi}_{3/2}} \mathfrak{d}_z \mathcal{I}_p - 3\, \mathcal{I}_p + \text{h.c.}\Big) \\
&+ \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \Big\{ \Big(M_4^2 (\langle \varphi_{a^*}^\dagger\rangle + \varphi_{a^*}^\dagger)(\langle \varphi^a\rangle + \varphi^a)\langle \Lambda^{a^*}{}_a\rangle + (\langle S_p^\dagger\rangle + S_p^\dagger)(\langle S^p\rangle + S^p)\Big)(\langle S^q\rangle + S^q)\mathcal{I}_q \\
&\qquad \times \Big(\bar{m}_{3/2} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S_r^\dagger \bar{\mathcal{I}}^r\Big) + \text{h.c.} \Big\} + M_4^2 \Big((\langle \varphi^a\rangle + \varphi^a)(\langle \varphi_{a^*}^\dagger\rangle + \varphi_{a^*}^\dagger)\Big) \\
&\times \Big(\big|m_{3/2}\big|^2 \mathcal{S}^{a^*}{}_a + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \big[\bar{m}_{3/2} S^p\, (\mathcal{S}_p)^{a^*}{}_a + \text{h.c.}\big] + \frac{1}{m_p^4} e^{\left|\langle z\rangle\right|^2} S^p S_q^\dagger\, (\mathcal{S}^q{}_p)^{a^*}{}_a\Big) \\
&+ \frac{1}{m_p^2} e^{\left|\langle z\rangle\right|^2} \Big(M_4^2 (\langle \varphi^a\rangle + \varphi^a)(\langle \varphi_{a^*}^\dagger\rangle + \varphi_{a^*}^\dagger)\langle \Lambda^{a^*}{}_a\rangle + (\langle S^p\rangle + S^p)(\langle S_p^\dagger\rangle + S_p^\dagger)\Big) \\
&\times \Big(\sum_r \big|\mathcal{I}_r\big|^2 + M_4^3 \bar{\mathcal{I}}^r \partial_r \omega_0 + M_4^3 \mathcal{I}_r \partial^r \bar{\omega}^0\Big) + \Big(\langle S^p S_p^\dagger\rangle + \langle M_4^2 \varphi_{a^*}^\dagger \Lambda^{a^*}{}_b \varphi^b\rangle\Big) \\
&\times \Big(3\big|m_{3/2}\big|^2 - \big|m'_{3/2}\big|^2 - \frac{1}{2m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \big(\bar{m}'_{3/2} S^q \mathfrak{d}^z \mathcal{I}_q + \bar{m}_{3/2} \mathcal{I}_q\big(\langle S^q\rangle - 2S^q\big) + \text{h.c.}\big)\Big) \\
&+ e^{\frac{1}{2}\left|\langle z\rangle\right|^2} \Big\{ \bar{m}_{3/2} M_4^3 R^b{}_a (\langle \varphi^a\rangle + \varphi^a)\partial_b \omega_0 + \frac{M_4^3}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} (R^p)^b{}_a (\langle \varphi^a\rangle + \varphi^a) S_p^\dagger \partial_b \omega_0 \\
&\qquad + \big[\bar{m}_{3/2} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S_q^\dagger \bar{\mathcal{I}}^q\big](\langle S^p\rangle + S^p)\big[\mathcal{I}_p + M_4^3 \partial_p \omega_0\big] + \frac{M_4^3}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} (\langle S_p^\dagger\rangle + S_p^\dagger)\bar{\mathcal{I}}^p \Delta\omega_0 + \text{h.c.} \Big\} \\
&+ e^{\frac{1}{2}\left|\langle z\rangle\right|^2} M_4^3 \Big(\Delta \mathfrak{d}_z \omega_0 \big[\frac{\bar{m}_{3/2}}{\bar{\xi}_{3/2}} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S_q^\dagger \mathfrak{d}^z \bar{\mathcal{I}}^q\big] - 3\Delta\omega_0 \big[\bar{m}_{3/2} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S_q^\dagger \bar{\mathcal{I}}^q\big] + \text{h.c.}\Big) \\
&+ \frac{1}{2m_p^4} \big|\mathcal{I}_p\big|^2 e^{\left|\langle z\rangle\right|^2} \Big(M_4^2 (\langle \varphi^a\rangle + \varphi^a)(\langle \varphi_{a^*}^\dagger\rangle + \varphi_{a^*}^\dagger)\langle \Lambda^{a^*}{}_a\rangle + (\langle S^p\rangle + S^p)(\langle S_p^\dagger\rangle + S_p^\dagger)\Big)^2 \\
&- M_4^2 \Big(\Big(m'_{3/2} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S^p \mathfrak{d}_z \mathcal{I}_p\Big)\langle \varphi^\dagger \partial^z \Lambda \varphi\rangle \Big(\bar{m}_{3/2} + \frac{1}{m_p^2} e^{\frac{1}{2}\left|\langle z\rangle\right|^2} S_q^\dagger \bar{\mathcal{I}}^q\Big) + \text{h.c.}\Big),
\end{aligned}$$

where we define $\xi_{3/2} = m_{3/2}/m'_{3/2}$ and

$$\mathcal{S}^{a^*}{}_a = \frac{1}{|\xi_{3/2}|^2}\Big(\langle \partial^z \Lambda^{a^*}{}_b (\Lambda^{-1})^b{}_{b^*} \partial_z \Lambda^{b^*}{}_a - \partial^z \partial_z \Lambda^{a^*}{}_a \rangle\Big) + \langle \Lambda^{a^*}{}_a \rangle \Big(\frac{1}{|\xi_{3/2}|^2} - 2\Big),$$
$$(\mathcal{S}_p)^{a^*}{}_a = \frac{1}{\bar{\xi}_{3/2}}\Big(\langle \partial^z \Lambda^{a^*}{}_b (\Lambda^{-1})^b{}_{b^*} \partial_z \Lambda^{b^*}{}_a - \partial^z \partial_z \Lambda^{a^*}{}_a \rangle\Big)\mathfrak{d}_z \mathcal{I}_p + \langle \Lambda^{a^*}{}_a \rangle \Big(\frac{1}{\bar{\xi}_{3/2}} \mathfrak{d}_z \mathcal{I}_p - 2\mathcal{I}_p\Big),$$
$$(\mathcal{S}^q{}_p)^{a^*}{}_a = \Big(\langle \partial^z \Lambda^{a^*}{}_b (\Lambda^{-1})^b{}_{b^*} \partial_z \Lambda^{b^*}{}_a - \partial^z \partial_z \Lambda^{a^*}{}_a \rangle\Big)\mathfrak{d}_z \mathcal{I}_p \mathfrak{d}^z \bar{\mathcal{I}}^q + \langle \Lambda^{a^*}{}_a \rangle (\mathfrak{d}_z \mathcal{I}_p \mathfrak{d}^z \bar{\mathcal{I}}^q - 2\mathcal{I}_p \bar{\mathcal{I}}^q\Big).$$

and

$$T = \frac{1}{|\xi_{3/2}|^2} - 2, \quad T_p = \frac{1}{\bar{\xi}_{3/2}} \mathfrak{d}_z \mathcal{I}_p - 2\mathcal{I}_p, \quad T^p{}_q = \mathfrak{d}_z \mathcal{I}_q \mathfrak{d}^z \bar{\mathcal{I}}^q - 2\mathcal{I}_q \bar{\mathcal{I}}^q, \tag{A.1}$$
$$R^a{}_b = \delta^a_b - \frac{1}{\bar{\xi}_{3/2}} \langle (\Lambda^{-1})^a{}_{b^*} \partial_z \Lambda^{b^*}{}_b \rangle, \quad (R^p)^a{}_b = \bar{\mathcal{I}}^p \delta^a{}_b - \mathfrak{d}^z \bar{\mathcal{I}}^p \langle (\Lambda^{-1})^a{}_{b^*} \partial^z \Lambda^{b^*}{}_b \rangle.$$

References

[1] R. Haag, J. T. Łopuszański and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. B **88**, 257 (1975), doi:10.1016/0550-3213(75)90279-5.

[2] S. K. Soni and H. A. Weldon, *Analysis of the supersymmetry breaking induced by N = 1 supergravity theories*, Phys. Lett. B **126**, 215 (1983), doi:10.1016/0370-2693(83)90593-2.

[3] G. Moultaka, M. R. de Traubenberg and D. Tant, *Low energy supergravity revisited (I)*, Int. J. Mod. Phys. A **34**, 1950004 (2019), doi:10.1142/S0217751X19500040.

[4] L. Girardello and M. T. Grisaru, *Soft breaking of supersymmetry*, Nucl. Phys. B **194**, 65 (1982), doi:10.1016/0550-3213(82)90512-0.

[5] A. Brignole, L. E. Ibáñez and C. Muñoz, *Soft supersymmetry-breaking terms from supergravity and superstring models*, in *Perspectives on supersymmetry*, World Scientific, Singapore, ISBN 9789810235536 (1998), doi:10.1142/9789812839657_0003.

[6] G. F. Giudice and A. Masiero, *A natural solution to the μ-problem in supergravity theories*, Phys. Lett. B **206**, 480 (1988), doi:10.1016/0370-2693(88)91613-9.

[7] P. Fayet, *Supersymmetry and weak, electromagnetic and strong interactions*, Phys. Lett. B **64**, 159 (1976), doi:10.1016/0370-2693(76)90319-1.

[8] R. Ducrocq, G. Moultaka and M. Rausch de Traubenberg, *Low energy supergravity revisited (II) (in preparation)*.

[9] U. Ellwanger, C. Hugonie and A. M. Teixeira, *The next-to-minimal supersymmetric Standard Model*, Phys. Rep. **496**, 1 (2010), doi:10.1016/j.physrep.2010.07.001.

An algebraic approach to intertwined quantum phase transitions in the Zr isotopes

Noam Gavrielov[⋆]

Center for Theoretical Physics, Sloane Physics Laboratory, Yale University,
New Haven, Connecticut 06520-8120, USA
Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

⋆ noam.gavrielov@yale.edu

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The algebraic framework of the interacting boson model with configuration mixing is employed to demonstrate the occurrence of intertwined quantum phase transitions (IQPTs) in the $_{40}$Zr isotopes with neutron number 52–70. The detailed quantum and classical analyses reveal a QPT of crossing normal and intruder configurations superimposed on a QPT of the intruder configuration from U(5) to SU(3) and a crossover from SU(3) to SO(6) dynamical symmetries.

Received 14-12-2022
Accepted 21-07-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.028

Contents

1 Introduction **2**

2 Theoretical framework **2**
2.1 The interacting boson model with configuration mixing 2
2.2 Wave functions structure 3
2.3 Geometry 4
2.4 QPTs and order parameters 4

3 QPTs in the Zr isotopes **5**
3.1 Model space 5
3.2 Hamiltonian and E2 transitions operator 5

4 Results **6**
4.1 Evolution of energy levels 6
4.2 Evolution of configuration content 7
4.3 Evolution of symmetry content 7
4.4 Evolution of order parameters 8
4.5 Classical analysis 9

5 Conclusions and Outlook 9

References 10

1 Introduction

Quantum phase transitions [1–3] are qualitative changes in the structure of a physical system that occur as a function of one (or more) parameters that appear in the quantum Hamiltonian describing the system. In nuclear physics [4], we vary the number of nucleons and examine mainly two types of quantum phase transitions (QPTs). The first describes shape phase transitions in a single configuration, denoted as Type I. When interpolating between two shapes, for example, the Hamiltonian can be written as a sum of two parts

$$\hat{H} = (1-\xi)\hat{H}_1 + \xi\hat{H}_2\,, \tag{1}$$

with ξ the control parameter. As we vary ξ with nucleon number from 0 to 1, the equilibrium shape and symmetry of the Hamiltonian vary from those of $\hat{H}_1$ to those of $\hat{H}_2$. QPTs of this type have been studied extensively in the framework of the interacting boson model (IBM) [4–7]. One example of such QPT is the $_{62}$Sm region with neutron number 84–94, where the shape evolves from spherical to axially-deformed, with a critical point at neutron number 90.

The second type of QPT occurs when the ground state configuration changes its character, typically from normal to intruder type of states, denoted as Type II QPT. In such cases, the Hamiltonian can be written in matrix form [8]. For two configurations A and B we have

$$\hat{H} = \begin{bmatrix} \hat{H}_A(\xi_A) & \hat{W}(\omega) \\ \hat{W}(\omega) & \hat{H}_B(\xi_B) \end{bmatrix}, \tag{2}$$

with ξ_i $(i = A, B)$, the control parameter of configuration (i), and $\hat{W}$, the coupling between them with parameter ω. QPTs of this type are manifested empirically near (sub-) shell closure, e.g. in the light Pb-Hg isotopes, with strong mixing between the configurations [9, 10].

Recently, we have introduced a new type of phase-transitions in even-even [11, 12] and odd-mass [13] nuclei called intertwined quantum phase transitions (IQPTs). The latter refers to a scenario where as we vary the control parameters (ξ_A, ξ_B, ω) in Eq. (2), each of the Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ undergoes a separate and clearly distinguished shape-phase transition (Type I), and the combined Hamiltonian simultaneously experiences a crossing of configurations A and B (Type II).

2 Theoretical framework

A convenient framework to study the different types of QPTs together is the extension of the IBM to include configuration mixing (IBM-CM) [14–16].

2.1 The interacting boson model with configuration mixing

The IBM for a single shell model configuration has been widely used to describe low-lying quadrupole collective states in nuclei in terms of N monopole ($s^\dagger$) and quadrupole ($d^\dagger$) bosons, representing valence nucleon pairs. The model has U(6) as a spectrum generating algebra, where the Hamiltonian is expanded in terms of its generators, $\{s^\dagger s, s^\dagger d_\mu, d^\dagger_\mu s, d^\dagger_\mu d_{\mu'}\}$, and consists of Hermitian, rotational-scalar interactions which conserve the total number of s- and d-bosons $\hat{N} = \hat{n}_s + \hat{n}_d = s^\dagger s + \sum_\mu d^\dagger_\mu d_\mu$. The boson number is fixed by the microscopic interpretation of the IBM [17] to be $N = N_\pi + N_\nu$, where N_π (N_ν) is the number of proton (neutron) particle or hole pairs counted from the nearest closed shell.

The solvable limits of the model correspond to dynamical symmetries (DSs) associated with chains of nested sub-algebras of U(6), terminating in the invariant SO(3) algebra. In the IBM there are three DS limits

$$\mathrm{U}(6) \supset \begin{cases} \mathrm{U}(5) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)\,, \\ \mathrm{SU}(3) \supset \mathrm{SO}(3)\,, \\ \mathrm{SO}(6) \supset \mathrm{SO}(5) \supset \mathrm{SO}(3)\,. \end{cases} \tag{3}$$

In a DS, the Hamiltonian is written in terms of Casimir operators of the algebras of a given chain. In such a case, the spectrum is completely solvable and resembles known paradigms of collective motion: spherical vibrator [U(5)], axially symmetric [SU(3)] and γ-soft deformed rotor [SO(6)]. In each case, the energies and eigenstates are labeled by quantum numbers that are the labels of irreducible representations (irreps) of the algebras in the chain. The corresponding basis states for each of the chains (3) are

$$\mathrm{U}(5): \quad |N, n_d, \tau, n_\Delta, L\rangle\,, \tag{4a}$$
$$\mathrm{SU}(3): \quad |N, (\lambda,\mu), K, L\rangle\,, \tag{4b}$$
$$\mathrm{SO}(6): \quad |N, \sigma, \tau, n_\Delta, L\rangle\,, \tag{4c}$$

where $N, n_d, (\lambda,\mu), \sigma, \tau, L$ label the irreps of U(6), U(5), SU(3), SO(6), SO(5) and SO(3), respectively, and n_Δ, K are multiplicity labels.

An extension of the IBM to include intruder excitations is based on associating the different shell-model spaces of 0p-0h, 2p-2h, 4p-4h, . . . particle-hole excitations, with the corresponding boson spaces with N, $N+2$, $N+4, \ldots$ bosons, which are subsequently mixed [15, 16]. For two configurations the resulting IBM-CM Hamiltonian can be transcribed in a form equivalent to that of Eq. (2)

$$\hat{H} = \hat{H}_A^{(N)} + \hat{H}_B^{(N+2)} + \hat{W}^{(N,N+2)}\,. \tag{5}$$

Here, the notations $\hat{O}^{(N)} = \hat{P}_N^\dagger \hat{O} \hat{P}_N$ and $\hat{O}^{(N,N')} = \hat{P}_N^\dagger \hat{O} \hat{P}_{N'}$, stand for an operator $\hat{O}$, with $\hat{P}_N$, a projection operator onto the N boson space. The Hamiltonian $\hat{H}_A^{(N)}$ represents the N boson space (normal A configuration) and $\hat{H}_B^{(N+2)}$ represents the $N+2$ boson space (intruder B configuration).

2.2 Wave functions structure

The eigenstates $|\Psi; L\rangle$ of the Hamiltonian (5) with angular momentum L, are linear combinations of the wave functions, Ψ_A and Ψ_B, in the two spaces $[N]$ and $[N+2]$,

$$|\Psi; L\rangle = a\,|\Psi_A; [N], L\rangle + b\,|\Psi_B; [N+2], L\rangle\,, \tag{6}$$

with $a^2 + b^2 = 1$. We note that each of the components in Eq. (6), $|\Psi_A; [N], L\rangle$ and $|\Psi_B; [N+2], L\rangle$, can be expanded in terms of the different DS limits with its corresponding boson number in the following manner

$$|\Psi_i; [N_i], L\rangle = \sum_\alpha C_\alpha^{(N_i,L)}\, |N_i, \alpha, L\rangle\,, \tag{7}$$

where $N_A = N$ and $N_B = N+2$, and $\alpha = \{n_d, \tau, n_\Delta\}, \{(\lambda,\mu), K\}, \{\sigma, \tau, n_\Delta\}$ are the quantum numbers of the DS eigenstates. The coefficients $C_\alpha^{(N,L)}$ give the weight of each component in the wave function. Using them, we can calculate the wave function probability of having definite quantum numbers of a given symmetry in the DS bases, Eq. (7), for its A or B parts

$$\mathrm{U(5)}: \quad P_{n_d}^{(N_i,L)} = \sum_{\tau,n_\Delta}[C_{n_d,\tau,n_\Delta}^{(N_i,L)}]^2\,, \qquad \mathrm{SO(6)}: \quad P_{\sigma}^{(N_i,L)} = \sum_{\tau,n_\Delta}[C_{\sigma,\tau,n_\Delta}^{(N_i,L)}]^2\,, \tag{8a}$$

$$\mathrm{SU(3)}: \quad P_{(\lambda,\mu)}^{(N_i,L)} = \sum_{K}[C_{(\lambda,\mu),K}^{(N_i,L)}]^2\,, \qquad \mathrm{SO(5)}: \quad P_{\tau}^{(N_i,L)} = \sum_{n_d,n_\Delta}[C_{n_d,\tau,n_\Delta}^{(N_i,L)}]^2\,. \tag{8b}$$

Here the subscripts $i = A, B$ denote the different configurations, i.e., $N_A = N$ and $N_B = N + 2$. Furthermore, for each eigenstate (6), we can also examine its coefficients a and b, which portray the probability of the normal-intruder mixing. They are evaluated from the sum of the squared coefficients of an IBM basis. For the U(5) basis, we have

$$P_a^{(N_A,L)} \equiv a^2 = \sum_{n_d,\tau,n_\Delta}|C_{n_d,\tau,n_\Delta}^{(N_A,L)}|^2\,, \qquad P_b^{(N_B,L)} \equiv b^2 = \sum_{n_d,\tau,n_\Delta}|C_{n_d,\tau,n_\Delta}^{(N_B,L)}|^2\,, \tag{9}$$

where the sum goes over all possible values of (n_d, τ, n_Δ) in the (N_i, L) space, $i = A, B$, and $a^2 + b^2 = 1$.

2.3 Geometry

To obtain a geometric interpretation of the IBM is we take the expectation value of the Hamiltonian between coherent (intrinsic) states [5,18] to form an energy surface

$$E_N(\beta,\gamma) = \langle \beta,\gamma; N|\hat{H}|\beta,\gamma; N\rangle\,. \tag{10}$$

The (β, γ) of Eq. (10) are quadrupole shape parameters whose values, $(\beta_{\mathrm{eq}}, \gamma_{\mathrm{eq}})$, at the global minimum of $E_N(\beta, \gamma)$ define the equilibrium shape for a given Hamiltonian. The values are $(\beta_{\mathrm{eq}} = 0)$, $(\beta_{\mathrm{eq}} = \sqrt{2}, \gamma_{\mathrm{eq}} = 0)$ and $(\beta_{\mathrm{eq}} = 1, \gamma_{\mathrm{eq}}$ arbitrary) for the U(5), SU(3) and SO(6) DS limits, respectively. Furthermore, for these values the ground-band intrinsic state, $|\beta_{\mathrm{eq}}, \gamma_{\mathrm{eq}}; N\rangle$, becomes a lowest weight state in the irrep of the leading subalgebra of the DS chain, with quantum numbers $(n_d = 0)$, $(\lambda, \mu) = (2N, 0)$ and $(\sigma = N)$ for the U(5), SU(3) and SO(6) DS limits, respectively.

For the IBM-CM Hamiltonian, the energy surface takes a matrix form [19]

$$E(\beta,\gamma) = \begin{bmatrix} E_A(\beta,\gamma;\xi_A) & \Omega(\beta,\gamma;\omega) \\ \Omega(\beta,\gamma;\omega) & E_B(\beta,\gamma;\xi_B) \end{bmatrix}, \tag{11}$$

where the entries are the matrix elements of the corresponding terms in the Hamiltonian (2), between the intrinsic states of each of the configurations, with the appropriate boson number. Diagonalization of this two-by-two matrix produces the so-called eigen-potentials, $E_\pm(\beta, \gamma)$.

2.4 QPTs and order parameters

The energy surface depends also on the Hamiltonian parameters and serves as the Landau potential whose topology determines the type of phase transition. In QPTs involving a single configuration (Type I), the ground state shape defines the phase of the system, which also identifies the corresponding DS as the phase of the system. Such Type I QPTs can be studied using a Hamiltonian as in Eq. (1), that interpolates between different DS limits (phases) by varying its control parameters ξ. The order parameter is taken to be the expectation value of the d-boson number operator, $\hat{n}_d$, in the ground state, $\langle \hat{n}_d\rangle_{0_1^+}$, and measures the amount of deformation in the ground state.

In QPTs involving multiple configurations (Type II), the dominant configuration in the ground state defines the phase of the system. Such Type II QPTs can be studied using a Hamiltonian as in Eq. (5), that interpolates between the different configurations by varying its control parameters ξ_A, ξ_B, ω. The order parameters are taken to be the expectation value of $\hat{n}_d$ in the ground state wave function, $|\Psi; L = 0_1^+\rangle$, and in its Ψ_A and Ψ_B components, Eq. (6), denoted by $\langle \hat{n}_d \rangle_{0_1^+}$, $\langle \hat{n}_d \rangle_A$ and $\langle \hat{n}_d \rangle_B$, respectively. The shape-evolution in each of the configurations A and B is encapsulated in $\langle \hat{n}_d \rangle_A$ and $\langle \hat{n}_d \rangle_B$, respectively. Their sum weighted by the probabilities of the Ψ_A and Ψ_B components $\langle \hat{n}_d \rangle_{0_1^+} = a^2 \langle \hat{n}_d \rangle_A + b^2 \langle \hat{n}_d \rangle_B$, portrays the evolution of the normal-intruder mixing.

3 QPTs in the Zr isotopes

Along the years, the $Z \approx 40$, $A \approx 100$ region was suggested by many works to have a ground state that is dominated by a normal spherical configuration for neutron numbers 50–58 and by an intruder deformed configuration for 60 onward. This dramatic change in structure is explained in the shell model by the isoscalar proton-neutron interaction between non-identical nucleons that occupy the spin-orbit partner orbitals $\pi 1g_{9/2}$ and $\nu 1g_{7/2}$ [20]. The crossing between configurations arises from the promotion of protons across the Z=40 subsell gap. The interaction energy results in a gain that compensates the loss in single-particle and pairing energy and a mutual polarization effect is enabled. Therefore, the single-particle orbitals at higher intruder configurations are lowered near the ground state normal configuration, which effectively reverses their order.

3.1 Model space

Using the framework of the IBM-CM, we consider ${}^{90}_{40}$Zr as a core and valence neutrons in the 50–82 major shell. The normal A configuration corresponds to having no active protons above $Z = 40$ sub-shell gap, and the intruder B configuration corresponds to two-proton excitation from below to above this gap, creating 2p-2h states. Therefore, the IBM-CM model space employed in this study, consists of $[N] \oplus [N+2]$ boson spaces with total boson number $N = 1, 2, \ldots 8$ for ${}^{92-106}$Zr and $\bar{N} = \bar{7}, \bar{6}$ for ^{108,110}Zr, respectively, where the bar over a number indicates that these are hole bosons.

3.2 Hamiltonian and E2 transitions operator

In order to describe the spectrum of the Zr isotopes, we take a Hamiltonian that has a form as in Eq. (5) with entries

$$\hat{H}_A(\epsilon_d^{(A)}, \kappa^{(A)}, \chi) = \epsilon_d^{(A)} \hat{n}_d + \kappa^{(A)} \hat{Q}_\chi \cdot \hat{Q}_\chi \,, \tag{12a}$$

$$\hat{H}_B(\epsilon_d^{(B)}, \kappa^{(B)}, \chi) = \epsilon_d^{(B)} \hat{n}_d + \kappa^{(B)} \hat{Q}_\chi \cdot \hat{Q}_\chi + \kappa'^{(B)} \hat{L} \cdot \hat{L} + \Delta_p \,, \tag{12b}$$

where the quadrupole operator is given by $\hat{Q}_\chi = d^\dagger s + s^\dagger \tilde{d} + \chi (d^\dagger \times \tilde{d})^{(2)}$, and $\hat{L} = \sqrt{10}(d^\dagger \tilde{d})^{(1)}$ is the angular momentum operator. Here $\tilde{d}_m = (-1)^m d_{-m}$ and standard notation of angular momentum coupling is used. The off-set energy between configurations A and B is Δ_p, where the index p denotes the fact that this is a proton excitation. The mixing term in Eq. (5) between configurations (A) and (B) has the form [14–16] $\hat{W} = \omega [(d^\dagger \times d^\dagger)^{(0)} + (s^\dagger)^2] + \text{H.c.}$, where H.c. stands for Hermitian conjugate. The parameters are obtained from a fit, elaborated in the appendix of Ref. [12].

The $E2$ operator for two configurations is written as $\hat{T}(E2) = e^{(A)}\hat{Q}_{\chi}^{(N)} + e^{(B)}\hat{Q}_{\chi}^{(N+2)}$, with $\hat{Q}_{\chi}^{(N)} = \hat{P}_N^{\dagger}\hat{Q}_{\chi}\hat{P}_N$ and $\hat{Q}_{\chi}^{(N+2)} = P_{N+2}^{\dagger}\hat{Q}_{\chi}\hat{P}_{N+2}$. The boson effective charges $e^{(A)}$ and $e^{(B)}$ are determined from the $2^+ \to 0^+$ transition within each configuration [12], and χ is the same parameter as in the Hamiltonian (12).

For the energy surface matrix (11), we calculate the expectation values of the Hamiltonians $\hat{H}_A$ (12a) and $\hat{H}_B$ (12b) in the intrinsic state of Section 2.3 with N and $N+2$ bosons respectively, and a non-diagonal matrix element of the mixing term $\hat{W}$ between them. The explicit expressions can be found in [12].

4 Results

In order to understand the change in structure of the Zr isotopes, it is insightful to examine the evolution of different properties along the chain.

4.1 Evolution of energy levels

In Fig. 1, we show a comparison between selected experimental and calculated levels, along with assignments to configurations based on Eq. (9) and to the closest DS based on Eq. (8), for each state. In the region between neutron number 50 and 56, there appear to be two configurations, one spherical (seniority-like), (A), and one weakly deformed, (B), as evidenced by the ratio $R_{4/2}$, which is $R_{4/2}^{(A)} \cong 1.6$ and $R_{4/2}^{(B)} \cong 2.3$ at at 52–56. From neutron number 58, there is a pronounced drop in energy for the configuration (B) states and at 60, the two configurations exchange their role, indicating a Type II QPT. At this stage, the B configuration appears to undergo a U(5)-SU(3) Type I QPT, similarly to case of the Sm region [14, 21, 22]. Beyond neutron number 60, the B configuration is strongly deformed, as evidenced by the small value of the excitation energy of the state 2_1^+, $E_{2_1^+} = 139.3$ keV and by the ratio $R_{4/2}^{(B)} = 3.24$ in ^{104}Zr. At still larger neutron number 66, the ground state band becomes γ-unstable (or triaxial) as

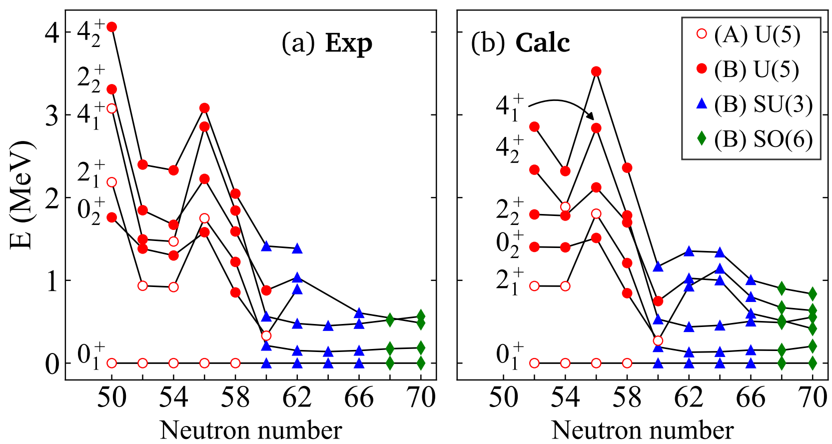


Figure 1: Comparison between (a) experimental and (b) calculated energy levels $0_1^+, 2_1^+, 4_1^+, 0_2^+, 2_2^+, 4_2^+$. Empty (filled) symbols indicate a state dominated by the normal A configuration (intruder B configuration), with assignments based on Eq. (9). The symbol [●, ▲, ♦], indicates the closest dynamical symmetry [U(5), SU(3), SO(6)] to the level considered, based on Eq. (8). Note that the calculated values start at neutron number 52, while the experimental values include the closed shell at 50. References for the data can be found in [12].

evidenced by the close energy of the states 2_2^+ and 4_1^+, $E_{2_2^+} = 607.0$ keV, $E_{4_1^+} = 476.5$ keV, in ^{106}Zr, and especially by the results $E_{4_1^+} = 565$ keV and $E_{2_2^+} = 485$ keV for ^{110}Zr of Ref. [23], a signature of the SO(6) symmetry. In this region, the B configuration undergoes a crossover from SU(3) to SO(6).

4.2 Evolution of configuration content

We examine the configuration change for each isotope, by calculating the evolution of the probability b^2, Eq. (9), of the 0_1^+ and 2_1^+ states. The left panels of Fig. 2 shows the percentage of the wave function within the B configuration as a function of neutron number across the Zr chain. The rapid change in structure of the 0_1^+ state (bottom left panel) from the normal A configuration in $^{92-98}$Zr (small b^2 probability) to the intruder B configuration in $^{100-110}$Zr (large b^2 probability) is clearly evident, signaling a Type II QPT. The configuration change appears however sooner in the 2_1^+ state (top left panel), which changes to configuration B already in ^{98}Zr, in line with [24]. Outside a narrow region near neutron number 60, where the crossing occurs, the two configurations are weakly mixed and the states retain a high level of purity, especially for neutron number larger than 60.

4.3 Evolution of symmetry content

We examine the changes in symmetry of the lowest 0^+ and 2^+ states within the B configuration, which undergoes a Type I QPT. In the right bottom panel of Fig. 2 the red dots represent the percentage of the U(5) $n_d = 0$ component in the wave function, $P^{(N+2,L=0)}_{n_d=0}$ of Eq. (8). It is large ($\approx$ 90%) for neutron number 52–58 and drops drastically ($\approx$ 30%) at 60. The drop means that other $n_d \neq 0$ components are present in the wave function and therefore this state becomes deformed. Above neutron number 60, the $n_d = 0$ component drops almost to zero (and rises again a little at 70), indicating the state is strongly deformed. To understand the type of DS associated with the deformation above neutron number 60, we add in blue triangles the percentage of the SU(3) $(\lambda,\mu) = (2N+4,0)$ component, $P^{(N+2,L=0)}_{(\lambda,\mu)=(2N+4,0)}$ of Eq. (8) for 60–66. For neutron number 60, it is moderately small ($\approx$ 35%), at neutron number 62 it jumps

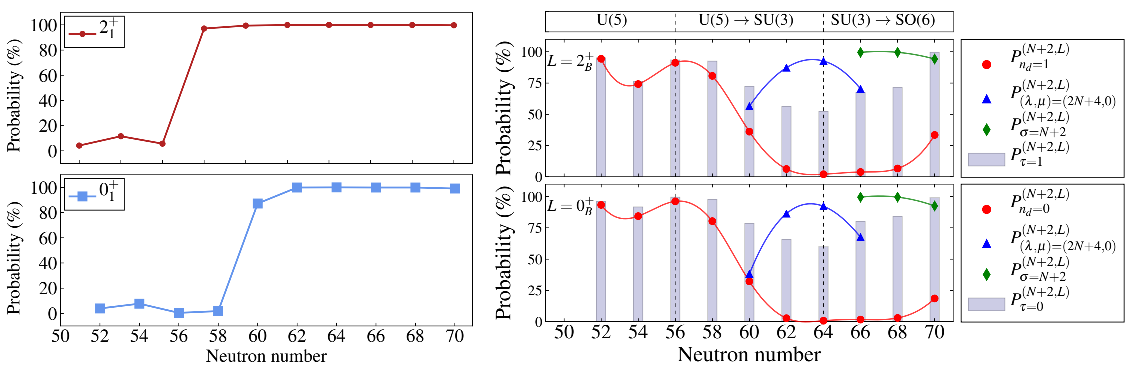


Figure 2: Left panels: percentage of the wave functions within the intruder B-configuration [the b^2 probability in Eq. (6)], for the ground 0_1^+ (bottom) and excited 2_1^+ (top) states in $^{92-110}$Zr. Right panels: evolution of symmetries for the lowest 0^+ (bottom) and 2^+ (top) state of configuration B along the Zr chain. Shown are the probabilities of selected components of U(5) (●), SU(3) (▲), SO(6) (◆) and SO(5) (▬), obtained from Eq. (8). For neutron numbers 52–58 (60–70), 0_B^+ corresponds to the experimental 0_2^+ (0_1^+) state. For neutron numbers 52–56 (58–70), 2_B^+ corresponds to the experimental 2_2^+ (2_1^+) state.

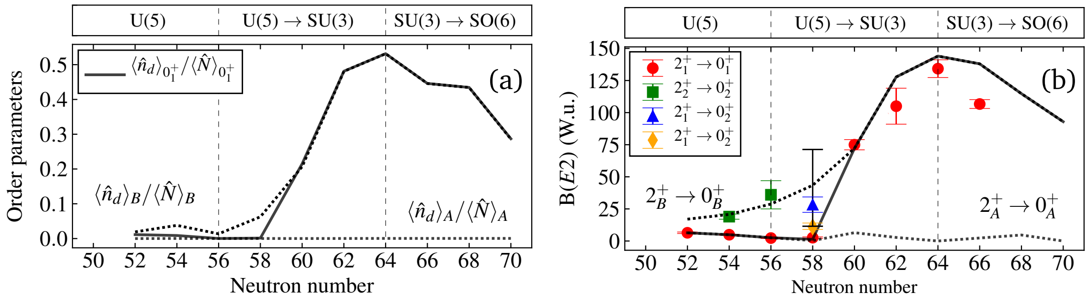


Figure 3: (a) Evolution of order parameters along the Zr chain, normalized (see text). (b) $B(E2)$ values in W.u. for $2^+ \to 0^+$ transitions in the Zr chain. The solid line (symbols ●, ■, ▲, ♦) denote calculated results (experimental results). Dotted lines denote calculated $E2$ transitions within a configuration. The data for ^{94}Zr, ^{96}Zr, ^{100}Zr, ^{102}Zr and (^{104}Zr, ^{106}Zr) are taken from [25], [26], [27], [28], [29], respectively. For ^{98}Zr (neutron number 58), the experimental values are from [30] (♦), from [31] (▲), and the upper and lower limits (black bars) are from [24, 27].

($\approx$ 85%) and becomes maximal at 64 ($\approx$ 92%). This serves as a clear evidence for a U(5)-SU(3) Type I QPT. At neutron number 66 the SU(3) $(\lambda,\mu)=(2N+4,0)$ component it is lowered, and one sees by the green diamonds the percentage of the SO(6) $\sigma = N+2$ component, $P^{(N+2,L=0)}_{\sigma=N+2}$ of Eq. (8). The latter becomes dominant for 66–70 ($\approx$ 99%), suggesting a crossover from SU(3) to SO(6).

In order to further elaborate the Type I QPT within configuration B from U(5) to SU(3) and the subsequent crossover to SO(6), we examine also the evolution of SO(5) symmetry. The gray histograms in the right panel of Fig. 2 depict the probability of the $\tau=0$ component of SO(5), $P^{(N+2,L=0)}_{\tau=0}$ of Eq. (8), for 0_B^+. For neutron numbers 52–56, the 0_B^+ state is composed mainly of a single $(n_d=0,\tau=0)$ component, appropriate for a state with good U(5) DS. For neutron number 58, the larger $\tau=0$ but smaller $n_d=0$ probabilities imply the presence of additional components with $(n_d\neq 0,\tau=0)$. For neutron numbers 60–64, the $\tau=0$ probability decreases, implying admixtures of components with $(n_d\neq 0,\tau\neq 0)$, appropriate for a state with good SU(3) DS. For neutron numbers 66–70, the $\tau=0$ probability increases towards its maximum value at 70, appropriate for a crossover to SO(6) structure with good SO(5) symmetry.

In the top right panel of Fig. 2 we observe a similar trend for the 2_B^+ state. For neutron numbers 52–58, it is dominated by a single $(n_d=1,\tau=1)$ component. For neutron number 60, $P^{(N+2,L=2_B^+)}_{n_d=1}$ is smaller than $P^{(N+2,L=2_B^+)}_{\tau=1}$, indicating the onset of deformation. For 62–64, $P^{(N+2,L=2_B^+)}_{n_d=1}$ is much smaller than $P^{(N+2,L=2_B^+)}_{\tau=1}$, implying admixtures of components with $(n_d\neq 1,\tau\neq 1)$. For neutron numbers 66–70, $P^{(N+2,L=2_B^+)}_{n_d=1}$ remains small but $P^{(N+2,L=2_B^+)}_{\tau=1}$ increases towards its maximum value at 70.

4.4 Evolution of order parameters

The configuration and symmetry analysis of Sections 4.2 and 4.3 suggest a situation of simultaneous occurrence of Type I and Type II QPTs. The order parameters can give further insight to these QPTs. Fig. 3(a) shows the evolution along the Zr chain of the order parameters ($\langle\hat{n}_d\rangle_A$, $\langle\hat{n}_d\rangle_B$ in dotted and $\langle\hat{n}_d\rangle_{0_1^+}$ in solid lines), normalized by the respective boson numbers, $\langle\hat{N}\rangle_A=N$, $\langle\hat{N}\rangle_B=N+2$, $\langle\hat{N}\rangle_{0_1^+}=a^2N+b^2(N+2)$. The order parameter $\langle\hat{n}_d\rangle_{0_1^+}$ is close to $\langle\hat{n}_d\rangle_A$ for neutron number 52–58 and coincides with $\langle\hat{n}_d\rangle_B$ at 60 and above. The clear jump

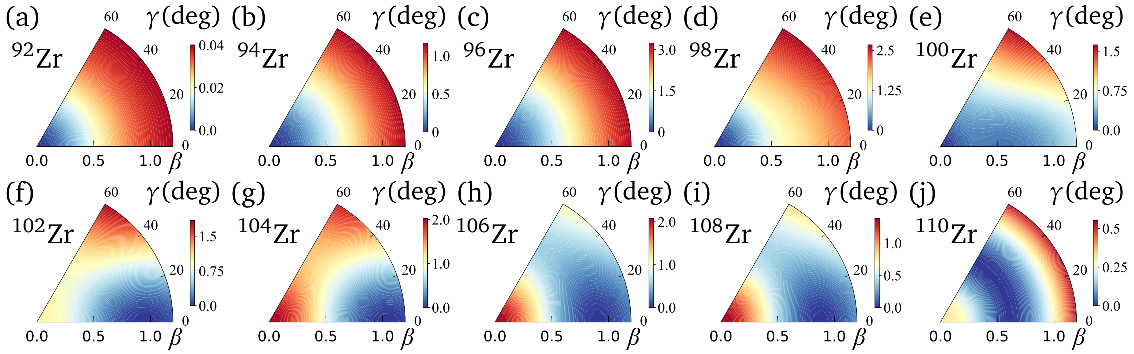


Figure 4: Contour plots in the (β, γ) plane of the lowest eigen-potential surface, $E_{-}(\beta, \gamma)$, for the $^{92-110}$Zr isotopes.

and change in configuration content from 58 to 60 indicates a Type II phase transition [8], with weak mixing between the configurations. Configuration A is spherical for all neutron numbers, and configuration B is weakly-deformed for neutron number 52–58. From neutron number 58 to 60 we see a sudden increase in $\langle \hat{n}_d \rangle_B$ that continues towards 64, indicating a U(5)-SU(3) Type I phase transition. Then, we observe a decrease from neutron number 66 onward, due in part to the crossover from SU(3) to SO(6) and in part to the shift from boson particles to boson holes after the middle of the major shell 50–82. These conclusions are stressed by an analysis of other observables [12], in particular, the $B(E2)$ values. As shown in Fig. 3(b), the calculated $B(E2)$'s agree with the experimental values and follow the same trends as the respective order parameters.

4.5 Classical analysis

In Fig. 4, we show the calculated lowest eigen-potential $E_{-}(\beta, \gamma)$, which is the lowest eigenvalue of the matrix Eq. (11). These classical potentials confirm the quantum results, as they show a transition from spherical ($^{92-98}$Zr), Figs. 4(a)-(d), to a double-minima potential that is almost flat-bottomed at ^{100}Zr, Fig. 4(e), to prolate axially deformed ($^{102-104}$Zr), Figs. 4(f)-(g), and finally to γ-unstable ($^{106-110}$Zr), Figs. 4(h)-(j).

5 Conclusions and Outlook

The algebraic framework of the IBM-CM allows us to examine QPTs using both quantum and classical analyses. We have employed this analysis to the Zr isotopes with A=92–110, which exhibit a complex structure that involves a shape-phase transition within the intruder configuration (Type I QPT) and a configuration-change between normal and intruder (Type II QPT), namely IQPTs. This was done by analyzing the energies, configuration and symmetry content of the wave functions, order parameters and $E2$ transition rates, and the energy surfaces. Further analysis of other observables supporting this scenario is presented in [12]. Recently, we have also exemplified the notion IQPTs in the odd-mass $_{41}$Nb isotopes [13] and it would be interesting to examine the notion of IQPTs in other even-even and odd-mass chains of isotopes in the $Z \approx 40$, $A \approx 100$ region and other physical systems.

Acknowledgments

This work was done in collaboration with F. Iachello (Yale university) and A. Leviatan (Hebrew university).

Funding information This work was supported in part by the US-Israel Binational Science Foundation Grant No. 2016032 and the Israel Academy of Sciences for a Postdoctoral Fellowship Program in Nuclear Physics.

References

[1] R. Gilmore and D. H. Feng, *Phase transitions in nuclear matter described by pseudospin Hamiltonians*, Nucl. Phys. A **301**, 189 (1978), doi:10.1016/0375-9474(78)90260-9.

[2] R. Gilmore, *The classical limit of quantum nonspin systems*, J. Math. Phys. **20**, 891 (1979), doi:10.1063/1.524137.

[3] L. D. Carr, *Understanding quantum phase transitions*, CRC press, Boca Raton, USA, ISBN 9781439802519 (2010).

[4] P. Cejnar, J. Jolie and R. F. Casten, *Quantum phase transitions in the shapes of atomic nuclei*, Rev. Mod. Phys. **82**, 2155 (2010), doi:10.1103/RevModPhys.82.2155.

[5] A. E. L. Dieperink, O. Scholten and F. Iachello, *Classical limit of the interacting-boson model*, Phys. Rev. Lett. **44**, 1747 (1980), doi:10.1103/PhysRevLett.44.1747.

[6] P. Cejnar and J. Jolie, *Quantum phase transitions in the interacting boson model*, Prog. Part. Nucl. Phys. **62**, 210 (2009), doi:10.1016/j.ppnp.2008.08.001.

[7] F. Iachello, *Quantum phase transitions in algebraic models*, Riv. Nuovo Cimento **34**, 617 (2011), doi:10.1393/NCR/I2011-10070-7.

[8] A. Frank, P. Van Isacker and F. Iachello, *Phase transitions in configuration mixed models*, Phys. Rev. C **73**, 061302 (2006), doi:10.1103/PhysRevC.73.061302.

[9] J. E. García-Ramos and K. Heyde, *The Pt isotopes: Comparing the interacting boson model with configuration mixing and the extended consistent-Q formalism*, Nucl. Phys. A **825**, 39 (2009), doi:10.1016/j.nuclphysa.2009.04.003.

[10] J. E. García-Ramos and K. Heyde, *Nuclear shape coexistence in Po isotopes: An interacting boson model study*, Phys. Rev. C **92**, 034309 (2015), doi:10.1103/PhysRevC.92.034309.

[11] N. Gavrielov, A. Leviatan and F. Iachello, *Intertwined quantum phase transitions in the Zr isotopes*, Phys. Rev. C **99**, 064324 (2019), doi:10.1103/PhysRevC.99.064324.

[12] N. Gavrielov, A. Leviatan and F. Iachello, *Zr isotopes as a region of intertwined quantum phase transitions*, Phys. Rev. C **105**, 014305 (2022), doi:10.1103/PhysRevC.105.014305.

[13] N. Gavrielov, A. Leviatan and F. Iachello, *Mixed configurations and intertwined quantum phase transitions in odd-mass nuclei*, Phys. Rev. C **106**, L051304 (2022), doi:10.1103/PhysRevC.106.L051304.

[14] F. Iachello and A. Arima, *The interacting boson model*, Cambridge University Press, Cambridge, UK, ISBN 9780521302821 (1987), doi:10.1017/CBO9780511895517.

[15] P. D. Duval and B. R. Barrett, *Configuration mixing in the interacting boson model*, Phys. Lett. B **100**, 223 (1981), doi:10.1016/0370-2693(81)90321-X.

[16] P. D. Duval and B. R. Barrett, *Quantitative description of configuration mixing in the interacting boson model*, Nucl. Phys. A **376**, 213 (1982), doi:10.1016/0375-9474(82)90061-6.

[17] F. Iachello and I. Talmi, *Shell-model foundations of the interacting boson model*, Rev. Mod. Phys. **59**, 339 (1987), doi:10.1103/RevModPhys.59.339.

[18] J. N. Ginocchio and M. W. Kirson, *Relationship between the Bohr collective Hamiltonian and the interacting-boson model*, Phys. Rev. Lett. **44**, 1744 (1980), doi:10.1103/PhysRevLett.44.1744.

[19] A. Frank, P. Van Isacker and C. E. Vargas, *Evolving shape coexistence in the lead isotopes: The geometry of configuration mixing in nuclei*, Phys. Rev. C **69**, 034323 (2004), doi:10.1103/PhysRevC.69.034323.

[20] P. Federman and S. Pittel, *Unified shell-model description of nuclear deformation*, Phys. Rev. C **20**, 820 (1979), doi:10.1103/PhysRevC.20.820.

[21] O. Scholten, F. Iachello and A. Arima, *Interacting boson model of collective nuclear states III. The transition from SU(5) to SU(3)*, Ann. Phys. **115**, 325 (1978), doi:10.1016/0003-4916(78)90159-8.

[22] M. A. Caprio et al., *Low-spin structure of ^{156}Dy through γ-ray spectroscopy*, Phys. Rev. C **66**, 054310 (2002), doi:10.1103/PhysRevC.66.054310.

[23] N. Paul et al., *Are there signatures of harmonic oscillator shells far from stability? First spectroscopy of ^{110}Zr*, Phys. Rev. Lett. **118**, 032501 (2017), doi:10.1103/PhysRevLett.118.032501.

[24] W. Witt et al., *Sub-shell closure and shape coexistence in the transitional nucleus ^{98}Zr*, Phys. Rev. C **98**, 041302 (2018), doi:10.1103/PhysRevC.98.041302.

[25] A. Chakraborty et al., *Collective structure in ^{94}Zr and subshell effects in shape coexistence*, Phys. Rev. Lett. **110**, 022504 (2013), doi:10.1103/PhysRevLett.110.022504.

[26] C. Kremer et al., *First measurement of collectivity of coexisting shapes based on type II shell evolution: The case of ^{96}Zr*, Phys. Rev. Lett. **117**, 172503 (2016), doi:10.1103/PhysRevLett.117.172503.

[27] S. Ansari et al., *Experimental study of the lifetime and phase transition in neutron-rich ^{98,100,102}Zr*, Phys. Rev. C **96**, 054323 (2017), doi:10.1103/PhysRevC.96.054323.

[28] D. De Frenne, *Nuclear data sheets for A = 102*, Nucl. Data Sheets **110**, 1745 (2009), doi:10.1016/J.NDS.2009.06.002.

[29] F. Browne et al., *Lifetime measurements of the first 2^+ states in ^{104,106}Zr: Evolution of ground-state deformations*, Phys. Lett. B **750**, 448 (2015), doi:10.1016/J.PHYSLETB.2015.09.043.

[30] V. Karayonchev et al., *Tests of collectivity in ^{98}Zr by absolute transition rates*, Phys. Rev. C **102**, 064314 (2020), doi:10.1103/PhysRevC.102.064314.

[31] P. Singh et al., *Evidence for coexisting shapes through lifetime measurements in ^{98}Zr*, Phys. Rev. Lett. **121**, 192501 (2018), doi:10.1103/PhysRevLett.121.192501.

Study of entanglement in symmetric multi-quDits systems through information diagrams

Julio Guerrero[1,2⋆], Antonio Sojo[1], Alberto Mayorgas[3] and Manuel Calixto[2,3]

1 Department of Mathematics, University of Jaén,
Campus Las Lagunillas s/n, 23071 Jaén, Spain
2 Institute Carlos I of Theoretical and Computational Physics (iC1),
University of Granada, Fuentenueva s/n, 18071 Granada, Spain
3 Department of Applied Mathematics, University of Granada,
Fuentenueva s/n, 18071 Granada, Spain

⋆ jguerrer@ujaen.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Along this paper, we analyze the entanglement properties of symmetric multi-quDits systems in a special type of states created from the generalization to $U(D)$ of the usual spin coherent states. By means of parity operators, we define what we call multicomponent Schrödinger cat states as parity adapted coherent states. Introducing the tool of information diagrams, i.e. representations of pairs of entropy measures, we analyze the correlation structure of this type of states and their M-wise reduced density matrices.

Received 18-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.029

1 Introduction

Information diagrams, simple representations based on a pair of entropic measures, are a helpful tool that can be used to analyze the correlation structure and mixture of a given quantum density matrix, and consequently the entanglement when applied to M-wise reduced density matrices.

On previous papers [1], we applied them to characterize the entanglement of parity adapted $U(D)$-spin coherent states (CSs) or DCATs. This work extends this analysis, computing information diagrams for the generalization to different parities of the purely even DCAT already studied.

To achieve this, we first review the concept of information diagrams in Section 2, then we define the generalization of DCAT using parity operators in Section 3, and eventually we combine both concepts to compute the information diagrams of the M-wise reduced density matrix (RDM) of these kind of states in Section 4.

2 Information diagrams and entropic measures

First, we will review the concept of information diagrams and its main properties. We will limit our scope since most details have already been given in [1] (and references therein).

Given a valid probability density function (PDF), we can compute the value of two different measures of information or entropy and plot a $2D$ point whose coordinates are those values. Information diagrams are the structure generated by this type of entropic representations when all valid PDF (or a subset thereof) are represented.

This idea can be generalized to quantum density matrices if we recall that their eigenvalues can be interpreted as probabilities, thus allowing the definition of a discrete PDF (for finite Hilbert spaces) that we can use to build up the information diagrams. In addition, if we select the two entropy measures as the normalized von Neumann $\mathcal{S}$ and linear entropies $\mathcal{L}$, we can directly compute them for a given density matrix ρ as:

$$\mathcal{S}(\rho) = -\mathrm{Tr}\,\rho \log_d \rho\,, \qquad \mathcal{L}(\rho) = \frac{d}{d-1}\left[1 - \mathrm{Tr}\,\rho^2\right], \tag{1}$$

where d is the dimension of the Hilbert space associated with our quantum system.

As defined, these entropic measures are normalized, i.e., their value is 0 for pure states and 1 for maximally mixed states, and therefore we can ensure that the set of points $(\mathcal{L}(\rho), \mathcal{S}(\rho))$ generated by all valid density matrices ρ is a bounded set; however, it does not completely fill the entire unit square. The boundaries can be computed by means of variational methods as the search of curves with maximal or minimal von Neunman entropy given a fixed value of the linear entropy and a density matrix rank k. This allows the subdivion of the set into $d-2$ rank-dependent subregions. This makes possible the classification of density matrices within each subregion by the minimum rank they have. Details about the explicit meaning of these curves and their parametrization can be found on [1].

In Figure 1, we plotted the basic structure of an information diagram for quantum density matrices with $d = 5$. The global boundaries (in black) are the global extremal curves that enclose the entire set of allowed local points, within there are extremal curves (in grey) that give information about matrix ranks. The bottom-most curve represent all density matrices with rank $k = 2$, while each extremal curve moving upwards for increasingly higher ranks $k = 3, ..., d$ set the minimum rank allowed on the whole area above it. Matrices with rank $k = 1$, i.e. pure states, are all located at the origin. It is also interesting to note that a density matrix located at the intersection point of the k-extremal curve with the global boundaries has k identical eigenvalues and $d - k$ zeros as it has maximal von Neumann entropy for its rank.

This rank based structure allows us to obtain an intuition about the level of entanglement and mixture of a family of density matrices. It is well-known [2] that density matrices which do not lie at the origin nor to the right of the NEMSs (Not Entangled Mixed States) Linear entropy[1] threshold (in dashed gray), all present entanglement. However, information diagrams do not give direct information about the absolute level of entanglement of a given density matrix.[2]

With all of this in mind, we will use information diagrams of RDMs to study the entanglement of what we call parity adapted coherent states of SU(D) or DCATS, a generalization of "Schrödinger cat states". In this case, information diagrams do provide useful information about entanglement and, in particular, the rank of the RDM is an entanglement monotone [3].

[1] A similar behaviour can be seen in the von Neumann Entropy.

[2] But if the system is divided in two parts and one of the parts is traced out, the location in the information diagram of the resulting RDM provides a good indicator of the level of entanglement of the original density matrix.

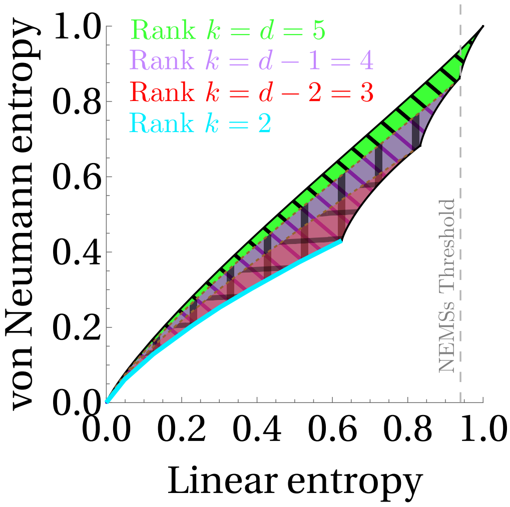


Figure 1: Information diagram for $d = 5$. Each region overlap with the ones below it

3 Parity adapted U(D) CSs in symmetric multi-quDit systems

We shall introduce here the definition of parity adapted $U(D)$-spin coherent states (DCATS) in symmetric multi-quDit system. Before that, establishing the required mathematical tools is necessary.

We consider a system of N identical indistinguishable particles, each of which has D possible states or levels, e.g., N D-level identical atoms. Thus, we can define the creation (annihilation) operator for each level: $\{a_i^\dagger\}_{i=0}^{D-1}$ $(\{a_i\}_{i=0}^{D-1})$. Note that we denote the ground level as $i = 0$. As usual, these operators create (destroy) a particle on the i-th level $|i\rangle$.

In its fully symmetric representation, the collective $U(D)$-spin operators can be expressed as bilinear products of creation and annihilation operators, that is $S_{ij} = a_i^\dagger a_j$, $0 \le i, j \le D-1$ (Schwinger representation).

Note that the diagonal operators S_{ii} correspond to the number operator of the i-th level, while off-diagonal operators S_{ij} $(i \neq j)$ are tunneling operators that move a particle from the j-th level $|j\rangle$ to the i-th level $|i\rangle$.

As this is the fully symmetric representation of $U(D)$, the associated space can be embedded into the Fock space $\mathcal{H}_F^{(N)}$ of dimension $d = \binom{N+D-1}{N}$, with a Bose-Einstein-Fock basis $\left\{|\vec{n}\rangle \in \mathcal{H}_F^{(N)} \,\middle|\, \|\vec{n}\|_1 = N\,,\, \langle \vec{n}|\vec{n}'\rangle = \delta_{\vec{n},\vec{n}'}\right\}$. Within this space, we pay special attention to the $U(D)$-spin coherent states (or DSCSs for short), which can be expressed as a multinomial form (see [1]) in terms of the $\{a_i^\dagger\}_{i=0}^{D-1}$ operators acting on the Fock vacuum.

However, for the sake of simplicity, we will use a more elementary (although equivalent) construction for DSCSs. Firstly, we define the one-particle state: $|\mathbf{z}\rangle^{(1)} = \frac{1}{\sqrt{1+|\mathbf{z}|^2}}\left[|0\rangle + \sum_{i=1}^{D-1} z_i\,|i\rangle\right]$, labeled with complex points $\mathbf{z} = (z_1, \ldots, z_{D-1}) \in \mathbb{C}^{D-1}$ without the coefficient $z_0 = 1$ (this election just represents the explicit choice of an specific local chart on the complex projective manifold defined by the normalized quantum states). The norm $|\mathbf{z}|$ is defined in terms of the scalar product $\mathbf{z}' \cdot \mathbf{z} = \sum_{i=1}^{D-1} \bar{z}'_i z_i$. The N particles $U(D)$-spin coherent states are simply defined as:

$$|\mathbf{z}\rangle^{(N)} = \bigotimes_{i=1}^{N} |\mathbf{z}\rangle_i^{(1)}\,, \tag{2}$$

where the superscript denotes number of particles, and the subscript represents the tag of each particle own space. It is obvious they are symmetric and that they do not present any entanglement as they are separable (tensor product states or TPS).

In addition, it is important to note that, in general, DSCSs are not orthogonal since their overlap is given by ${}^{(N)}\langle \mathbf{z}'|\mathbf{z}\rangle^{(N)} = \left[\frac{1+\mathbf{z}'\cdot\mathbf{z}}{\sqrt{(1+|\mathbf{z}'|^2)(1+|\mathbf{z}|^2)}}\right]^N$. This means that, even though they do not form a basis, they are an overcomplete continuous set of states that spans the entire space $\mathcal{H}_F^{(N)}$, i.e., they form a frame [4].

These non-entangled states can be combined to form highly entangled states by means of parity operators. These parity operators can be defined as $\Pi_j = \exp\{i\pi S_{jj}\}$, $j = 0, \ldots, D-1$ with $i^2 = -1$ being the imaginary unit. They measure the parity of the number of particles in the j-th level of a specific state. Note that $\Pi_i = \Pi_i^{-1}$. These operators generate the parity group of symmetric quDits systems $\mathbb{Z}_2 \times \overset{D}{\ldots} \times \mathbb{Z}_2 = \mathbb{Z}_2^D$. However, as the number of particles N is fixed, $\Pi_0 \cdot \ldots \cdot \Pi_{D-1} = (-1)^N$ impose a constrain on the total parity $(-1)^N$, allowing us to discard one copy of $\mathbb{Z}_2$ and its corresponding parity operator. In particular, we select the one corresponding to the ground level (in correspondence with the choice $z_0 = 1$).

Parity operators act upon CSs such that they change the sign of the corresponding $\mathbf{z}$ components,

$$\Pi_i |\mathbf{z}\rangle^{(N)} = \Pi_i |z_1, \ldots., z_i, \ldots., z_{D-1}\rangle^{(N)} = |z_1, \ldots., -z_i, \ldots., z_{D-1}\rangle^{(N)} . \quad (3)$$

We define the following parity operators: $\Pi_j^{b_j} = \exp\{i\pi b_j S_{jj}\}$ and $\Pi^{\mathbb{b}} = \Pi_1^{b_1}\Pi_2^{b_2} \cdot \ldots \cdot \Pi_{D-1}^{b_{D-1}}$, where $\mathbb{b} = (b_1, b_2, \ldots, b_{D-1}) \in \{0,1\}^{D-1}$ is a binary string of length $D-1$ that labels elements of the parity group $\mathbb{Z}_2^{D-1}$. They allow us to define the definite parity subspace projectors as the Fourier transform in the $\mathbb{Z}_2^{D-2}$ group,

$$\Pi_{\mathbb{c}} = 2^{1-D} \sum_{\mathbb{b}\in\{0,1\}^{D-1}} (-1)^{\mathbb{b}\cdot\mathbb{c}} \Pi^{\mathbb{b}}, \quad (4)$$

where $\mathbb{c} = [c_1, c_2, \ldots, c_{D-1}] \in \{0,1\}^{D-1}$ is the parity of the subspace that this projector projects to. They can be used to project states to a defined parity $\mathbb{c}$ subspace, thus allowing us to define the $\mathbb{c}$-parity adapted CSs states or $\mathbb{c}$-parity DCAT as:

$$|\text{DCAT}_{\mathbb{c}}(\mathbf{z})\rangle^{(N)} = \frac{1}{\mathcal{N}_{\mathbb{c}}^{(N)}(z)} \Pi_{\mathbb{c}} |\mathbf{z}\rangle^{(N)} = \frac{2^{1-D}}{\mathcal{N}_{\mathbb{c}}(\mathbf{z})} \sum_{\mathbb{b}\in\{0,1\}^{D-1}} (-1)^{\mathbb{b}\cdot\mathbb{c}} \left|\mathbf{z}^{\mathbb{b}}\right\rangle^{(N)}, \quad (5)$$

where $\mathbf{z}^{\mathbb{b}} := ((-1)^{b_1} z_1, \ldots., (-1)^{b_i} z_i, \ldots., (-1)^{b_{D-1}} z_{D-1})$ and

$$\left[\mathcal{N}_{\mathbb{c}}^{(N)}(\mathbf{z})\right]^2 = 2^{1-D} \sum_{\mathbb{b}\in\{0,1\}^{D-1}} (-1)^{\mathbb{b}\cdot\mathbb{c}} \frac{[1+\mathbf{z}^{\mathbb{b}}\cdot\mathbf{z}]^N}{(1+|\mathbf{z}|^2)^N}, \quad (6)$$

is a normalization factor or the norm of the unnormalized DCAT.

As an example, we provide the explicit expression of the $\mathbb{c}$-parity DCAT state and $\mathcal{N}_{\mathbb{c}}(\mathbf{z})$ for $D = 2, 3$ (qubits and qutrits).

For $D = 2$, there are only two possible levels, and one component of $\mathbf{z}$ and parities; thus $\mathbf{z} = \alpha$ and $\mathbb{c} = c$. Any one-particle coherent state $|\mathbf{z}\rangle^{(1)} := |\alpha\rangle^{(1)}$ is written as $|\alpha\rangle^{(1)} = \frac{|0\rangle + \alpha|1\rangle}{\sqrt{1+|\alpha|^2}}$, and the corresponding N-particle 2CAT_c is expressed as

$$|2\text{CAT}_c(\alpha)\rangle^{(N)} = \frac{1}{2\mathcal{N}_c^{(N)}(\alpha)} \left[|\alpha\rangle^{(N)} + (-1)^c |-\alpha\rangle^{(N)}\right], \quad (7)$$

with:

$$\left[\mathcal{N}_c^{(N)}(\alpha)\right]^2 = \frac{1}{2} \frac{(1+|\alpha|^2)^N + (-1)^c (1-|\alpha|^2)^N}{(1+|\alpha|^2)^N} = \frac{1}{2}\left[1 + (-1)^c \left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^N\right]. \quad (8)$$

It is easy to check that $\Pi^b |2\text{CAT}_c(\alpha)\rangle^{(N)} = (-1)^{c\cdot b} |2\text{CAT}_c(\alpha)\rangle^{(N)}$ as it corresponds to its parity. It is also important to note that, for $c = 1$, the $2\text{CAT}_1(0)$ can only be defined as the limit $\alpha \to 0$ of the normalized DCAT, since the normalization factor tends to zero as the unnormalized state itself also does.

For $D = 3$, there are three possible levels and two component of $\mathbf{z}$ and parities; thus $\mathbf{z} = (\alpha, \beta)$ and $\mathbb{c} = [c_1, c_2]$. A one-particle coherent state $|\mathbf{z}\rangle^{(1)} := |\alpha,\beta\rangle^{(1)}$ is written as $|\alpha,\beta\rangle^{(1)} = \frac{|0\rangle+\alpha|1\rangle+\beta|1\rangle}{\sqrt{1+|\alpha|^2+|\beta|^2}}$, with the corresponding N-particle $3\text{CAT}_\mathbb{c}$ expressed as:

$$
\begin{aligned}
|3\text{CAT}_\mathbb{c}(\alpha,\beta)\rangle^{(N)} = \frac{1}{4\mathcal{N}_\mathbb{c}^{(N)}(\alpha,\beta)}\Big[\, &|\alpha,\beta\rangle^{(N)} + (-1)^{c_1}|-\alpha,\beta\rangle^{(N)} \\
&+ (-1)^{c_2}|\alpha,-\beta\rangle^{(N)} + (-1)^{c_1+c_2}|-\alpha,-\beta\rangle^{(N)}\,\Big],
\end{aligned} \tag{9}
$$

with:

$$
\begin{aligned}
\left[\mathcal{N}_\mathbb{c}^{(N)}(\alpha,\beta)\right]^2 = \frac{1}{4} &+ \frac{(-1)^{c_1}(1-|\alpha|^2+|\beta|^2)^N + (-1)^{c_2}(1+|\alpha|^2-|\beta|^2)^N}{4(1+|\alpha|^2+|\beta|^2)^N} \\
&+ \frac{(-1)^{c_1+c_2}(1-|\alpha|^2-|\beta|^2)^N}{4(1+|\alpha|^2+|\beta|^2)^N}\,.
\end{aligned} \tag{10}
$$

It is easy to check that $\Pi^{\mathbb{b}} |3\text{CAT}_\mathbb{c}(\alpha,\beta)\rangle^{(N)} = (-1)^{\mathbb{c}\cdot\mathbb{b}} |3\text{CAT}_\mathbb{c}(\alpha,\beta)\rangle^{(N)}$ as it corresponds to the partial parity of the given pair.

These expressions will be used in Section 4 to compute the eingenvalues of the corresponding M-wise RDM $\rho_\mathbb{c}^{(M)}(\mathbf{z}) = \mathrm{Tr}_{(N-M)}[|\text{DCAT}_\mathbb{c}(\mathbf{z})\rangle\langle\text{DCAT}_\mathbb{c}(\mathbf{z})|]$, which allows us to analyze the entanglement structure of the DCATs using information diagrams.

4 Entropic measures of reduced density matrices

As we have seen in Section (2), we can compute von Neumann and linear entropies associated to any quantum state using the eingenvalues of the corresponding density matrix operator.

This allow us to compute the entropies associated to the M-wise RDM of a DCAT state (i.e. partial tracing $N-M$ out of the N particles). Using the explicit expression of DCATs in terms of the Fock basis [1], one can easily compute the RDM taking partial trace. This RDM can be diagonalized and then its entropy computed [6].

For the sake of simplicity, we will limit ourselves, again, to $D = 2, 3$. As it is not relevant here, we omit the diagonalization and directly provide the eingenvalues of the RDM. We consider the partial trace of $N-M > N/2$ particles, leaving $M \leq N/2$ on our state (cases where $M > N/2$ are symmetric to the case $M \leq N/2$ by interchanging $M \to N-M$). This means that the M-wise RDM is associated with the Fock space $\mathcal{H}_F^{(M)} \neq \mathcal{H}_F^{(N)}$ with dimension $d_M = \binom{M+D-1}{M} < d_N = \binom{N+D-1}{N}$.

For $D = 2$, we provide the eingenvalues of the RDM as a vector, and we omit the expression of eigenvectors, which are not relevant here,

$$
\rho_c^{(M)}(\alpha) = \left(\frac{\left[L_+^{N-M} + (-1)^c L_-^{N-M}\right]\left[L_+^M + L_-^M\right]}{2L_+^N\left[\mathcal{N}_c^{(N)}\right]^2}, \frac{\left[L_+^{N-M} - (-1)^c L_-^{N-M}\right]\left[L_+^M - L_-^M\right]}{2L_+^N\left[\mathcal{N}_c^{(N)}\right]^2}, \vec{0} \right), \tag{11}
$$

where $L_\pm := 1 \pm |\alpha|^2$ and $\vec{0}$ is a vector that pads the diagonal with zeros until the required dimension is reached. In general, we only have two non-zero eingenvalues.

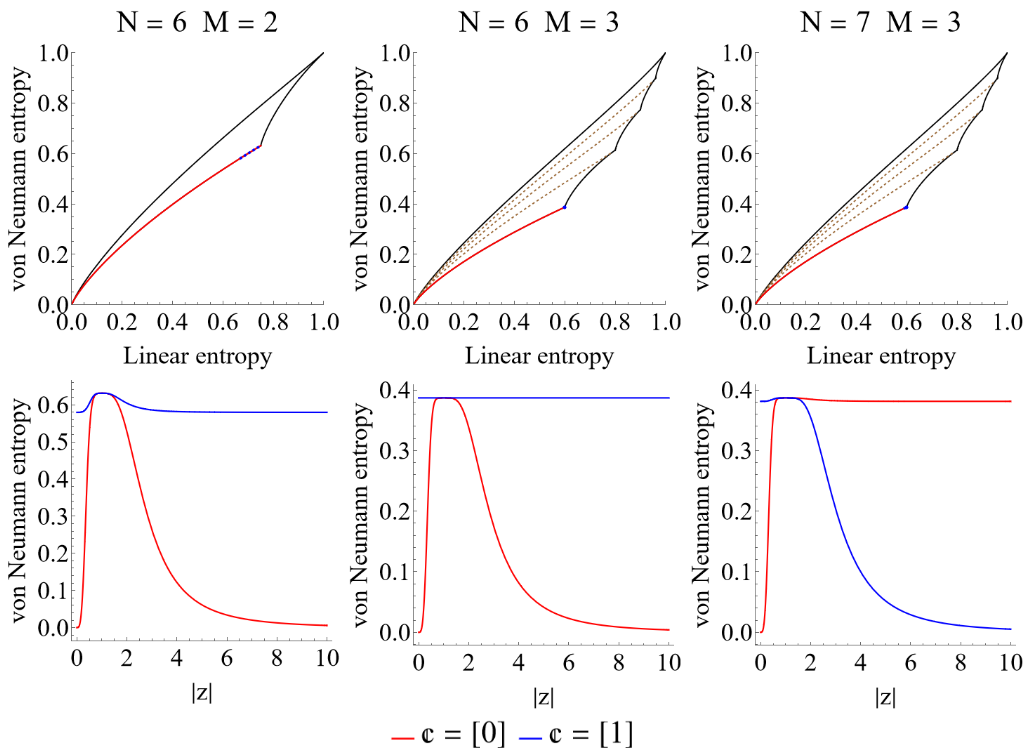


Figure 2: Information diagrams (upper line) for $D = 2$, $N = 6,7$ and $M = 2,3$, and the corresponding plots (bottom line) of the von Neumann entropy as a function of the parameter $|z|$.

In Figure (2), we plotted the von Neumann entropy as a function of the absolute value of the complex parameter z as well as the associated information diagrams for three pairs of (N, M) and both possible parities $\mathbb{c} = [0], [1]$.

At $z = 0$, the $\mathbb{c} = [0]$ RDM presents no entropy, while $\mathbb{c} = [1]$ starts from a high value of the entropy. As $|z|$ increases, so does the entropy for both cases and any pair (N, M) of particles until the peak value is reached at $|z| = 1$. At this point, it can be shown that any derivative of order $< M$ vanishes. This means that the higher M is, the flatter the peak will be. From this point, the entropy decreases for both parities with one of them consistently over the other. For even N, $\mathbb{c} = [1]$ remains at the top while this behaviour reverses if N is odd.

Using the previous information, we can give an interpretation to each corresponding information diagram. For $D = 2$, the maximum value of the rank is $k = 2$ for all $M > 1$ since all RDM lie on the bottom-most curve. However, $k = 1$ is reached for RDM with no entropy and, therefore, no entanglement (pure states). For even N, only $\mathbb{c} = [0]$ covers the entire possible entropy range (entropy is normalized to dimension $d_M = M$ for $D = 2$), while $\mathbb{c} = 1$ only spans a narrow interval of possible values. On the other hand, for odd N,the entire range is available for both parities.

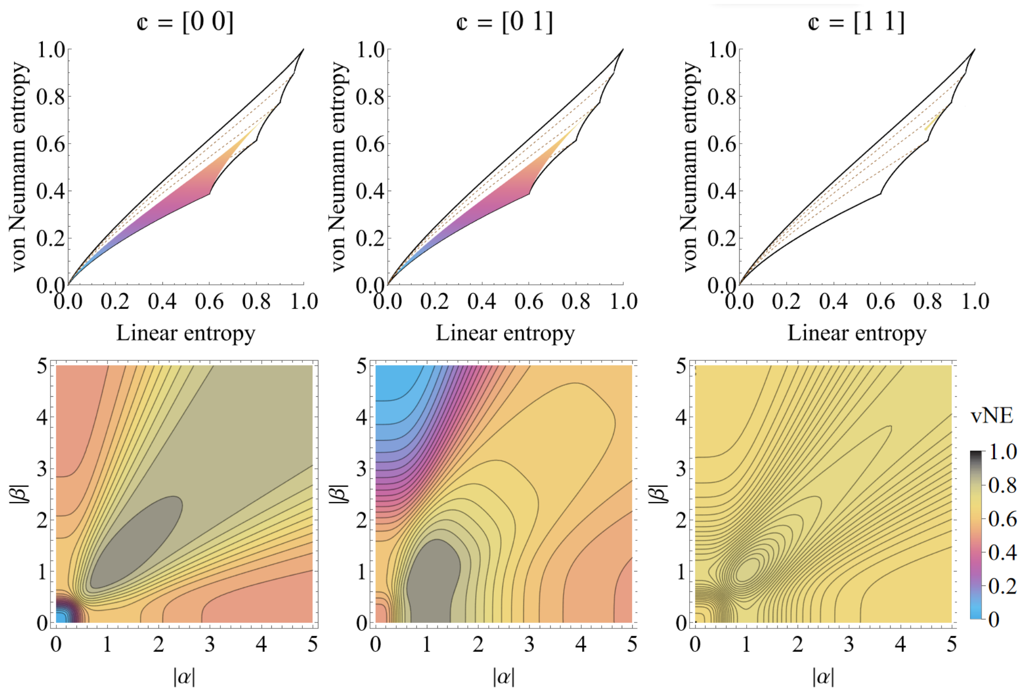


Figure 3: Information diagrams (upper row) for $D = 3$, $N = 7$, $M = 2$, and all relevant parities, and the corresponding contour plots (bottom row) representation of the von Neumann entropy as a function of the parameters $(|\alpha|, |\beta|)$

For $D = 3$, we have:

$$\begin{aligned}\rho_{\mathbb{c}}^{(M)}(\alpha,\beta) = &\frac{1}{4L_{++}^{N}\,[\mathcal{N}_{\mathbb{c}}(\alpha,\beta)]^2}\\ \times\Big(&\big[L_{++}^{N-M}+(-1)^{c_1}L_{-+}^{N-M}+(-1)^{c_2}L_{+-}^{N-M}+(-1)^{c_1+c_2}L_{--}^{N-M}\big]\big[L_{++}^{M}+L_{-+}^{M}+L_{+-}^{M}+L_{--}^{M}\big],\\ &\big[L_{++}^{N-M}-(-1)^{c_1}L_{-+}^{N-M}+(-1)^{c_2}L_{+-}^{N-M}-(-1)^{c_1+c_2}L_{--}^{N-M}\big]\big[L_{++}^{M}-L_{-+}^{M}+L_{+-}^{M}-L_{--}^{M}\big],\\ &\big[L_{++}^{N-M}+(-1)^{c_1}L_{-+}^{N-M}-(-1)^{c_2}L_{+-}^{N-M}-(-1)^{c_1+c_2}L_{--}^{N-M}\big]\big[L_{++}^{M}+L_{-+}^{M}-L_{+-}^{M}-L_{--}^{M}\big],\\ &\big[L_{++}^{N-M}-(-1)^{c_1}L_{-+}^{N-M}-(-1)^{c_2}L_{+-}^{N-M}+(-1)^{c_1+c_2}L_{--}^{N-M}\big]\big[L_{++}^{M}-L_{-+}^{M}-L_{+-}^{M}+L_{--}^{M}\big],\vec{0}\Big),\end{aligned} \tag{12}$$

where $L_{\sigma_1\sigma_2} := 1+\sigma_1|\alpha|^2+\sigma_2|\beta|^2$ with $\sigma_i \in \{-1,+1\}$.

Now, in Figure 3, we show a contour plot of the von Neumann entropy as a function of the point $(|\alpha|, |\beta|)$, as well as the corresponding information diagram for fixed $N = 7$ and $M = 2$ and all relevant parities (the $\mathbb{c} = [1, 0]$ case is symmetric to $\mathbb{c} = [0, 1]$ with the interchange $\alpha \longleftrightarrow \beta$). The color scale represent the von Neumann entropy in both contour plots and information diagrams.

For $D = 3$, the behaviour is similar to $D = 2$ but the number of possible cases increases exponentially. For even N, the higher the number of 1's a parity has, the higher the associated average entropy is (higher on the information diagram), and the narrower the range is occupied. For odd N and all parities except $\mathbb{c} = [1, 1]$, the entire possible entropy range is occupied taking into account the maximum rank available. Each one of the two axis $(|\alpha|, |\beta|)$ has an intimate relation with its corresponding parity component. If the parity components are not equal, an asymmetry between both axis is created as not all directions are equivalent, decreasing the entropy at points near the axises with corresponding non-zero parity component.

Again, all the properties can be directly visualized on the information diagrams in Figure

3. Now, the possible ranks scale from $k = 1$ to $k = 4$ for any M, and completely even parities occupy higher areas (with non null measure) on the diagrams.

5 Conclusion

Parity adapted coherent states or DCATS can be physically produced on Kerr materials [5] or as products of Bose-Einstein condensates and have applications on models such as the Lipkin-Meshkov-Glick model [1]. Therefore the entanglement structure of this kind of states is of high interest.

This work is limited in scope but in [6] the ideas seen here are further developed, generalizing the decomposition of the RDM of DCATS into their eigenvectors using Group theoretical methods and more sophisticated interpretations.

Funding information We thank the support of the Spanish MICINN through the projects PGC2018-097831-B-I00 and PID2022-138144NB-I00, and Junta de Andalucía through the projects UHU-1262561, FQM-381 and FEDER/UJA-1381026. AM thanks the Spanish MIU for the FPU19/06376 predoctoral fellowship. AS thanks University of Jaén for a contract under the project FEDER/UJA-1381026.

References

[1] J. Guerrero, A. Mayorgas and M. Calixto, *Information diagrams in the study of entanglement in symmetric multi-quDit systems and applications to quantum phase transitions in Lipkin-Meshkov-Glick D-level atom models*, Quantum Inf. Process. **21**, 223 (2022), doi:10.1007/s11128-022-03524-7.

[2] K. Życzkowski, P. Horodecki, A. Sanpera and M. Lewenstein, *Volume of the set of separable states*, Phys. Rev. A **58**, 883 (1998), doi:10.1103/physreva.58.883.

[3] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009), doi:10.1103/RevModPhys.81.865.

[4] A. Vourdas, *Analytic representations in quantum mechanics*, J. Phys. A: Math. Gen. **39**, R65 (2006), doi:10.1088/0305-4470/39/7/R01.

[5] S. Arjika, M. Calixto and J. Guerrero, *Quantum statistical properties of multiphoton hypergeometric coherent states and the discrete circle representation*, J. Math. Phys. **60**, 103506 (2019), doi:10.1063/1.5099683.

[6] J. Guerrero, A. Sojo, A. Mayorgas and M. Calixto, *Schmidt decomposition of parity adapted coherent states for symmetric multi-quDits*, J. Phys. A: Math. Theor. **56**, 355304 (2023), doi:10.1088/1751-8121/aceae0.

Hyperquaternions and physics

Patrick R. Girard[1⋆], Romaric Pujol[2], Patrick Clarysse[1] and Philippe Delachartre[1]

1 CREATIS, Université de Lyon, CNRS UMR5520, INSERM U1294, INSA-Lyon, France
2 Pôle de Mathématiques, INSA-Lyon, France

⋆ patrick.girard@creatis.insa-lyon.fr

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The paper develops, within a new representation of Clifford algebras in terms of tensor products of quaternions called hyperquaternions, several applications. The first application is a quaternion 2D representation in contradistinction to the frequently used 3D one. The second one is a new representation of the conformal group in (1+2) space (signature $+--$) within the Dirac algebra $C_5(2,3) \simeq \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{H}$ subalgebra of $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$. A numerical example and a canonical decomposition into simple planes are given. The third application is a classification of all hyperquaternion algebras into four types, providing the general formulas of the signatures and relating them to the symmetry groups of physics.

Received 18-11-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.030

Contents

1 Introduction **2**

2 Preliminaries: Clifford algebras and hyperquaternions **2**
2.1 Quaternions and biquaternions 2
2.2 Clifford algebras and hyperquaternions 2

3 Quaternion 2D representation **3**

4 Hyperquaternionic conformal group in (1+2) space **3**
4.1 Algebraic structure 3
4.2 Restricted conformal group 4
4.3 Numerical example 4

5 Classification of hyperquaternion algebras **5**

6 Conclusion **7**

A Multivector structure of $C_5(2,3)$ **8**

References **8**

1 Introduction

A new hyperquaternionic representation of Clifford algebras [1, 2] has been introduced recently [3–7], hyperquaternion algebras being defined as tensor products of quaternion algebras (or subalgebra thereof). This paper develops several hyperquaternionic applications.

Throughout this paper, $\mathbb{H}^{\otimes m}$ will denote the tensor product of m quaternion algebras, i.e. $\mathbb{H}^{\otimes m} = \mathbb{H} \otimes \mathbb{H} \otimes \cdots \otimes \mathbb{H}$ (m terms). The structure of the paper is as follows.

In the preliminaries, the historical origins and basic concepts of hyperquaternion Clifford algebras are examined. In the third section, a quaternion $2D$ representation is proposed in contradistinction to a widely used $3D$ representation. In the fourth section, the conformal group of the $(1+2)$ space (signature $+--$) is developed within the Dirac algebra $C_5(2,3) \simeq \mathbb{C} \otimes \mathbb{H}^{\otimes 2}$ considered as a subalgebra of $\mathbb{H}^{\otimes 3} \simeq C_6(2,4)$. The choice of the $(1+2)$ space is motivated by its use in quantum gravity [8]. A numerical example together with a canonical decomposition into simple planes is provided. Finally, the fifth section gives a classification of all hyperquaternion algebras into four types with general formulas of the signatures and associated symmetry groups of physics.

2 Preliminaries: Clifford algebras and hyperquaternions

2.1 Quaternions and biquaternions

The quaternion group [9, 10] was discovered in 1843 by W. R. Hamilton and is constituted by the elements $(\pm 1, \pm i, \pm j, \pm k)$ satisfying the formula

$$i^2 = j^2 = k^2 = ijk = -1\,. \tag{1}$$

The quaternion algebra $\mathbb{H}$ is defined as a set of four real numbers q_i, called quaternions $q = q_0 + q_1 i + q_2 j + q_3 k$. The conjugate q_c of q is defined by $q_c = q_0 - q_1 i - q_2 j - q_3 k$. Hamilton was to give a $3D$ (if not $4D$) interpretation of quaternions which was to lead to the classical vector calculus still in use today. He also introduced complex quaternions which he named biquaternions.

2.2 Clifford algebras and hyperquaternions

Clifford in 1878, introduced his algebras as tensor products of quaternion algebras [11]. He proved the following theorem

$$C_{2m} \simeq \mathbb{H}^{\otimes m}\,, \quad C_{2m-1} \simeq \mathbb{C} \otimes \mathbb{H}^{\otimes m-1}\,. \tag{2}$$

Lipschitz in 1880, derived the rotation formula of nD Euclidean spaces [12]

$$x' = axa^{-1}\,, \quad a \in C^+\,. \tag{3}$$

He thereby rediscovered the (even) Clifford algebras. In 1922, Moore [13] was to call Lipschitz's algebras: hyperquaternions, a term which we shall extend to all Clifford algebras. A major success of Clifford algebras in physics was the Dirac algebra and the spinor calculus. Recent developments in Clifford algebras seem to have somewhat neglected if not totally ignored the hyperquaternionic filiation.

In terms of generators, the Clifford algebra $C_n(p,q)$ has $n = p + q$ generators e_i such that $e_i e_j + e_j e_i = 0$ $(i \neq j)$, $e_i^2 = +1$ (p generators) and $e_i^2 = -1$ (q generators). The total number of elements is 2^n. The algebra contains scalars (S), vectors (V) e_i, bivectors (B) $e_i e_j$ $(i \neq j)$, etc.

C^+ is the (even) subalgebra constituted by products of an even number of e_i. It is to be noticed that the hyperquaternion product is independent of the choice of the generators whereas the multivector structure depends on it.

Examples of hyperquaternion Clifford algebras are: quaternions $\mathbb{H}$ ($e_1 = i, e_2 = j$), biquaternions $\mathbb{C} \otimes \mathbb{H}$ ($e_1 = iI, e_2 = jI, e_3 = kI, I = 1 \otimes i$), $\mathbb{H} \otimes \mathbb{H}$ ($e_0 = j, e_1 = kI, e_2 = kJ, e_3 = kK$) with $(I, J, K) = 1 \otimes (i, j, k)$.

3 Quaternion 2D representation

In contradistinction to Hamilton who gave a $3D$ interpretation of quaternions which is still widely used today, we shall provide a $2D$ plane representation below since quaternions constitute a Clifford algebra with only two generators

$$e_1 = i\,, \quad e_2 = j\,, \quad e_1 e_2 = k \quad \left(e_1^2 = e_2^2 = -1\right). \tag{4}$$

Interior and exterior products can be defined with $x = x_1 i + x_2 j \in V$, $B = bk$ bivector ($b \in \mathbb{R}$) by

$$x.y = -(xy + yx)/2 = x_1 y_1 + x_2 y_2 \in S\,, \tag{5}$$
$$x \wedge y = (xy - yx)/2 = (x_1 y_2 - x_2 y_1)\, k \in B\,, \tag{6}$$
$$x.B = -(xB - Bx)/2 = b\,(-x_2 i + x_1 j) \in V\,, \tag{7}$$
$$x \wedge B = (xB + Bx)/2 = 0\,. \tag{8}$$

The rotation group $SO(2)$ is expressed by

$$x' = r x r_c = (x_1 \cos\theta - x_2 \sin\theta)\, i + (x_1 \sin\theta + x_2 \cos\theta)\, j\,, \tag{9}$$

with

$$r = e^{k\theta/2} = (\cos\theta/2 + k \sin\theta/2) \in B\,. \tag{10}$$

The modeling of an Euclidean $3D$ space can be realized similarly with biquaternions [14].

4 Hyperquaternionic conformal group in (1+2) space

The conformal group of the $(1+3)$ space has been examined within the algebra $\mathbb{H}^{\otimes 3} \simeq C_6(2,4)$ in [4]. Here, we consider the $(1+2)$ subspace within the subalgebra $C_5(2,3) \simeq \mathbb{C} \otimes \mathbb{H}^{\otimes 2} \simeq \mathbb{C}(4)$ isomorphic to the Dirac algebra. This space has received much attention in particular with respect to quantum gravity [8]. We first introduce the algebraic structure, then the restricted conformal group and a numerical example including a canonical decomposition into simple planes.

4.1 Algebraic structure

As generators of the subalgebra $C_5(2,3) \simeq \mathbb{C} \otimes \mathbb{H}^{\otimes 2}$, we take

$$e_a = kI\,, \quad e_0 = kJ\,, \quad e_1 = kKl\,, \quad e_2 = kKm\,, \quad e_b = j\,, \tag{11}$$

with

$$\mathbb{H}^{\otimes 3} = (i, j, k) \otimes (I, J, K) \otimes (l, m, n)\,, \tag{12}$$

and $(l,m,n) = 1 \otimes 1 \otimes (i,j,k)$. A general element A of $\mathbb{H}^{\otimes 3}$ can be viewed as a set of 16 quaternions $[q_i] = a_i + b_i l + c_i m + d_i n$

$$A = [q_1] + I[q_2] + J[q_3] + K[q_4] + i[q_5] + iI[q_6] + iJ[q_7] + iK[q_8] + j[q_9] + jI[q_{10}] + jJ[q_{11}] + jK[q_{12}] + k[q_{13}] + kI[q_{14}] + kJ[q_{15}] + kK[q_{16}]\,. \tag{13}$$

The explicit multivector structure of $C_5(2,3)$ is given in Appendix [A]. The algebra has $2^5 = 32$ elements with 10 parameters for the bivectors. The product is implemented in http://www.notebookarchive.org/2021-08-6z1zbda/.

4.2 Restricted conformal group

The restricted conformal group in $(1+2)$ space is obtained via the procedure described in [1].

First, one constructs an affine space within $C_5(2,3)$. Let X be a five dimensional vector

$$X = \frac{(x^2-1)}{2}e_a + x + \frac{(x^2+1)}{2}e_b = x^2\varepsilon_1 + x + \varepsilon_2\,, \tag{14}$$

with $x = x_0e_0 + x_1e_1 + x_2e_2 \in E_3$, $\quad X^2 = 0$ and

$$\varepsilon_1 = \frac{e_a + e_b}{2}\,, \quad \varepsilon_2 = \frac{e_b - e_a}{2}\,, \quad \varepsilon_1^2 = \varepsilon_2^2 = 0\,. \tag{15}$$

The restricted conformal group is then expressed by the transformations

$$X' = aXa_c \quad (aa_c = 1\,, \quad a \in C_5^+(2,3))\,. \tag{16}$$

They are composed of

- spatial rotations $a = e^{n\frac{\theta}{2}}$,
- boosts $a = e^{B\frac{\theta}{2}}$, $\quad B \in (Il, Im)$,
- translations $a = e^{\varepsilon_1 u} = 1 + \varepsilon_1 u \quad (u \in E_3)$,
- transversions $a = e^{\varepsilon_2 v} = 1 + \varepsilon_2 v \quad (v \in E_3)$,
- dilations $a = e^{e_a e_b \frac{\varphi}{2}} = e^{-iI\frac{\varphi}{2}} = \cosh\frac{\varphi}{2} - iI\sinh\frac{\varphi}{2}$.

The total number of parameters is $\frac{(n+2)(n+1)}{2} = 10$ $(n=3)$. Through combinations, one obtains the general transformations

$$X' = fXf_c \quad (ff_c = 1\,, \quad f \in C_5^+(2,3))\,. \tag{17}$$

The Lie algebra is given in [4]

4.3 Numerical example

Here, we present a numerical example consisting of a set of transformations together with a canonical decomposition thereof.

As transformation $X' = fXf_c$ we shall consider a dilation ($e^{-\varphi} = 1/3$) followed by a unit translation ($u = e_1$) and a rotation ($\theta = \pi/2$ in the plane $e_{12} = n$). The combination of these transforms yields the hyperquaternion $f \in C^+$

$$
\begin{aligned}
f &= e^{n\frac{\theta}{2}} e^{\varepsilon_1 u} e^{-iI\frac{\varphi}{2}} && (18)\\
&= (\cos\theta/2 + n\sin\theta/2)(1+\varepsilon_1 u)(\cosh\varphi/2 - iI\sinh\varphi/2) && (19)\\
&= \left(\frac{1}{\sqrt{2}} + n\frac{1}{\sqrt{2}}\right)\left[1 + \frac{(kI+j)}{2}kKl\right]\left(\frac{2}{\sqrt{3}} - iI\frac{1}{\sqrt{3}}\right) && (20)\\
&= \left(\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{6}}iI\right)(1+n) + \frac{1}{2}\sqrt{\frac{2}{3}}(J+iK)(l+m)\,, && (21)
\end{aligned}
$$

with $\tan\frac{\theta}{2} = 1, \tanh\frac{\varphi}{2} = \frac{1}{2}\left(e^{\varphi} = \frac{1+th\varphi/2}{1-th\varphi/2} = \frac{3/2}{1/2} = 3\right)$.

The bivector part B of f generating the transformation, divided by the scalar $\sqrt{\frac{2}{3}}$ is

$$
B = n - \frac{iJ}{2} + \frac{3}{4}(J+iK)(l+m)\,. \quad (22)
$$

The canonical decomposition [4] of B and f into simple, orthogonal and commuting planes (B_1, B_2) with $b_1 = \tan\frac{\Phi_1}{2} = 1, b_2 = \tanh\frac{\Phi_2}{2} = \frac{1}{2}$ leads to

$$
B = b_1B_1 + b_2B_2\,, \quad f = e^{\frac{\Phi_1}{2}B_1} e^{\frac{\Phi_2}{2}B_2}\,, \quad (23)
$$

with

$$
\begin{aligned}
B_1 &= n + \frac{3}{10}(J+iK)(3l+m)\,, \quad B_1^2 = -1\,, && (24)\\
B_2 &= \frac{3}{10}(J+iK)(-l+3m) - IK\,, \quad B_2^2 = 1\,. && (25)
\end{aligned}
$$

The two invariants of the transformation are

$$
S_1 = B.B = -\frac{3}{4}\,, \quad S_2 = [(B\wedge B).B].B = -1\,. \quad (26)
$$

The conformal transformation with $X = e_a + e_1 + e_b$ ($x_0 = 0, x_1 = 1, x_2 = 0$) is obtained either directly

$$
e_1 \xrightarrow{D} e_1/3 \xrightarrow{T} (1/3+1)e_1 = (4/3)e_1 \xrightarrow{R} (4/3)e_2\,, \quad (27)
$$

or by computation:

$$
\begin{aligned}
X' &= fXf_c = x'_a e_a + x' + x'_b e_b && (28)\\
&= -\frac{25}{6}e_a + 4e_2 - \frac{7}{6}e_b\,, && (29)
\end{aligned}
$$

yielding the final transform

$$
x \rightarrow y(x) = \frac{x'}{x'_b - x'_a} = \frac{4}{3}e_2\,. \quad (30)
$$

5 Classification of hyperquaternion algebras

Table 1 lists a few hyperquaternion algebras and their signature (p,q) obtained via the generators given in [7]. The table shows the importance of the parameter $s = p - q$ [2, 15]. It reveals four classes of hyperquaternions: the algebras $\mathbb{H}^{\otimes r}$ (r even or odd) and the subalgebras C^+. From (n,s) one deduces $p=(n+s)/2, q=(n-s)/2$ yielding the general formulas for m integer $(m \geqslant 1)$

$$\begin{aligned}
\mathbb{H}^{\otimes 2m} &\simeq C_{4m}(2m+1,2m-1), \quad (s=2),\\
\mathbb{C}\otimes\mathbb{H}^{\otimes(2m-1)} &\simeq C_{4m-1}(2m+1,2m-2), \quad (s=3),\\
\mathbb{H}^{\otimes(2m-1)} &\simeq C_{4m-2}(2m-2,2m), \quad (s=-2),\\
\mathbb{C}\otimes\mathbb{H}^{\otimes(2m-2)} &\simeq C_{4m-3}(2m-2,2m-1), \quad (s=-1).
\end{aligned}$$

All signatures of hyperquaternion algebras can be derived from the first four ones via the formula

$$C_{n+4}(p+2,q+2)=C_n(p,q)\otimes\mathbb{H}^{\otimes 2},$$

resulting from the double application of the general formula

$$C_{n+2}(p+1,q+1)=C_n(p,q)\otimes C_2(1,1),$$

together with $C_2(1,1)\simeq\mathbb{R}(2), \mathbb{R}(2)\otimes\mathbb{R}(2)\simeq\mathbb{R}(4)\simeq\mathbb{H}^{\otimes 2}$. Furthermore, since

$$\mathbb{H}^{\otimes 2}\simeq\mathbb{R}(4), \quad \mathbb{C}\otimes\mathbb{H}^{\otimes 2}\simeq\mathbb{C}(4), \quad \mathbb{H}^{\otimes 3}\simeq\mathbb{H}(4),$$

one obtains all square real, complex and quaternionic matrices. Concerning the matrix representation of hyperquaternion algebras, which is beyond the scope of this paper, the above isomorphisms show that $\mathbb{H}^{\otimes 2}$ can be represented either by a reducible real matrix $\mathbb{R}(16)$ (real 16×16 matrix) or by an irreducible $\mathbb{R}(4)$ matrix ($\mathbb{H}$ being represented by an irreducible $\mathbb{R}(4)$ matrix). Similarly, $\mathbb{H}^{\otimes 3}$ and its subalgebra $\mathbb{C}\otimes\mathbb{H}^{\otimes 2}$ can be represented either by a reducible matrix $\mathbb{R}(64)$ or by an irreducible matrix $\mathbb{R}(16)$. A classification of real irreducible representations of quaternionic Clifford algebras can be found in [16, 17].

A hyperconjugation defined as

$$A_H=(i_c,j_c,k_c)\otimes(I_c,J_c,K_c)\otimes(l_c,m_c,n_c),$$

yields the matrix transposition, adjunction and transpose quaternion conjugate. Finally, writing $\omega=e_1\cdots e_n$, one obtains for all hyperquaternion algebras with the above values of s and

Table 1: Hyperquaternion algebras (SR: special relativity, RQM: relativistic quantum mechanics, usp: unitary symplectic physics, sm: standard model).

$C_n(p,q)$	n	p	q	$s=p-q$	*Group*	*Physics*
$\mathbb{C}$	1	0	1	−1	$U(1)$	$1D$
$\mathbb{H}$	2	0	2	−2	$USp(1)$	$2D$
$\mathbb{C}\otimes\mathbb{H}$	3	3	0	3	$SU(2)$	$3D$
$\mathbb{H}^{\otimes 2}\simeq\mathbb{R}(4)$	4	3	1	2	$SO(3,1)$	*SR*
$\mathbb{C}\otimes\mathbb{H}^{\otimes 2}\simeq\mathbb{C}(4)$	5	2	3	−1	$SU(4)$	*RQM*
$\mathbb{H}^{\otimes 3}\simeq\mathbb{H}(4)$	6	2	4	−2	$USp(4)$	*usp*
$\mathbb{C}\otimes\mathbb{H}^{\otimes 3}$	7	5	2	3	$SU(8)$	*sm*
$\mathbb{H}^{\otimes 4}$	8	5	3	2	$SO(5,3)$	
$\mathbb{C}\otimes\mathbb{H}^{\otimes 4}$	9	4	5	−1	$SU(16)$	
$\mathbb{H}^{\otimes 5}$	10	4	6	−2	$USp(16)$	
$\mathbb{C}\otimes\mathbb{H}^{\otimes 5}$	11	7	4	3	$SU(32)$	
$\mathbb{H}^{\otimes 6}$	12	7	5	2	$SO(7,5)$	

the classical derivation (developing $n = p + q$ and using $(-1)^{pq} = (-1)^{-pq}$)

$$\omega^2 = (-1)^{\frac{n(n-1)}{2}} e_1^2 ... e_n^2 = (-1)^{\frac{n(n-1)}{2}+q} = (-1)^{\frac{s(s-1)}{2}} = -1\,.$$

Though hyperquaternion algebras have been neglected in the past, recent algebraic software like `Mathematica` and numerical computing have opened perspectives for the hyperquaternion calculus. An advantage of the hyperquaternion representation of Clifford algebras, is that the product is defined independently of the choice of the generators. Furthermore, hyperquaternion algebras single out specific Clifford algebras which seem to be closely related to symmetry group of physics as indicated in the table above. Thus they might constitute a step towards a greater unification as proposed in [18].

6 Conclusion

The paper has developed applications of a new hyperquaternionic representation of Clifford algebras in terms of tensor products of quaternion algebras. One advantage of hyperquaternion algebras is a uniquely defined product, independent of the choice of generators. Though, hyperquaternions have been somewhat neglected so far, they have become more accessible due to the introduction of algebraic and numerical computing. As applications, the paper has examined the quaternion $2D$ representation, the conformal group in $(1+2)$ space together with a numerical example and implementation. Finally, a classification of all hyperquaternion algebras into four types has been given, with general formulas of the signatures and the associated symmetry groups. We hope to have shown that the hyperquaternion algebras might constitute a useful unifying tool for physics.

Acknowledgements

The paper is dedicated to the memory of our colleague and friend Prof. Dr. Robert Goutte (1932-2022) who greatly furthered and collaborated on research on quaternions and hyperquaternions.

The authors acknowledge and thank an anonymous referee for comments clarifying the matrix quaternionic Clifford algebra representation issue.

Funding information This work was supported by the LABEX PRIMES (ANR-11-LABX-0063) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

A Multivector structure of $C_5(2,3)$

($e_{012} = e_0 e_1 e_2$, etc..)

$$\left[\begin{array}{llll} 1 & 0 & 0 & n = e_{12} \\ \hline 0 & I\ l = e_{10} & I\ m = e_{20} & 0 \\ 0 & J\ l = e_{a1} & J\ m = e_{a2} & 0 \\ K = e_{0a} & 0 & 0 & Kn = e_{0a12} \\ 0 & il = e_{0a1b} & im = e_{0a2b} & 0 \\ iI = e_{ba} & 0 & 0 & iI\ n = e_{1a2b} \\ iJ = e_{b0} & 0 & 0 & iJ\ n = e_{102b} \\ 0 & iKl = e_{b1} & iKm = e_{b2} & 0 \\ j = e_b & 0 & 0 & jn = e_{12b} \\ 0 & jI\ l = e_{10b} & jI\ m = e_{b20} & 0 \\ 0 & jJ\ l = e_{a1b} & jJ\ m = e_{a2b} & 0 \\ jK = e_{0ab} & 0 & 0 & jKn = e_{0a12b} \\ 0 & kl = e_{a01} & km = e_{a02} & 0 \\ kI = e_a & 0 & 0 & kI\ n = e_{a12} \\ kJ = e_0 & 0 & 0 & kJ\ n = e_{012} \\ 0 & kKl = e_1 & kKm = e_2 & 0 \end{array}\right]$$

References

[1] P. Anglès, *Conformal groups in geometry and spin structures*, Birkhäuser, Boston, USA, ISBN 9780817635121 (2008), doi:10.1007/978-0-8176-4643-1.

[2] M. Rausch de Traubenberg, *Clifford algebras in physics*, Adv. Appl. Clifford Algebras **19**, 869 (2009), doi:10.1007/s00006-009-0191-2.

[3] G. L. Panga, *Hyperquaternionic representations of conic sections*, J. Appl. Math. Phys. **10**, 2989 (2022), doi:10.4236/jamp.2022.1010200.

[4] P. R. Girard, P. Clarysse, R. Pujol, R. Goutte and P. Delachartre, *Hyperquaternion conformal groups*, Adv. Appl. Clifford Algebras **31**, 56 (2021), doi:10.1007/s00006-021-01159-y.

[5] P. R. Girard, P. Clarysse, R. Pujol, R. Goutte and P. Delachartre, *Dual hyperquaternion Poincaré groups*, Adv. Appl. Clifford Algebras **31**, 15 (2021), doi:10.1007/s00006-021-01120-z.

[6] P. R. Girard, P. Clarysse, R. Pujol, R. Goutte and P. Delachartre, *Hyperquaternions: An efficient mathematical formalism for geometry*, in *Geometric science of information*, Springer, Cham, Switzerland, ISBN 9783030269791 (2019), doi:10.1007/978-3-030-26980-7_13.

[7] P. R. Girard, P. Clarysse, R. Pujol, R. Goutte and P. Delachartre, *Hyperquaternions: A new tool for physics*, Adv. Appl. Clifford Algebras **28**, 68 (2018), doi:10.1007/s00006-018-0881-8.

[8] S. Carlip, *Quantum gravity in* $2+1$ *dimensions*, Cambridge University Press, Cambridge, UK, ISBN 9780521564083 (1998), doi:10.1017/CBO9780511564192.

[9] P. R. Girard, *Quaternions, Clifford algebras and relativistic physics*, Birkhäuser, Basel, Switzerland, ISBN 9783764377908 (2007), doi:10.1007/978-3-7643-7791-5.

[10] P. R. Girard, *Quaternions, algèbre de Clifford et physique relativiste*, Presses Polytechniques et Universitaires Romandes, Lausanne, Switzerland, ISBN 9782880746063 (2004).

[11] W. K. Clifford, *Applications of Grassmann's extensive algebra*, Am. J. Math. **1**, 350 (1878), doi:10.2307/2369379.

[12] R. Lipschitz, *Principes d'un calcul algébrique qui contient comme espèces particulières le calcul des quantitiés imaginaires et des quaternions*, C. R. Acad. Sci. Paris **91**, 619 (1880).

[13] C. L. E. Moore, *Hyperquaternions*, J. Math. Phys. **1**, 63 (1922).

[14] P. R. Girard, P. Clarysse, R. Pujol, L. Wang and P. Delachartre, *Differential geometry revisited by biquaternion Clifford algebra*, in *Curves and surfaces*, Springer, Cham, Switzerland, ISBN 9783319228037 (2015), doi:10.1007/978-3-319-22804-4_17.

[15] D. W. Marks, *Binary encoded recursive generation of quantum space-times*, Adv. Appl. Clifford Algebras **32**, 51 (2022), doi:10.1007/s00006-022-01235-x.

[16] S. Okubo, *Real representations of finite Clifford algebras. I. Classification*, J. Math. Phys. **32**, 1657 (1991), doi:10.1063/1.529277.

[17] H. L. Carrion, M. Rojas and F. Toppan, *Quaternionic and octonionic spinors. A classification*, J. High Energy Phys. **04**, 040 (2003), doi:10.1088/1126-6708/2003/04/040.

[18] M. Tegmark, *Our mathematical Universe: My quest for the ultimate nature of reality*, Penguin Books, London, UK, ISBN 9780241954638 (2015)

Group invariants for Feynman diagrams

Idrish Huet[1], Michel Rausch de Traubenberg[2] and Christian Schubert[3⋆]

1 Facultad de Ciencias en Física y Matemáticas, Universidad Autónoma de Chiapas, Ciudad Universitaria, Tuxtla Gutiérrez 29050, Mexico
2 Université de Strasbourg, CNRS, IPHC UMR7178, F-67037 Strasbourg Cedex, France
3 Centro Internacional de Ciencias A.C. Campus UNAM-UAEM Avenida Universidad 1001, Cuernavaca, Morelos, Mexico C.P. 62100

⋆ christianschubert137@gmail.com

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

It is well-known that the symmetry group of a Feynman diagram can give important information on possible strategies for its evaluation, and the mathematical objects that will be involved. Motivated by ongoing work on multi-loop multi-photon amplitudes in quantum electrodynamics, here I will discuss the usefulness of introducing a polynomial basis of invariants of the symmetry group of a diagram in Feynman-Schwinger parameter space.

Received 18-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.031

1 Introduction: Schwinger parameter representation of Feynman diagrams

The most universal approach to the calculation of Feynman diagrams uses Feynman-Schwinger parameters x_i, introduced through the exponentiation of the (Euclidean) scalar propagator,

$$\frac{1}{p^2+m^2} = \int_0^\infty dx\, \mathrm{e}^{-x(p^2+m^2)}\,.$$

For scalar diagrams, one finds the following universal structure for an arbitrary graph G with n internal lines and l loops in D dimensions:

$$I_G = \Gamma(n-lD/2)\int_{x_i\geq 0} d^n x\,\delta\Big(1-\sum_{i=1}^n x_i\Big)\frac{\mathcal{U}^{n-(l+1)\frac{D}{2}}}{\mathcal{F}^{n-lD/2}}\,.$$

$\mathcal{U}$ and $\mathcal{F}$ are polynomials in the x_i called the first and second Symanzyk (graph) polynomials. There exist graphical methods for their construction.

For more general theories (involving not only scalar particles) the same graph will involve, in addition to these two polynomials, also a *numerator polynomial* $\mathcal{N}(x_1,\dots,x_n)$. Such numerator polynomials can become extremely large, and in the present talk I would like to point out the universal option of rewriting them as polynomials in a basis of invariants of the symmetry group of the graph. After introducing the symmetries of Feynman diagrams and some related facts of invariant theory in section 2, sections 3 to 8 are devoted to the discussion and motivation of our main example, the three-loop effective Lagrangian in two-dimensional QED, where we have found this procedure to lead to significantly more manageable expressions. We come back to the group-theoretical aspects of this calculation in section 9, before summarizing our findings in sections 10 and 11.

2 Symmetries of Feynman diagrams

Many Feynman diagrams possess a non-trivial symmetry group, generated by interchanges of the internal lines that leave the topology of the graph unchanged.[1] Then all its graph polynomials must be invariant under the natural action of the group on the set of polynomials $\mathbb{R}[x_1,\dots,x_n]$, $g.P(X)\equiv P(g.X)$.

A nice example is the $l-loop$ *banana graph* shown in Fig. 1.

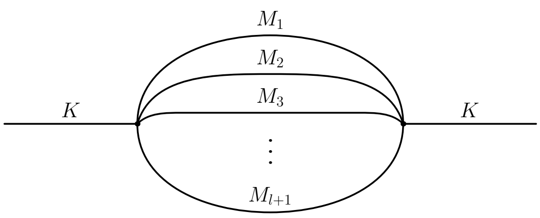


Figure 1: l-loop banana graph.

When all the masses M_i are equal, this graph has full permutation symmetry in all the internal lines, so the symmetry group is S_{l+1}, and its graph polynomials must be symmetric functions of $x_1,\dots,x_{l+1}$. As is well-known, this implies that they can be rewritten as polynomials in the *elementary symmetric polynomials* $S_1,\dots,S_n$.

Perhaps less known is that this generalizes to the case of a general symmetry group as follows [1]:

Theorem 1: Let G be a finite group and let Γ be an n−dimensional (real) representation.

1. There exist $n=\dim(\Gamma)$ algebraically invariant polynomials $P_1,\cdots,P_n$, called the *primitive invariants*, such that the Jacobian $\frac{\partial(P_1,\dots,P_n)}{\partial(x_1,\dots,x_n)}\neq 0$.

2. Denote $d_k=\deg(P_k)$ and $\mathcal{R}=\mathbb{R}[P_1,\cdots,P_n]$ the subalgebra of polynomial invariants generated by the primitive invariants.

3. There exist $m=d_1\cdots d_n/|G|$ secondary invariant polynomials $S_1,\cdots,S_m$.

4. The subalgebra of invariants $\mathbb{R}[x_1,\cdots,x_n]^\Gamma$ is a free $\mathcal{R}$−module with basis $(S_1,\cdots,S_m)$. In particular this means that any invariant $I\in\mathbb{R}[x_1,\cdots,x_n]^\Gamma$ can be uniquely written

[1] We do not consider here the exchange of external lines, since for such an exchange to leave a graph invariant would require the two external momenta to be equal, which is not a natural condition. The exception is the case where all the external momenta go to zero, which is what effectively happens in our main example below.

as $I = \sum_{i=1}^{m} f_i(P_1, \cdots, P_n) S_i$, where $f_i(P_1, \cdots, P_n), i = 1, \cdots, m$ belong to $\mathcal{R}$, *i.e.*, are polynomials in $(P_1, \cdots, P_n)$.

There exist computer algebra systems for the computation of $P_1, \ldots, P_n$ such as SINGULAR [2].

3 The Euler-Heisenberg Lagrangian at one loop

In 1936 Heisenberg and Euler obtained their famous representation of the one-loop QED effective Lagrangian in a constant field ("Euler-Heisenberg Lagrangian" or "EHL")

$$\mathcal{L}^{(1)}(a,b) = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3}\, \mathrm{e}^{-m^2 T} \left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2 - b^2)T^2 - 1 \right]. \tag{1}$$

Here m is the electron mass, and a, b are the two invariants of the Maxwell field, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E} \cdot \mathbf{B}$. This Lagrangian holds the information on the QED N - photon amplitudes in the low-energy limit where all photon energies are small compared to the electron mass, $\omega_i \ll m$. It corresponds to the diagrams shown in Fig. 2.

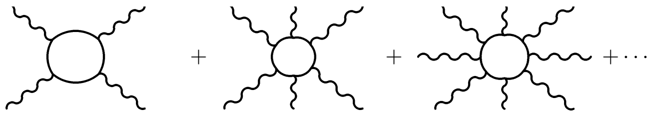


Figure 2: Feynman diagrams corresponding to the EHL.

For the extraction of the amplitudes from the effective Lagrangian, one expands it in powers of the Maxwell invariants,

$$\mathcal{L}(a,b) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l}\,, \tag{2}$$

then fixes a helicity assignment and uses spinors helicity techniques [3].

4 Imaginary part of the effective action

Except for the purely magnetic case where $b = 0$, the proper-time integral in (1) has poles on the integration contour at $ebT = k\pi$ which create an imaginary part. For the purely electric case one gets [4]

$$\mathrm{Im}\mathcal{L}^{(1)}(E) = \frac{m^4}{8\pi^3} \beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta} \right],$$

($\beta = eE/m^2$). We note:

- The kth term relates to coherent creation of k pairs in one Compton volume.
- $\mathrm{Im}\mathcal{L}(E)$ depends on E non-perturbatively (non-analytically), which is consistent with Sauter's [5] interpretation of pair creation as vacuum tunnelling (Fig. 3).

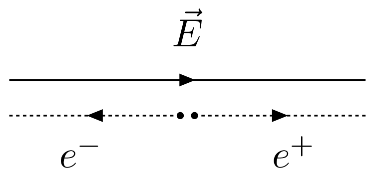


Figure 3: Pair creation by an external field as vacuum tunnelling.

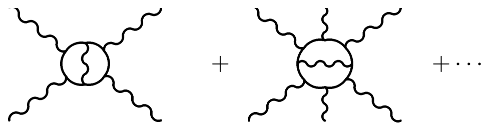


Figure 4: Feynman diagrams contributing to the 2-loop EHL.

5 Beyond one loop

The two-loop correction to the EHL due to one internal photon exchange (Fig. 4) has been analyzed [6–8], and turned out to contain important information on the Sauter tunnelling picture [9], on-shell versus off-shell renormalization [6, 10], and the asymptotic properties of the QED photon S-matrix [11].

It leads to rather intractable two-parameter integrals. However, in the electric case its imaginary part $\mathrm{Im}\mathcal{L}^{(2)}(E)$ permits a decomposition analogous to Schwinger's (3) [9]. For single-pair production, this is now interpreted as a tunnelling process where, in the process of turning real, the electron-positron pair is already interacting at the one-photon exchange level.

Even for the imaginary part no completely explicit formulas are available. However, it simplifies dramatically in the weak-field limit, where it just becomes an $\alpha\pi$ correction to the one-loop contribution:

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\big(1+\alpha\pi\big)\,\mathrm{e}^{-\frac{\pi}{\beta}}\,.$$

This suggests that higher loop orders might lead to an exponentiation, and indeed Lebedev and Ritus [9] provided strong support for this hypothesis by showing that, assuming that

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) + \mathrm{Im}\mathcal{L}^{(3)}(E) + \ldots \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\exp\Big[-\frac{\pi}{\beta}+\alpha\pi\Big] = \mathrm{Im}\mathcal{L}^{(1)}(E)\,\mathrm{e}^{\alpha\pi}\,,$$

then the result can be interpreted in the tunnelling picture as the corrections to the Schwinger pair creation rate due to the pair being created with a negative Coulomb interaction energy

$$m(E) \approx m + \delta m(E)\,, \qquad \delta m(E) = -\frac{\alpha}{2}\frac{eE}{m}\,.$$

Moreover, the resulting field-dependent mass-shift $\delta m(E)$ is identical with the *Ritus mass shift*, originally derived by Ritus in [13] from the crossed process of one-loop electron propagation in the field (Fig. 5).

Unbeknownst to the authors of [9], for scalar QED the corresponding conjecture had already been established two years earlier by Affleck, Alvarez and Manton [12] using Feynman's worldline path integral formalism and a semi-classical *worldline instanton* approximation.

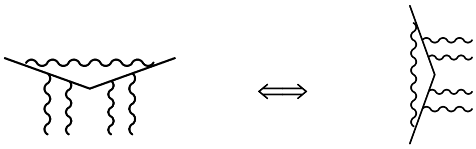


Figure 5: Photon-corrected pair-creation vs. electron propagation in the field.

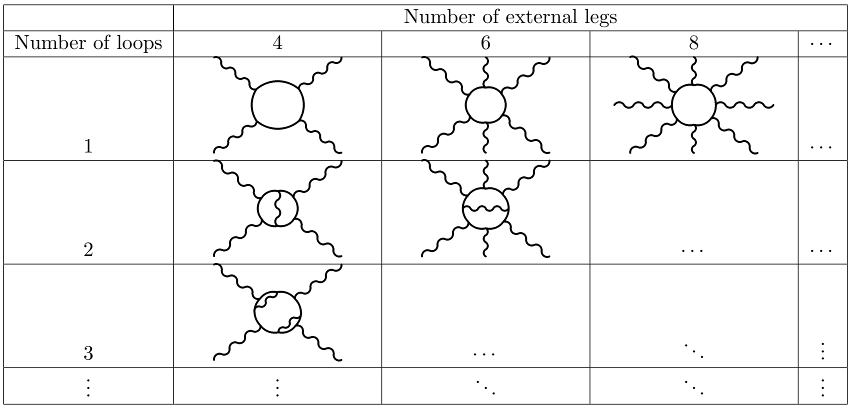


Figure 6: Feynman diagrams contributing to the exponentiation hypothesis.

Diagrammatically, we note the following features of the exponentiation formula (see Fig. 6):

- It Involves diagrams with any numbers of loops and legs.
- Although not shown, also all the counter-diagrams from mass renormalization must contribute.
- It does *not* include diagrams with more than one fermion loop (those get suppressed in the weak-field limit [12]).
- Horizontal summation produces the Schwinger exponential $e^{-\frac{\pi}{\beta}}$.
- Vertical summation produces the Ritus-Lebedev/Affleck-Alvarez-Manton exponential $e^{\alpha\pi}$.

6 QED in 1+1 dimensions

The exponentiation conjecture has so far been verified only at two loops. A three-loop check is in order, but calculating the three-loop EHL in $D = 4$ is presently hardly feasible. Motivated by work by Krasnansky [14] on the EHL in various dimensions, in 2010 two of the authors with D.G.C. McKeon started investigating the analogous problem in $2D$ QED. In [15] we used the worldline instanton method to generalize the exponentiation conjecture to the $2D$ case, resulting in

$$\mathrm{Im}\mathcal{L}_{2D}^{(all-loop)} \sim e^{-\frac{m^2\pi}{eE}+\tilde{\alpha}\pi^2\kappa^2}, \tag{3}$$

where $\kappa = m^2/(2ef)$, $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and $\tilde{\alpha} = \frac{2e^2}{\pi m^2}$ is the two-dimensional analogue of the fine-structure constant. Defining the weak-field expansion coefficients in $2D$ by

$$\mathcal{L}^{(l)(2D)}(\kappa) = \frac{m^2}{2\pi}\sum_{n=1}^{\infty}(-1)^{l-1}c_{2D}^{(l)}(n)(i\kappa)^{-2n}, \tag{4}$$

we then used Borel analysis to derive from (3) a formula for the limits of ratios of l - loop to one - loop coefficients:

$$\lim_{n\to\infty}\frac{c_{2D}^{(l)}(n)}{c_{2D}^{(1)}(n+l-1)}=\frac{(\tilde{\alpha}\pi^2)^{l-1}}{(l-1)!}\,. \tag{5}$$

Moreover, we calculated the $2D$ EHL at one and two loops,

$$\mathcal{L}^{(1)}(f)=-\frac{m^2}{4\pi}\frac{1}{\kappa}\left[\ln\Gamma(\kappa)-\kappa(\ln\kappa-1)+\frac{1}{2}\ln\left(\frac{\kappa}{2\pi}\right)\right], \tag{6}$$

$$\mathcal{L}^{(2)}(f)=\frac{m^2}{4\pi}\frac{\tilde{\alpha}}{4}\left[\tilde{\psi}(\kappa)+\kappa\tilde{\psi}'(\kappa)+\ln(\lambda_0 m^2)+\gamma+2\right], \tag{7}$$

where $\tilde{\psi}(x)\equiv\psi(x)-\ln x+\frac{1}{2x}$, $\psi(x)=\Gamma'(x)/\Gamma(x)$, and the constant λ_0 comes from an IR cutoff. One finds from (6) and (7) that

$$c_{2D}^{(1)}(n)=(-1)^{n+1}\frac{B_{2n}}{4n(2n-1)}\,, \tag{8}$$

$$c_{2D}^{(2)}(n)=(-1)^{n+1}\frac{\tilde{\alpha}}{8}\frac{2n-1}{2n}B_{2n}\,. \tag{9}$$

Using properties of the Bernoulli numbers B_n it is then easy to verify that

$$\lim_{n\to\infty}\frac{c_{2D}^{(2)}(n)}{c_{2D}^{(1)}(n+1)}=\tilde{\alpha}\pi^2\,,$$

in accordance with (5).

7 Three-loop EHL in 2D: Diagrams

At three loops, we face the task of computing the two diagrams shown in Fig. 7 (there are also diagrams involving more than one fermion-loop, including several that involve Gies-Karbstein tadpoles [16], but those can be shown to be subdominant in the asymptotic limit).

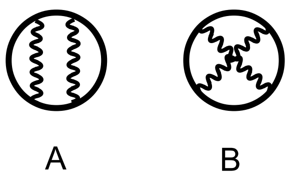


Figure 7: Three-loop diagrams contributing to the exponentiation conjecture.

The fermion propagators in these diagrams are the exact ones in the constant external field. Thus, although they are depicted as vacuum diagrams, they are equivalent to the full set of ordinary diagrams of the given topology with any number of zero momentum photons attached to them in all possible ways.

Due to the super-renormalizability of $2D$ QED these diagrams are already UV finite. They suffer from spurious IR - divergences, but those can be removed by going to the *traceless gauge* $\xi=-2$ [17]. The calculation of diagram A is relatively straightforward, thus we focus on the much more substantial task of computing diagram B and its weak-field expansion coefficients.

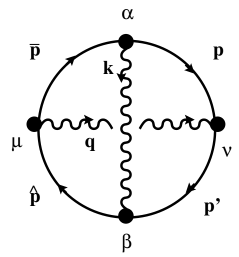


Figure 8: Parametrization of diagram B.

Introducing Schwinger parameters for this diagram as shown in Fig. 8 leads to the integral representation [17]

$$\mathcal{L}^{3B}(f) = \frac{\tilde{\alpha}^2 m^2}{128\pi} \int_0^\infty dw dw' d\hat{w} d\bar{w}\, I_B\, \mathrm{e}^{-a},$$

$$I_B = \frac{\rho^3}{\cosh^2 \rho w \cosh^2 \rho w' \cosh^2 \rho \hat{w} \cosh^2 \rho \bar{w}} \frac{B}{A^3 C} - \rho \frac{\cosh(\rho \tilde{w})}{\cosh \rho w \cosh \rho w' \cosh \rho \hat{w} \cosh \rho \bar{w}} \left[\frac{1}{A} - \frac{C}{G^2} \ln\left(1 + \frac{G^2}{AC}\right)\right],$$

where

$$B = (\tanh^2 z + \tanh^2 \hat{z})(\tanh z' + \tanh \bar{z}) + (\tanh^2 z' + \tanh^2 \bar{z})(\tanh z + \tanh \hat{z}),$$
$$C = \tanh z \tanh z' \tanh \hat{z} + \tanh z \tanh z' \tanh \bar{z} + \tanh z \tanh \hat{z} \tanh \bar{z} + \tanh z' \tanh \hat{z} \tanh \bar{z},$$
$$G = \tanh z \tanh \hat{z} - \tanh z' \tanh \bar{z}$$

($z = \rho w$ etc.). Although for a three-loop diagram this is a fairly compact representation, an exact calculation is out of the question, and a straightforward expansion in powers of the external field to get the weak-field expansion coefficients turns out to create huge numerator polynomials. To deal with those, we will now take advantage of the high symmetry of the diagram.

8 Integration-by-parts algorithm

Introduce the operator $\tilde{d} \equiv \frac{\partial}{\partial w} - \frac{\partial}{\partial w'} + \frac{\partial}{\partial \hat{w}} - \frac{\partial}{\partial \bar{w}}$ which acts simply on the trigonometric building blocks of the integrand. Integrating by parts with this operator, it is possible to write the integrand of β_n, the n-th coefficient of the expansion of I_B as a power series in ρ, as a total derivative $\beta_n = \tilde{d}\theta_n$. Then, using once more the symmetry of the graph,

$$\int_0^\infty dw dw' d\hat{w} d\bar{w}\, \mathrm{e}^{-a} \beta_n = \int_0^\infty dw d\bar{w} d\hat{w} dw' \tilde{d}\, \mathrm{e}^{-(w+w'+\hat{w}+\bar{w})} \theta_n = 4 \int_0^\infty dw dw' d\hat{w}\, \mathrm{e}^{-(w+w'+\hat{w})}\, \theta_n|_{\bar{w}=0}.$$

The remaining threefold integrals are already of a fairly standard type.

9 Using the polynomial invariants of D_4

Diagram B has the symmetries

$$\begin{aligned} w &\longleftrightarrow \hat{w}\,, \\ w' &\longleftrightarrow \bar{w}\,, \\ (w,\hat{w}) &\longleftrightarrow (w',\bar{w})\,. \end{aligned}$$

Those generate the dihedral group D_4. After a slight generalization to the inclusion of semi-invariants (invariants up to a sign) [18], Theorem 1 can be used to deduce that the numerator polynomials can be rewritten as polynomials in the variable $\tilde{w} = w - w' + \hat{w} - \bar{w}$ with coefficients that are polynomials in the four D_4 - invariants a, v, j, h,

$$\begin{aligned} a &= w + w' + \hat{w} + \bar{w}\,, \\ v &= 2(w\hat{w} + w'\bar{w}) + (w + \hat{w})(w' + \bar{w})\,, \\ j &= a\tilde{w} - 4(w\hat{w} - w'\bar{w})\,, \\ h &= a(ww'\hat{w} + ww'\bar{w} + w\hat{w}\bar{w} + w'\hat{w}\bar{w}) + (w\hat{w} - w'\bar{w})^2\,. \end{aligned}$$

These invariants are moreover chosen such that they are annihilated by $\tilde{d}$. Thus they are well-adapted to the integration-by-parts algorithm. This rewriting leads to a very significant reduction in the size of the expressions generated by the expansion in the field.

10 Results

In this way we obtained the first two coefficients of the weak-field expansion analytically,

$$\begin{aligned} \Gamma_0^B &= -\frac{3}{2} + \frac{7}{4}\zeta(3)\,, \\ \Gamma_1^B &= -\frac{251}{120} + \frac{35}{16}\zeta(3)\,, \end{aligned}$$

and five more coefficients numerically (the coefficients Γ_n are related to the ones introduced in (4) by $4^n c_n^{(3)} = \frac{\tilde{\alpha}^2}{64}\Gamma_n$). For a definite conclusion concerning the exponentiation conjecture this is still insufficient, and the computation of further coefficients is in progress.

11 Outlook

- Writing Feynman graph polynomials in terms of invariant polynomials is a universal option that, to the best of our knowledge, has not previously been used, but we expect that it will be found very useful for multiloop calculations involving diagrams with nontrivial symmetry groups and a large number of propagators.
- In particular, this is the case for the weak-field expansion of the QED effective Lagrangian starting from three loops (in any dimension).

References

[1] B. Sturmfels, *Algorithms in invariant theory*, Springer, Vienna, Austria, ISBN 9783211774168 (2008), doi:10.1007/978-3-211-77417-5.

[2] W. Decker and C. Lossen, *Computing in algebraic geometry: A quick start using SINGULAR*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540289920 (2006), doi:10.1007/3-540-28993-3.

[3] L. C. Martin, C. Schubert and V. M. Villanueva Sandoval, *On the low-energy limit of the QED N - photon amplitudes*, Nucl. Phys. B **688**, 335 (2003), doi:10.1016/j.nuclphysb.2018.07.026.

[4] J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. **82**, 664 (1951), doi:10.1103/PhysRev.82.664.

[5] F. Sauter, *Über das verhalten eines elektrons im homogenen elektrischen feld nach der relativistischen theorie diracs*, Z. Phys. **69**, 742 (1931), doi:10.1007/BF01339461.

[6] V. I. Ritus, *Lagrangian of an intense electromagnetic field and quantum electrodynamics at short distances*, Zh. Eksp. Teor. Fiz **69**, 1517 (1975).

[7] W. Dittrich and M. Reuter, *Effective Lagrangians in quantum electrodynamics*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540151821 (1985), doi:10.1007/3-540-15182-6.

[8] M. Reuter, M. G. Schmidt and C. Schubert, *Constant external fields in gauge theory and the spin 0, 1/2, 1 path integrals*, Ann. Phys. **259**, 313 (1997), doi:10.1006/aphy.1997.5716.

[9] S. L. Lebedev and V. I. Ritus, *Virial representation of the imaginary part of the Lagrangian function of an electromagnetic field*, Zh. Eksp. Teor. Fiz. **86**, 408 (1984).

[10] V. I. Ritus, *Connection between strong-field quantum electrodynamics with short-distance quantum electrodynamics*, Zh. Eksp. Teor. Fiz **73**, 807 (1977).

[11] I. Huet, M. Rausch de Traubenberg and C. Schubert, *Asymptotic behavior of the QED perturbation series*, Adv. High Energy Phys. 6214341 (2017), doi:10.1155/2017/6214341.

[12] I. K. Affleck, O. Alvarez and N. S. Manton, *Pair production at strong coupling in weak external fields*, Nucl. Phys. B **197**, 509 (1982), doi:10.1016/0550-3213(82)90455-2.

[13] V. I. Ritus, *Method of eigenfunctions and mass operator in quantum electrodynamics of a constant field*, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978).

[14] M. Krasnansky, *Two-loop vacuum diagrams in background field and Heisenberg-Euler effective action in different dimensions*, Int. J. Mod. Phys. A **23**, 5201 (2008), doi:10.1142/S0217751X08042572.

[15] I. Huet, D. G. C. McKeon and C. Schubert, *Euler-Heisenberg Lagrangians and asymptotic analysis in* $1+1$ *QED. Part I: Two-loop*, J. High Energy Phys. **12**, 036 (2010), doi:10.1007/JHEP12(2010)036.

[16] H. Gies and F. Karbstein, *An addendum to the Heisenberg-Euler effective action beyond one loop*, J. High Energy Phys. **03**, 108 (2017), doi:10.1007/JHEP03(2017)108.

[17] I. Huet, M. Rausch de Traubenberg and C. Schubert, *Three-loop Euler-Heisenberg Lagrangian in* $1+1$ *QED, part 1: Single fermion-loop part*, J. High Energy Phys. **03**, 167 (2019), doi:10.1007/JHEP03(2019)167.

[18] I. Huet, M. Rausch de Traubenberg and C. Schubert, *Dihedral invariant polynomials in the effective Lagrangian of QED*, J. Phys.: Conf. Ser. **1194**, 012046 (2019), doi:10.1088/1742-6596/1194/1/012046.

Quantum cylindrical integrability in magnetic fields

Ondřej Kubů⋆ and Libor Šnobl

Czech Technical University in Prague, Faculty of Nuclear Sciences
and Physical Engineering, Prague, Czech Republic

⋆ ondrej.kubu@fjfi.cvut.cz

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We present the classification of quadratically integrable systems of the cylindrical type with magnetic fields in quantum mechanics. Following the direct method used in classical mechanics by [F Fournier et al 2020 J. Phys. A: Math. Theor. 53 085203] to facilitate the comparison, the cases which may *a priori* differ yield 2 systems without any correction and 2 with it. In all of them the magnetic field B coincides with the classical one, only the scalar potential W may contain a $\hbar^2$-dependent correction. Two of the systems have both cylindrical integrals quadratic in momenta and are therefore not separable. These results form a basis for a prospective study of superintegrability.

Received 10-10-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.032

1 Introduction

This article is a contribution to the study of integrable and superintegrable Hamiltonian systems with magnetic fields on the 3D Euclidean space $\mathbb{E}_3$ in quantum mechanics. More specifically, we assume a Hamiltonian of the form (using units where $e = -1,\ m = 1$)

$$H = \frac{1}{2}\left(\vec{p}^{\,2} + A_j(\vec{x})p_j + p_jA_j(\vec{x}) + A_j(\vec{x})^2\right) + W(\vec{x}), \tag{1}$$

with implicit summation over repeated indices $j = 1,2,3$ (in the whole paper), $\vec{p} = -i\hbar\vec{\nabla}$ is the momentum operator and $\vec{A} = (A_1(\vec{x}),A_2(\vec{x}),A_3(\vec{x}))$ and $W(\vec{x})$ are the vector and scalar potentials of the electromagnetic field.

Integrability then entails the existence of two algebraically independent integrals of motion X_1, X_2 (further specified below) mutually in involution, i.e.

$$[H,X_1] = [H,X_2] = [X_1,X_2] = 0\,. \tag{2}$$

They are usually considered to be polynomials in the momenta p_j, for computational feasibility usually of a low order (typically 2).

Integrable (and especially superintegrable) systems are rare and distinguished by the possibility to obtain the solution to their equations of motion in a closed form. They are subsequently invaluable for gaining physical intuition and serve as a starting point for modelling more complicated systems. Finding and classifying these systems is therefore of utmost importance.

The case without the vector potential $\vec{A}$ has been widely studied. The quadratic integrable systems were classified in 1960s and the 1:1 correspondence with orthogonal separation of variables of the Schrödinger (or, in classical context, the Hamilton-Jacobi) equation was found [1–3]. This leads to the 11 classes of scalar potentials V studied by Eisenhart [4]. Higher order superintegrability followed, see e.g. [5] and references therein.

Despite its physical relevance, integrability with magnetic fields was mostly ignored due to its computational difficulty. The first systematic result remedying this omission was the article by Shapovalov on separable systems [6], followed by the articles in $\mathbb{E}_2$ [7, 8]. Subsequent articles in $\mathbb{E}_3$ assumed first order integrals [9] or separation of variables [10–13]. Marchesiello et al. [9] found a quadratic superintegrable system with an integral not connected to separation of variables, which was recently followed up by [14, 15].

Here we present in an abridged form the classification of quadratically integrable systems of the cylindrical type (see (9)) in quantum mechanics obtained in O. Kubů's Master thesis [16], which closely followed Fournier et al.'s [13] classical analysis to highlight the differences arising in quantum mechanics.

In Section 2 we introduce the differential form formalism for magnetic fields in cylindrical coordinates, derive the determining equations for cylindrical–type integrals and reduce them to a simpler form. The calculations separate into several cases depending on the rank of the matrix in equation (15). In the case that may *a priori* differ from the classical one from [13] only ranks 2 and 1 are relevant. We present the corresponding results in Sections 3 and 4, respectively. We draw our conclusions in Section 5.

2 Cylindrical–type system

Before we specify the corresponding integrals X_1, X_2, we have to introduce the formalism used for magnetic field in curvilinear coordinates in classical mechanics, cf. [13, 17].

Defining the cylindrical coordinates

$$x = r\cos(\phi), \quad y = r\sin(\phi), \quad z = Z, \tag{3}$$

we represent the vector potential A as a 1-form

$$A = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z = A_r \mathrm{d}r + A_\phi \mathrm{d}\phi + A_Z \mathrm{d}Z. \tag{4}$$

Hence, we obtain the following transformations

$$A_x = \cos(\phi)A_r - \frac{\sin(\phi)}{r}A_\phi, \qquad A_y = \sin(\phi)A_r + \frac{\cos(\phi)}{r}A_\phi, \qquad A_z = A_Z. \tag{5}$$

As a part of the canonical 1-form $\lambda = p_j \mathrm{d}x^J$, the momenta p_j transform in the same way and we can define the covariant momenta by $p_j^A = p_j + A_j$ in both Cartesian and cylindrical coordinates.

Components of the magnetic field 2-form $B = \mathrm{d}A$ are

$$\begin{aligned} B &= B^x(\vec{x})\,\mathrm{d}y \wedge \mathrm{d}z + B^y(\vec{x})\,\mathrm{d}z \wedge \mathrm{d}x + B^z(\vec{x})\,\mathrm{d}x \wedge \mathrm{d}y \\ &= B^r(r,\phi,Z)\,\mathrm{d}\phi \wedge \mathrm{d}Z + B^\phi(r,\phi,Z)\,\mathrm{d}Z \wedge \mathrm{d}r + B^Z(r,\phi,Z)\,\mathrm{d}r \wedge \mathrm{d}\phi, \end{aligned} \tag{6}$$

which leads to the following transformation

$$
\begin{aligned}
B^x(\vec{x}) &= \frac{\cos(\phi)}{r} B^r(r,\phi,Z) - \sin(\phi) B^\phi(r,\phi,Z), \\
B^y(\vec{x}) &= \frac{\sin(\phi)}{r} B^r(r,\phi,Z) + \cos(\phi) B^\phi(r,\phi,Z), \\
B^z(\vec{x}) &= \frac{1}{r} B^Z(r,\phi,Z).
\end{aligned} \tag{7}
$$

We can use the same formalism and notation in quantum mechanics as well, we just have to quantize the equations and the integrals (properly symmetrized) in Cartesian coordinates and subsequently transform our equations into cylindrical ones. For example, the transformed momenta read

$$
p_x = -i\hbar\left(\cos(\phi)\partial_r - \frac{\sin(\phi)}{r}\partial_\phi\right), \quad p_y = -i\hbar\left(\sin(\phi)\partial_r + \frac{\cos(\phi)}{r}\partial_\phi\right), \quad p_z = -i\hbar\partial_Z, \tag{8}
$$

i.e. the transformation is the same as (5) upon defining $p_{r,\phi,Z} = -i\hbar\partial_{r,\phi,Z}$.

We can now introduce integrals of motion of the cylindrical type, i.e. integrals that imply separation of Schrödinger (or, classically, Hamilton-Jacobi) equation in the cylindrical coordinates in the limit of vanishing magnetic field $\vec{B}$. Expressed in the cylindrical coordinates they read

$$
\begin{aligned}
X_1 &= \left(p_\phi^A\right)^2 + \frac{1}{2}\sum_{\alpha=r,\phi,Z}\left(s_1^\alpha(r,\phi,Z)p_\alpha^A + p_\alpha^A s_1^\alpha(r,\phi,Z)\right) + m_1(r,\phi,Z), \\
X_2 &= \left(p_Z^A\right)^2 + \frac{1}{2}\sum_{\alpha=r,\phi,Z}\left(s_2^\alpha(r,\phi,Z)p_\alpha^A + p_\alpha^A s_2^\alpha(r,\phi,Z)\right) + m_2(r,\phi,Z).
\end{aligned} \tag{9}
$$

The functions $s_{1,2}^{\{r,\phi,Z\}}, m_{1,2}$ are to be determined from the integrability conditions (2) together with the electromagnetic field B, W.

Our form of integrals allows us to separate the integrability conditions (2) into coefficients of momenta, e.g. $p_r p_Z$, which must all vanish, yielding the so-called determining equations. The second order ones can be solved in terms of 5 auxiliary functions of one variable each, namely

$$
\begin{aligned}
&s_1^r = \frac{\mathrm{d}}{\mathrm{d}\phi}\psi(\phi), \qquad s_1^\phi = -\frac{\psi(\phi)}{r} - r^2\mu(Z) + \rho(r), \qquad s_1^Z = \tau(\phi), \\
&s_2^r = 0, \qquad s_2^\phi = \mu(Z), \qquad s_2^Z = -\frac{\tau(\phi)}{r^2} + \sigma(r),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
B^r &= -\frac{r^2}{2}\frac{\mathrm{d}}{\mathrm{d}Z}\mu(Z) + \frac{1}{2r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\tau(\phi), \qquad B^\phi = \frac{\tau(\phi)}{r^3} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\sigma(r), \\
B^Z &= \frac{-\psi(\phi)}{2r^2} + r\mu(Z) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\rho(r) - \frac{1}{2r^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\psi(\phi),
\end{aligned} \tag{11}
$$

cf. [13]. We further use primes for derivatives of these functions with respect to their variable.

We substitute this result into the remaining determining equations and the corresponding Clairaut compatibility conditions $\partial_{ba}m_j = \partial_{ab}m_j$ and after some calculations we obtain the following reduced equations. (Indexes of W mean partial derivatives.)

$$
\psi'(\phi)\left(r^3\sigma'(r) + 2\tau(\phi)\right) - \tau'(\phi)\left(r\rho(r) - \psi(\phi)\right) = 0, \tag{12}
$$

$$
\mu(Z)\psi'(\phi) + r^3\sigma(r)\mu'(Z) = 0, \tag{13}
$$

$$\begin{aligned}
W_{r\phi} &= -\frac{2}{r}W_\phi + \frac{1}{4r^5}\left[\psi'(\phi)\left(r^3(\rho''(r)-\mu(Z))-r^2\rho'(r)+r\rho(r)-3\psi''(\phi)-4\psi(\phi)\right)\right.\\
&\qquad\left. +\,\tau'(\phi)\left(r^3\sigma'(r)+2\tau(\phi)\right)-2r^4\tau(\phi)\mu'(Z)-\psi'''(\phi)(\psi(\phi)-r\rho(r))\right],\\
W_{\phi Z} &= -\frac{1}{4r^2}\left[r^2\mu''(Z)\left(\tau(\phi)-r^2\sigma(r)\right)+\tau''(\phi)\mu(Z)\right], \qquad (14)\\
W_{rZ} &= \frac{1}{4r^3}\left[r\mu'(Z)\left(r^2\rho'(r)+\psi(\phi)-2r^3\mu(Z)\right)+2\mu(Z)\tau'(\phi)\right],
\end{aligned}$$

$$\begin{pmatrix} 0 & r^2\mu(Z) & r^2\sigma(r)-\tau(\phi) \\ \psi'(\phi) & \rho(r)-r^2\mu(Z)-\frac{\psi(\phi)}{r} & \tau(\phi) \\ 0 & 4r^7\mu(Z) & -4r^5\tau(\phi) \end{pmatrix}\cdot\begin{pmatrix} W_r \\ W_\phi \\ W_Z \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\hbar^2(\psi'''(\phi)+\psi'(\phi))}{4r^3} \\ 0 \end{pmatrix}. \qquad (15)$$

We denote the matrix in (15) by M.

The only change with respect to the classical case is the non-zero RHS in equation (15), corresponding to equation (39) in [13].

We proceed depending on the rank of the matrix M: rank 0 implies vanishing magnetic field and rank 3 is inconsistent with the other reduced equations (12)–(14), we therefore consider ranks 1 and 2 only.

For both these ranks we have to consider the following 2 cases,

a) $\psi'(\phi)=0$,

b) $\psi'(\phi)\neq 0$, and $\mu(Z)=0$.

Case a) implies vanishing quantum correction in (15), i.e. the systems are characterized by the same functions B, W, s and m in the classical and quantum mechanics. The reader can find those in [13]. Hereafter we present only the results for case b) for brevity, details can be found in [16].

3 rank $(M)=2$

Here the matrix equation (15) implies that the coordinate Z is cyclical in a suitable gauge and the integral X_2 reduces to a first order one $p_Z = p_z$ in that gauge.

We obtain 2 systems. The dynamics of the first one splits to a free motion in the z direction and a 2D system with perpendicular magnetic field, so some details were extracted from the consideration of 2D systems in [7, 8]. The other system cannot be separated in this way due to a more complicated magnetic field (but p_z remains an integral).

In what follows we use $r=\sqrt{x^2+y^2}$ for brevity in Cartesian coordinates as well and expressions like ρ_1, ψ_2 and W_0 are (usually nonvanishing) constants unless a variable is explicitly indicated.

1. The first system is classical, i.e. the quantum correction vanishes. The electromagnetic field in Cartesian coordinates reads

$$\begin{aligned}
&B^x=0,\quad B^y=0,\quad B^z=-6\rho_2 r^2+\rho_1,\\
&W=-2\rho_2(\psi_1 x+\psi_2 y)-\rho_2^2 r^6+\frac{\rho_2\rho_1}{2}r^4-\rho_2 W_0 r^2.
\end{aligned} \qquad (16)$$

The cylindrical integral of motion X_1 in Cartesian coordinates is

$$\begin{aligned}
X_1 = \left(L_z^A\right)^2 &+ \left(3\rho_2 r^4-\rho_1 r^2+W_0\right)L_z^A-\psi_2 p_x^A+\psi_1 p_y^A\\
&+(2\rho_2 r^2-\rho_1)(\psi_1 x+\psi_2 y)\\
&+\tfrac{1}{4}\left(3\rho_2 r^4-\rho_1 r^2+2W_0\right)\left(3\rho_2 r^2-\rho_1\right)r^2.
\end{aligned} \qquad (17)$$

2. This time the reduced equations (12)–(15) were not completely solved. The electromagnetic field listed below features a function $\beta(\phi) = \psi(\phi) - \rho_0$ which must satisfy the following ODE

$$\beta'(\phi)\big(7\beta(\phi)\beta''(\phi) + 4\beta'(\phi)^2 + 12\beta(\phi)^2 + f_1\big) + \beta(\phi)^2\beta'''(\phi) = 0\,, \tag{18}$$

where f_1 is a parameter, or equivalently its reduced form with integrating constants β_1, β_2

$$4\beta(\phi)^4\beta'(\phi)^2 + 4\beta(\phi)^6 - 4\beta_1\beta(\phi)^2 + f_1\beta(\phi)^4 = \beta_2\,. \tag{19}$$

We were not able to solve this autonomous first order differential equation in general in an explicit form. As was noted in [7], a hodograph transformation leads to $\phi(\beta)$ obtainable by a quadrature in terms of elliptic integrals. However, the subsequent inversion to find $\beta(\phi)$ is either impossible or not illuminating, therefore the authors of [7] proceed only with some special solutions. e.g. by choosing $\beta_2 = 0$, $f_1 < 0$, and $-\frac{f_1^2}{64} < \beta_1 < 0$, we obtain a well-defined solution

$$\beta(\phi) = \sqrt{\frac{\sqrt{64\beta_1 + f_1^2}\,\sin(2(\phi - \phi_0)) - f_1}{8}}\,. \tag{20}$$

Using the ODEs above to eliminate derivatives of $\beta(\phi)$, the electromagnetic field for any solution $\beta(\phi)$ of (19) is given by

$$\begin{aligned}
B^r &= -\tau_1\frac{\sqrt{4\beta_1\beta(\phi)^2 + \beta_2 - 4\beta(\phi)^6 - f_1\beta(\phi)^4}}{2r^2\beta(\phi)^5}\,,\\
B^\phi &= \frac{\tau_1}{r^3\beta(\phi)^2}\,, \qquad B^Z = \frac{2\beta_1\beta(\phi)^2 + \beta_2}{4r^2\beta(\phi)^5}\,,\\
W &= \frac{W_0}{r^2\beta(\phi)^2} - \frac{(4\tau_1^2 + \beta_2)}{32r^4\beta(\phi)^4} + \hbar^2\frac{f_1\beta(\phi)^4 - 12\beta_1\beta(\phi)^2 - 5\beta_2}{32r^2\beta(\phi)^6}\,.
\end{aligned} \tag{21}$$

The sign of the square root depends on the branch chosen while substituting for $\beta'(\phi)$ from (19).

We note that the magnetic field is the same classically and quantum mechanically in both cases, only the scalar potential W obtains an $\hbar^2$-proportional correction depending on β. This system admits $\tau_1 = 0$, which leads to its separation into free 1D motion plus 2D motion in a perpendicular magnetic field.

The lower order terms of the cylindrical integral X_1 from (9) are determined by

$$\begin{aligned}
s_1^r &= \frac{\sqrt{4\beta_1\beta(\phi)^2 + \beta_2 - 4\beta(\phi)^6 - \beta(\phi)^4 f_1}}{2\beta(\phi)^2}\,,\\
s_1^\phi &= -\frac{\beta(\phi)}{r}, \qquad s_1^Z = \tau_0 + \frac{\tau_1}{\beta(\phi)^2}\,,\\
m_1 &= \frac{2W_0}{\beta(\phi)^2} - \frac{4\beta(\phi)^2\tau_0\tau_1 + 2\beta_1\beta(\phi)^2 + 4\tau_1^2 + \beta_2}{8\beta(\phi)^4 r^2} + \hbar^2\frac{f_1\beta(\phi)^4 - 12\beta_1\beta(\phi)^2 - 5\beta_2}{16r^2\beta(\phi)^6}\,,
\end{aligned} \tag{22}$$

the integral X_2 reduces to the first order one $\tilde{X}_2 = p_Z$ in a suitable gauge.

4 rank$(M) = 1$

Here the general form of the fields is as follows

$$\begin{aligned} B^x &= 0\,, \quad B^y = 0\,, \quad B^z = B^z(x,y)\,, \\ W &= W_{12}(x,y) + W_3(z)\,. \end{aligned} \tag{23}$$

This implies that the system again separates into the 1D motion in the z direction, influenced by the scalar potential $W_3(z)$ and no magnetic field ($B^x = B^y = 0$), and the motion in the xy direction determined by a perpendicular magnetic field $B^z(x,y)$ and a scalar potential $W_{12}(x,y)$ containing *a priori* a quantum correction.

The presence of the scalar potential $W_3(z)$, which is not constrained further, implies that the cylindrical integral,

$$X_2 = \left(p_z^A\right)^2 + 2W_3(z)\,, \tag{24}$$

does not reduce to a first order one and the motion in the z direction is no longer free.

The remaining 2D motion, studied earlier in [7,8], is integrable due to the integral X_1 from (9). The relevant magnetic field $B^z(x,y)$, the 2D scalar potential $W_{12}(x,y)$ and the functions $s_1^{r,\phi,Z}, m_1$ coincide with the results of Section 3, namely case 1, see (16), and case 2 with $\tau_1 = 0$, see (21). In both cases neither of the cylindrical integrals reduces to a first order one (with potential exceptions for some special solutions of $\beta(\phi)$), therefore the results of [6,18] imply that these systems are in general not separable.

5 Conclusions

In this article we presented the classification of quantum quadratically integrable systems of cylindrical type obtained in O. Kubů's Master thesis [16]. We proceeded by directly solving the determining equations (2). Despite our focus on the quantum case, we followed the analysis from the classical case [13] to facilitate comparison and because only the zeroth order equations contain a correction, see the reduced equation (15).

Noting that the results of case a) coincide with the classical ones known from [13], we analyze further only case b) where the quantum correction is *a priori* non–trivial. We find that in all remaining subcases the magnetic fields coincide with their classical counterparts and only the scalar potential W is modified by a $\hbar^2$-proportional correction. However, even here it may vanish due to the consistency conditions on the scalar potential W, leaving us with 2 systems with a correction, namely system 2 in Section 3 and its counterpart in Section 4 (in the latter $\tau_1 = 0$ is necessary).

In all cases there is at least one free parameter, a constant or even a function. It is therefore probable that some superintegrable systems can be found by imposing further restrictions. This has been done in classical mechanics for separable systems [19] and on the intersection with other integrable systems [20], but only for first order integrals in quantum mechanics [16]. Easing these restrictions is necessary as well as going beyond integrals connected to orthogonal separation of variables as was shown in [14].

Acknowledgements

OK thanks late Pavel Winternitz, his supervisor and host during his ERASMUS+ stay at Université de Montréal, where the work on his Master thesis started.

Funding information The research in OK's Master thesis, on which the article is based, was supported by the Czech Science Foundation (Grant Agency of the Czech Republic), project 17-11805S, and Grant Agency of the Czech Technical University in Prague, grant No. SGS19/183/OHK4/3T/14. OK's presentation at Group 34 conference and recent research is supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS22/178/OHK4/3T/14.

References

[1] P. Winternitz, J. A. Smorodinsky, M. Uhlíř and I. Friš, *Symmetry groups in classical and quantum mechanics*, Soviet J. Nucl. Phys. **4**, 444 (1967).

[2] A. A. Makarov, J. A. Smorodinsky, K. Valiev and P. Winternitz, *A systematic search for nonrelativistic systems with dynamical symmetries*, Nuovo Cimento A **52**, 1061 (1967), doi:10.1007/BF02755212.

[3] J. Friš, V. Mandrosov, Y. A. Smorodinsky, M. Uhlíř and P. Winternitz, *On higher symmetries in quantum mechanics*, Phys. Lett. **16**, 354 (1965), doi:10.1016/0031-9163(65)90885-1.

[4] L. P. Eisenhart, *Separable systems of stackel*, Ann. Math. **35**, 284 (1934), doi:10.2307/1968433.

[5] A. M. Escobar-Ruiz, J. C. López Vieyra and P. Winternitz, *Fourth order superintegrable systems separating in polar coordinates. I. Exotic potentials*, J. Phys. A: Math. Theor. **50**, 495206 (2017), doi:10.1088/1751-8121/aa9203.

[6] V. N. Shapovalov, V. G. Bagrov and A. G. Meshkov, *Separation of variables in the stationary Schroedinger equation*, Sov. Phys. J. **15**, 1115 (1972), doi:10.1007/bf00910289.

[7] J. Bérubé and P. Winternitz, *Integrable and superintegrable quantum systems in a magnetic field*, J. Math. Phys. **45**, 1959 (2004), doi:10.1063/1.1695447.

[8] E. McSween and P. Winternitz, *Integrable and superintegrable Hamiltonian systems in magnetic fields*, J. Math. Phys. **41**, 2957 (2000), doi:10.1063/1.533283.

[9] A. Marchesiello, L. Šnobl and P. Winternitz, *Three-dimensional superintegrable systems in a static electromagnetic field*, J. Phys. A: Math. Theor. **48**, 395206 (2015), doi:10.1088/1751-8113/48/39/395206.

[10] A. Zhalij, *Quantum integrable systems in three-dimensional magnetic fields: The Cartesian case*, J. Phys.: Conf. Ser. **621**, 012019 (2015), doi:10.1088/1742-6596/621/1/012019.

[11] A. Marchesiello and L. Šnobl, *An infinite family of maximally superintegrable systems in a magnetic field with higher order integrals*, Symmetry Integr. Geom.: Methods Appl. **14**, 092 (2018), doi:10.3842/SIGMA.2018.092.

[12] S. Bertrand and L. Šnobl, *On rotationally invariant integrable and superintegrable classical systems in magnetic fields with non-subgroup type integrals*, J. Phys. A: Math. Theor. **52**, 195201 (2019), doi:10.1088/1751-8121/ab14c2.

[13] F. Fournier, L. Šnobl and P. Winternitz, *Cylindrical type integrable classical systems in a magnetic field*, J. Phys. A: Math. Theor. **53**, 085203 (2020), doi:10.1088/1751-8121/ab64a6.

[14] A. Marchesiello and L. Šnobl, *Pairs of commuting quadratic elements in the universal enveloping algebra of Euclidean algebra and integrals of motion*, J. Phys. A: Math. Theor. **55**, 145203 (2022), doi:10.1088/1751-8121/ac515e.

[15] O. Kubů, A. Marchesiello and L. Šnobl, *New classes of quadratically integrable systems in magnetic fields: The generalized cylindrical and spherical cases* Ann. Phys. **451**, 169264 (2022), doi:10.1016/j.aop.2023.169264.

[16] O. Kubů, *Integrable and superintegrable systems of cylindrical type in magnetic fields*, MSc thesis, Czech Technical University in Prague, Czech Republic, (arXiv preprint) doi:10.48550/arXiv.2210.02393.

[17] A. Marchesiello, L. Šnobl and P. Winternitz, *Spherical type integrable classical systems in a magnetic field*, J. Phys. A: Math. Theor. **51**, 135205 (2018), doi:10.1088/1751-8121/aaae9b.

[18] S. Benenti, C. Chanu and G. Rastelli, *Variable separation for natural Hamiltonians with scalar and vector potentials on Riemannian manifolds*, J. Math. Phys. **42**, 2065 (2001), doi:10.1063/1.1340868.

[19] O. Kubů, A. Marchesiello and L. Šnobl, *Superintegrability of separable systems with magnetic field: The cylindrical case*, J. Phys. A: Math. Theor. **54**, 425204 (2021), doi:10.1088/1751-8121/ac2476.

[20] S. Bertrand, O. Kubů and L. Šnobl, *On superintegrability of 3D axially-symmetric non-subgroup-type systems with magnetic fields*, J. Phys. A: Math. Theor. **54**, 015201 (2020), doi:10.1088/1751-8121/abc4b8.

SO(1,3) Yang-Mills solutions on Minkowski space via cosets

Kaushlendra Kumar*°

Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstraße 2, 30167 Hannover, Germany

* kaushal.kumar224@gmail.com

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We present a novel family of Yang–Mills solutions, with gauge group SO(1,3), on Minkowski space that are geometrically distinguished into two classes, viz. interior and exterior of the lightcone. We achieve this by foliating the former with SO(1,3)/SO(3) cosets and the latter with SO(1,3)/SO(1,2) cosets and analytically solving the Yang–Mills equation of an SO(1,3)-invariant gauge field. The resulting fields and their stress-energy tensor, when translated to the Minkowski space, diverge at the lightcone, but we demonstrate how this stress-energy tensor could be regularised due to its unique algebraic structure.

Received 05-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.033

1 Introduction

Analytic solutions of Yang-Mills theory with compact gauge groups, such as SU(2), and finite action are very few, like the ones presented in [1–3] or more recently in [4–6]. Here we improve upon this situation by presenting new solutions [7], albeit with a non-compact gauge group G=SO(1,3), i.e. the Lorentz group. The latter appears in a gauge theory formulation of general relativity and could be relevant for emergent/modified theories of gravity including supergravity and matrix models.

The construction of these solutions relies on the fact that, owing to the natural action of the Lorentz group on Minkowski space $\mathbb{R}^{1,3}$, there exists foliations of $\mathbb{R}^{1,3}$ into G-orbits that are reductive and symmetric coset spaces G/H. Specifically, on the inside of the lightcone we have H=SO(3) and on the outside of the lightcone we have H=SO(1,2). The former is the Riemannian two-sheeted hyperbolic space H^3, foliating the future and past of the lightcone with timelike parameter u, while the latter is a pseudo-Riemmanian de Sitter space dS_3, foliating the exterior of lightcone with spacelike parameter u. The Yang–Mills dynamics on these

° Address since July 1, 2023: Institute for Physics, Humboldt Universität, Zum Großen Windkanal 2, 12489 Berlin, Germany.

spaces are separately studied by considering a G-invariant gauge connection $\mathcal{A}$ and employing dimensional reduction on $\mathbb{R}\times G/H$ that yields a Newton's particle, parameterized by u and subject to an inverted double-well potential, which admits analytic solutions.

These field configurations are then pulled back to the respective domains of the Minkowski space, yielding the color-electromagnetic fields—diverging at the lightcone—that we then use to compute the stress-energy tensor. The latter turns out to have the same form in both the domains; this form, curiously, takes the shape of a pure *improvement term*. This fact can be used to regularize the stress-energy tensor across the lightcone so that one can have matching of the fields, defined on two domains of the spacetime, at the lightcone.

2 Minkowskian geometry and its foliations

We can foliate the Minkowski space $\mathbb{R}^{1,3}$, with metric $(\eta_{\mu\nu})=(-,+,+,+)$ for $\mu,\nu=0,1,2,3$, in two parts as depicted in Figure 1a: (a) the lightcone-interior $\mathcal{T}$ with two-sheeted hyperbolic space H^3 and (b) the lightcone-exterior $\mathcal{S}$ with single-sheeted de Sitter space dS_3.

For the first case, the hyperbolic space H^3 is embedded in $\mathbb{R}^{1,3}$ algebraically as

$$y\cdot y\equiv\eta_{\mu\nu}\,y^\mu\,y^\nu\;=\;-1\,, \tag{1}$$

and foliates—with a timelike parameter u obeying $\mathrm{e}^u=\sqrt{|x\cdot x|}$—the lightcone-interior $\mathcal{T}$ as[1]

$$\begin{aligned}
\varphi_{\mathcal{T}}:\;&\mathbb{R}\times H^3\to\mathcal{T}\,,\quad (u,y^\mu)\mapsto x^\mu:=\mathrm{e}^u\,y^\mu\,,\\
\varphi_{\mathcal{T}}^{-1}:\;&\mathcal{T}\to\mathbb{R}\times H^3\,,\quad x^\mu\mapsto(u,y^\mu):=\left(\ln\sqrt{|x\cdot x|}\,,\,\frac{x^\mu}{\sqrt{|x\cdot x|}}\right).
\end{aligned} \tag{2}$$

With this, the metric on $\mathcal{T}$ becomes conformal to a Lorentzian cylinder $\mathbb{R}\times H^3$:

$$\mathrm{d}s^2_{\mathcal{T}}\;=\;\mathrm{e}^{2u}\left(-\mathrm{d}u^2+\mathrm{d}s^2_{H^3}\right)\,, \tag{3}$$

where $\mathrm{d}s^2_{H^3}$ is the flat metric on H^3 arising from (1).

In the second case, we can embed dS_3 inside the Minkowski space $\mathbb{R}^{1,3}$ by

$$y\cdot y\equiv\eta_{\mu\nu}\,y^\mu\,y^\nu\;=\;1\,. \tag{4}$$

The foliation of $\mathcal{S}$ follows analogous to the previous case, albeit with a spacelike foliation parameter u satisfying $\mathrm{e}^u=\sqrt{|x\cdot x|}\equiv\sqrt{r^2-t^2}$:

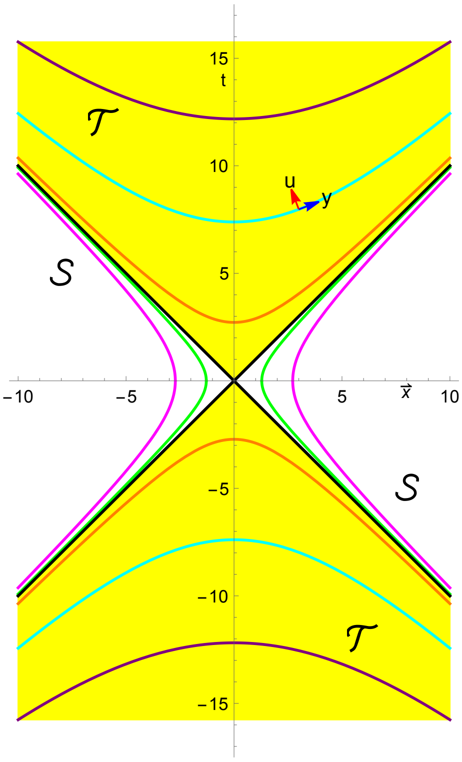
$$\begin{aligned}
\varphi_{\mathcal{S}}:\;&\mathbb{R}\times\mathrm{dS}_3\to\mathcal{S}\,,\quad (u,y^\mu)\mapsto x^\mu:=\mathrm{e}^u\,y^\mu\,,\\
\varphi_{\mathcal{S}}^{-1}:\;&\mathcal{S}\to\mathbb{R}\times\mathrm{dS}_3\,,\quad x^\mu\mapsto(u,y^\mu):=\left(\ln\sqrt{|x\cdot x|},\,\frac{x^\mu}{\sqrt{|x\cdot x|}}\right),
\end{aligned} \tag{5}$$

such that the metric on $\mathcal{S}$ becomes conformal to the metric on a cylinder $\mathbb{R}\times\mathrm{dS}_3$,

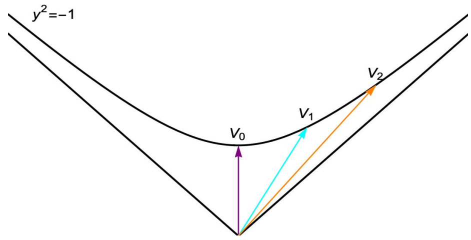
$$\mathrm{d}s^2_{\mathcal{S}}\;=\;\mathrm{e}^{2u}\left(\mathrm{d}u^2+\mathrm{d}s^2_{\mathrm{dS}_3}\right)\,, \tag{6}$$

where the flat dS_3-metric $\mathrm{d}s^2_{\mathrm{dS}_3}$ is induced from (4).

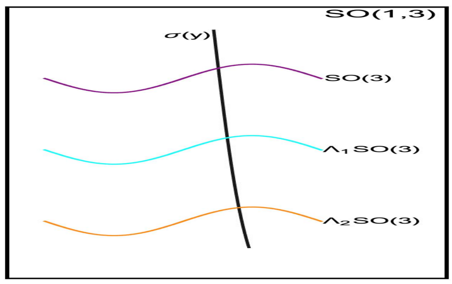
[1] We employ standard conventions $x^0{=}t, x^1{=}x, x^2{=}y, x^3{=}z$, $\vec{x}:=(x^1,x^2,x^3)$ and $r=\sqrt{\vec{x}\cdot\vec{x}}$ in this article.



(a) Foliation of interior (yellow region) and exterior of the lightcone with H^3- and dS_3-slices respectively. Every slicing has internal coordinate y and foliation parameter u.



(b) Every H^3 vector V_α is related to $V_0 \sim (1,0,0,0)^\top$ with a unique boost Λ_α, yielding its stability subgroup: $\Lambda_\alpha\,\mathrm{SO}(3)\,\Lambda_\alpha^{-1}$.



(c) The above H^3 vectors V_α lies in one-to-one correspondence with cosets $\Lambda_\alpha\,\mathrm{SO}(3)$ so that H^3, as a 3-dimensional submanifold $\sigma(y)$ inside SO(1,3), arises through a choice of representative σ in each of these cosets.

Figure 1: Minkowski foliations and demonstration of $H^3 \cong \mathrm{SO}(1,3)/\mathrm{SO}(3)$.

3 Algebra/geometry of symmetric SO(1,3)-cosets

We consider here reductive coset spaces G/H with 6-dimensional Lie group $G = \mathrm{SO}(1,3)$ that are also symmetric. This yields homogeneous spaces H^3 and dS_3 with stablity subgroups H=SO(3) and H=$SO(1,2)$ respectively. The equivalence of SO(1,3)/SO(3) with H^3 is geometrically illustrated through Figures 1b and 1c.

The six generators $\{I_A\}$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ are nothing but the canonical rotation (J_a) and boost (K_a) generators of the Lorentz group:

$$J_1{=}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\ J_2{=}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix},\ J_3{=}\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\ K_1{=}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ K_2{=}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\ K_3{=}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}. \tag{7}$$

Moreover, for reductive cosets, $\mathfrak{g}$ splits into a Lie subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$ and its orthogonal complement[2] $\mathfrak{m}$; this is also reflected in the splitting of the generators $\{I_A\}$ as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \Longrightarrow \quad \{I_A\} = \{I_i\} \cup \{I_a\}, \quad \text{with} \quad i = 4,5,6, \quad \text{and} \quad a = 1,2,3, \tag{8}$$

where $\{I_i\}$ spans $\mathfrak{h}$ and $\{I_a\}$ spans $\mathfrak{m}$. They satisfy following commutation relations

$$[I_i, I_j] = f_{ij}{}^k I_k, \qquad [I_i, I_a] = f_{ia}{}^b I_b, \quad \text{and} \quad [I_a, I_b] = f_{ab}{}^i I_i. \tag{9}$$

[2]The orthogonality here is with respect to the Cartan–Killing metric, given by the trace of adjoint representation of these generators $\{I_A\}$.

Similarly, the left-invariant one-forms of SO(1,3) split into $\{e^A\} = \{e^i\} \cup \{e^a\}$, such that the following structure equations—with same structure coefficients as in (9)—are satisfied

$$\mathrm{d}e^a + f_{ib}{}^a\, e^i \wedge e^b \;=\; 0\,, \quad \text{and} \quad \mathrm{d}e^i + \tfrac{1}{2} f_{jk}{}^i\, e^j \wedge e^k + \tfrac{1}{2} f_{ab}{}^i\, e^a \wedge e^b \;=\; 0\,. \tag{10}$$

Here e^a yields the metric on G/H while $e^i = e^i_{\ a}\, e^a$ are linearly dependent.

3.1 Case I: $H^3 \cong$ SO(1,3)/SO(3)

For SO(1,3)/SO(3) the splitting (8) and structure coefficients (9) are given by

$$I_i = J_{i-3}\,, \;\; I_a = K_a \quad \Longrightarrow \quad f_{ij}{}^k \;=\; \varepsilon_{i-3\, j-3\, k-3}\,, \;\; f_{ia}{}^b \;=\; \varepsilon_{i-3\, a\, b}\,, \;\; f_{ab}{}^i \;=\; -\varepsilon_{a\, b\, i-3}\,. \tag{11}$$

The identification of coset space SO(1,3)/SO(3) with H^3 is seen from the following maps

$$\begin{aligned} \alpha_{\mathcal{T}} &: \; \mathrm{SO}(1,3)/\mathrm{SO}(3) \to H^3\,, \quad [\Lambda_{\mathcal{T}}] \mapsto y^\mu = (\Lambda_{\mathcal{T}})^\mu{}_0\,, \\ \alpha_{\mathcal{T}}^{-1} &: \; H^3 \to \mathrm{SO}(1,3)/\mathrm{SO}(3)\,, \quad y^\mu \mapsto [\Lambda_{\mathcal{T}}]\,, \end{aligned} \tag{12}$$

where the representative $\Lambda_{\mathcal{T}}$ of the coset $[\Lambda_{\mathcal{T}}] := \{\Lambda = \Lambda_{\mathcal{T}} h : h \in SO(3)\}$ is given by

$$\Lambda_{\mathcal{T}} \;=\; \begin{pmatrix} \gamma & \gamma\, \boldsymbol{\beta}^\top \\ \gamma\, \boldsymbol{\beta} & \mathbb{1} + (\gamma-1)\frac{\boldsymbol{\beta}\otimes\boldsymbol{\beta}}{\boldsymbol{\beta}^2} \end{pmatrix}, \quad \text{with} \quad \beta^a = \frac{y^a}{y^0}\,, \quad \gamma = \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} = y^0\,, \tag{13}$$

and $\boldsymbol{\beta}^2 = \delta_{ab}\, \beta^a \beta^b \geq 0$. It is straightforward to verify that the above map $\alpha_{\mathcal{T}}$ is well-defined and, in fact, $\Lambda_{\mathcal{T}}$ is a generic boost obtained from coset generators $I_a \in \mathfrak{m}$ as

$$\Lambda_{\mathcal{T}} \;=\; \exp(\eta^a\, I_a)\,, \quad \text{with} \quad \beta^a = \frac{\eta^a}{\sqrt{\boldsymbol{\eta}^2}} \tanh \sqrt{\boldsymbol{\eta}^2}\,, \quad \text{for} \quad \boldsymbol{\eta}^2 = \delta_{ab}\, \eta^a \eta^b\,. \tag{14}$$

We can now obtain the left-invariant one-forms by employing the Maurer–Cartan prescription:

$$\Lambda_{\mathcal{T}}^{-1}\, \mathrm{d}\Lambda_{\mathcal{T}} \;=\; e^a\, I_a + e^i\, I_i\,, \quad e^a = \Big(\delta^{ab} - \frac{y^a\, y^b}{y^0(1+y^0)}\Big)\mathrm{d}y^b\,, \quad \text{and} \quad e^i = \varepsilon_{i-3\, a\, b}\, \frac{y^a}{1+y^0}\, \mathrm{d}y^b\,, \tag{15}$$

such that e^a reproduces the metric on H^3 while e^i become linearly dependent as follows

$$\mathrm{d}s^2_{H^3} \;=\; \delta_{ab}\, e^a \otimes e^b\,, \quad \text{and} \quad e^i = e^i_{\ a}\, e^a\,, \quad \text{with} \quad e^i_{\ a} = \varepsilon_{a\, i-3\, b}\, \frac{y^b}{1+y^0}\,. \tag{16}$$

3.2 Case II: $\mathrm{dS}_3 \cong$ SO(1,3)/SO(1,2)

For the coset space SO(1,3)/SO(1,2) we chose the splitting (8) as follows

$$I_i \in \{K_1, K_2, J_3\}\,, \quad \text{and} \quad I_a \in \{J_1, J_2, K_3\}\,, \tag{17}$$

such that the structure coefficients (9) comes out to be

$$f_{ij}{}^k \;=\; \varepsilon_{i-3\, j-3\, k-3}(1-2\,\delta_{k6})\,, \qquad f_{ia}{}^b \;=\; \varepsilon_{i-3\, a\, b}(1-2\,\delta_{a3})\,, \quad \text{and} \quad f_{ab}{}^i \;=\; \varepsilon_{a\, b\, i-3}\,, \tag{18}$$

where no summation convention is used inside the brackets. As before, we demonstrate the equivalence between dS_3 and SO(1,3)/SO(1,2) through following well-defined maps

$$\begin{aligned} \alpha_{\mathcal{S}} &: \; \mathrm{SO}(1,3)/\mathrm{SO}(1,2) \to \mathrm{dS}_3\,, \quad [\Lambda_{\mathcal{S}}] \mapsto y^\mu := (\Lambda_{\mathcal{S}})^\mu{}_3\,, \\ \alpha_{\mathcal{S}}^{-1} &: \; \mathrm{dS}_3 \to \mathrm{SO}(1,3)/\mathrm{SO}(1,2)\,, \quad y^\mu \mapsto [\Lambda_{\mathcal{S}}]\,, \end{aligned} \tag{19}$$

where the representative left-coset element $\Lambda_{\mathcal{S}}$ is again obtained though exponentiation with coset generators $\{J_1, J_2, K_3\}$. The resultant Maurer–Cartan one-forms look like

$$\Lambda_{\mathcal{S}}^{-1}\,\mathrm{d}\Lambda_{\mathcal{S}} \;=\; e^a\,I_a + e^i\,I_i\,, \quad e^a = \mathrm{d}y^{3-a} - \frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^3\,, \quad \text{and} \quad e^i = -\varepsilon_{i-3\,a\,b}\,\frac{y^{3-a}}{1+y^3}\,\mathrm{d}y^{3-b}\,. \tag{20}$$

These one-forms behave as expected with (η_{ab})=$(-,+,+)$ due to the stabilizer SO(1,2):

$$\mathrm{d}s^2_{\mathrm{dS}_3} \;=\; \eta_{ab}\,e^a \otimes e^b\,, \quad \text{and} \quad e^i = e^i_a\,e^a\,, \quad \text{with} \quad e^i_a = \varepsilon_{i-3\,a\,b}\,\frac{y^{3-b}}{1+y^3}\,. \tag{21}$$

3.3 The lightcone exception

Before we move on, let us make a remark on the lightcone itself that sits here as an exception in the following manner. It can be shown that both future and past of the lightcone are individually isomorphic to SO(1,3)/ISO(2), where the stability subgroup is a Euclidean group E(2)=ISO(2) generated by two translations and one rotation. However, this coset space is non-reductive and, on top of that, this does not give rise to any foliation here, making this case unsuitable to study Yang–Mills dynamics as discussed in Section 4.

4 Yang–Mills fields from dimensional reduction

The study of Yang–Mills dynamics on $\mathbb{R} \times G/H$ via dimensional reduction is a well-known topic with excellent review in [8]. Given an orthonormal frame $\{e^u{:=}\mathrm{d}u, e^a\}$ on the cylinder $\mathbb{R} \times G/H$, we can write a generic connection one-form $\mathcal{A}$ in the "temporal" gauge $\mathcal{A}_u$=0 and its curvature two-form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ as follows

$$\mathcal{A} \;=\; \mathcal{A}_a\,e^a \qquad \Longrightarrow \qquad \mathcal{F} = \mathcal{F}_{ua}\,e^u \wedge e^a + \tfrac{1}{2}\mathcal{F}_{ab}\,e^a \wedge e^b\,. \tag{22}$$

Next, we expand the gauge field $\mathcal{A}_a$ in terms of full SO(1,3)-generators (8) as $\mathcal{A}_a = \mathcal{A}^i_a\,I_i + \mathcal{A}^b_a\,I_b$ and impose G-invariance on this, yielding following two conditions[3]

$$\mathcal{A}^i_a \;=\; e^i_a\,, \quad \text{and} \quad \mathcal{A}^a_b = \mathcal{A}^a_b(u)\,, \quad \text{with} \quad f_{ia}{}^c\,\mathcal{A}^a_b \;=\; f_{ib}{}^a\,\mathcal{A}^c_a\,. \tag{23}$$

Furthermore, for the symmetric spaces that concerns us, one finds that $\mathcal{A}^a_b(u) = \phi(u)\,\delta^a_b$ such that our G-invariant gauge field $\mathcal{A}$ depends on a single real function ϕ:

$$\mathcal{A} \;=\; I_i\,e^i + \phi(u)\,I_a\,e^a\,. \tag{24}$$

The components of the field strength $\mathcal{F}$, using (23) and (10), computes to

$$\mathcal{F}_{ua} \;=\; \dot{\phi}\,I_a\,, \quad \text{and} \quad \mathcal{F}_{ab} \;=\; (\phi^2{-}1)f_{ab}{}^i\,I_i\,, \quad \text{with} \quad \dot{\phi} := \partial_u \phi\,, \tag{25}$$

yielding the the color-electric field $\mathcal{E}_a = \mathcal{F}_{au} \in \mathfrak{m}$ and -magnetic field $\mathcal{B}_a = \frac{1}{2}\varepsilon_{abc}\mathcal{F}_{bc} \in \mathfrak{h}$ on the cylinder. Finally, to work out the dynamics of $\phi(u)$ we look at the Yang–Mills action

$$S_{\mathrm{YM}} \;=\; -\frac{1}{4g^2}\int \mathrm{tr}_{\mathrm{ad}}(\mathcal{F} \wedge *\mathcal{F})\,, \tag{26}$$

which simplifies drastically in both cases, viz. interior of the lightcone $\mathcal{T}$ with $M^3 := H^3$ and exterior of the lightcone with $M^3 := \mathrm{dS}_3$, as follows

$$S_{\mathrm{YM}} \;=\; \frac{6}{g^2}\int_{\mathbb{R}\times M^3} \mathrm{dvol}\,\left(\tfrac{1}{2}\dot{\phi}^2 - V(\phi)\right), \qquad V(\phi) \;=\; -\tfrac{1}{2}(\phi^2{-}1)^2\,, \tag{27}$$

[3] We can write the second relation more succinctly as $[I_i, \widetilde{\mathcal{A}}_a] = f_{ia}{}^b\,\widetilde{\mathcal{A}}_b$ for $\widetilde{\mathcal{A}}_a := \mathcal{A}^a_b\,I_a \in \mathfrak{m}$.

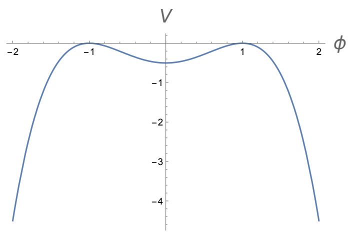


Figure 2: Plot of $V(\phi)$.

where $\mathrm{dvol} = \frac{1}{3!}\varepsilon_{abc}\, du \wedge e^a \wedge e^b \wedge e^c$ is the volume form. We immediately observe that the above action represents a mechanical particle $\phi(u)$ in an *inverted* double-well potential $V(\phi)$ depicted in Figure 2, which yields the equation of motion for an anharmonic oscillator,

$$\ddot{\phi} = -\frac{\partial V}{\partial \phi} = 2\phi\,(\phi^2-1). \tag{28}$$

This admits analytic solutions in terms of Jacobi elliptic functions. For example, in the bounded case where the mechanical energy $\frac{1}{2}\dot{\phi}^2 + V(\phi) =: \epsilon \in [-\frac{1}{2}, 0]$ we have

$$\phi_{\epsilon,u_0}(u) = f_-(\epsilon)\,\mathrm{sn}\big(f_+(\epsilon)(u-u_0), k\big), \quad \text{with} \quad f_\pm(\epsilon) = \sqrt{1 \pm \sqrt{-2\epsilon}}, \quad k^2 = \frac{f_-(\epsilon)}{f_+(\epsilon)}, \tag{29}$$

and a 'time'-shift parameter u_0. This also include special cases, such as a "kink":

$$\phi = \begin{cases} 0, & \text{for} \quad \epsilon = -\frac{1}{2}, \\ \tanh(u-u_0), & \text{for} \quad \epsilon = 0, \\ \pm 1, & \text{for} \quad \epsilon = 0. \end{cases} \tag{30}$$

Now in order to pull these solution back to $\mathcal{T}$ we transform the orthonormal frame $\{e^u, e^a\}$ on $\mathbb{R} \times H^3$, using the map $\varphi_{\mathcal{T}}$ (2) and abbreviation $|x| := \sqrt{|x \cdot x|}$, as follows

$$e^u := du = \frac{t\,dt - r\,dr}{t^2 - r^2}, \quad \text{and} \quad e^a = \frac{1}{|x|}\left(dx^a - \frac{x^a}{|x|}dt + \frac{x^a}{|x|(|x|+t)}r\,dr\right). \tag{31}$$

The SO(1,3)-invariant gauge field $\mathcal{A} \equiv A$ (24) can be casted into a Minkowski one-form

$$A = \frac{1}{|x|}\left\{\frac{\varepsilon_{ab}^{\ \ k-3}\,x^a}{|x|+t}dx^b I_k + \phi(x)\Big(dx^a - \frac{x^a}{|x|}dt + \frac{x^a}{|x|(|x|+t)}r\,dr\Big)I_a\right\}, \tag{32}$$

where $\phi(x) := \phi_{\epsilon,u_0}(u(x))$. We can then find the field strength $F = F_{\mu\nu}\,dx^\mu \wedge dx^\nu$ on $\mathcal{T}$ from its cylinder version $\mathcal{F}$ (25) using vierbein components $e^u = e^u_\mu\,dx^\mu$ and $e^a = e^a_\mu\,dx^\mu$ (31). The corresponding color-electric $E_i := F_{0i}$ and -magnetic $B_i := \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields read

$$\begin{aligned} E_a &= \frac{1}{|x|^3}\left\{(\phi^2-1)\,\varepsilon_{ab}^{\ \ i-3}\,x^b I_i - \dot{\phi}\Big(t\,\delta^{ab} - \frac{x^a x^b}{|x|+t}\Big)I_b\right\}, \\ B_a &= -\frac{1}{|x|^3}\left\{(\phi^2-1)\Big(t\,\delta^{a\,i-3} - \frac{x^a x^{i-3}}{|x|+t}\Big)I_i + \dot{\phi}\,\varepsilon_{ab}^{\ \ c}\,x^b I_c\right\}. \end{aligned} \tag{33}$$

An interesting feature of these fields is the presence of color-electromagnetic duality, i.e. $E_a \to B_a$ and $B_a \to -E_a$, which works when we simultaneously interchange $\dot{\phi} \leftrightarrow (\phi^2-1)$ and switch the generators as follows: $I_i \to I_a$ and $I_a \to -I_i$ (plus some obvious index adjustment). More importantly, we notice that the gauge field $\mathcal{A}$ (32) along with the electric E_i and magnetic B_i fields (33) become singular at the lightcone $t = \pm r$. One can find out the fields on $\mathcal{S}$ using (5) and following the same recipe as above. We restrain from reproducing these results here owing to space constraint and refer the reader to [7] for explicit form of such fields.

5 The stress-energy tensor

We can compute the stress-energy tensor, given by the expression

$$T_{\mu\nu} \;=\; -\tfrac{1}{2g^2}\,\mathrm{tr}_{\mathrm{ad}}\big(F_{\mu\alpha}\,F_{\nu\beta}\,\eta^{\alpha\beta} - \tfrac14\eta_{\mu\nu}F^2\big)\,, \quad \text{with} \quad F^2 = F_{\mu\nu}\,F^{\mu\nu}\,, \tag{34}$$

for such Yang–Mills fields in a straightforward manner. Interestingly, we find that the computation yields the same form of stress-energy tensor on both sides of the lightcone that reads

$$T \;=\; \frac{\epsilon}{g^2(r^2{-}t^2)^3}\begin{pmatrix} 3t^2{+}r^2 & -4tx & -4ty & -4tz \\ -4tx & t^2{+}4x^2{-}r^2 & 4xy & 4xz \\ -4ty & 4xy & t^2{+}4y^2{-}r^2 & 4yz \\ -4tz & 4xz & 4yz & t^2{+}4z^2{-}r^2 \end{pmatrix}. \tag{35}$$

It is worth emphasising here that the explicit form of ϕ, like in (29), is irrelevant here as T only depends on the total mechanical energy ϵ. Moreover, this has a vanishing trace and presence of lightcone singularity, as expected. Surprisingly, this admits a nice compact form that can be recasted into a pure "improvement" term as follows,

$$T_{\mu\nu} \;=\; \partial^\rho S_{\rho\mu\nu}\,, \quad \text{with} \quad S_{\rho\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{x_\rho\eta_{\mu\nu} - x_\mu\eta_{\rho\nu}}{(x{\cdot}x)^2}\,, \tag{36}$$

where the term $S_{\rho\mu\nu}$ can be expressed using abbreviation $(\tilde{S}_\rho)_{\mu\nu} := \frac{g^2(x\cdot x)^2}{\epsilon}S_{\rho\mu\nu}$ as

$$\tilde{S}_0 \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & -t & 0 & 0 \\ y & 0 & -t & 0 \\ z & 0 & 0 & -t \end{pmatrix}, \quad \tilde{S}_1 \;=\; \begin{pmatrix} -x & t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -y & x & 0 \\ 0 & -z & 0 & x \end{pmatrix}, \quad \tilde{S}_2 \;=\; \begin{pmatrix} -y & 0 & t & 0 \\ 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -z & y \end{pmatrix}, \quad \tilde{S}_3 \;=\; \begin{pmatrix} -z & 0 & 0 & t \\ 0 & z & 0 & -x \\ 0 & 0 & z & -y \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

Naturally, one hopes to glue the two expressions for stress-energy tensors to find a single expression valid across the Minkowski spacetime. The price to pay here is the singularity at the lightcone, which can be remedied through the following regularization procedure

$$S^{\mathrm{reg}}_{\rho\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{x_\rho\eta_{\mu\nu} - x_\mu\eta_{\rho\nu}}{(x{\cdot}x + \delta)^2} \qquad\Longrightarrow\qquad T^{\mathrm{reg}}_{\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{4\,x_\mu x_\nu - \eta_{\mu\nu}x{\cdot}x + 3\,\delta\,\eta_{\mu\nu}}{(x{\cdot}x+\delta)^3}\;. \tag{38}$$

This nonsingular improvement term (with a finite regularization parameter δ) yields vanishing energy and momenta as their fall-off behaviour at spatial infinity is fast enough.

An alternate route to regularization could be to directly shift only the denominator of $T_{\mu\nu}$ in (36) via $x{\cdot}x \mapsto x{\cdot}x + \delta$. We can then improve the resultant stress-energy tensor up to the term in (38) so as to obtain the following energy-momentum tensor candidate that is regular and that also vanishes as $\delta \to 0$,

$$T^{\delta}_{\mu\nu} \;=\; \frac{\epsilon}{g^2}\,\frac{4\,x_\mu x_\nu - \eta_{\mu\nu}x{\cdot}x}{(x{\cdot}x+\delta)^3} \;\sim\; \frac{\epsilon}{g^2}\,\frac{-3\,\delta\,\eta_{\mu\nu}}{(x{\cdot}x+\delta)^3}\,. \tag{39}$$

6 Conclusion

Starting from the geometry of the Minkowski foliations with H^3- (interior of the lightcone) and dS_3-slices (exterior of the lightcone) and exploring the origin of these symmetric spaces through cosets of the gauge group SO(1,3), we have obtained analytic solutions of Yang–Mills equation on Minkowski space that, however, diverge at the lightcone. We achieved this by first solving a SO(1,3)-invariant configuration on the cylinder $\mathbb{R}\times\mathrm{SO}(1,3)/H$, with $H{=}\mathrm{SO}(3)$ on the interior and $H{=}\mathrm{SO}(1,2)$ on the exterior of the lightcone, using dimensional reduction technique of gauge theory and then translating these solutions to two different domains of Minkowski spacetime, seperated by the lightcone, with their respective foliation maps. We then computed the stress-energy tensor in both cases and found out that they have the same form. Not only this, when written compactly, we were able to cast it into a pure improvement term, a fact that helped us in finding a regularized candidate for the stress-energy tensor, defined throughout the spacetime; how this modified stress-energy tensor arise from a source term remains an open question though.

Acknowledgements

The author thanks Olaf Lechtenfeld, Gabriel Picanço Costa and Jona Röhrig for contribution in [7]. The author is also grateful to OL for a careful reading of the draft and suggesting improvements.

Funding information This work was supported by the Deutscher Akademischer Austauschdienst (DAAD) with grant number 57381412.

References

[1] V. De Alfaro, S. Fubini and G. Furlan, *A new classical solution of the Yang-Mills field equations*, Phys. Lett. B **65**, 163 (1976), doi:10.1016/0370-2693(76)90022-8.

[2] M. Lüscher, *SO(4)-symmetric solutions of Minkowskian Yang-Mills field equations*, Phys. Lett. B **70**, 321 (1977), doi:10.1016/0370-2693(77)90668-2.

[3] B. M. Schechter, *Yang-Mills theory on the hypertorus*, Phys. Rev. D **16**, 3015 (1977), doi:10.1103/PhysRevD.16.3015.

[4] T. A. Ivanova, O. Lechtenfeld and A. D. Popov, *Solutions to Yang-Mills equations on four-dimensional de Sitter space*, Phys. Rev. Lett. **119**, 061601 (2017), doi:10.1103/PhysRevLett.119.061601.

[5] T. A. Ivanova, O. Lechtenfeld and A. D. Popov, *Finite-action solutions of Yang-Mills equations on de Sitter dS_4 and anti-de Sitter AdS_4 spaces*, J. High Energy Phys. **11**, 017 (2017), doi:10.1007/JHEP11(2017)017.

[6] T. A. Ivanova and O. Lechtenfeld, *Yang–Mills instantons and dyons on group manifolds*, Phys. Lett. B **670**, 91 (2008), doi:10.1016/j.physletb.2008.10.027.

[7] K. Kumar, O. Lechtenfeld, G. Picanço Costa and J. Röhrig, *Yang–Mills solutions on Minkowski space via non-compact coset spaces*, Phys. Lett. B **835**, 137564 (2022), doi:10.1016/j.physletb.2022.137564.

[8] D. Kapetanakis, *Coset space dimensional reduction of gauge theories*, Phys. Rep. **219**, 4 (1992), doi:10.1016/0370-1573(92)90101-5.

Spin degrees of freedom incorporated in conformal group: Introduction of an intrinsic momentum operator

Seiichi Kuwata*★*

Graduate School of Information Sciences, Hiroshima City University,
3-4-1 Ozuka, Asaminami-ku, Hiroshima 731-3194, Japan

★ kuwata@hiroshima-cu.ac.jp

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Considering spin degrees of freedom incorporated in the conformal generators, we introduce an intrinsic momentum operator π_μ, which is feasible for the Bhabha wave equation. If a physical state ψ_{ph} for spin s is annihilated by the π_μ, the degree of ψ_{ph}, $\deg\psi_{\text{ph}}$, should equal twice the spin degrees of freedom, $2(2s+1)$ for a massive particle, where the multiplicity 2 indicates the chirality. The relation $\deg\psi_{\text{ph}} = 2(2s+1)$ holds in the representation $\text{R}_5(s,s)$, irreducible representation of the Lorentz group in five dimensions.

Received 16-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.034

1 Introduction

Conformal symmetry [1] has many applications in string theory and critical phenomena in condensed matter and statistical physics. For a scalar field, the conformal generators are composed of dilatation D, momentum P_μ, special conformal K_μ, and angular momentum $L_{\mu\nu}$. For a multicomponent field Φ, where spin degrees of freedom is incorporated as $L_{\mu\nu} \to L_{\mu\nu} + s_{\mu\nu}$, the D and K_μ are generalized as $D \to D + \Delta$ and $K_\mu \to K_\mu + \kappa_\mu$, while the P_μ, in an ordinary context [1], remains unchanged as $P_\mu \to P_\mu$. The unchangeability of P_μ may be because Φ transforms as a scalar under spacetime translation. If we assume that $\Phi(x) \to \Phi'(x') = \Phi(x)$ under $x \to x' = x + a$, that is, $\Phi'(x) = \Phi(x-a) = \mathrm{e}^{-a\cdot P}\Phi(x)$, we find it unnecessary to introduce an intrinsic momentum operator π_μ as $P_\mu \to P_\mu + \pi_\mu$. Even if we admit the scalar property of $\Phi(x)$ under $x \to x + a$, we can introduce π_μ in such a way that the π_μ may annihilate physical states.

This paper aims to introduce such an intrinsic momentum operator π_μ, to find that π_μ can realize for a matrix structure in parafermion-based Dirac-like equations, such as spin-1 Kemmer equation [2], and more generally, Bhabha equation [3]. In Sec. 2, we give some preliminaries concerning the conformal algebra, together with its Casimir operator. In Secs. 3-5, we deal with the π_μ in the case of spin $\frac{1}{2}, 1, \frac{3}{2}$, respectively. We devote Sec. 6 to the summary.

2 Preliminaries

We begin with the commutation relations between the intrinsic conformal generators Δ, π_μ, κ_μ, and $s_{\mu\nu}$, corresponding to D, P_μ, K_μ, and $L_{\mu\nu}$, respectively. If the intrinsic conformal generators satisfy the same commutation relations as ordinary conformal generators, we can write the non-vanishing commutation relations as

$$[\Delta, \pi_\mu] = \mathrm{i}\pi_\mu\,, \quad [\Delta, \kappa_\mu] = -\mathrm{i}\kappa_\mu\,, \quad [\kappa_\mu, \pi_\nu] = 2\mathrm{i}(g_{\mu\nu}\Delta - s_{\mu\nu})\,, \tag{1}$$

$$[\pi_\rho, s_{\mu\nu}] = \mathrm{i}(g_{\rho\mu}\pi_\nu - g_{\rho\nu}\pi_\mu)\,, \quad [\kappa_\rho, s_{\mu\nu}] = \mathrm{i}(g_{\rho\mu}\kappa_\nu - g_{\rho\nu}\kappa_\mu)\,, \tag{2}$$

$$[s_{\mu\nu}, s_{\rho\sigma}] = \mathrm{i}(g_{\nu\rho}s_{\mu\sigma} + g_{\mu\sigma}s_{\nu\rho} - g_{\mu\rho}s_{\nu\sigma} - g_{\nu\sigma}s_{\mu\rho})\,, \tag{3}$$

while the vanishing commutation relations are given by

$$[\Delta, s_{\mu\nu}] = [\pi_\mu, \pi_\nu] = [\kappa_\mu, \kappa_\nu] = 0\,. \tag{4}$$

It should be remarked that (1)-(4) are invariant under the scaling of π_μ and κ_μ, and also under the substitution between π_μ and κ_μ as

$$(\Delta, \pi_\mu, \kappa_\mu, s_{\mu\nu}) \to (\Delta, \lambda\pi_\mu, \lambda^{-1}\kappa_\mu, s_{\mu\nu})\,, \tag{5}$$

$$(\Delta, \pi_\mu, \kappa_\mu, s_{\mu\nu}) \to (-\Delta, \kappa_\mu, \pi_\mu, s_{\mu\nu})\,, \tag{6}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, and use has been made of $s_{\nu\mu} = -s_{\mu\nu}$ in (6). Note that (5) represents the "chiral" transformation $g \to g' = \mathrm{e}^{\theta\Delta} g \mathrm{e}^{-\theta\Delta}$ ($g \in \{\Delta, \pi_\mu, \kappa_\mu, s_{\mu\nu}\}$), where $\lambda = \mathrm{e}^{\mathrm{i}\theta}$.

To check the irreducibility of the representation for the conformal group, it may be available to obtain the Casimir operator C. Note that although the C is invariant under (5) due to the chiral transformation, the invariance of C under (6) is somewhat naive. For simplicity, we consider $(3+1)$ spacetime dimensions, where the conformal algebra is isomorphic to $\mathfrak{so}(4,2)$ [1]. In this case, the order of C is given by $2, 3, 4$, as in the case of $\mathfrak{so}(6)$ [4]. Explicitly, we have $C = C_2, C_3, C_4$ (the index i in C_i represents the order) as [5]

$$\begin{aligned} C_2 &= \frac{1}{2}s_{\mu\nu}s^{\mu\nu} + \frac{1}{2}\{\kappa_\mu, \pi^\mu\} - \Delta^2\,, \qquad C_3 = \epsilon^{\mu\nu\rho\sigma}\big(\Delta s_{\mu\nu} + \{\kappa_\mu, \pi_\nu\}\big)s_{\rho\sigma}\,, \\ C_4 &= \frac{1}{2}\mathcal{J}_{\mu\nu}\mathcal{J}^{\mu\nu} - \frac{1}{2}\{\mathcal{J}_{K,\mu}, \mathcal{J}_P^\mu\} - \frac{1}{16}\mathcal{J}^2\,, \end{aligned} \tag{7}$$

where $\mathcal{J}^{\mu\nu}, \mathcal{J}_K^\mu, \mathcal{J}_P^\mu$, and $\mathcal{J}$ are given by $\mathcal{J}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\left(\Delta s_{\rho\sigma} + \frac{1}{2}\{\kappa_\rho, \pi_\sigma\}\right)$, $\mathcal{J}_K^\mu = \epsilon^{\mu\nu\rho\sigma}\kappa_\nu s_{\rho\sigma}$, $\mathcal{J}_P^\mu = \epsilon^{\mu\nu\rho\sigma}\pi_\nu s_{\rho\sigma}$, and $\mathcal{J} = \epsilon^{\mu\nu\rho\sigma}s_{\mu\nu}s_{\rho\sigma}$, with $\epsilon^{\mu\nu\rho\sigma}$ the totally anti-symmetric Levi-Civita tensor ($\epsilon^{0123} = 1$), and $\{A, B\} = AB + BA$. It confirms that all the C's are invariant under (5). If the $\epsilon^{\mu\nu\rho\sigma}$ remains invariant under (6), the C_i's transform as $(C_2, C_3\, C_4) \to (C_2, -C_3, C_4)$. However, the invariance of $\epsilon^{\mu\nu\rho\sigma}$ under (6) is not so trivial, which will be discussed at the end of the next section and afterward.

3 Spin $\frac{1}{2}$

This section deals with the Dirac equation, which describes a spin-$\frac{1}{2}$ particle. In this case, the spin operator $s_{\mu\nu}$, which satisfies (3), can be written using the gamma matrix γ_μ as $s_{\mu\nu} = \mathrm{i}\frac{1}{4}[\gamma_\mu, \gamma_\nu]$, where $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}$. The next thing is to obtain π_μ from the first equality in (2) and $[\pi_\mu, \pi_\nu] = 0$. Considering that $[\gamma_\rho, s_{\mu\nu}] = \mathrm{i}(g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu)$, one may suspect that π_μ may be given by $\pi_\mu = \lambda\gamma_\mu$ ($\lambda \in \mathbb{C}$), which, however, would not be appropriate due to $[\pi_\mu, \pi_\nu] \neq 0$. This conclusion is not the end of the story. For an even spacetime dimension, there is a matrix γ_5 such that $\gamma_5^2 = \mathbb{1}$ and $\{\gamma_5, \gamma_\mu\} = 0$. Under the existence of γ_5, the choice of $\pi_\mu = \lambda(\gamma_\mu \pm \gamma_5\gamma_\mu)$ satisfies the first equality in (2) and $[\pi_\mu, \pi_\nu] = 0$. In a similar way, we obtain $\kappa_\mu = \lambda'(\gamma_\mu \pm \gamma_5\gamma_\mu)$ from the second equality in (2) and $[\kappa_\mu, \kappa_\nu] = 0$. The relation between λ and λ', along with the remaining generator Δ, can be derived from (1). To summarize, we have

$$\Delta = \pm\frac{1}{2}\mathrm{i}\gamma_5\,, \quad \pi_\mu = M\left(\frac{\mathbb{1}\pm\gamma_5}{2}\right)\gamma_\mu\,, \quad \kappa_\mu = \frac{1}{M}\left(\frac{\mathbb{1}\mp\gamma_5}{2}\right)\gamma_\mu\,, \quad s_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu]\,, \tag{8}$$

where the multiplier $M \in \mathbb{C} \setminus \{0\}$ corresponds to λ in (5). Note that the substitution (6) can be interpreted as $\gamma_5 \to -\gamma_5$. Note also that $[\Delta, s_{\mu\nu}] = 0$.

The fundamental property of π_μ (or κ_μ) is the nilpotence of order two. Let $a_\mu^\pm := (\mathbb{1}\pm\gamma_5)\gamma_\mu$. Then it follows that

$$a_\nu^+ a_\mu^+ = 0 = a_\nu^- a_\mu^-\,. \tag{9}$$

To be more exact, we can show that

$$\begin{cases} a_\mu^+ \mathrm{P}_1 = 0\,, \\ a_\mu^- \mathrm{P}_2 = 0\,, \end{cases} \qquad \begin{cases} a_\mu^+ \mathrm{P}_2 = 2\mathrm{P}_1\gamma_\mu\,, \\ a_\mu^- \mathrm{P}_1 = 2\mathrm{P}_2\gamma_\mu\,, \end{cases} \tag{10}$$

where $\mathrm{P}_1 = \frac{1}{2}(\mathbb{1}+\gamma_5)$ and $\mathrm{P}_2 = \frac{1}{2}(\mathbb{1}-\gamma_5)$ represent the projection operators such that $\mathrm{P}_1 + \mathrm{P}_2 = \mathbb{1}$ and $\mathrm{P}_i\mathrm{P}_j = \delta_{ij}\mathrm{P}_i$. In the Dirac theory, it is well known that P_1 and P_2 are employed in the chiral decomposition. In this sense, (10) can be derived without recognizing the concept of the intrinsic momentum operator π_μ; the existence of π_μ will play a substantial role in higher spin states.

Now we give some properties concerning the Casimir operators C_i's in (7). First, we discuss the invariance of C_3 under (6). Recalling that the substitution (6) corresponds to $\gamma_5 \to -\gamma_5$, and that $\gamma_5 = -\frac{1}{4!}\mathrm{i}\,\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$, we find that $\gamma_5 \to -\gamma_5$ implies that $\epsilon^{\mu\nu\rho\sigma} \to -\epsilon^{\mu\nu\rho\sigma}$. In this sense, C_3 remains invariant under (6). Next, we obtain the relation between C_2 and C_4. Note that $\mathcal{J}^{\mu\nu}$ can be rewritten as $3\Delta\epsilon^{\mu\nu\rho\sigma}s_{\rho\sigma}$, which leads to $\mathcal{J}_{\mu\nu}\mathcal{J}^{\mu\nu} = 9s_{\mu\nu}s^{\mu\nu}$. In a similar way, we have $\{\mathcal{J}_{K,\mu}, \mathcal{J}_P^\mu\} = -9\{\kappa_\mu, \pi^\mu\}$ and $\frac{1}{16}\mathcal{J}^2 = 9\Delta^2$. Thus we obtain $C_4 = 9C_2$. Anyway, there is no such operator (except a scalar multiple of identity $\mathbb{1}$) that is commutative with all the γ_μ's, so that the C_i's are given by a multiple of identity $\mathbb{1}$ as $(C_2, C_3, C_4) = \frac{15}{4}(1, 2^2, 3^2)\mathbb{1}$.

4 Spin 1

This section deals with relativistically invariant wave equations for spin $s = 1$. For the sake of simplicity, spacetime dimension d is restricted to $(3+1)$. We summarize the wave functions for a free massive particle in Table 1, to find that the π_μ is allowed for the KDP equation but not for the Proca and the WSG equations. This is because the $n \times n$ matrix π_μ such that $[\pi_\rho, s_{\mu\nu}] = \mathrm{i}(g_{\rho\mu}\pi_\nu - g_{\rho\nu}\pi_\mu)$ is allowed for $n = 10$, but not for $n = 4, 6$. In what follows, we concentrate on the KDP equation, where the β_μ's satisfy the trilinear relations

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\rho + g_{\rho\nu}\beta_\mu \qquad (\mu, \nu, \rho \in \{0,1,2,3\})\,. \tag{11}$$

Note that β_i $(i = 1,2,3)$ can be identified with the non-relativistic spin-1 operator s_i in the sense that the s_i's satisfy $s_is_js_k + s_ks_js_i = \delta_{ij}s_k + \delta_{kj}s_i$.

For $n = 10$, it is known that [2] there is a matrix ω $(= \beta_5)$ which is given by extending (11) to those for $\mu, \nu, \rho \in \{0,1,2,3,5\}$ with $g_{5\mu} = g_{\mu 5} = \delta_{5\mu}$. Explicitly, we have

$$\omega^3 = \omega\,, \qquad \begin{cases} \{\omega^2, \beta_\mu\} = \beta_\mu\,, \\ \omega\beta_\mu\omega = 0\,, \end{cases} \qquad \begin{cases} \beta_\mu\omega\beta_\nu + \beta_\nu\omega\beta_\mu = 0\,, \\ \omega\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\omega = g_{\mu\nu}\omega\,. \end{cases} \tag{12}$$

Table 1: Lorentz invariant wave equations for $s = 1$ and $d = 3+1$. For the Proca equation, the upperscript in $\psi = (A^0, A^1, A^2, A^3)$ represents the Lorentz vector component, and $\Lambda_{\mu\nu}$ represents the generator of the Lorentz transformation. For the WSG equation, s_i $(i = 1,2,3)$ is given by the (3×3) representation matrix for the non-relativistic spin-1 operator.

Name	Equation	Degree of ψ	$s_{\mu\nu}$	π_μ
Proca	$(\Box + m^2)A^\mu = \partial^\mu(\partial\cdot A)$	4	$\Lambda_{\mu\nu}$	NA
WSG [6,7]	$(\Box + \gamma_{\mu\nu}\partial^\mu\partial^\nu)\psi = 2m_0^2\psi$	6	$\begin{cases} s_{0i} = \frac{1}{\mathrm{i}}\sigma_3\otimes s_i \\ s_{ij} = \mathbb{1}\otimes\epsilon_{ijk}s_k \end{cases}$	NA
KDP [2,8,9]	$(\mathrm{i}\beta_\mu\partial^\mu + m)\psi = 0$	10	$\mathrm{i}[\beta_\mu,\, \beta_\nu]$	✓

Then the intrinsic conformal generators are given by

$$\Delta = \pm\mathrm{i}\omega\,, \quad \pi_\mu = M\left(\beta_\mu \pm [\omega,\, \beta_\mu]\right)\,, \quad \kappa_\mu = \frac{1}{M}\left(\beta_\mu \mp [\omega,\, \beta_\mu]\right)\,, \quad s_{\mu\nu} = \mathrm{i}[\beta_\mu,\, \beta_\nu]\,. \tag{13}$$

Note that (13) reduces to (8) under $(\beta_\mu, \omega) \to \frac{1}{2}(\gamma_\mu, \gamma_5)$. It is not so difficult to obtain from (11) and (12) the nilpotence of π_μ as

$$\alpha^+_\mu\alpha^+_\nu\alpha^+_\rho = 0 = \alpha^-_\mu\alpha^-_\nu\alpha^-_\rho\,, \tag{14}$$

where $\alpha^\pm_\mu := \beta_\mu \pm [\omega,\, \beta_\mu]$. To be more exact, we have the following relations:

$$\begin{cases} \alpha^+_\mu \mathrm{P}_1 = 0\,, \\ \alpha^-_\mu \mathrm{P}_3 = 0\,, \end{cases} \quad \begin{cases} \alpha^+_\mu \mathrm{P}_2 = 2\mathrm{P}_1\beta_\mu\,, \\ \alpha^-_\mu \mathrm{P}_2 = 2\mathrm{P}_3\beta_\mu\,, \end{cases} \quad \begin{cases} \alpha^+_\nu\alpha^+_\mu \mathrm{P}_3 = 2\mathrm{P}_1 A_{\mu\nu}\,, \\ \alpha^-_\nu\alpha^-_\mu \mathrm{P}_1 = 2\mathrm{P}_3 A_{\mu\nu}\,, \end{cases} \tag{15}$$

where $A_{\mu\nu} = \{\beta_\mu, \beta_\nu\} - g_{\mu\nu}\mathbb{1}$, and P_i represents a projection operators as $\mathrm{P}_1 = \frac{1}{2}\omega(\omega + \mathbb{1})$, $\mathrm{P}_2 = \mathbb{1} - \omega^2$, and $\mathrm{P}_3 = \frac{1}{2}\omega(\omega - \mathbb{1})$, so that $\sum_{i=1}^3 \mathrm{P}_i = \mathbb{1}$ and $\mathrm{P}_i\mathrm{P}_j = \delta_{ij}\mathrm{P}_i$. Notice that in (15), the lower relations can derive from the corresponding upper ones through the substitution $\omega \to -\omega$. Notice further that $A_{\mu\nu}$ anticommutes with ω, that is

$$\{A_{\mu\nu},\, \omega\} = 0\,. \tag{16}$$

The relation (16) leads to $[A^\mu_\mu,\, \omega^2] = 0$. Note that A^μ_μ and ω are Lorentz invariant in the sense that $[s_{\alpha\beta}, A^\mu_\mu] = 0 = [s_{\alpha\beta},\, \omega]$. This relation implies that A^μ_μ can be written as $A^\mu_\mu = \sum_{i=0}^2 c_i\omega^i$ $(c_i \in \mathbb{C})$, where c_i $(i \geq 3)$ is not necessary due to $\omega^3 = \omega$. Here we have assumed that there is no Lorentz invariant other than $\mathbb{1}, \omega$, and ω^2. In this case, we find that $c_0 + c_2 = 0 = c_1$ from $\{A^\mu_\mu,\, \omega\} = 0$ by (16), and that $c_0 = 2$ from $\{\beta_\nu,\, \beta_\mu\beta^\mu\} = 5\beta_\nu$ by (11) and $\{\beta_\nu,\, \omega^2\} = \beta_\nu$ by (12). Eventually, we have

$$\beta_\mu\beta^\mu = \mathrm{P}_2 + 2\mathbb{1}\,. \tag{17}$$

Actually, the relation (17) holds in the ten-dimensional representation [2] for (11) and (12), which corresponds to the adjoint representation of the Lorentz group in five dimensions (for the adjoint representation, we have $\binom{5}{2} = 10$ Lorentz group generators). For later convenience, we rewrite $\frac{1}{2}s_{\mu\nu}s^{\mu\nu}$ using P_2 as

$$\frac{1}{2}s_{\mu\nu}s^{\mu\nu} = 4\mathbb{1} - \mathrm{P}_2\,, \tag{18}$$

where we have used (17), together with $\mathrm{P}_2^2 = \mathrm{P}_2$.

As was mentioned in Sec. 1, the π_μ should annihilate the physical state. To check the validity, we show that the rank of P_k (or equivalently, the trace of P_k) for $k = 1,3$ equals the spin degrees of freedom. In the ten-dimensional representation, the eigenvalues of ω are given by $1, 0, -1$ appearing $3, 4, 3$ times, respectively. Thus, we obtain

$$\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_3) = 3\,, \qquad \mathrm{Rank}(\mathsf{P}_2) = 4\,.$$

This result is quite reasonable because the number "3" equals the spin degree of freedom for a massive particle for $s = 1$. To confirm the validity, we calculate the 3-dimensional spin magnitude $\langle s\rangle^2 := s_{12}^2 + s_{23}^2 + s_{31}^2$. Let $|\psi^+_{\mathrm{ph}}\rangle = \mathsf{P}_1|\psi\rangle$, $|\psi^-_{\mathrm{ph}}\rangle = \mathsf{P}_3|\psi\rangle$, and $|\psi_{\mathrm{un}}\rangle = \mathsf{P}_2|\psi\rangle$, in which we have $\alpha^\pm_\mu|\psi^\pm_{\mathrm{ph}}\rangle = 0$. Recalling that $\langle s\rangle^2$ $(= \frac{1}{4}s_{\mu\nu}s^{\mu\nu}) = 2\mathbb{1} - \frac{1}{2}\mathsf{P}_2$ by (18), and that $\mathsf{P}_i\mathsf{P}_j = \delta_{ij}\mathsf{P}_i$, we obtain $\langle s\rangle^2|\psi^\pm_{\mathrm{ph}}\rangle = s(s+1)|\psi^\pm_{\mathrm{ph}}\rangle$ $(s = 1)$ and $\langle s\rangle^2|\psi_{\mathrm{un}}\rangle = \frac{3}{2}|\psi_{\mathrm{un}}\rangle$. These relations indicate that $|\psi^\pm_{\mathrm{ph}}\rangle$ represents the spin-1 state, while $|\psi_{\mathrm{un}}\rangle$ does not. Bearing these findings in mind, we can regard $|\psi^\pm_{\mathrm{ph}}\rangle$ and $|\psi_{\mathrm{un}}\rangle$ as physical and unphysical states, respectively.

Finally, we give some properties of the Casimir operator C. As in the case of $s = \frac{1}{2}$, the invariance of C_3 under (6) is guaranteed by the statement that $(\omega \to -\omega) \Longrightarrow (\epsilon^{\mu\nu\rho\sigma} \to -\epsilon^{\mu\nu\rho\sigma})$ by $\omega = -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$ [10, 11]. After a somewhat tedious calculation, we can write the C_i's in (7) as $(C_2, C_3, C_4) = (9, 48, 144)\mathbb{1}$, which confirms the irreducibility of the ten-dimensional representation.

5 Spin $\frac{3}{2}$

In this section, we consider the $(3+1)$-dimensional Minkowski space, as in the case of $s = 1$. Although the Rarita-Schwinger equation is well known as a relativistic invariant wave equation for $s = \frac{3}{2}$, the intrinsic momentum operator is not allowed, as in the case of the Proca equation. Instead, we adopt a Dirac-like wave equation for parafermion of order 3, namely (massive) Bhabha wave equation [3] (see Table 2).

Extending the polynomial relations among the non-relativistic spin operators s_i's $(i=1,2,3)$ to those among s_μ's $(\mu = 0, 1, 2, 3)$ in a relativistically covariant way, we obtain

$$\begin{cases} s_\mu s_\nu s_\alpha + s_\alpha s_\nu s_\mu + g_{\mu\alpha}s_\nu = s_\mu s_\alpha s_\nu + s_\nu s_\alpha s_\mu + g_{\mu\nu}s_\alpha\,, \\ 0 = \left(s_\mu s_\nu s_\alpha s_\beta - \frac{5}{4}\{s_\mu, s_\nu\}g_{\alpha\beta} + \frac{9}{16}g_{\mu\nu}g_{\alpha\beta}\right) + (\text{perm. of } \mu, \nu, \alpha, \beta)\,. \end{cases} \tag{19}$$

It may be convenient to rewrite the first relation of (19) as $[s_\mu, [s_\nu, s_\alpha]] = g_{\mu\nu}s_\alpha - g_{\mu\alpha}s_\nu$. Note that $\frac{1}{2}\gamma_\mu$ satisfies both relations in (19). This implies that there should exist a polynomial relation such that $p(s_0, s_1, s_2, s_3) = 0$ with $p(s_0, s_1, s_2, s_3)|_{s_\mu \to \frac{1}{2}\gamma_\mu} \neq 0$. However, we neglect, for the time being, such a polynomial relation because it is not irrelevant to the following discussion. Suppose that there exists an operator s_5 which satisfies (19) for $\mu, \nu, \alpha, \beta \in \{0, 1, 2, 3, 5\}$, with

Table 2: Lorentz invariant wave equations for $s = \frac{3}{2}$. For the Rarita equation, ψ is composed of four Dirac spinors as $\psi := (\psi_0, \psi_1, \psi_2, \psi_3)$, where the subscript represents the Lorentz vector component, so that Λ $(= \{\Lambda_{\mu\nu}\}) : \psi \mapsto \psi'$ acts as $(\psi')_\mu = \Lambda_\mu^{\ \nu}\psi_\nu$.

Name	Equation	Degree of ψ	$s_{\mu\nu}$	π_μ
Rarita-Schwinger	$(\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho + mg^{\mu\sigma})\psi_\sigma = 0$	4×4	$\Lambda_{\mu\nu} + \frac{i}{4}[\gamma_\mu, \gamma_\nu]$	NA
Bhabha	$(\mathrm{i}s_\mu\partial^\mu + m)\psi = 0$	20	$\mathrm{i}[s_\mu, s_\nu]$	✓

$g_{5\mu} = g_{\mu 5} = \delta_{5\mu}$. Then the intrinsic conformal generators are given, as is analogous to the case of $s = \frac{1}{2}, 1$, by

$$\Delta = \pm \mathrm{i} s_5\,, \quad \pi_\mu = M\left(s_\mu \pm [s_5, s_\mu]\right), \quad \kappa_\mu = \frac{1}{M}\left(s_\mu \mp [s_5, s_\mu]\right), \quad s_{\mu\nu} = \mathrm{i}[s_\mu, s_\nu]\,. \tag{20}$$

Note that the first equality in (19), together with the existence of s_5, is sufficient for (20); the second equality in (19) is not necessary for (20). Recalling that the first relation in (19) is satisfied for $s_\mu \to \frac{1}{2}\gamma_\mu$ ($s = \frac{1}{2}$) and for $s_\mu \to \beta_\mu$ ($s = 1$), we find it natural that the relation (20) is the same form as (8) and (13). For later convenience, we obtain some operators which anti-commute with s_5. Such operators are exemplified as

$$\{s_5, A_\mu\} = 0 = \{s_5, A_{\rho\nu\mu} + (\text{perm. of } \rho, \nu, \mu)\}\,, \tag{21}$$

where $A_\mu = s_5 s_\mu s_5 - \frac{3}{4}s_\mu$, and $A_{\rho\nu\mu} = s_\rho s_\nu s_\mu - \frac{7}{4}g_{\rho\nu}s_\mu$.

The projection operators P_i's ($i = 1,2,3,4$) can be written using the minimum polynomial $f(x)$ with respect to s_5 as $\mathsf{P}_i = \frac{1}{f'(\lambda_i)}\frac{f(s_5)\mathbb{1}}{s_5 - \lambda_i\mathbb{1}}$, where $f(x) = \prod_{i=1}^4 (x - \lambda_i)$, with $\lambda_1 = \frac{3}{2}$, $\lambda_2 = \frac{1}{2}$, $\lambda_3 = -\frac{1}{2}$, $\lambda_4 = -\frac{3}{2}$. Let $s_\mu^\pm := s_\mu \pm [s_5, s_\mu]$. Then it follows that (see Appendix A)

$$\begin{cases} s_\mu^+ \mathsf{P}_1 = 0\,, \\ s_\mu^- \mathsf{P}_4 = 0\,, \end{cases} \quad \begin{cases} s_\mu^+ \mathsf{P}_2 = 2\mathsf{P}_1 X_\mu\,, \\ s_\mu^- \mathsf{P}_3 = 2\mathsf{P}_4 X_\mu\,, \end{cases} \quad \begin{cases} s_\nu^+ s_\mu^+ \mathsf{P}_3 = 2\mathsf{P}_1 X_{\nu\mu}\,, \\ s_\nu^- s_\mu^- \mathsf{P}_2 = 2\mathsf{P}_4 X_{\nu\mu}\,, \end{cases} \quad \begin{cases} s_\rho^+ s_\nu^+ s_\mu^+ \mathsf{P}_4 = \frac{4}{3}\mathsf{P}_1 X_{\rho\nu\mu}\,, \\ s_\rho^- s_\nu^- s_\mu^- \mathsf{P}_1 = \frac{4}{3}\mathsf{P}_4 X_{\rho\nu\mu}\,, \end{cases} \tag{22}$$

where X_μ, $X_{\nu\mu}$ and $X_{\rho\nu\mu}$ are given by

$$X_\mu = s_\mu\,, \quad X_{\nu\mu} = \{s_\nu, s_\mu\} - s g_{\nu\mu}\mathbb{1}\,, \quad X_{\rho\nu\mu} = \left[Y_{\rho\nu\mu} + (\text{perm. of } \rho, \nu, \mu)\right],$$

with $s = \frac{3}{2}$ and $Y_{\rho\nu\mu} := s_\rho s_\nu s_\mu - g_{\rho\nu}(s s_\mu + \frac{1}{2s}s_5 s_\mu s_5) \to A_{\rho\nu\mu} - \frac{1}{3}g_{\rho\nu}A_\mu \quad (s = \frac{3}{2})$. The relations (22) lead to $s_\mu^+ s_\nu^+ s_\rho^+ s_\sigma^+ \mathsf{P}_i = 0 = s_\mu^- s_\nu^- s_\rho^- s_\sigma^- \mathsf{P}_i$ ($i = 1,2,3,4$), from which, together wirh $\sum_{i=1}^4 \mathsf{P}_i = \mathbb{1}$, we obtain the nilpotence of $s_\mu^\pm$ (of order 4) as

$$s_\mu^+ s_\nu^+ s_\rho^+ s_\sigma^+ = 0 = s_\mu^- s_\nu^- s_\rho^- s_\sigma^-\,. \tag{23}$$

Note that by (21), not only have we the anti-commutativity

$$\{X_{\rho\nu\mu}, s_5\} = 0\,,$$

but also the anti-commutativities $\{\gamma_\mu, \gamma_5\} = 0$ and (16) can be rewritten using X_μ and $X_{\nu\mu}$ as

$$\{X_\mu^{(\frac{1}{2})}, \gamma_5\} = 0 = \{X_{\nu\mu}^{(1)}, \omega\}\,, \tag{24}$$

where $X_\mu^{(\frac{1}{2})}$ and $X_{\nu\mu}^{(1)}$, more generally, $X_{\nu\mu\ldots}^{(s)}$ represents the corresponding $X_{\nu\mu\ldots}$ for a given spin s. For example, we have $Y_{\rho\nu\mu}^{(\frac{1}{2})} = \frac{1}{8}\gamma_\rho\gamma_\nu\gamma_\mu - \frac{1}{8}g_{\rho\nu}\gamma_\mu$, and $Y_{\rho\nu\mu}^{(1)} = \beta_\rho\beta_\nu\beta_\mu - g_{\rho\nu}\beta_\mu$ by replacing $(s_\rho, s_\nu, s_\mu; s)$ in $Y_{\rho\nu\mu}$ with $\frac{1}{2}(\gamma_\rho, \gamma_\nu, \gamma_\mu; 1)$ and $(\beta_\rho, \beta_\nu, \beta_\mu; 1)$, respectively. Note further that we have the following vanishing relations:

$$X_{\nu\mu}^{(\frac{1}{2})} = X_{\rho\nu\mu}^{(\frac{1}{2})} = 0\,, \qquad X_{\rho\nu\mu}^{(1)} = 0\,,$$

which, in vew of (22), are due to the relations (9) and (14), respectively.

Now we discuss whether or not physical states can be given by $\mathsf{P}_k|\psi\rangle$ ($k = 1,4$) by calculating the rank of P_k. In the Bhabha theory [3] for $s = \frac{3}{2}$, we have two irreducible representations $\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})$ and $\mathrm{R}_5(\frac{3}{2}, \frac{1}{2})$, where $\mathrm{R}_5(s, \tilde{s})$ represents the spin-s Lorentz group representation in five dimensions. Let $S := \{s_1, s_2, s_3, \mathrm{i}s_0\}$. For $\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})$, the eigenvalues of $x \in S$

are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ appearing 4, 6, 6, 4 times, respectively; while for $\mathrm{R}_5(\frac{3}{2}, \frac{1}{2})$, the eigenvalues of $x \in S$ are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ appearing 2, 6, 6, 2 times, respectively. If s_5 realizes, the eigenvalues of s_5 are identical with those of $x \in S$, so that

$$\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_4) = \begin{cases} 4 & \left(\mathrm{R}_5\left(\frac{3}{2}, \frac{3}{2}\right)\right), \\ 2 & \left(\mathrm{R}_5\left(\frac{3}{2}, \frac{1}{2}\right)\right), \end{cases} \qquad \mathrm{Rank}(\mathsf{P}_2) = \mathrm{Rank}(\mathsf{P}_3) = \begin{cases} 6 & \left(\mathrm{R}_5\left(\frac{3}{2}, \frac{3}{2}\right)\right), \\ 6 & \left(\mathrm{R}_5\left(\frac{3}{2}, \frac{1}{2}\right)\right). \end{cases}$$

Thus we obtain in the representation $\mathrm{R}_5(\frac{3}{2}, \frac{3}{2})$, the relation $\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_4) = 4$, the spin degrees of freedom for a spin-$\frac{3}{2}$ massive particle.

The analogous relation holds for a general spin s. Note that by a fundamental property of the projector, we have $\mathrm{Rank}(\mathsf{P}_i) = N_i$, where N_i represents the number of the eigenvalue $(s + 1 - i)$ of s_5. Note also that in the representaion $\mathrm{R}_5(s, \tilde{s})$ $(\tilde{s} = s, s-1, \ldots)$, the maximum and minimum eigenvalues of s_5 [that is, s and $(-s)$, respectively] occur $(2\tilde{s} + 1)$ times [3]. Considering these two remarks, we obtain in the representation $\mathrm{R}_5(s, s)$, the relation $\mathrm{Rank}(\mathsf{P}_1) = \mathrm{Rank}(\mathsf{P}_{2s+1}) = 2s+1$, the spin degrees of freedom. To confirm that $|\psi^+_{\mathrm{ph}}\rangle = \mathsf{P}_1|\psi\rangle$ and $|\psi^-_{\mathrm{ph}}\rangle = \mathsf{P}_{2s+1}|\psi\rangle$, in which we have $s^{\pm}_{\mu}|\psi^{\pm}_{\mathrm{ph}}\rangle = 0$, can be regarded as physical states, we should further show $\langle s\rangle^2|\psi^{\pm}_{\mathrm{ph}}\rangle = s(s+1)|\psi^{\pm}_{\mathrm{ph}}\rangle$, which, however, will be discussed elsewhere.

6 Conclusion

We have found that the intrinsic momentum operator $\pi_\mu = s^+_\mu, s^-_\mu$, which we do not introduce in the ordinary conformal group, is feasible for the Bhabha wave equation, provided that s_5, corresponding to $\frac{1}{2}\gamma_5$ $(s = \frac{1}{2})$ and ω $(s = 1)$, exists. For a general spin s, we can write the intrinsic conformal generators as the same relations as (20) and those where $s_5 \to (-s_5)$, satisfying the invariance under (5) and (6). The fundamental property of π_μ is the nilpotence of order $(2s+1)$. To be more exact, let P_i's $(i = 1, 2, \ldots, 2s+1)$ be the projection operators concerning the s_5 as $\mathsf{P}_i = \frac{1}{f'(\lambda_i)} \frac{f(s_5)\mathbb{1}}{s_5 - \lambda_i \mathbb{1}}$, where $f(x) = \prod_{i=1}^{2s+1}(x - \lambda_i)$, $\lambda_i = s + 1 - i$. Then we have the same hierarchical relation as (22), where $X^{(\frac{1}{2})}_\mu$, $X^{(1)}_{\mu\nu}, \ldots$ anti-commute with γ_5, $\omega, \ldots$, respectively. As long as the wave function transforms as a scalar under the spacetime translation, either s^+_μ or s^-_μ should annihilate a physical state, so that the relation $\mathrm{Rank}(\mathsf{P}_k) = 2s+1$ $(k = 1, 2s+1)$ is required for a massive particle. Fortunately, this relation holds in the representation $\mathrm{R}_5(s, s)$, irreducible representation of the Lorentz group in five dimensions.

Acknowledgements

The author is indebted to N. Aizawa, H. Fujisaka, and T. Kobayashi for valuable comments and discussions.

Funding information HCU Grant and JSPS KAKENHI (JP19K04394) partially supported S. K.

A Derivation of (22)

It is not so difficult to obtain X_μ and $X_{\nu\mu}$ by rewriting $s^+_\mu \mathsf{P}_2$ and $s^+_\nu s^+_\mu \mathsf{P}_3$ in such a way that s_5 is located as leftward as possible. However, this procedure is not practical for the calculation of $X_{\rho\nu\mu}$ because $X_{\rho\nu\mu}$ hinges on s_5 so that we may not represent $X_{\rho\nu\mu}$ uniquely due to some relations between s_5 and s_μ's. In this sense, it would be better to adopt another approach. We start with the following relation:

$$s^+_\mu \mathsf{P}_4 = 2X_\mu \mathsf{P}_4 \qquad (X_\mu = s_\mu)\,. \tag{A.1}$$

Keeping the form of (A.1) without rearranging s_5 leftward, and applying s^+_ν to both sides of (A.1) from the left, then we find it rather simple to obtain

$$s^+_\nu s^+_\mu \mathsf{P}_4 = 2X_{\nu\mu}\mathsf{P}_4 \quad \left(X_{\nu\mu} = \{s_\nu, s_\mu\} - s\mathbb{1}\,, \quad s = \frac{3}{2}\right),$$

where we have used $[s^+_\nu, s_\mu] = [s_\nu, s_\mu] + g_{\nu\mu}s_5$, together with the relation $s_5\mathsf{P}_4 = -s\mathsf{P}_4$. Further application of s^+_ρ leads to the relation

$$s^+_\rho s^+_\nu s^+_\mu \mathsf{P}_4 = \tfrac{4}{3}X_{\rho\nu\mu}\mathsf{P}_4 \quad \left(X_{\rho\nu\mu} = Y_{\rho\nu\mu} + (\text{perm. of } \rho, \nu, \mu)\right),$$

where $Y_{\rho\nu\mu} = s_\rho s_\nu s_\mu - g_{\rho\nu}(ss_\mu + \frac{1}{2s}s_5 s_\mu s_5)$. A similar calculation yields $s^-_\rho s^-_\nu s^-_\mu \mathsf{P}_1 = \frac{4}{3}X_{\rho\nu\mu}\mathsf{P}_1$. Recalling that $\{s_5, X_{\rho\nu\mu}\} = 0$ by (21) and noticing that $\mathsf{P}_1 \longleftrightarrow \mathsf{P}_4$ under the substitution $s_5 \to -s_5$, we finally get the last relation in (22).

References

[1] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Springer, New York, USA, ISBN 9781461274759 (1997), doi:10.1007/978-1-4612-2256-9.

[2] N. Kemmer, *The particle aspect of meson theory*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **173**, 91 (1939), doi:10.1098/rspa.1939.0131.

[3] H. J. Bhabha, *Relativistic wave equations for the proton*, Proc. Indian Acad. Sci. A **21**, 241 (1945).

[4] F. Iachello, *Lie algebras and applications*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540362364 (2006), doi:10.1007/3-540-36239-8.

[5] Y. Murai, *On the group of transformations in six-dimensional space*, Prog. Theor. Phys. **9**, 147 (1953), doi:10.1143/ptp/9.2.147.

[6] S. Weinberg, *Feynman rules for any spin*, Phys. Rev. **133**, B1318 (1964), doi:10.1103/PhysRev.133.B1318.

[7] D. Shay and R. H. Good, *Spin-one particle in an external electromagnetic field*, Phys. Rev. **179**, 1410 (1969), doi:10.1103/PhysRev.179.1410.

[8] R. J. Duffin, *On the characteristic matrices of covariant systems*, Phys. Rev. **54**, 1114 (1938), doi:10.1103/PhysRev.54.1114.

[9] G. Petiau, *Contribution à la théorie des équations d'ondes corpusculaires*, in *Memoires de la Classe des sciences*, Palais des Académies, Brussels, Belgium (1936).

[10] M. Harish-Chandra, *The correspondence between the particle and the wave aspects of the meson and the photon*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **186**, 502 (1946), doi:10.1098/rspa.1946.0061.

[11] Y. A. Markov and M. A. Markova, *Generalization of Geyer's commutation relations with respect to the orthogonal group in even dimensions*, Eur. Phys. J. C **80**, 1153 (2020), doi:10.1140/epjc/s10052-020-08605-4.

Vinberg's T-algebras: From exceptional periodicity to black hole entropy

Alessio Marrani★

Instituto de Física Teorica, Dep.to de Física, Universidad de Murcia,
Campus de Espinardo, E-30100, Spain

★ alessio.marrani@um.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We introduce the so-called Magic Star (MS) projection within the root lattice of finite-dimensional exceptional Lie algebras, and relate it to rank-3 simple and semi-simple Jordan algebras. By relying on the Bott periodicity of reality and conjugacy properties of spinor representations, we present the so-called Exceptional Periodicity (EP) algebras, which are finite-dimensional algebras, violating the Jacobi identity, and providing an alternative with respect to Kac-Moody infinite-dimensional Lie algebras. Remarkably, also EP algebras can be characterized in terms of a MS projection, exploiting special Vinberg T-algebras, a class of generalized Hermitian matrix algebras introduced by Vinberg in the '60s within his theory of homogeneous convex cones. As physical applications, we highlight the role of the invariant norm of special Vinberg T-algebras in Maxwell-Einstein-scalar theories in 5 space-time dimensions, in which the Bekenstein-Hawking entropy of extremal black strings can be expressed in terms of the cubic polynomial norm of the T-algebras.

Received 24-06-2023
Accepted 29-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.035

Contents

1 Projecting root lattices onto the magic star 2

2 Spinor content of exceptional Lie algebras and Fierz identities in $8+q$ dimensions 3

3 From Bott periodicity to exceptional periodicity 6

4 Vinberg special T-algebras and Bekenstein-Hawking entropy 7

References 9

1 Projecting root lattices onto the magic star

Within the r-dimensional root lattice of $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ (with $r = 2,4,6,7,8$, resp.), one can find a plane (defined by the two Cartans of an $\mathfrak{a}_2$ subalgebra) on which the projection of the roots results into the so-called “Magic Star" (MS) (reported in Fig. 1). To the best of our knowledge, the MS was firstly observed in late ’90s by Mukai[1] [2], and later re-discovered and treated in some detail by Truini [3] (see also [4]), within a different approach relying Jordan Pairs [5]; see also [1].

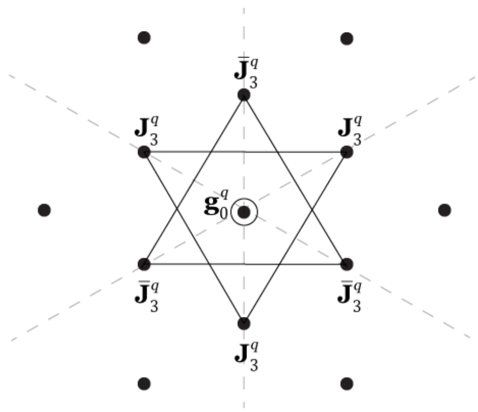


Figure 1: The Magic Star of exceptional Lie algebras [2, 3]. $\mathbf{J}_3^q$ denotes a rank-3 simple Jordan algebra, realized as matrix algebra of 3×3 Hermitian matrices over Hurwitz’s division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (of real dimension $q = \dim_{\mathbb{R}} \mathbb{A} = 1,2,4,8$, resp.). The limit case of $\mathfrak{g}_2$ (corresponding to $q = -2/3$) corresponds to a trivial Jordan algebra, given by the identity element only: $\mathbf{J}_3^{-2/3} \equiv \mathbb{I} := diag(1,1,1)$.

The existence of the MS relies on the so-called (not necessarily maximal, generally non-symmetric) MS embedding/decomposition[2]

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) \supset \mathfrak{a}_2 \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right), \tag{1}$$

where $\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$ and $\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$ stand for the *quasi-conformal* resp. the *reduced structure* Lie algebra of $\mathbf{J}_3^q$ (see e.g. [13, 14] for basic definitions, and a list of Refs.).

Over $\mathbb{C}$, (1) implies [3, 4]

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) = \mathfrak{a}_2 \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right) \oplus \mathbf{3} \times \mathbf{J}_3^q \oplus \overline{\mathbf{3}} \times \overline{\mathbf{J}_3^q}. \tag{2}$$

Upon setting $q = 8,4,2,1,0,-2/3,-1$, one obtains the exceptional sequence (or exceptional series) Table 1, cf. e.g. [8].[3]

$\mathbf{J}_3^q$ stands for the rank-3 simple Jordan algebra [10] (cfr. e.g. [9], and Refs. therein) associated to the parameter q, which for $q = 8,4,2,1$ is the real dimension of the division algebra $\mathbb{A}$ on which the corresponding Jordan algebra is realized as a 3×3 generalized matrix

[1] Mukai used the name “$\mathfrak{g}_2$ *decomposition*".

[2] For an application to supergravity, see [6] (where MS embedding was named Jordan pairs’ embedding), as well as [7], in which the MS embedding was elucidated to be nothing but the $D = 5$ instance of the so-called super-Ehlers embedding.

[3] Note that we consider $\mathfrak{b}_3$, corresponding to $q = -1/3$ and absent in [8].

Table 1

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_8$	$\mathfrak{e}_7$	$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{d}_4$	$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathfrak{a}_2$
$\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_6$	$\mathfrak{a}_5$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	0	$-$

algebra with the property of $\mathbb{A}$-Hermiticity: $q = \dim_{\mathbb{R}} \mathbb{A} = 8,4,2,1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, resp., and $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}} \equiv H_3(\mathbb{A})$ are equivalent notations. Remarkably, $\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$ and $\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$ span the entries of the fourth resp. second row/column of the Freudenthal-Tits Magic Square [11, 12] when setting $q = 8,4,2,1$. From the classification of finite-dimensional, semi-simple cubic Jordan algebras [10], $\mathbf{J}_3^0 \equiv \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ is the completely factorized (triality symmetric) rank-3 Jordan algebra, whereas $\mathbf{J}_3^{-1/3} \equiv \mathbb{C} \oplus \mathbb{C}$ and $\mathbf{J}_3^{-2/3} \equiv \mathbb{C}$ are its partial and total diagonal degenerations, respectively.

Within this report, we will consider things over $\mathbb{R}$. In this case, there are *at least* two non-compact real forms of the "enlarged" exceptional sequence $\left\{\mathfrak{qconf}\left(\mathbf{J}_3^q\right)\right\}_{q=8,4,2,1,0,-1/3,-2/3,-1}$ which can be easily interpreted in terms of symmetries of rank-3 real Jordan algebras: they are given in Tables 2 and Table 3. and they both pertain to the following non-compact, real form of (2)): $\mathfrak{qconf}\ \mathfrak{e}_8$

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) = \mathfrak{sl}_{3,\mathbb{R}} \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right) \oplus \mathbf{3} \times \mathbf{J}_3^q \oplus \mathbf{3}' \times \mathbf{J}_3^{q\prime} . \tag{3}$$

Table 2: The split real form of the exceptional sequence. In this case, for $q = 8,4,2,1$, $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}_s} \equiv H_3(\mathbb{A}_s)$, where $\mathbb{A}_s$ is the split form of $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}$, respectively.

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathfrak{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}_{6,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}} \oplus \mathfrak{sl}_{3,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	0	$-$

Table 3: Another (non-split) non-compact real form of the exceptional sequence.

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathfrak{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{su}_6^*$	$(\mathfrak{sl}_{3,\mathbb{C}})_{\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	0	$-$

2 Spinor content of exceptional Lie algebras and Fierz identities in $8 + q$ dimensions

The following maximal, Jordan algebraic embeddings

$$\begin{aligned} \mathbf{J}_3^{\mathbb{A}} &\supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}} , \\ \mathbf{J}_3^{\mathbb{A}_s} &\supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}_s} , \end{aligned} \tag{4}$$

enjoy the following matrix realization as ($r_i \in \mathbb{R}$, $A_i \in \mathbb{A}$ or $\mathbb{A}_s$, $i = 1,2,3$)

$$\mathbf{J}_3^{\mathbb{A}} \ni J = \begin{pmatrix} r_1 & A_1 & \overline{A}_2 \\ \overline{A}_1 & r_2 & A_3 \\ A_2 & \overline{A}_3 & r_3 \end{pmatrix} \Rightarrow J' = \begin{pmatrix} r_1 & A_1 & 0 \\ \overline{A}_1 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \in \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}} , \tag{5}$$

where the bar denotes the conjugation in $\mathbb{A}$ or in $\mathbb{A}_s$. Usually, the matrix elements r_1 and r_2 are associated to lightcone degrees of freedom, i.e.

$$r_1 := x_+ + x_-, \; r_2 := x_+ - x_- . \tag{6}$$

Furthermore, the following algebraic isomorphisms hold (cf. e.g. [15]):

$$\mathbf{J}_2^{\mathbb{A}} \sim \mathbf{\Gamma}_{1,q+1}, \tag{7}$$
$$\mathbf{J}_2^{\mathbb{A}_s} \sim \mathbf{\Gamma}_{q/2+1,q/2+1}, \tag{8}$$

where $\mathbf{\Gamma}_{1,q+1}$ and $\mathbf{\Gamma}_{q/2+1,q/2+1}$ are (generally simple) Jordan algebras of rank 2 with a quadratic form of (Lorentian resp. Kleinian) signature $(1, q+1)$ resp. $(q/2+1, q/2+1)$, i.e. the Clifford algebras of $O(1, q+1)$ resp. $O(q/2+1, q/2+1)$; for this reason, it is customary to refer to (4) as to the the *spin-factor* embeddings.

By setting $\mathbb{A} = \mathbb{O}$, i.e. $q = 8$, in (4), and considering the various symmetries of Jordan algebras, one obtains the graded structure of suitable real forms of finite-dimensional exceptional Lie algebras with respect to the corresponding pseudo-orthogonal Lie algebras, thus obtaining the spinor content of the exceptional algebras themselves:

1. For what concerns the derivations $\mathfrak{der}$ (namely, the Lie algebra of the automorphism group) of the rank-3 Jordan algebras, one obtains the 2-graded structure of the real, compact form of $\mathfrak{f}_4$, namely:

$$\mathfrak{der}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{der}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{f}_{4(-52)} \supset^{m,s} \mathfrak{so}_9 , \\ \\ \mathfrak{f}_{4(-52)} = \mathfrak{so}_9 \oplus \mathbf{16} , \end{cases} \tag{9}$$

where $\mathbf{16}$ is the Majorana spinor irrepr. of $\mathbf{so}_9$, and the upperscripts "m" and "s" respectively indicate maximality and symmetric nature. The fact that the 2-graded vector space $\mathfrak{so}_9 \oplus \mathbf{16}$ can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{16}$), relies on a remarkable Fierz identity for $\mathfrak{so}_9$ gamma matrices.

2. At the level of the *reduced structure* Lie algebra $\mathfrak{str}_0$, one obtains the 3-graded structure of the real, minimally non-compact form of $\mathfrak{e}_6$, namely:

$$\mathfrak{str}_0\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{str}_0\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{6(-26)} \supset^{m,s} \mathfrak{so}_{9,1} \oplus \mathbb{R} , \\ \\ \mathfrak{e}_{6(-26)} = \mathbf{16}'_{-1} \oplus \left(\mathfrak{so}_{9,1} \oplus \mathbb{R}\right)_0 \oplus \mathbf{16}_1 , \\ \qquad \text{or} \\ \mathfrak{e}_{6(-26)} = \mathbf{16}_{-1} \oplus \left(\mathfrak{so}_{9,1} \oplus \mathbb{R}\right)_0 \oplus \mathbf{16}'_1 , \end{cases} \tag{10}$$

where $\mathbf{16}$ and $\mathbf{16}'$ are the Majorana-Weyl (MW) spinors of $\mathbf{so}_{9,1}$, which constitute an example of *Jordan pair* which is not a pair of Jordan algebras (see e.g. [5], as well as [3, 4] for a recent treatment); also, the indeterminacy denoted by "or" depends on the *spinor polarization* of the embedding [16]. The fact that the 3-graded vector space(s) in the r.h.s. of (10) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{16} \oplus \mathbf{16}'$), relies on a remarkable Fierz identity for $\mathfrak{so}_{9,1}$ gamma matrices. Note that $\mathfrak{str}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathfrak{str}_0\left(\mathbf{J}_3^{\mathbb{O}}\right) \oplus \mathbb{R}$ is isomorphic to the Lie algebra of the automorphism group $\mathrm{Aut}\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)$ of the *Jordan pair* $\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)$:

$$\mathfrak{str}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathrm{Lie}\left(\mathrm{Aut}\left(\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)\right)\right) \simeq \mathfrak{der}\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right) . \tag{11}$$

3. At the level of the *conformal* Lie algebra $\mathfrak{conf}$, one obtains

$$\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{conf}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{7(-25)} \supset^{m,s} \mathfrak{so}_{10,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}\,, \\ \mathfrak{e}_{7(-25)} = \mathfrak{so}_{10,2} \oplus \mathfrak{sl}_{2,\mathbb{R}} \oplus \left(\mathbf{32}^{(\prime)}, \mathbf{2}\right), \end{cases} \tag{12}$$

where $\mathbf{32}$ is the MW spinor of $\mathfrak{so}_{10,2}$, and the possible priming (denoting spinor conjugation) depends on the choice of the spinor polarization [16]. By further branching the $\mathfrak{sl}_{2,\mathbb{R}}$, one obtain a 5-grading of contact type (recently reconsidered within the classification worked out in [17]) of the real, minimally non-compact form of $\mathfrak{e}_7$, namely:

$$\begin{aligned} &\mathfrak{e}_{7(-25)} \supset \mathfrak{so}_{10,2} \oplus \mathbb{R}\,, \\ &\mathfrak{e}_{7(-25)} = \mathbf{1}_{-2} \oplus \mathbf{32}_{-1}^{(\prime)} \oplus \left(\mathfrak{so}_{10,2} \oplus \mathbb{R}\right)_0 \oplus \mathbf{32}_{1}^{(\prime)} \oplus \mathbf{1}_2. \end{aligned} \tag{13}$$

The fact that the 5-graded vector space(s) in the r.h.s. of (13) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{32}^{(\prime)} \oplus \mathbf{32}^{(\prime)}$), relies on a remarkable Fierz identity for $\mathfrak{so}_{10,2}$ gamma matrices. Note that $\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right)$ is isomorphic to the Lie algebra of the automorphism group $\mathrm{Aut}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right)$ of the reduced Freudenthal triple system constructed over $\mathbf{J}_3^{\mathbb{O}}$:

$$\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathrm{Lie}\left(\mathrm{Aut}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right)\right) \simeq \mathfrak{der}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right). \tag{14}$$

4. Finally, at the level of the *quasi-conformal* Lie algebra[4] $\mathfrak{qconf}$ [13, 14], one obtains the 2-graded structure of the real, minimally non-compact form of $\mathfrak{e}_8$, namely:

$$\mathfrak{qconf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{qconf}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{8(-24)} \supset^{m,s} \mathfrak{so}_{12,4}\,, \\ \mathfrak{e}_{8(-24)} = \mathfrak{so}_{12,4} \oplus \mathbf{128}^{(\prime)}, \end{cases} \tag{15}$$

where $\mathbf{128}$ is the MW spinor of $\mathfrak{so}_{12,4}$, and, again, the possible priming (standing for spinorial conjugation) relates to the choice of the spinor polarization [16]. Further decomposition of $\mathfrak{so}_{12,4}$ yields to a 5-grading of “extended Poincaré" type [17]:

$$\begin{aligned} &\mathfrak{e}_{8(-24)} \supset \mathfrak{so}_{11,3} \oplus \mathbb{R}\,, \\ &\mathfrak{e}_{8(-24)} = \begin{cases} \mathbf{14}_{-2} \oplus \mathbf{64}'_{-1} \oplus \left(\mathfrak{so}_{11,3} \oplus \mathbb{R}\right)_0 \oplus \mathbf{64}_1 \oplus \mathbf{14}_2\,, \\ \qquad\qquad\text{or} \\ \mathbf{14}_{-2} \oplus \mathbf{64}_{-1} \oplus \left(\mathfrak{so}_{11,3} \oplus \mathbb{R}\right)_0 \oplus \mathbf{64}'_1 \oplus \mathbf{14}_2\,, \end{cases} \end{aligned} \tag{16}$$

where $\mathbf{64}$ is the MW spinor of $\mathfrak{so}_{11,3}$ and the “or” indeterminacy depends on the spinor polarization [16]. The fact that the 2-graded vector space $\mathfrak{so}_{12,4} \oplus \mathbf{128}^{(\prime)}$ can be endowed with the structure of a (simple,exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{128}^{(\prime)}$), relies on a remarkable Fierz identity for $\mathfrak{so}_{12,4}$ gamma matrices. Equivalently, the fact that the 5-graded vector space(s) in the r.h.s. of (16) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{64} \oplus \mathbf{64}'$), relies on a remarkable Fierz identity for $\mathfrak{so}_{11,3}$ gamma matrices.

[4] We recall that the quasi-conformal realization of $\mathfrak{e}_{8(-24)}$ concerns a non-linear action on an *extended* derived Freudenthal triple system $\mathfrak{E}\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathbb{R} \oplus \mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)$ [13].

3 From Bott periodicity to exceptional periodicity

Thus, we have related the existence of (finite-dimensional, simple) exceptional Lie algebras to some remarkable Fierz identities holding in $q+8$ dimensions (in particular, with signature $9+0$, $9+1$, $10+2$,and $12+4$, for $q=1,2,4$ and 8, respectively).

Now, by observing that the reality properties of spinors and the existence and symmetry of invariant spinor bilinears are periodic *mod* 8 (Bott periodicity), one can define some algebras which (for the moment, formally) generalize the spinor content of the real forms of exceptional Lie algebras discussed above: these are the so-called *"Exceptional Periodicity"* (EP) *algebras* [1, 18], and, as vector spaces, they are defined as follows ($n \in \mathbb{N} \cup \{0\}$ throughout[5]):

1. Level $\mathfrak{der}$:
$$\mathfrak{f}^{(n)}_{4(-52)} := \mathfrak{so}_{9+8n} \oplus \psi_{\mathbf{so}_{9+8n}}\,, \tag{17}$$
where $\psi_{\mathbf{so}_{9+8n}} \equiv \mathbf{2}^{4+4n}$ is the Majorana spinor of $\mathfrak{so}_{9+8n}$.

2. Level $\mathfrak{str}_0$:
$$\mathfrak{e}^{(n)}_{6(-26)} := \psi'_{\mathbf{so}_{9+8n,1},-1} \oplus \left(\mathfrak{so}_{9+8n,1} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{9+8n,1},1}\,, \tag{18}$$
where $\psi_{\mathbf{so}_{9+8n,1}} \equiv \mathbf{2}^{4+4n}$ is the MW spinor of $\mathfrak{so}_{9+8n,1}$.

3. Level $\mathfrak{conf}$:
$$\begin{aligned}\mathfrak{e}^{(n)}_{7(-25)} &:= \left(\mathfrak{so}_{10+8n,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}\right) \oplus \left(\psi_{\mathbf{so}_{10+8n,2}}, \mathbf{2}\right) \\ &= \mathbf{1}_{-2} \oplus \psi_{\mathbf{so}_{10+8n,2},-1} \oplus \left(\mathfrak{so}_{10+8n,2} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{10+8n,2},1} \oplus \mathbf{1}_2\,,\end{aligned} \tag{19}$$
where $\psi_{\mathbf{so}_{10+8n,2}} \equiv \mathbf{2}^{5+4n}$ is the MW spinor of $\mathfrak{so}_{10+8n,2}$.

4. Level $\mathfrak{qconf}$:
$$\begin{aligned}\mathfrak{e}^{(n)}_{8(-24)} &:= \mathfrak{so}_{12+8n,4} \oplus \psi_{\mathbf{so}_{12+8n,4}} \\ &= (\mathbf{14+8n})_{-2} \oplus \psi'_{\mathbf{so}_{11+8n,3},-1} \oplus \left(\mathfrak{so}_{11+8n,3} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{11+8n,3},1} \oplus (\mathbf{14+8n})_2\,,\end{aligned} \tag{20}$$
where $\psi_{\mathbf{so}_{12+8n,4}} \equiv \mathbf{2}^{7+4n}$ and $\psi_{\mathbf{so}_{11+8n,3}} \equiv \mathbf{2}^{6+4n}$ respectively denote the MW spinors of $\mathfrak{so}_{12+8n,4}$ and of $\mathfrak{so}_{11+8n,3}$.

A rigorous algebraic definition of the above EP algebras has been given in [18] (see also [1]) by introducing the notion of *generalized* roots, and by defining the structure constants in terms of (a suitably generalized) Kac's *asymmetry function* [19, 20]. In this report, we confine ourselves to remark that EP algebras are *not* simply non-reductive nor semisimple, spinor-affine extensions of (pseudo-)orthogonal Lie algebras, but their spinor generators are non-translational (i.e., non-Abelian), as are the spinor generators of[6] $\mathfrak{f}_{4(-52)} \equiv \mathfrak{f}^{(n=0)}_{4(-52)}$, $\mathfrak{e}_{6(-26)} \equiv \mathfrak{e}^{(n=0)}_{6(-26)}$, $\mathfrak{e}_{7(-25)} \equiv \mathfrak{e}^{(n=0)}_{7(-25)}$, and $\mathfrak{e}_{8(-24)} \equiv \mathfrak{e}^{(n=0)}_{8(-24)}$. This yields to the *violation* of the Jacobi identity when considering three spinor generators as input in the Jacobiator [18]. As of today, a rigorous, axiomatic treatment of EP algebras is missing: can EP algebras be defined in terms of some characterizing identities, going beyond Jacobi? This remains an open problem.

[5] Note that there has been a shift of unity with respect to the notation of [1] and [18]: the index n used here is actually $n-1$ of such Refs.

[6] The treatment on $\mathbb{R}$ given here is based on the EP generalization of the various symmetry Lie algebras of the Albert algebra $\mathbf{J}_3^{\mathbb{O}}$, and it yielded to some specific real forms of $\mathfrak{f}_4^{(n)}$, $\mathfrak{e}_6^{(n)}$, $\mathfrak{e}_7^{(n)}$ and $\mathfrak{e}_8^{(n)}$. Starting from $\mathbb{C}$, a rigorous definition of all real forms of EP algebras, by means of the introduction of suitable involutive morphisms within the corresponding EP generalized root lattices [18], will be the object of forthcoming works.

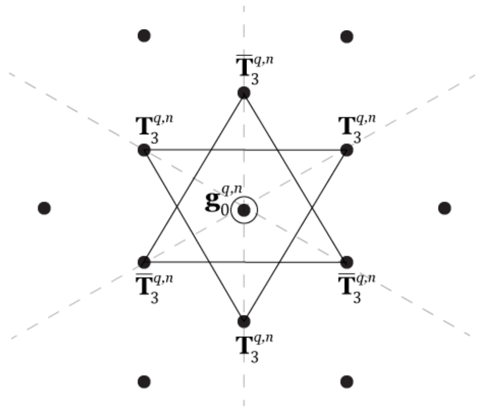


Figure 2: The *Magic Star* structure of the $\mathfrak{a}_2$-projection of the generalized root lattices of EP algebras. finite-dimensional [18]. $\mathbf{T}_3^{q,n}$ stands for a Vinberg T-algebra of rank-3 and of *special type* [22], parametrized by $q = 1, 2, 4, 8$ and $n \in N \cup \{0\}$, corresponding to $\mathfrak{f}_4^{(n)}$, $\mathfrak{e}_6^{(n)}$, $\mathfrak{e}_7^{(n)}$, $\mathfrak{e}_8^{(n)}$, respectively.

The crucial result, which motivates and renders all the above construction and the corresponding construction in the EP lattices non-trivial, is the following [18]: for $n > 0$, all EP algebras admit a $\mathfrak{a}_2$ subalgebra, such that the projection of their generalized root lattices onto the 2-dimensional plane defined by the Cartans of such $\mathfrak{a}_2$ has a *Magic Star structure*, with those generalized roots corresponding to the degeneracies on the tips of such EP-generalized Magic Star which can be endowed with an algebraic structure, denoted by $\mathbf{T}_3^{q,n}$, generalizing the rank-3 simple Jordan algebras $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}} \equiv H_3(\mathbb{A})$ mentioned above. The resulting, EP-generalized Magic Star is depicted in Fig. 2. Remarkably, such a generalization is[7] the *unique* possible one, and it is provided by the Hermitian part of (a class of) rank-3 T-algebras *of special type*. Such algebras were introduced some time ago by Vinberg [22], and they recently appeared in [23–25], in which they have been named *Vinberg special T-algebras*.

4 Vinberg special T-algebras and Bekenstein-Hawking entropy

The real forms of EP algebras resulting from the treatment given above, i.e. $\mathfrak{f}_{4(-52)}^{(n)}$, $\mathfrak{e}_{6(-26)}^{(n)}$, $\mathfrak{e}_{7(-25)}^{(n)}$, and $\mathfrak{e}_{8(-24)}^{(n)}$ (corresponding to $\mathfrak{Der}$, $\mathfrak{str}_0$, $\mathfrak{conf}$ and $\mathfrak{qconf}$ levels, or, equivalently - by the symmetry of the Freudenthal-Tits Magic Square [11, 12] - to $q = 1, 2, 4$ and 8, respectively), the 3×3 generalized matrix algebras $\mathbf{T}_3^{q,n}$ corresponding to the set of generalized roots degenerating to a point on each of the tips of the EP-generalized Magic Star (depicted in Fig. 2) can be realized as follows:

$$\mathbf{T}_3^{q,n} := \begin{pmatrix} r_1 & \mathbf{V}_{\mathfrak{so}_{q+8n}} & \psi_{\mathfrak{so}_{q+8n}} \\ \overline{\mathbf{V}}_{\mathfrak{so}_{q+8n}} & r_2 & \psi'_{\mathfrak{so}_{q+8n}} \\ \overline{\psi}_{\mathfrak{so}_{q+8n}} & \overline{\psi'}_{\mathfrak{so}_{q+8n}} & r_3 \end{pmatrix}, \tag{21}$$

[7]Within a set of reasonable and intuitive assumptions [22].

where[8]

$$\mathbf{V}_{\mathfrak{so}_{q+8n}} := (\boldsymbol{q+8n}, \mathbf{1}) , \tag{22}$$

$$\psi_{\mathfrak{so}_{q+8n}} := \left(\mathbf{2}^{[(q+1)/2]+4n-1+\delta_{q,1}}, \mathbf{Fund}\left(\mathcal{S}_q\right)\right) , \tag{23}$$

are irreducible representation spaces of the Lie algebra

$$\mathfrak{so}_{q+8n} \oplus \mathcal{S}_q , \tag{24}$$

with

$$\mathcal{S}_q := \mathfrak{tri}_{\mathbb{A}} \ominus \mathfrak{so}_{\mathbb{A}} = 0, \mathfrak{so}_2, \mathfrak{su}_2, 0 , \quad \text{for } q = 1,2,4,8 \quad (\text{i.e., for } \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}, \text{ resp.}) , \tag{25}$$

denoting the coset algebra of the *triality* symmetry $\mathbf{tri}_{\mathbb{A}}$ of $\mathbb{A}$ [26]:

$$\mathfrak{tri}_{\mathbb{A}} : = \{(A,B,C)\,|A(xy) = B(x)y + xC(y),\ A,B,C \in \mathfrak{so}_{\mathbb{A}},\ x,y \in \mathbb{A}\} \tag{26}$$

$$= 0, \mathfrak{so}_2^{\oplus 2}, \mathfrak{so}_3{}^{\oplus 3}, \mathfrak{so}_8 , \quad \text{for } \mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} , \tag{27}$$

modded by the *norm-preserving* symmetry $\mathfrak{so}_{\mathbb{A}}$of $\mathbb{A}$:

$$\mathfrak{so}_{\mathbb{A}} := \mathfrak{so}_q = 0, \mathfrak{so}_2, \mathfrak{so}_4, \mathfrak{so}_8 , \quad \text{for } \mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} . \tag{28}$$

Actually, $\mathcal{S}_q$ is related to the reality properties of the spinors of $\mathfrak{so}_{q+8n}$, and in Physics it is named $\mathcal{R}$-symmetry. Furthermore, $\mathbf{Fund}\left(\mathcal{S}_q\right)$ denotes the smallest non-trivial representation of the simple Lie algebra $\mathcal{S}_q$ (if any):

$$\mathbf{Fund}\left(\mathcal{S}_q\right) = -, \mathbf{2}, \mathbf{2}, - , \quad \text{for } q = 1,2,4,8 , \tag{29}$$

with real dimension

$$\text{fund}_q := \dim_{\mathbb{R}} \mathbf{Fund}\left(\mathcal{S}_q\right) = 1,2,2,1 , \quad \text{for } q = 1,2,4,8 . \tag{30}$$

Thus, the total real dimension of $\mathbf{T}_3^{q,n}$ is

$$\dim_{\mathbb{R}}(\mathbf{T}_3^{q,n}) = q + 3 + 8n + \text{fund}_q \cdot 2^{[(q+1)/2]+4n+\delta_{q,1}} . \tag{31}$$

As mentioned above, $\mathbf{T}_3^{q,n}$ (21) is the Hermitian part of a certain class of generalized matrix algebras going under the name of rank-3 T-algebras, introduced sometime ago by Vinberg as a unique,consistent generalization of rank-3, simple Jordan algebras, within its theory of homogeneous convex cones [22]: more precisely, $\mathbf{T}_3^{q,n}$ has been dubbed *exceptional* T-algebra in Sec. 4.3 of [1]. Upon a slight generalization (i.e., by including $P + \dot{P}$ copies of spinor irreprs., and correspondingly extending $\mathcal{S}_q$ to the "full-fledged" $\mathcal{R}$-symmetry $\mathcal{S}_q\left(P,\dot{P}\right)$), $\mathbf{T}_3^{q,n}$ gets generalized to $\mathbf{T}_3^{q,n,P,\dot{P}}$ (with $P,\dot{P} \in \mathbb{N}\cup\{0\}$), which occur in the study of so-called homogeneous *real special* manifolds.[9] These are non-compact Riemannian spaces occurring as (vector multiplets') scalar manifolds of $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 4+1$ space-time dimensions, firstly discussed to some extent by Cecotti [28]. More recently, $\mathbf{T}_3^{q,n,P,\dot{P}}$ have appeared under the name of *Vinberg special T- algebras* in works on Vinberg's theory of homogeneous cones (and generalizations thereof) and on its relation to the entropy of extremal black holes in $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 3+1$ space-time dimensions [23–25].

[8] $[\cdot]$ denotes the integer part throughout.

[9] And, of course, in their images under R-map and c-map (cfr. e.g. [27], and Refs. therein).

The *unique* invariant structure of the algebra[10] $\mathbf{T}_3^{q,n} \equiv \mathbf{T}_3^{q,n,P,\dot{P}}\Big|_{P=1,\dot{P}=0}$ given by (21) is provided by its (formal) "determinant". In order to define it, let us introduce ($\mu = 0, 1, ..., q+1+8n$)

$$V^{\mu} := \left(r_1, r_2, \mathbf{V}_{\mathbf{so}_{q+8n}}\right), \tag{32}$$

which, by recalling (6), is recognized to be a vector module of $Spin(q+1+8n,1)$; we also denote the corresponding spinor of $\mathfrak{so}_{q+1+8n,1}$ (which is chiral for $q = 2,4,8$), of real dimension $\text{fund}_q \cdot 2^{[(q+1)/2]+4n+\delta_{q,1}}$, by $\Psi^{\alpha A}$ (where $\alpha = 1, ..., 2^{[(q+1)/2]+4n+\delta_{q,1}}$ and $A = 1, .., \text{fund}_q$). Then, the "determinant" of the generalized Hermitian matrix algebra $\mathbf{T}_3^{q,n}$, which defines the cubic norm $\mathbf{N}$ of $\mathbf{T}_3^{q,n}$ itself, is defined as

$$\mathbf{N}\left(\mathbf{T}_3^{q,n}\right) := \frac{1}{2}\eta_{\mu\nu}\left[r_3 V^{\mu}V^{\nu} + \gamma^{\mu}_{\alpha\beta}\Psi^{\alpha A}\Psi^{\beta}_{A}V^{\nu}\right], \tag{33}$$

where $\eta_{\mu\nu}$ is the symmetric bilinear invariant of the vector module V (32) of $\text{Spin}(q+1+8n,1)$, and $\gamma^{\mu}_{\alpha\beta}$ are the gamma matrices of $\mathfrak{so}_{q+1+8n,1}$.

Remarkably, Ferrar's classification [29] of elements of a rank-3 Jordan algebras in terms of *invariant rank*$= 0,1,2,3$ can be generalized to the classification of the elements of $\mathbf{T}_3^{q,n}$ depending on their invariant rank as well, defined as follows [18]:

$$\begin{aligned} &\text{rank-3}: \quad \mathbf{N} \neq 0\,, \\ &\text{rank-2}: \quad \mathbf{N} = 0\,, \\ &\text{rank-1}: \quad \partial\mathbf{N} = 0\,. \end{aligned} \tag{34}$$

In those (ungauged) $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 4+1$ space-time dimensions based on $\mathbf{T}_3^{q,n}$ [28], the magnetic charges of extremal black strings (with near-horizon geometry $AdS_3 \otimes S^2$) fit into $\mathbf{T}_3^{q,n}$ itself, and its Bekenstein-Hawking entropy S_{BS} enjoys the interestingly simple expression

$$S_{BS} = \pi\sqrt{|\mathbf{N}|}\,. \tag{35}$$

We conclude this report by pointing out that the entropy of the extremal dyonic black holes in the corresponding (ungauged) $(3+1)$-dimensional supergravity theory (obtained by compactifying the fourth spacial dimension on S^1 and keeping the massless sector) has been recently discussed in [24]. Analogue formulæ hold when considering the most general case $\mathbf{T}_3^{q,n,P,\dot{P}}$ (with $P, \dot{P} \in \mathbb{N} \cup \{0\}$).

References

[1] P. Truini, A. Marrani and M. Rios, *Magic star and exceptional periodicity: An approach to quantum gravity*, J. Phys.: Conf. Ser. **1194**, 012106 (2019), doi:10.1088/1742-6596/1194/1/012106.

[2] S. Mukai, *Simple Lie algebra and Legendre variety*, Nagoya Sūri Forum **3**, 1 (1996), https://www.math.nagoya-u.ac.jp/~mukai/paper/warwick15.pdf.

[3] P. Truini, *Exceptional Lie algebras, SU(3), and Jordan pairs*, Pacific J. Math. **260**, 227 (2012), doi:10.2140/pjm.2012.260.227.

[10] Correspondingly, $\mathcal{S}_q \equiv \mathcal{S}_q(P,\dot{P})\Big|_{P=1,\dot{P}=0}$.

[4] A. Marrani and P. Truini, *Exceptional Lie algebras, SU(3) and Jordan pairs: Part 2. Zorn-type representations*, J. Phys. A: Math. Theor. **47**, 265202 (2014), doi:10.1088/1751-8113/47/26/265202.

[5] O. Loos, *Jordan pairs*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540071662 (1975), doi:10.1007/BFb0080843.

[6] S. Ferrara, A. Marrani and B. Zumino, *Jordan pairs, E_6 and U-duality in five dimensions*, J. Phys. A: Math. Theor. **46**, 065402 (2013), doi:10.1088/1751-8113/46/6/065402.

[7] S. Ferrara, A. Marrani and M. Trigiante, *Super-Ehlers in any dimension*, J. High Energy Phys. **11**, 068 (2012), doi:10.1007/JHEP11(2012)068.

[8] J. M. Landsberg and L. Manivel, *Triality, exceptional Lie algebras and Deligne dimension formulas*, Adv. Math. **171**, 59 (2002), doi:10.1006/aima.2002.2071.

[9] K. McCrimmon, *Jordan algebras and their applications*, Bull. Amer. Math. Soc. **84**, 612 (1978), doi:10.1090/S0002-9904-1978-14503-0.

[10] P. Jordan, J. v. Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29 (1934), doi:10.2307/1968117.

[11] J. Tits, *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Indag. Math. **24**, 530 (1962), doi:10.1016/s1385-7258(62)50051-6.

[12] H. Freudenthal, *Sur le groupe exceptionnel E_7*, in *Hans Freudenthal: Selecta*, European Mathematical Society, Zurich, Switzerland, ISBN 9783037190586 (2009).

[13] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and quasiconformal realizations of exceptional Lie groups*, Commun. Math. Phys. **221**, 57 (2001), doi:10.1007/PL00005574.

[14] M. Gunaydin and O. Pavlyk, *Minimal unitary realizations of exceptional U-duality groups and their subgroups as quasiconformal groups*, J. High Energy Phys. **01**, 019 (2005), doi:10.1088/1126-6708/2005/01/019.

[15] M. Günaydin, *Lectures on spectrum generating symmetries and U-duality in supergravity, extremal black holes, quantum attractors and harmonic superspace*, in *The attractor mechanism*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642107351 (2010), doi:10.1007/978-3-642-10736-8_2.

[16] A. N. Minchenko, *The semisimple subalgebras of exceptional Lie algebras*, Trans. Moscow Math. Soc. **67**, 225 (2006), doi:10.1090/S0077-1554-06-00156-7.

[17] N. Cantarini, A. Ricciardo and A. Santi, *Classification of simple linearly compact Kantor triple systems over the complex numbers*, J. Algebra **514**, 468 (2018), doi:10.1016/j.jalgebra.2018.08.009.

[18] P. Truini, M. Rios, A. Marrani, *The magic star of exceptional periodicity*, in *Nonassociative mathematics and its applications*, American Mathematical Society, Providence, USA, ISBN 9781470442453 (2019), doi:10.1090/conm/721.

[19] W. A. de Graaf, *Lie algebras: Theory and algorithms*, Elsevier, Amsterdam, Netherlands, ISBN 9780444501165 (2000), doi:10.1016/s0924-6509(00)x8001-4.

[20] V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, UK, ISBN 9780521372152 (1990), doi:10.1017/CBO9780511626234.

[21] R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40**, 101 (2001), doi:10.1016/S0393-0440(01)00023-7.

[22] È. B. Vinberg, *The theory of convex homogeneous cones*, Trudy Moskov. Mat. Obšč. **12**, 303 (1963), http://mathscinet.ams.org/mathscinet-getitem?mr=158414.

[23] D. V. Alekseevsky and V. Cortés, *Special vinberg cones*, Transform. Groups **26**, 377 (2021), doi:10.1007/s00031-021-09649-w.

[24] D. V. Alekseevsky, A. Marrani and A. Spiro, *Special Vinberg cones and the entropy of BPS extremal black holes*, J. High Energy Phys. **11**, 100 (2021), doi:10.1007/JHEP11(2021)100.

[25] D. V. Alekseevsky, A. Marrani and A. Spiro, *Special Vinberg cones, invariant admissible cubics and special real manifolds*, (arXiv preprint) doi:10.48550/arXiv.2301.01168.

[26] C. H. Barton and A. Sudbery, *Magic squares and matrix models of Lie algebras*, Adv. Math. **180**, 596 (2003), doi:10.1016/S0001-8708(03)00015-X.

[27] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. B **400**, 463 (1993), doi:10.1016/0550-3213(93)90413-J.

[28] S. Cecotti, *Homogeneous Kähler manifolds and T-algebras in $N = 2$ supergravity and superstrings*, Commun. Math. Phys. **124**, 23 (1989), doi:10.1007/BF01218467.

[29] J. C. Ferrar, *Strictly regular elements in Freudenthal triple systems*, Trans. Amer. Math. Soc. **174**, 313 (1972), doi:10.1090/S0002-9947-1972-0374223-1.

Mixed permutation symmetry quantum phase transitions of critical three-level atom models

Alberto Mayorgas[1⋆], J. Guerrero[2,3] and Manuel Calixto[1,2]

1 Department of Applied Mathematics, University of Granada, Fuentenueva s/n, 18071 Granada, Spain
2 Institute Carlos I of Theoretical and Computational Physics, University of Granada, Fuentenueva s/n, 18071 Granada, Spain
3 Department of Mathematics, University of Jaen, Campus Las Lagunillas s/n, 23071 Jaen, Spain

⋆ albmayrey97@ugr.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We define the concept of Mixed Symmetry Quantum Phase Transition (MSQPT), considering each permutation symmetry sector μ of an identical particles system, as singularities in the properties of the lowest-energy state into each μ when shifting a Hamiltonian control parameter λ. A three-level Lipkin-Meshkov-Glick (LMG) model is chosen to typify our construction. Firstly, we analyze the finite number N of particles case, proving the presence of MSQPT precursors. Then, in the thermodynamic limit $N \to \infty$, we calculate the lowest-energy density inside each sector μ, augmenting the control parameter space by μ, and showing a phase diagram with four different quantum phases.

Received 22-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.036

1 Introduction

When studying quantum systems of identical particles, e.g. bosons and fermions, permutation symmetry becomes crucial. A nontrivial example is that of N identical particles distributed in a set of L levels ($\mathcal{H}_L^{\otimes N}$ as Hilbert space) and a second quantized Hamiltonian describing pair correlations [1]. In particular, the condition of identical atoms allows to use permutation symmetry S_N to decompose $\mathcal{H}_L^{\otimes N}$ into a "Clebsh-Gordan" direct sum of unitary irreducible representations (unirreps or sectors) of $\mathrm{U}(L)$. We shall use Young tableaux as a useful graphical method to depict this decomposition.

It is common in the literature the restriction to the totally symmetric unirrep or sector when studying quantum phase transitions (QPTs) of critical quantum systems in the thermodynamic limit $N \to \infty$, like in Refs. [2–4], reducing Hilbert space $\mathcal{H}_L^{\otimes N}$ dimension from L^N to, for example, $\binom{N+L-1}{N} = N+1$ for $L=2$. This means to make the particles indistinguishable, which is a broadly assumed procedure without any evident physical justification (usually for computational benefit). Therefore, we are devoted to study the role of these often disregarded mixed permutation symmetry sectors in this work. As a paradigmatic case, we will use the Lipkin-Meshkov-Glick (LMG) Hamiltonian for $L=3$ levels (2), where λ will be the control parameter used to detect critical phenomena (QPTs). The case $L=2$ (see [5]) is not considered because all sectors can be reduced to the symmetric one, and the cases $L>3$ provide an extra difficulty when minimizing the energy surface of the Hamiltonian. We address the reader to Ref. [6] for more information.

The organization of this article is the following; in Section 2 we focus on a simplified version of the Hamiltonian for $L=3$ levels, and examine the numerical/exact lowest-energy state inside different permutation symmetry sectors for a finite number of particles N. In Section 3, we find mixed symmetry quantum phase transitions (MSQPTs) in the thermodynamic limit $N\to\infty$ using variational states. At the end, in Sec. 4, we give the conclusions.

2 The 3-level LMG model. U(L) unirreps and QPT precursors

Models describing pairing correlations are usually described by a Hamiltonian in the second quantization form

$$H_L = \sum_{i=1}^{L}\sum_{\mu=1}^{N}\varepsilon_i c_{i\mu}^\dagger c_{i\mu} - \sum_{i,j,k,l=1}^{L}\sum_{\mu,\nu=1}^{N}\lambda_{ij}^{kl} c_{i\mu}^\dagger c_{j\mu} c_{k\nu}^\dagger c_{l\nu}\,, \tag{1}$$

where $c_{i\mu}$ ($c_{i\mu}^\dagger$) destroys (creates) a particle in the μ state of the level i. Precisely, there is a finite number N of identical particles distributed over L energy levels (N-fold degenerate). Pairs of particles are scattered between the L levels when considering the two-body residual interactions of strength λ, so that the total number of particles remains constant.

In our case, we focus on $L=3$ level systems and apply the following list of restrictions to the Hamiltonian (1): Firstly, we define U(L) generators as $S_{ij} = \sum_{\mu=1}^{N} c_{i\mu}^\dagger c_{j\mu}$ according to the Jordan-Schwinger map [7, 8]. Secondly, we disregard interactions for particles in the same level and consider equal interactions in different levels, i.e. $\lambda_{ij}^{kl} = \frac{\lambda}{N(N-1)}\delta_{ik}\delta_{jl}(1-\delta_{ij})$. Thirdly, we transform the Hamiltonian into an energy density (intensive quantity) by separating the interaction strength λ by the total particle pairs $N(N-1)$. Fourthly, we place the levels symmetrically about the level $i=2$, $\varepsilon_3 = -\varepsilon_1 = \epsilon/N$ and $\varepsilon_2 = 0$. Eventually, the Hamiltonian turns into the 3-level simplified version of the LMG Hamiltonian,

$$H_3 = \frac{\epsilon}{N}(S_{33}-S_{11}) - \frac{\lambda}{N(N-1)}\sum_{i\neq j=1}^{3} S_{ij}^2\,. \tag{2}$$

It can be regarded as an extension of the paradigmatic $L=2$ levels LMG Hamiltonian used in the shell model [9, 10].

An interesting property of the 3-level LMG Hamiltonian (2) is that the λ-interaction only scatters pairs of particles, and therefore, conserves the parity $\Pi_i = \exp(i\pi S_{ii})$ of the population S_{ii} in each level $i=1,2,3$. Consequently, the parity symmetry is described by the parity group $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ with the constraint $\Pi_1\Pi_2\Pi_3 = (-1)^N$. This symmetry will be spontaneously broken in the thermodynamic limit $N\to\infty$ leading to a highly degenerated ground state [11]. In addition, if we choose basis vectors adapted to irreducible representations of the Lie group $U(3)$, the Hamiltonian matrix (2) will be block diagonal, and hence the procedure presented in the following paragraphs.

We want to focus on the decomposition of the N-fold tensor product Hilbert space $\mathcal{H}_L^{\otimes N}$ of N L-level atoms into U(L) unirreps. In particular, we shall use Young tableaux and Gelfand-Tsetlin (GT) patterns along this article since they are powerful diagrammatic methods (see [7, 12] for more details and definitions). The fundamental $L \times L$ representation of U(L) is given by a Young box $\boxed{}$, and states of one particle by Weyl patterns/tableaux, $\boxed{1} = |1\rangle$, $\boxed{2} = |2\rangle$, $\boxed{3} = |3\rangle$, ... In the case $L = 3$, we apply the Gram-Schmidt orthonormalization procedure to the columns of a complex triangular matrix T in order to obtain unitary matrices of U(3)

$$T = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix} \xrightarrow{\text{G-S}} V = \begin{pmatrix} \frac{1}{\sqrt{\ell_1}} & \frac{-\bar{\alpha}-\gamma\bar{\beta}}{\sqrt{\ell_1\ell_2}} & \frac{-\bar{\beta}+\bar{\alpha}\bar{\gamma}}{\sqrt{\ell_2}} \\ \frac{\alpha}{\sqrt{\ell_1}} & \frac{1+\beta\bar{\beta}-\alpha\gamma\bar{\beta}}{\sqrt{\ell_1\ell_2}} & \frac{-\bar{\gamma}}{\sqrt{\ell_2}} \\ \frac{\beta}{\sqrt{\ell_1}} & \frac{\gamma-\beta\bar{\alpha}+\gamma\alpha\bar{\alpha}}{\sqrt{\ell_1\ell_2}} & \frac{1}{\sqrt{\ell_2}} \end{pmatrix}, \tag{3}$$

which is parameterized by the complex parameters $\alpha, \beta, \gamma \in \mathbb{C}$, where $\ell_1 = |T^\dagger T|_1 = 1 + \alpha\bar{\alpha} + \beta\bar{\beta}$ and $\ell_2 = |T^\dagger T|_2 = 1+\gamma\bar{\gamma}+(\beta-\alpha\gamma)(\bar{\beta}-\bar{\alpha}\bar{\gamma})$. Actually, the addition of the three Cartan phases $u_j = e^{i\theta_j} \in U(1), j = 1, 2, 3$ completes the parameterization as $U = V \cdot \mathrm{diag}(u_1, u_2, u_3) \in$U(3). This parameterization is chosen for convenience and is derived from the Bruhat decomposition, which is a general version of the Gauss-Jordan elimination and is related to the Schubert cell decomposition of flag manifolds [13]. However, there are many others relevant parameterizations in the field of spin coherent states such as [14–17].

The L^N-dimensional Hilbert space $\mathcal{H}_L^{\otimes N}$ is represented by the N-fold tensor product representation $\boxed{} \otimes \overset{(N)}{\cdots} \otimes \boxed{}$. The Hilbert space is reducible into invariant subspaces, which are graphically represented by Young frames of $h_1 + \cdots + h_L = N$ boxes labeled by $h = [h_1, \ldots, h_L]$, where h_i is the number of boxes in a row $i = 1, \ldots, L$, fulfilling $h_1 \geq \cdots \geq h_L$.

We shall remind that Weyl patterns symbolize the different vectors of a given representation (Young frame). They are in semistandard form when labels (numbers) inside the pattern increase from the right to the left, and strictly increase from the top to the bottom. An important result is that the number of semistandard form Weyl patterns is the dimension of the unirrep. Another useful definition is the weight of a Weyl pattern, which is the vector $w = (w_1, \ldots, w_L)$ whose components w_k are the population of level k, with $w_1 + \cdots + w_L = N$. The lexicographical rule states that a state of weight w has lower weight than another with weight w' if the first non-zero coefficient of $w - w'$ is positive. Notably, the highest weight (HW) vector of a unirrep $h = [h_1, h_2, h_3]$ of $U(3)$ is $w = (h_1, h_2, h_3)$.

The semistandard form Weyl patterns are in one-to-one correspondence with Gelfand-Tsetlin (GT) patterns [7], another useful diagrammatic method to express the vectors spanning U(L) unirreps. GT patterns are labeled by vectors $|\mathbb{m}\rangle$, and are useful for obtaining the eigenvalues and matrix elements $\langle \mathbb{m}|S_{ij}|\mathbb{m}'\rangle$ of the collective operators S_{ij} in each unirrep h. This is called the Gelfand-Tsetlin method [18, 19].

From this point on, we shall study the symmetry classification of the LMG U(3) Hamiltonian (2) eigenstates, and some QPT precursors. The free LMG U(3) Hamiltonian is obtained by taking $\lambda = 0$ in (2), $H^{(0)} = \frac{\epsilon}{N}(S_{33} - S_{11})$, $\epsilon > 0$. According to the Lieb-Mattis theorem [20, 21], the lowest-energy eigenstate is the highest weight vector of the fully symmetric unirrep $h = [N, 0, 0]$, which corresponds to arrange all the particles in the level $i = 1$, $|\psi_0\rangle = |\mathbb{m}_{\mathrm{hw}}\rangle = \boxed{1 \cdots 1}$ (N boxes). The excited states have an energy $E_n = \frac{n-N}{N}\epsilon$, $n = 1, \ldots, 2N$, and are highly degenerated, except for E_0 and E_{2N}. For instance, the states $\boxed{1 \cdots 1\,|\,2}$ and $\begin{array}{|c|c|c|}\hline 1 & \cdots & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$.

The two-body interactions governed by λ lift the degeneracy of the eigenstates. For instance, the lowest energy in the unirrep $h = [3, 1, 0]$ is below the third lowest energy in $h = [4, 0, 0]$ for $\lambda < 1$, hence mixed symmetry sectors (such as $h = [3, 1, 0]$) should not be disregarded in general when studying excited states and their energies.

At this point, it is convenient to define the concept of Mixed Symmetry Quantum Phase Transition (MSQPT) in a nutshell. We want to analyze critical behavior into each Hilbert subspace $\mathcal{H}_h$ corresponding to a unirrep h of $U(3)$, as Hamiltonian evolution does not mix different sectors h. Consequently, we choose the lowest-energy vector $|\psi_0^h\rangle$ inside each $\mathcal{H}_h$, and seek abrupt changes in its structure when shifting λ in the thermodynamic limit $N \to \infty$. But before doing that, we should consider QPT precursors for finite N (exact eigenstates), which are calculated with exact/numerical Hamiltonian eigenstates and can anticipate the approximate situation of critical points. One of them is the fidelity [22, 23], measuring how similar (overlap) two states are in the vicinity ($\delta\lambda \ll 1$) of λ, $F_\psi(\lambda, \delta\lambda) = |\langle\psi(\lambda)|\psi(\lambda+\delta\lambda)\rangle|^2$. The fidelity reaches a minimum in the proximity of a critical point $\lambda^{(0)}$, when the state $|\psi(\lambda)\rangle$ suffers a drastic change of its structure. Another precursor, which is less sensitive to the step size $\delta\lambda$, is the susceptibility

$$\chi_\psi(\lambda, \delta\lambda) = 2\frac{1 - F_\psi(\lambda, \delta\lambda)}{(\delta\lambda)^2}, \tag{4}$$

which reaches a maximum in the vicinity of the critical point $\lambda^{(0)}$.

Figure 1a shows the susceptibility of the exact/numerical ground state (GS) of the LMG U(3) model for different number of particles N. We have done the calculations numerically, giving a matrix form to the S_{ij} operators using the GT basis $|\mathbb{m}\rangle$ in each unirrep. In particular, thanks to the Lieb-Mattis theorem [20, 21], we know that the GS belongs to the fully symmetric irrep, reducing the computations to $h = [N, 0, 0]$ in this case. The susceptibility is sharper as N increases, predicting a critical point around $\lambda \simeq 0.55\epsilon$ for the highest $N = 100$ curve, which is a precursor of the QPT eventually occurring exactly at $\lambda^{(0)} = 0.5\epsilon$ as we will see in Section 3.

On the other hand, Figure 1b displays the susceptibility of the exact lowest-energy vector inside different mixed symmetry sectors (unirreps h) for a fixed number of particles $N = 30$. Now, the would-be critical points (maximum of the susceptibility) move along the different sectors; they shift to the right from $h = [30, 0, 0]$ to $h = [20, 10, 0]$ (cyan dashed line), and to the left from $h = [20, 10, 0]$ to $h = [15, 15, 0]$ (magenta dashed line). Consequently, the figure envisages a quadruple point at the unirrep $h = [2N/3, N/3, 0]$. The maxima at the right in the figure are precursor of another QPT at $\lambda \simeq 1.5\epsilon$, but it is in a different scale and requires a higher N to be properly characterized.

3 Thermodynamic limit and MSQPTs

We shall start this section talking about coherent states. They are excellent variational (semi-classical) states, as they reproduce the structure and mean energy density of lowest-energy states inside each symmetry sector h at $N \to \infty$. For a detailed explanation, see the reference [24], and [5] for the U(2) case. In our case, we follow the Perelomov's construction [25, 26] of the coherent states in a given unirrep h of U(L). Namely, we rotate the HW vector state $|\mathbb{m}_{\text{hw}}\rangle$ of a unirrep h by a unitary matrix $U \in$U(3) parameterized as in (3), $|h, U\rangle = K_h(U)|h; \alpha, \beta, \gamma\}$, where $|h; \alpha, \beta, \gamma\} = e^{\beta S_{31}} e^{\alpha S_{21}} e^{\gamma S_{32}} |\mathbb{m}_{\text{hw}}\rangle$, and $K_h(U)$ is a normalization factor. For the totally symmetric unirrep $h = [N, 0, 0]$, the highest weight state is invariant under a U(2) subgroup, thus, any one of the exponential factors can be eliminated to properly define a U(3) CS. The coherent state expectation values $s_{ij} = \langle h, U|S_{ij}|h, U\rangle$ of the basic symmetry operators S_{ij} can be easily calculated in the differential representation (see the Appendix A of [6] for a detailed calculation).

From now on, it is convenient to relabel U(3) unirreps $h = [h_1, h_2, h_3]$ by parameters μ, ν (we only need two parameters because of the constraint $h_1 + h_2 + h_3 = N$.). More ex-

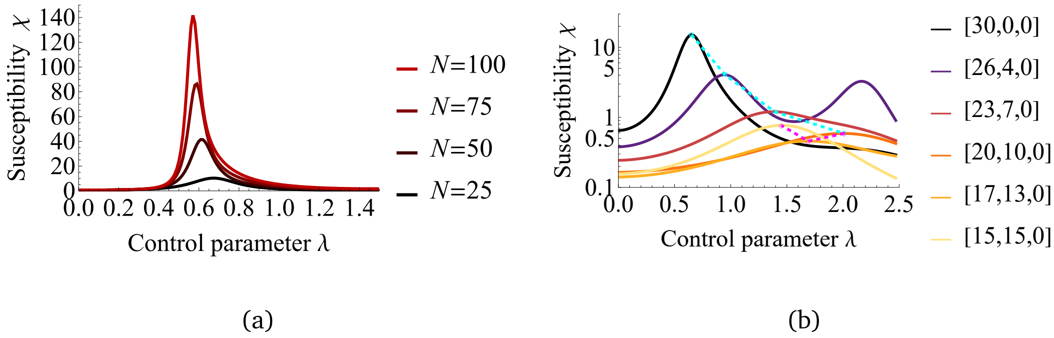


Figure 1: (a) Susceptibility χ_ψ of the ground state $\psi_0^{h=[N,0,0]}$ of the 3-level LMG Hamiltonian (2) for different values of the control parameter λ and the total number of particles N. It predicts a QPT whose critical point is around $\lambda^{(0)} \simeq 0.55$. (b) Susceptibility $\chi_{\psi_0^h}$ (in logarithmic scale) of the lowest-energy vector ψ_0^h into different sectors h for a fixed number of $N = 30$ atoms. The dashed lines interpolate between the maxima of the susceptibilities, which are precursors of the would-be critical points dividing phase I from phase II (cyan), and phase I from phase IV (magenta) (see later on Figure 2 for the different phases). The turn away point, where both dashed lines meet, corresponds to the unirrep $h = [20, 10, 0]$, where four phases will coincide (see later on Sec. 3). We use ϵ units for λ and a step size $\delta\lambda = 0.01$ in both figures.

plicitly, $h_3 = \nu N$, $h_2 = (1-\mu)(1-\nu)N$, $h_1 = \mu(1-\nu)N$, for all $\nu \in [0, \frac{1}{3}]$, $\mu \in [\frac{1}{2}, \frac{1-2\nu}{1-\nu}]$, becoming continuous parameters in the thermodynamic limit. Then, we are able to define the energy surface of a Hamiltonian density H into the Hilbert space sector (μ, ν) as $E^U_{\mu,\nu}(\epsilon, \lambda) = \lim_{N\to\infty} \langle h, U|H|h, U\rangle$. That is, the coherent state expectation value of the Hamiltonian density in the thermodynamic limit ($N \to \infty$). In the LMG U(3) case,

$$E^U_{\mu,\nu}(\epsilon,\lambda) = \lim_{N\to\infty}\left(\frac{\epsilon(s_{33}-s_{11})}{N} - \frac{\lambda\sum_{i\neq j=1}^{3} s_{ij}^2}{N(N-1)}\right), \tag{5}$$

which depends on the type of unirrep (μ, ν), the complex coordinates of U (α, β and γ), and the control parameters ϵ and λ. We fix ϵ and measure the energy surface and λ in ϵ units, since $E^U_{\mu,\nu}(\epsilon, \lambda) = \epsilon E^U_{\mu,\nu}(1, \lambda/\epsilon)$. In addition, we benefit from $h = [h_1, h_2, h_3]$ and $h' = [h_1 - h_3, h_2 - h_3, 0]$ being equivalent SU(3) unirreps and obtain the expression $E^U_{\mu,\nu}(\epsilon, \lambda) = (1-3\nu)E^U_{\tilde{\mu},0}(\epsilon, (1-3\nu)\lambda)$, $\tilde{\mu} = \frac{\mu(1-\nu)-\nu}{1-3\nu}$, so we restrict to the study of the parent case $\nu = 0, \mu \in [\frac{1}{2}, 1]$. For $\mu = 1$, we have the totally symmetric representations, with a four-dimensional phase space $\alpha, \beta \in \mathbb{C}$ and an energy surface

$$E^{(\alpha,\beta)}_{1,0}(\epsilon,\lambda) = \epsilon\frac{\beta\bar{\beta}-1}{\alpha\bar{\alpha}+\beta\bar{\beta}+1} - \lambda\frac{\alpha^2\left(\bar{\beta}^2+1\right)+\left(\beta^2+1\right)\bar{\alpha}^2+\bar{\beta}^2+\beta^2}{\left(\alpha\bar{\alpha}+\beta\bar{\beta}+1\right)^2}, \tag{6}$$

which is invariant under $\alpha \to -\alpha$, $\beta \to -\beta$, thus preserving the discrete parity symmetry inherited from the Hamiltonian). For $\mu = 1/2$, the representations are linked to rectangular Young tableaux ($h_1 = N/2 = h_2$), and the energy surface $E^U_{\frac{1}{2},0}(\epsilon, \lambda) = \frac{1}{2}E^{(\gamma,\beta')}_{1,0}(\epsilon, \frac{\lambda}{2})$, $\beta' = \beta - \alpha\gamma$, can be obtained from the totally symmetric case. The intermediate values $\mu \in (\frac{1}{2}, 1)$ give a six-dimensional phase space (flag manifold structure [13]) $\alpha, \beta, \gamma \in \mathbb{C}$, whose explicit energy surface expression is bulky.

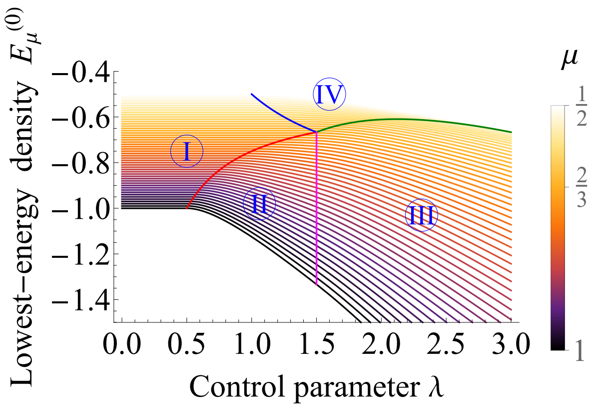


Figure 2: Lowest-energy density $E_\mu^{(0)}(\epsilon,\lambda)$ of the parental case (7) for different values of the control parameter λ and the unirrep continuous parameter μ, varying from $\mu = 1$ (black curve) to $\mu = 1/2$ (light yellow curve), with a step size of $\delta\mu = 0.01$. There are four different quantum phases in the phase diagram, which coincide at a quadruple point $(\lambda,\mu)_q = (3/2, 2/3)$. The phases are separated by curves of critical points in color red, magenta, green, and blue. Both axes are in ϵ units.

Henceforward we minimize in the phase space coordinates the energy density of the parent case (take $\nu = 0, \mu \in [\frac{1}{2}, 1]$ in (5)), i.e. we find the minimum energy

$$E_\mu^{(0)}(\epsilon,\lambda) = \min_{U\in \mathrm{U}(3)} E_{\mu,0}^{U}(\epsilon,\lambda), \quad \forall\, \mu \in \left[\frac{1}{2}, 1\right]. \tag{7}$$

As we can see in Figure 2, the representation label μ behaves as an additional control parameter, differentiating four different quantum phases (I, II, III and IV) in the λ-μ plane (color lines). The transitions between phases for $\mu \neq 1$ can be understood as MSQPTs. We can also find the aforementioned quadruple point at $(\lambda,\mu)_q = (3\epsilon/2, 2/3)$, where the four phases coexist. The MSQPTs are second-order phase transitions as the second derivatives $\partial_{\mu\mu}E_\mu^{(0)}(\epsilon,\lambda)$, $\partial_{\lambda\lambda}E_\mu^{(0)}(\epsilon,\lambda)$, and $\partial_{\mu\lambda}E_\mu^{(0)}(\epsilon,\lambda)$ are discontinuous at critical points.

The minimization gives many critical points $\alpha_0, \beta_0, \gamma_0$ in the phase space with the same $E_\mu^{(0)}$, so the lowest-energy state for a general μ is highly degenerated. This behavior is easier to show in the fully symmetric case $\mu = 1$ (lowest lines of Figure 2), where there are three different phases and two second-order QPTs at $\lambda^{(0)}_{\mathrm{I}\leftrightarrow\mathrm{II}} = \epsilon/2$ and $\lambda^{(0)}_{\mathrm{II}\leftrightarrow\mathrm{III}} = 3\epsilon/2$. The critical values of α and β which make the energy surface minimum are real numbers which have the properties $\alpha_0^{\pm}(\epsilon,\lambda) = 0\ \forall 0 \le \lambda \le \frac{\epsilon}{2}$, and $\beta_0^{\pm}(\epsilon,\lambda) = 0\ \forall\, 0 \le \lambda \le \frac{3\epsilon}{2}$ (check the reference [6] for an explicit expression of the minimum energy surface and the critical points). Therefore, there is a single minimum in phase I, $0 \le \lambda/\epsilon \le 1/2$, located at $\alpha = \beta = 0$; a double minimum in phase II, $1/2 \le \lambda/\epsilon \le 3/2$,, with $\beta = 0$; and a quadruple minimum in phase III, $\lambda/\epsilon \ge 3/2$. This degenerated minima effect is due to the spontaneous breakdown of the discrete parity symmetry of the Hamiltonian, as in the limit $N \to \infty$, the four coherent states $|\alpha_0^{\pm}, \beta_0^{\pm}\rangle$ reach the same minimum energy $E_1^{(0)}$ (minimization of the symmetric case $\mu = 1$, $\nu = 0$ in (5)). The parity restoration of the GS is discussed in the references [27, 28].

4 Conclusion

QPTs research in many-body systems usually presuppose the particle indistinguishability, restricting the scope to the fully symmetric representation ($\mu = 1$), which is often not a general procedure. That is why we have defined MSQPTs as QPTs of the lowest-energy state in a particular symmetry sector μ. As a test model, we have chosen an extension of the ubiquitous LMG model to $L = 3$ levels.

Firstly, we have done numerical calculations for a finite number of particles N to obtain QPT precursors, such as the susceptibility, which anticipate the QPT in the thermodynamic limit $N \to \infty$. In general, the precursors give a better approximation to the critical points when increasing N.

Secondly, using coherent (semiclassical) states, we have considered the thermodynamic limit $N \to \infty$ and minimized the energy surface in different unirreps. The critical points $\lambda^{(0)}$, where the MSQPTs occur, turn out to depend on the representation index μ. Therefore, we have extended the phase diagram in an extended control parameter space (λ, μ). In addition, there are evidences of a quadruple point where four different phases coincide at $\mu = 2/3$. We have also discussed that the lowest-energy state for general representation μ is degenerated, because of the spontaneous breakdown of the discrete parity symmetry of the Hamiltonian in the limit $N \to \infty$.

To conclude, we propose for further research the possible overlap between MSQPT and ESQPT [29], and the exploitation of permutation symmetry in the realm of quantum technologies [30].

Acknowledgments

We thank the organizers of the 34th edition of the ICGTMP for their hospitality and the preparation of the congress. We are also glad to have shared the interesting discussions with J.P. Gazeau, M.A. del Olmo, M.A. Lledó, P. Van Isacker, D. Schuch, and F.J. Herranz during the sessions. We thank O. Castaños and E. Perez-Romero for their collaboration in this work.

Funding information We thank the support of the Spanish MICINN through the project PGC2018-097831-B-I00 and Junta de Andalucía through the projects UHU-1262561, FQM-381 and FEDER/UJA-1381026. AM thanks the Spanish MIU for the FPU19/06376 predoctoral fellowship.

References

[1] J.-Q. Chen, J. Ping and F. Wang, *Group representation theory for physicists*, World Scientific, Singapore, ISBN 9789812380654 (2002), doi:10.1142/5019.

[2] S. Gnutzmann and M. Kus, *Coherent states and the classical limit on irreducible representations*, J. Phys. A: Math. Gen. **31**, 9871 (1998), doi:10.1088/0305-4470/31/49/011.

[3] S. Gnutzmann, F. Haake and M. Kus, *Quantum chaos of SU(3) observables*, J. Phys. A: Math. Gen. **33**, 143 (1999), doi:10.1088/0305-4470/33/1/309.

[4] D. C. Meredith, S. E. Koonin and M. R. Zirnbauer, *Quantum chaos in a schematic shell model*, Phys. Rev. A **37**, 3499 (1988), doi:10.1103/PhysRevA.37.3499.

[5] O. Castaños, R. López-Peña, J. G. Hirsch and E. López-Moreno, *Classical and quantum phase transitions in the Lipkin-Meshkov-Glick model*, Phys. Rev. B **74**, 104118 (2006), doi:10.1103/PhysRevB.74.104118.

[6] M. Calixto, A. Mayorgas and J. Guerrero, *Role of mixed permutation symmetry sectors in the thermodynamic limit of critical three-level Lipkin-Meshkov-Glick atom models*, Phys. Rev. E **103**, 012116 (2021), doi:10.1103/PhysRevE.103.012116.

[7] L. C. Biedenharn, J. D. Louck and P. A. Carruthers, *Angular momentum in quantum physics*, Cambridge University Press, Cambridge, UK, ISBN 9780521302289 (1984), doi:10.1017/CBO9780511759888.

[8] J. J. Sakurai, *Modern quantum mechanics*, Addison-Wesley, Boston, USA, ISBN 9780201539295 (1994).

[9] H. J. Lipkin, N. Meshkov and A. J. Glick, *Validity of many-body approximation methods for a solvable model*, Nucl. Phys. **62**, 188 (1965), doi:10.1016/0029-5582(65)90862-X.

[10] A. J. Glick, H. J. Lipkin and N. Meshkov, *Validity of many-body approximation methods for a solvable model*, Nucl. Phys. **62**, 211 (1965), doi:10.1016/0029-5582(65)90864-3.

[11] Y. Huang, T. Li and Z.-q. Yin, *Symmetry-breaking dynamics of the finite-size Lipkin-Meshkov-Glick model near ground state*, Phys. Rev. A **97**, 012115 (2018), doi:10.1103/PhysRevA.97.012115.

[12] W. Greiner and B. Müller, *Quantum mechanics*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540580805 (1994), doi:10.1007/978-3-642-57976-9.

[13] W. Fulton and J. Harris, *Representation theory*, Springer, New York, USA, ISBN 9783540005391 (2004), doi:10.1007/978-1-4612-0979-9.

[14] A. B. Klimov and H. de Guise, *General approach to $\mathfrak{SU}(n)$ quasi-distribution functions*, J. Phys. A: Math. Theor. **43**, 402001 (2010), doi:10.1088/1751-8113/43/40/402001.

[15] A. B. Klimov, J. L. Romero and H. de Guise, *Generalized SU(2) covariant Wigner functions and some of their applications*, J. Phys. A: Math. Theor. **50**, 323001 (2017), doi:10.1088/1751-8121/50/32/323001.

[16] K. Nemoto, *Generalized coherent states for SU(n) systems*, J. Phys. A: Math. Gen. **33**, 3493 (2000), doi:10.1088/0305-4470/33/17/307.

[17] M. Byrd, *Differential geometry on SU(3) with applications to three state systems*, J. Math. Phys. **39**, 6125 (1998), doi:10.1063/1.532618.

[18] I. M. Gelfand, R. A. Minlos and Z. Y. Shapiro, *Representations of the rotation and Lorentz groups and their applications*, Dover Publications, Mineola, USA, ISBN 9780486823850 (2018).

[19] A. Alex, M. Kalus, A. Huckleberry and J. von Delft, *A numerical algorithm for the explicit calculation of SU(N) and SL(N,ℂ) Clebsch-Gordan coefficients*, J. Math. Phys. **52**, 023507 (2011), doi:10.1063/1.3521562.

[20] E. Lieb and D. Mattis, *Theory of ferromagnetism and the ordering of electronic energy levels*, Phys. Rev. **125**, 164 (1962), doi:10.1103/PhysRev.125.164.

[21] M. Calixto, A. Mayorgas and J. Guerrero, *Hilbert space structure of the low energy sector of U(N) quantum Hall ferromagnets and their classical limit*, Symmetry **14**, 872 (2022), doi:10.3390/sym14050872.

[22] S.-J. Gu, *Fidelity approach to quantum phase transitions*, Int. J. Mod. Phys. B **24**, 4371 (2010), doi:10.1142/S0217979210056335.

[23] P. Zanardi, P. Giorda and M. Cozzini, *Information-theoretic differential geometry of quantum phase transitions*, Phys. Rev. Lett. **99**, 100603 (2007), doi:10.1103/PhysRevLett.99.100603.

[24] R. Gilmore, *Catastrophe theory for scientists and engineers*, Wiley, Hoboken, USA, ISBN 9780471050643 (1981).

[25] A. Perelomov, *Generalized coherent states and their applications*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642648915 (1986), doi:10.1007/978-3-642-61629-7.

[26] J. Gazeau, *Coherent states in quantum physics*, Wiley, Hoboken, USA, ISBN 9783527407095 (2009), doi:10.1002/9783527628285.

[27] M. Calixto, A. Mayorgas and J. Guerrero, *Entanglement and U(D)-spin squeezing in symmetric multi-qudit systems and applications to quantum phase transitions in Lipkin-Meshkov-Glick D-level atom models*, Quantum Inf. Process. **20**, 304 (2021), doi:10.1007/s11128-021-03218-6.

[28] J. Guerrero, A. Mayorgas and M. Calixto, *Information diagrams in the study of entanglement in symmetric multi-qudit systems and applications to quantum phase transitions in Lipkin-Meshkov-Glick D-level atom models*, Quantum Inf. Process. **21**, 223 (2022), doi:10.1007/s11128-022-03524-7.

[29] M. A. Caprio, P. Cejnar and F. Iachello, *Excited state quantum phase transitions in many-body systems*, Ann. Phys. **323**, 1106 (2008), doi:10.1016/j.aop.2007.06.011.

[30] D. Bacon, I. L. Chuang and A. W. Harrow, *Efficient quantum circuits for Schur and Clebsch-Gordan transforms*, Phys. Rev. Lett. **97**, 170502 (2006), doi:10.1103/PhysRevLett.97.170502.

Relativistic kinematics in flat and curved space-times

Patrick Moylan⋆

Department of Physics, The Pennsylvania State University,
Abington College, Abington, PA 19001 USA

⋆ pjm11@psu.edu

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Almost immediately after the seminal papers of Poincaré (1905,1906) and Einstein (1905) on special relativity, wherein Poincaré established the full covariance of the Maxwell-Lorentz equations under the scale-extended Poincaré group and Einstein explained the Lorentz transformation using his assumption that the one-way speed of light in vacuo is constant and the same for all inertial observers (Einstein's second postulate), attempts were made to get at the Lorentz transformations from basic properties of space and time but avoiding Einstein's second postulate. Various such approaches usually involve general consequences of the relativity principle, such as a group structure to the set of all admissible inertial transformations and also assumptions about causality and/or homogeneity of space-time combined with isotropy of space. The first such attempt is usually attributed to von Ignatowsky in 1911. It was followed shortly thereafter by a paper of Frank and Rothe published in the same year. Since then, papers have continued to be written on the subject even up to the present. We elaborate on some of the results of such papers paying special attention to a 1968 paper of Bacri and Lévy-Leblond where possible kinematical groups include the de Sitter and anti-de Sitter groups and lead to special relativity in de Sitter and anti-de Sitter spaces.

Received 20-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.037

1 Introduction

On Sept. 24, 1904, in a powerful and prophetic address to the *International Congress of Arts and Sciences* in St. Louis, Missouri, Henri Poincaré ushered in the *new relativity theory*.[1] Its

[1]The International Congress took place that year in St. Louis along with the other festivities of the 1904 World's Fair (Louisiana Purchase Exposition) celebrating the 100th anniversary of the Louisiana purchase of 1803.

foundation was the "*Principle of Relativity*," which, according to Poincaré, is [1]:

> *"That principle according to which the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation, so that we have not and could not have any means of discerning whether or not we are carried along in such a motion."*

Poincaré's lecture, appropriately delivered in the "New World," elevated the principle of relativity to a general law of physics on an equal footing with conservation of energy, which necessarily entails its universality. That this is such was boldly reasserted by him in Ref. [2], declaring that we shall "admit this law ... and admit [it] without restriction." The St. Louis lecture was published and widely read in academic circles worldwide in the ensuing months. A year later, in Albert Einstein's first paper on special relativity [3], there is found *without reference* a somewhat weaker and less precise rewording of Poincaré's statement of the relativity principle. In contrast to Poincaré, Einstein made no claim as to its universality.

Space-time is a four-dimensional Hausdorff manifold M with a smooth differentiable structure on it. Points in M correspond to events and curves to world lines of particles. Following Ehlers, Pirani and Schild (EPS) [4], [5] we take the curved analogs of straight world lines in affine space to be those curves (geodesics) which are "world lines of freely falling particles" and "behave *infinitesimally* like the straight lines of projective (or affine) four-space." A *symmetric affine connection* specifies the family of geodesics, with geodesics being curves whose "tangent directions" are "autoparallel." Translations are one-parameter groups of transformations with possibly only local C^∞ action on M, the geodesics being orbits of points under the action of the one-parameter translation subgroups.

With geodesics representing world lines of inertial observers and with inertial transformations being mappings between such geodesics, defined possibly only locally in some cases, our generalization of the relativity principle to curved space can be formulated in essentially the same way as Poincaré's statement of it. Just as in the affine case, the relativity principle demands that: (i) inertial transformations from one inertial frame to another take geodesics to geodesics and preserve parallelism of geodesics and (ii) "a group structure for the set of all inertial transformations" [6] at least in a local sense. We call the set of all such inertial transformations the relativity group or kinematical group of M. For the global formulation of Lie groups of transformations acting on a manifold, due in its local form to Sophus Lie, we refer the reader to [7].

2 Classification of possible kinematical groups

We assume that the kinematical or relativity group contains the rotation group $SO(3)$ as a subgroup. Furthermore, with translations defined as above and inertial boosts being defined as "uniform movements of translation," the kinematical group should be a subgroup of the Lie group of transformations formed out of *rotations*, *translations*, *inertial boosts* and *scale transformation* with (skew-symmetric) infinitesimal generators $\mathbf{L}_{ij}$, $\mathbf{P}_i$, $\mathbf{L}_{0i}$ $(i,j=1,2,3)$ and $\mathbf{S}$, respectively. Assume scale transformations commute with rotations and inertial boosts, and assuming rotational invariance, which implies that $\mathbf{P}_i$, $\mathbf{L}_{0i}$ are $SO(3)$ vector operators, we obtain

$$[\mathbf{L}_{ij},\mathbf{L}_{k\ell}]=-\delta_{ik}\mathbf{L}_{j\ell}-\delta_{i\ell}\mathbf{L}_{jk}+\delta_{jk}\mathbf{L}_{i\ell}+\delta_{j\ell}\mathbf{L}_{ik}\,, \tag{1}$$

$$[\mathbf{L}_{ij},\mathbf{L}_{0k}]=-\delta_{ik}\mathbf{L}_{0j}+\delta_{jk}\mathbf{L}_{0i}\,,\ [\mathbf{L}_{ij},\mathbf{P}_0]=0\,,\qquad [\mathbf{L}_{ij},\mathbf{P}_k]=-\delta_{ik}\mathbf{P}_j+\delta_{jk}\mathbf{P}_i\,, \tag{2}$$

$$[\mathbf{S},\mathbf{L}_{ij}]=0\,,\qquad [\mathbf{S},\mathbf{L}_{0i}]=0\,. \tag{3}$$

For the other brackets we have [8]:

$$[\mathbf{P}_0,\mathbf{P}_i]=\omega_i\mathbf{P}_0+\gamma_{ij}\mathbf{P}_j+\frac{1}{2}\epsilon_{ik}\epsilon_{kmn}\mathbf{L}_{mn}+\alpha_{ik}\mathbf{L}_{0k}+\kappa_i\mathbf{S}\,, \tag{4}$$

$$[\mathbf{P}_i,\mathbf{P}_j]=\iota_{ij}\mathbf{P}_0+\nu_{ijk}\mathbf{P}_k+\frac{1}{2}\mu_{ijk}\epsilon_{kmn}\mathbf{L}_{mn}+\psi_{ijk}\mathbf{L}_{0k}+\kappa_{ij}\mathbf{S}\,, \tag{5}$$

$$[\mathbf{L}_{0i},\mathbf{P}_0]=\chi_i\mathbf{P}_0+\lambda_{ij}\mathbf{P}_j+\frac{1}{2}\zeta_{ik}\epsilon_{kmn}\mathbf{L}_{mn}+\eta_{ik}\mathbf{L}_{0k}+\lambda_i\mathbf{S}\,, \tag{6}$$

$$[\mathbf{L}_{0i},\mathbf{P}_j]=\rho_{ij}\mathbf{P}_0+\pi_{ijk}\mathbf{P}_k+\frac{1}{2}\sigma_{ijk}\epsilon_{kmn}\mathbf{L}_{mn}+\tau_{ijk}\mathbf{L}_{0k}+\omega_{ij}\mathbf{S}\,, \tag{7}$$

$$[\mathbf{L}_{0i},\mathbf{L}_{0j}]=\xi_{ij}\mathbf{P}_0+\beta_{ijk}\mathbf{P}_k+\frac{1}{2}\lambda_{ijk}\epsilon_{kmn}\mathbf{L}_{mn}+\nu_{ijk}\mathbf{L}_{0k}+\tau_{ij}\mathbf{S}\,, \tag{8}$$

$$[\mathbf{S},\mathbf{P}_0]=\alpha\mathbf{P}_0+\beta_i\mathbf{P}_i+\frac{1}{2}\gamma_i\epsilon_{imn}\mathbf{L}_{mn}+\delta_i\mathbf{L}_{0i}+\zeta\mathbf{S}\,, \tag{9}$$

$$[\mathbf{S},\mathbf{P}_i]=\alpha_i\mathbf{P}_0+\beta_{ij}\mathbf{P}_j+\frac{1}{2}\gamma_{ik}\epsilon_{kmn}\mathbf{L}_{mn}+\delta_{ij}\mathbf{L}_{0j}+\eta_i\mathbf{S}\,. \tag{10}$$

The other brackets can be simplifed by further exploiting *rotational invariance* and *spatial isotropy* which implies that they must be expressible as linear combinations of the basic generators with the (rotationally) covariant tensor δ_{ij} and pseudo-tensor ϵ_{ijk} where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the totally antisymmetric symbol. By using these facts we can rewrite Eqns. (4) to (10) as [8]

$$[\mathbf{P}_0,\mathbf{P}_i]=\omega_i\mathbf{P}_0+\gamma\mathbf{P}_i+\frac{1}{2}\varepsilon\,\epsilon_{imn}\mathbf{L}_{mn}+\alpha\mathbf{L}_{0i}+\kappa_i\mathbf{S}\,, \tag{11}$$

$$[\mathbf{P}_i,\mathbf{P}_j]=\iota\delta_{ij}\mathbf{P}_0+\nu\epsilon_{ijk}\mathbf{P}_k+\mu\mathbf{L}_{ij}+\psi\epsilon_{ijk}\mathbf{L}_{0k}+\kappa\delta_{ij}\mathbf{S}\,, \tag{12}$$

$$[\mathbf{L}_{0i},\mathbf{P}_0]=\chi_i\mathbf{P}_0+\lambda\mathbf{P}_i+\frac{1}{2}\zeta\epsilon_{imn}\mathbf{L}_{mn}+\eta\mathbf{L}_{0i}+\lambda_i\mathbf{S}\,, \tag{13}$$

$$[\mathbf{L}_{0i},\mathbf{P}_j]=\rho\delta_{ij}\mathbf{P}_0+\pi\epsilon_{ijk}\mathbf{P}_k+\sigma\mathbf{L}_{ij}+\tilde{\tau}\epsilon_{ijk}\mathbf{L}_{0k}+\omega\delta_{ij}\mathbf{S}\,, \tag{14}$$

$$[\mathbf{L}_{0i},\mathbf{L}_{0j}]=\xi\delta_{ij}\mathbf{P}_0+\beta\epsilon_{ijk}\mathbf{P}_k+\tilde{\lambda}\mathbf{L}_{ij}+\tilde{\nu}\epsilon_{ijk}\mathbf{L}_{0k}+\tau\delta_{ij}\mathbf{S}\,, \tag{15}$$

$$[\mathbf{S},\mathbf{P}_0]=\alpha\mathbf{P}_0+\beta_i\mathbf{P}_i+\frac{1}{2}\gamma_i\epsilon_{imn}\mathbf{L}_{mn}+\delta_i\mathbf{L}_{0i}+\zeta\mathbf{S}\,, \tag{16}$$

$$[\mathbf{S},\mathbf{P}_i]=\alpha_i\mathbf{P}_0+\tilde{\beta}\delta_{ij}\mathbf{P}_j+\frac{1}{2}\tilde{\gamma}\epsilon_{imn}\mathbf{L}_{mn}+\delta\mathbf{L}_{0i}+\eta_i\mathbf{S}\,. \tag{17}$$

Next consider the following automorphisms of the relativity group, G: the parity operator, Π, with action on $\mathfrak{g}$, the Lie algebra of G, given by

$$\Pi(\mathbf{P}_0)=\mathbf{P}_0\,,\quad \Pi(\mathbf{P}_i)=-\mathbf{P}_i\,,\quad \Pi(\mathbf{L}_{ij})=\mathbf{L}_{ij}\,,\quad \Pi(\mathbf{L}_{0i})=-\mathbf{L}_{0i}\,,\quad \Pi(\mathbf{S})=\mathbf{S}\,, \tag{18}$$

and the time reversal operator, Θ, with action on $\mathfrak{g}$ given by

$$\Theta(\mathbf{P}_0)=-\mathbf{P}_0\,,\quad \Theta(\mathbf{P}_i)=\mathbf{P}_i\,,\quad \Theta(\mathbf{L}_{ij})=\mathbf{L}_{ij}\,,\quad \Theta(\mathbf{L}_{0i})=-\mathbf{L}_{0i}\,,\quad \Theta(\mathbf{S})=\mathbf{S}\,. \tag{19}$$

Application of these automorphisms to the commutators, Eqns. (11) to (17) gives [8]

$$[\mathbf{P}_0,\mathbf{P}_i]=\alpha\mathbf{L}_{0i}\,, \tag{20}$$

$$[\mathbf{P}_i,\mathbf{P}_j]=\mu\mathbf{L}_{ij}+\kappa\delta_{ij}\mathbf{S}=\mu\mathbf{L}_{ij}\,, \tag{21}$$

$$[\mathbf{L}_{0i},\mathbf{P}_0]=\lambda\mathbf{P}_i\,, \tag{22}$$

$$[\mathbf{L}_{0i},\mathbf{P}_j]=\rho\delta_{ij}\mathbf{P}_0\,, \tag{23}$$

$$[\mathbf{L}_{0i},\mathbf{L}_{0j}]=\tilde{\lambda}\mathbf{L}_{ij}+\tau\delta_{ij}\mathbf{S}=\tilde{\lambda}\mathbf{L}_{ij}\,, \tag{24}$$

$$[\mathbf{S},\mathbf{P}_0]=\tilde{\alpha}\mathbf{P}_0\,, \tag{25}$$

$$[\mathbf{S},\mathbf{P}_i]=\tilde{\beta}\mathbf{P}_i\,, \tag{26}$$

where we used the fact that the bracket is skew-symmetric to obtain $\kappa=\tau=0$.

***Proposition* 1**

$$\tilde{\alpha} = \tilde{\beta} = 1\,. \tag{27}$$

Proof: *Making use of commutators of Eqns. (20) to (26) we obtain*

$$\tilde{\beta}\mathbf{P}_i = [\mathbf{S},\mathbf{P}_i] = \frac{1}{\lambda}[\mathbf{S},[\mathbf{L}_{0i},\mathbf{P}_0]] = -\frac{1}{\lambda}[\mathbf{L}_{0i},[\mathbf{P}_0,\mathbf{S}]] = \frac{\tilde{\alpha}}{\lambda}[\mathbf{L}_{0i},\mathbf{P}_0] = \tilde{\alpha}\mathbf{P}_i\,,$$

which implies $\tilde{\alpha} = \tilde{\beta}$ *. To obtain* $\tilde{\alpha} = 1$*, use* $[\mathbf{L}_{01},\mathbf{P}_1] = \rho\mathbf{P}_0$ *to get*

$$\tilde{\alpha}\mathbf{P}_0 = [\mathbf{S},\mathbf{P}_0] = \frac{\tilde{\alpha}}{\rho}[\mathbf{S},[\mathbf{L}_{01},\mathbf{P}_1]] = \frac{\tilde{\alpha}}{\rho}[\mathbf{L}_{01},[\mathbf{S},\mathbf{P}_1]] = \frac{\tilde{\alpha}^2}{\rho}[\mathbf{L}_{01},\mathbf{P}_1] = \tilde{\alpha}^2\mathbf{P}_0\,.$$

Proposition* 2** **(Bacry, Lévy-Leblond [8]***)**

$$\mu - \rho\alpha = 0\,, \tag{28}$$

$$\tilde{\lambda} - \rho\lambda = 0\,. \tag{29}$$

Proof: *Eq. (28) follows from the Jacobi identity*

$$[\mathbf{P}_i,[\mathbf{P}_j,\mathbf{L}_{0k}]] + [\mathbf{P}_j,[\mathbf{L}_{0k},\mathbf{P}_i]] + [\mathbf{L}_{0k},[\mathbf{P}_i,\mathbf{P}_j]] = 0\,,$$

together with $[\mathbf{L}_{ij},\mathbf{L}_{0k}] = -\delta_{ik}\mathbf{L}_{0j} + \delta_{jk}\mathbf{L}_{0i}$ *and the commutators before Proposition I. For Eq. (29) first use the the Jacobi identity*

$$[\mathbf{P}_0,[\mathbf{P}_i,\mathbf{L}_{0j}]] + [\mathbf{P}_i,[\mathbf{L}_{0j},\mathbf{P}_0]] + [\mathbf{L}_{0j},[\mathbf{P}_0,\mathbf{P}_i]] = 0\,,$$

and the commutators before Proposition I to obtain

$$\alpha\tilde{\lambda} - \lambda\mu = 0\,.$$

Then use this together with Eq. (28) to obtain Eq. (29).

One can show that the remaining Jacobi identities do not lead to any further independent constraints on the parameters in Eqns. (20) to (26) [8].

Propositions 1 and 2 imply the classification of admissible $\mathfrak{g}$ depends upon three independent real parameters ρ, α and λ. Let $\mathfrak{g} = \mathfrak{g}_{(\rho,\alpha,\lambda)}$. Then any admissible $\mathfrak{g}_{(\rho,\alpha,\lambda)}$ is isomorphic to $\mathfrak{g}_{(\rho,\alpha,\lambda)}$ with ρ, α, λ taking values 1 or 0. The explicit isomorphism is obtained by an appropriate scaling of generators, e.g. $\tilde{\mathbf{L}}_{0i} = \phi_\lambda(\mathbf{L}_{0i}) = \lambda^{-1/2}\mathbf{L}_{0i}$ with $\lambda > 0$ so that $[\tilde{\mathbf{L}}_{0i},\tilde{\mathbf{L}}_{0j}] = \frac{1}{\lambda}[\mathbf{L}_{0i},\mathbf{L}_{0j}] = \mathbf{L}_{ij}$. Thus, up to such isomorphisms, it suffices to restrict ρ, α, λ to values of 0 or 1. Following [8] we are led to the following cases:
Class R (relative time): $\rho = 1$:

$R1$. ($\alpha = 1$, $\lambda = 1$) From Eqns. (28) and (29) we have $\mu \neq 0$ and $\tilde{\lambda} \neq 0$ and from the commutation relations Eqns. (20) to (26) we obtain

$$\mathfrak{g}_{(1,1,1)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \mathfrak{so}(5)\,,$$

with three possible real forms

$$\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(5)\,, \qquad \mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(1,4)\,, \qquad \mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(2,3)\,,$$

where $\mathbb{C}\mathbf{S}$ and $\mathbb{R}\mathbf{S}$ respectively denote the one-dimensional Lie algebras over $\mathbb{C}$ and $\mathbb{R}$ generated by $\mathbf{S}$. ($\oplus_\tau$ means semidirect sum.)

*R*2. $(\alpha = 0,\ \lambda = 1 \Rightarrow \mu = 0,\ \tilde{\lambda} = 1)$

$$\mathfrak{g}_{(1,0,1)} \cong (\mathfrak{so}(4) \oplus \mathbb{C}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,,$$

where $\mathfrak{t}^4$ is the four dimensional abelian Lie algebra (ideal) over $\mathbb{C}$ generated by the $\mathbf{P}_0$, $\mathbf{P}_i$ $(i = 1,2,3)$. Up to isomorphism, all permissible real forms are[2]

$$(\mathfrak{so}(4) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,, \qquad \text{and} \qquad (\mathfrak{so}(1,3) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,,$$

where now $\mathfrak{so}(4)$ and $\mathfrak{t}^4$ are real Lie algebras. The case $(\mathfrak{so}(1,3) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4$ describes standard Lorentzian relativity exended by scale.

*R*3. $(\alpha = 1,\ \lambda = 0 \Rightarrow \mu = 1,\ \tilde{\lambda} = 0)$

$$\mathfrak{g}_{(1,1,0)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \left\{ \widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right\},$$

where $\widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4$ is the semidirect sum of the $\mathfrak{so}(4)$ generated by $\mathbf{L}_{ij}$ and $\mathbf{P}_i$ $(i = 1,2,3)$ with an abelian Lie algebra $\tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)}$ over $\mathbb{C}$ generated by $\mathbf{P}_0$ and the $\mathbf{L}_{0i}$ $(i = 1,2,3)$. Permissible real forms (up to isomorphism) are

$$\mathbb{R}\mathbf{S} \oplus_\tau \left(\widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right), \qquad \text{and} \qquad \mathbb{R}\mathbf{S} \oplus_\tau \underbrace{\left(\widetilde{\mathfrak{so}}(1,3)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right)}_{\textit{para-Poincaré Lie algebra}}.$$

*R*4. $(\alpha = 0,\ \lambda = 0 \Rightarrow \mu = 0,\ \tilde{\lambda} = 0)$ Lie algebra of the (scale-extended) Carroll Group: [8]

$$\mathfrak{g}_{(1,0,0)} \cong (\mathfrak{so}(3) \oplus_\tau \tilde{\mathfrak{t}}^3_{\mathbf{L}_{0i}} \oplus \mathbb{C}\mathbf{S}) \oplus_{\tilde{\tau}} \mathfrak{t}^4_{\mathbf{P}_\mu}\,.$$

There is only one acceptable real form. It is obtained by restricting $\mathfrak{g}_{(1,0,0)}$ to the reals. It is the (real) Lie algebra of the (scale-extended) Carroll group.

Class $\tilde{A}$ (absolute time): $\rho = 0$ $(\Rightarrow \mu = \tilde{\lambda} = 0)$:[3]

$\tilde{A}1$. $(\alpha = 0,\ \lambda = 1)$

$$\mathfrak{g}_{(0,0,1)} \cong \underbrace{\{\mathfrak{g}_{(\mathbf{L}_{ij},\mathbf{L}_{0k})} \oplus \mathbb{C}\mathbf{S}\}}_{\substack{\text{homogeneous Galilei Lie}\\ \text{algebra (scale extended)}}} \oplus_\tau \mathfrak{t}^4_{\mathbf{P}_0,\mathbf{P}_i}\,.$$

There is only one acceptable real form obtained by restricting $\mathfrak{g}_{(0,0,1)}$ to the reals. It is the (real) Lie algebra of the (scale-extended) inhomogeneous Galilei group.

$\tilde{A}2$. $(\alpha = 1,\ \lambda = 0)$

$$\mathfrak{g}_{(0,1,0)} \cong \{\{\mathfrak{so}(3)_{\mathbf{L}_{ij}} \oplus \mathbb{C}\mathbf{S}\} \oplus_\tau \mathfrak{t}^3_{\mathbf{P}_i}\} \oplus_\tau \mathfrak{t}^4_{(\mathbf{P}_0,\mathbf{L}_{0i})}\,.$$

It's easy to see that $\mathfrak{g}_{(0,1,0)} \cong \mathfrak{g}_{(0,0,1)}$ and that there is only one acceptable real form obtained by restricting $\mathfrak{g}_{(0,1,0)}$ to the reals.

$\tilde{A}3$. $(\alpha = 1,\ \lambda = 1)$

$$\mathfrak{g}_{(0,1,1)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \mathfrak{n}\,.$$

$\mathfrak{n}$ is the ideal generated by $\mathbf{L}_{ij}$, $\mathbf{L}_{0i}$, $\mathbf{P}_0$, $\mathbf{P}_i$ and its two admissible real forms are the Lie algebras of the two Newton-Hooke groups [8].

$\tilde{A}4$. $(\alpha = 0,\ \lambda = 0)$

$$\mathfrak{g}_{(0,0,0)} \cong (\mathfrak{so}(3)_{\mathbf{L}_{ij}} \oplus \mathbb{C}\mathbf{S}) \oplus_\tau \mathfrak{t}^7_{(\mathbf{L}_{0i},\mathbf{P}_0,\mathbf{P}_i)}\,,$$

where $\mathfrak{t}^7_{(\mathbf{L}_{0i},\mathbf{P}_0,\mathbf{P}_i)}$ is the 7 dimensional abelian ideal generated by the $\mathbf{L}_{0i}$, $\mathbf{P}_0$, $\mathbf{P}_i$ $(i = 1,2,3)$. $\mathfrak{g}_{(0,0,0)}$ is the Lie algebra, extended by scale, of what is called the *static group* or *Aristotle group*.

[2]The reason why the real form containing $\mathfrak{so}(2,2)$ is not permitted is due to our assumption of rotational symmetry, which we made at the very start and which implies that admissible Lie algebras must contain the subalgebra $\mathfrak{so}(3)$.

[3]Our description of this class differs slighty from that in [8] so we put tildes on the *A*'s to distinguish them from Bacry and Lévy-Lebond's *A*'s in Ref. [8], i.e. $\tilde{A}1$ instead of *A*1 of Bacry, Lévy-Lebond etc.

3 Reduction of symmetry and non-compactness of the $e^{\nu \mathbf{L}_{0i}}$

The classification just given is at the Lie algebra level. The corresponding kinematical groups are obtained by suitable "exponentiation" [7] and restriction to subgroups [2]. As in [8], we consider only those cases for which any one-parameter subgroup of boosts "in any given direction forms a noncompact subgroup," i.e. the subgroups $e^{\nu \mathbf{L}_{0i}}$ are noncompact subgroups. This eliminates several of the listed real forms in the above classification. In particular, the first real form in the $R2$ case, which is $\{\mathfrak{so}(4)_{(\mathbf{L}_{ij},\mathbf{L}_{0i})} \oplus \mathbb{R}\mathbf{S}\} \oplus_\tau \mathfrak{t}^4$, is excluded as a possible kinematical group, since $\mathfrak{so}(4)_{(\mathbf{L}_{ij},\mathbf{L}_{0i})}$ is compact.

Following Poincaŕe in Ref. [2], we must, due to physical requirements, restrict the kinematical group to a subgroup. It should be a subgroup of the scale extended group with scale transformations depending upon the boost parameter, ν: "...we should consider only certain transformations in this group; we must assume that λ [the scale transformation] is a function of ν, and it is a question of choosing this function in such a way that this part of the group, which will be denoted by P, is itself a group [2]." For standard Lorentzian relativity (the second real form in the $R2$ case) this leads us to the result that $\lambda = \lambda_\nu = \pm 1$ (cf. [2]). Poincaré's argument for reduction of the scale extended Lorentz group to $SO_0(1,3) \times \Sigma_2 \cong SO(1,3)$ with $\Sigma_2 = \{\mathbb{I}_4, -\mathbb{I}_4\}$ runs as follows. Let

$$\Lambda(\nu) = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

($\beta = \operatorname{arctanh}\nu$). "Any [homogeneous] transformation of the group P may be regarded as a transformation of the form $\lambda_\nu \Lambda(\nu)$ preceded and followed by suitable rotations" (KAK decomposition for scale extended $SO_0(1,3)$ restricted to the homogeneous part of P). We easily show that $R_\pi \lambda_\nu \Lambda(\nu) R_\pi^{-1} = \lambda_\nu \Lambda(-\nu)$ where R_π is a rotation about the y axis by π. Since the homogeneous part of P consists of all matrices of the form $\lambda_\nu \Lambda(R\nu) R'$ with $\nu \in \mathbb{R}$ and $R, R' \in SO(3)$, $\lambda_\nu \Lambda(-\nu)$ is in P. It will equal $\lambda_{-\nu}\Lambda(-\nu)$ for $\lambda_\nu = \lambda_{-\nu}$. So **$\boldsymbol{\lambda_\nu}$ should be an even function of $\boldsymbol{\nu}$.**

Now the inverse of $\lambda_\nu \Lambda(\nu)$ is ${\lambda_\nu}^{-1}\Lambda(-\nu)$. In order for this to be in P it must equal $\lambda_{-\nu}\Lambda(-\nu) = \lambda_\nu \Lambda(-\nu)$ which leads to $\lambda_\nu^{-1} = \lambda_\nu$. Hence $\lambda_\nu^2 = 1 \Rightarrow \lambda_\nu = \pm 1$ and, with $\rtimes$ denoting semidirect product, we have:

Theorem 1 *(Poincaré [2])*
Reduction of symmetry for the $SO_0(1,3)$ real form of Case R2 (scale extended $SO_0(1,3) \rtimes \mathbb{T}^4$) leads to $P = SO(1,3) \rtimes \mathbb{T}^4$, the proper inhomogeneous Lorentz group, as the kinematical group of special relativity. P contains space-time inversion $-\mathbb{I}_4$ and its connected component is $SO_0(1,3) \rtimes \mathbb{T}^4$.

Even though the homogeneous part of the Galilean group is not semisimple, Poincaré's arguments leading to Theorem 1 carry over to the scale extended Galilean group (Case $\tilde{A}1$) and they lead to the same conclusions, namely that $\{(SO(3) \rtimes N_3) \times \Sigma_2\} \rtimes \mathbb{T}^4$ is the kinematical group, where N_3 is the 3 dimensional subgroup of Galilean boosts and $\Sigma_2 = \{\mathbb{I}_4, -\mathbb{I}_4\}$, with $-\mathbb{I}_4$ being space-time inversion.

For the real forms $\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(1,4)$ and $\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(2,3)$ of Case $R1$, the situtation regarding reduction of scale is even more interesting. It is due to the fact that the connected components of the Lie groups associated with $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$ have group decompositions into subgroups which involve $SO(1,3)$ instead of $SO_0(1,3)$ as one of the factors [9]. Since $SO(1,3)$ has two disconnected components, the generalization of Poincaré's argument to these cases is more complicated. It again leads to $\lambda_v = \pm 1$. However, we are free to set λ_v as $+1$ or -1 on either component. This leads to several choices for the relativity group, involving improper $O(1,4)$ or $O(2,3)$ transformations. Such additional structures could possibly lead to novel results in descriptions of elementary systems [10], [11] for relativistic quantum mechanics on de Sitter or anti-de Sitter space based on projective representations of $O(1,4)$ or $O(2,3)$, respectively. Although this is surely something well worth exploring, page limitations do not permit us to go further into the matter.

4 Other approaches and conclusion

There are other approaches to describing possible space-time structures and associated kinematical groups. The causality approach starts with a partial ordering on space time, M. Causal automorphisms on M are automorphisms which preserve the partial ordering. The set of all causal automorphisms forms a group, which, for M being an affine space, turns out to be the scale extended inhomogeneous orthochronous Lorentz group (Alexandrov-Zeeman result) [12], [13]. Lalan's 1937 classification [14] of all possible linear kinematics in two space-time dimensions compatible with the relativity principle is based on the Frank and Rothe paper [15]. Another very interesting approach going back to V. Gorini [16] rests on a physical assumption which essentially means that the set of inertial transformations taking frames at rest to frames at rest is the group $O(3)$. He proves that the only subgroups of $GL(4,\mathbb{R})$ satisfying this physical assumption are the proper orthochronous Galilean group and the proper orthochronous Lorentz group, along with isomorphic copies of it obtained by a rescaling of the boost generators [16].

In conclusion, incorporating scale symmetry into the analysis of classifications of possible kinematical groups leads to more interesting possible structures regarding discrete transformations like time reversal and spatial inversion, especially for the cases involving the de Sitter and anti-de Sitter groups.

References

[1] H. Poincaré, *L'état actuel et l'avenir de la physique mathématique*, Bull. Sci. Math. **28**, 302 (1904), https://gallica.bnf.fr/ark:/12148/bpt6k486258r/f476.item.

[2] H. Poincaré, *Sur la dynamique de l'électron*, Rend. Circ. Matem. Palermo **21**, 129 (1906), doi:10.1007/BF03013466.

[3] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys. **322**, 891 (1905), doi:10.1002/andp.19053221004.

[4] J. Ehlers, F. A. E. Pirani, A. Schild, *General relativity; papers in honour of J. L. Synge*, Clarendon Press, Oxford, UK, ISBN 9780198511267 (1972).

[5] J. Ehlers and A. Schild, *Geometry in a manifold with projective structure*, Commun. Math. Phys. **32**, 119 (1973), doi:10.1007/BF01645651.

[6] J.-M. Lévy-Leblond, *One more derivation of the Lorentz transformation*, Am. J. Phys **44**, 271 (1976), doi:10.1119/1.10490.

[7] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. **22**, 10 (1957), doi:10.1090/memo/0022.

[8] H. Bacry and J.-M. Lévy-Leblond, *Possible kinematics*, J. Math. Phys. **9**, 1605 (1968), doi:10.1063/1.1664490.

[9] P. Moylan, *Parallelizations for induced representations of $SO_0(2,q)$*, in *Quantum theory and symmetries with lie theory and its applications in physics*, Springer, Singapore, ISBN 9789811327148 (2018), doi:10.1007/978-981-13-2715-5_12.

[10] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40**, 149 (1939), doi:10.2307/1968551.

[11] V. Bargmann and E. P. Wigner, *Group theoretical discussion of relativistic wave equations*, Proc. Natl. Acad. Sci. **34**, 211 (1948), doi:10.1073/pnas.34.5.211.

[12] A. D. Alexandrov, V. V. Ovchinnikova, *Notes on the foundations of relativity theory*, Vestnik Leningrad. Univ., **11** 95 (1953).

[13] E. C. Zeeman, *Causality implies the Lorentz group*, J. Math. Phys. **5**, 490 (1964), doi:10.1063/1.1704140.

[14] V. Lalan, *Sur les postulats qui sont à la base des cinemátiques* Bull. Soc. Math. France **65**, 83 (1937), http://eudml.org/doc/86692.

[15] P. Frank and H. Rothe, *Über die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme*, Ann. Phys. **339**, 825 (1911), doi:10.1002/andp.19113390502.

[16] V. Gorini, *Linear kinematical groups*, Commun. Math. Phys. **21**, 150 (1971), doi:10.1007/BF01646749.

Unitary Howe dualities in fermionic and bosonic algebras and related Dirac operators

Guner Muarem⋆

University of Antwerp, Antwerp, Belgium

⋆ guner.muarem@uantwerpen.be

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

In this paper we use the canonical complex structure $\mathbb{J}$ on $\mathbb{R}^{2n}$ to introduce a twist of the symplectic Dirac operator. This can be interpreted as the bosonic analogue of the Dirac operators on a Hermitian manifold. Moreover, we prove that the algebra of these Dirac operators is isomorphic to the Lie algebra $\mathfrak{su}(1,2)$ which leads to the Howe dual pair $(\mathrm{U}(n), \mathfrak{su}(1,2))$.

Received 17-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.038

1 Introduction

The CCR (canonical commuting relation) and CAR (canonical anticommuting relation) algebras are fundamental algebras in theoretical physics used for the study of bosons and fermions. From a mathematical viewpoint, these algebras are named the Weyl algebra (or symplectic Clifford algebra) and Clifford algebra. These algebras can be constructed in a very analogous way. The Clifford algebra is constructed on a vector space V equipped with a symmetric bilinear form B, whereas the Weyl algebra requires an even dimensional vector space equipped with a skew-symmetric bilinear form (or symplectic form) ω. In both cases, one then constructs the tensor algebra $T(V)$ where an ideal $I(V)$ is divided out. In the orthogonal setting, this is the ideal $I_B(V)$ with elements subject to the relation $\{u, v\} = 2B(u, v)$. In the symplectic setting, this is the ideal $I_\omega(V)$ generated by $[u, v] = -\omega(u, v)$.

There is, however, a fundamental difference: the Clifford algebra is finite-dimensional, whereas the Weyl algebra is infinite-dimensional. For the spinors (orthogonal versus symplectic) the same infinite-dimensional principle holds as for the Clifford algebras. As a matter of fact, the symplectic spinors are the smooth vectors in the metaplectic representation [2]. Using the generators of the Clifford (resp. Weyl) algebra, one can associate a natural first order spin (resp. metaplectic) invariant differential operator by contracting the Clifford algebra elements using the bilinear form B (resp. the symplectic form ω) with derivatives. This gives rise to the Dirac operator $\partial_x = \sum_{k=1}^n e_k \partial_{x_k}$ where $\{e_j, e_k\} = -2\delta_{ij}$ and the symplectic Dirac operator $\sum_{k=1}^n \left(iq_k \partial_{y_k} - \partial_{q_k} \partial_{x_k}\right)$ where $[\partial_{q_j}, iq_k] = i\delta_{jk}$ are the Heisenberg relations (see [1,2,4]).

The theory which studies the solutions of the Dirac operator is known as Clifford analysis and can be seen as a hypercomplex function theory. Moreover, quite some generalisations have occurred in the last two decades. This involves e.g. Clifford analysis on superspace and Clifford analysis on (hyper)Kähler spaces. It is in the latter framework in which this paper is situated, but then from a symplectic point of view. More precisely, we provide the foundations of what we will call a *Hermitian variant of symplectic Clifford analysis*, where we incorporate the additional datum of a compatible complex structure $\mathbb{J}$ on the flat symplectic space $\mathbb{R}^{2n}$. This leads to the study of the symplectic Dirac operator on a Kähler manifold as was already initiated in [2,3]. However, the underlying invariance symmetry and the algebra generated by these type of operators (and their duals) was never investigated.

2 Rudiments of symplectic Clifford analysis

Let us consider the canonical symplectic space $\mathbb{R}^{2n}$ with coordinates (x, y) and the usual symplectic form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ which has the matrix representation $\Omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Recall that the symplectic group $\mathsf{Sp}(2n, \mathbb{R})$ is the group given by invertible linear transformations preserving the non-degenerate skew-symmetric bilinear form from above and is given (in terms of matrices) by

$$\mathsf{Sp}(2n, \mathbb{R}) = \{M \in \mathsf{GL}(2n, \mathbb{R}) \mid M^T \Omega_0 M = \Omega_0\}\,.$$

The group is non-compact and has dimension $2n^2+n$. Moreover, the corresponding Lie algebra is denoted by $\mathfrak{sp}(2n, \mathbb{R})$. The main difference with the orthogonal case, lies in the fact that the metaplectic group (the double cover of the symplectic group) does *not* admit a finite-dimensional representation (it is not a matrix group). This is a strong contrast with the spin representation in the orthogonal case. Moreover, the orthogonal spinors $\mathbb{S}$ are realised as a idempotent left ideal in the Clifford algebra, which is not the case for the symplectic spinors. As mentioned, the symplectic equivalent of the spin representations are infinite dimensional, which means that one needs to work with the theory of unitary representations.

2.1 The Schwartz space and metaplectic representation

For further convenience, we fix notation and define the Schwartz space, which plays a crucial role in the construction of the metaplectic representation. On the space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$ we define (using the multi-index notation) the norm $||f||_{\alpha,\beta} := \sup_{q\in\mathbb{R}^n} |q^\alpha (D^\beta f)(q)|$ for all $\alpha, \beta \in \mathbb{N}^n$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the subspace of $L^p(\mathbb{R}^n)$ (for $1 \leq p \leq \infty$) consisting of rapidly decreasing functions and is given by

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}) : ||f||_{\alpha,\beta} < \infty\,,\ \text{for all } \alpha, \beta \in \mathbb{N}^n\}\,.$$

We now describe (following [2]) the infinite-dimensional Segal-Shale-Weil representation (also oscillator or metaplectic representation) of the metaplectic group. The smooth vectors of the unitary representation $\mathfrak{m} : \mathsf{Mp}(2n) \to \mathsf{U}(L^2(\mathbb{R}^n))$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and are a model for the symplectic spinors $\mathbb{S}^\infty$. Due to Stone-Von Neumann theorem the representation is unique (up to unitary equivalence).

2.2 The symplectic Clifford algebra and the related Dirac operator

Let (V,ω) be a symplectic vector space. The *symplectic Clifford algebra* $\mathsf{Cl}_s(V,\omega)$ is defined as the quotient algebra of the tensor algebra $T(V)$ of V, by the two-sided ideal

$$\mathcal{I}_\omega := \{v\otimes u - u\otimes v + \omega(v,u) : u,v\in V\}.$$

In other words $\mathsf{Cl}_s(V,\omega) := T(V)/\mathcal{I}_\omega$ is the algebra generated by V in terms of the relation $[v,u] = -\omega(v,u)$, where we have omitted the tensor product symbols. We refer to the symplectic Clifford algebra on $\mathbb{R}^{2n}$ as the nth Weyl algebra $\mathcal{W}_n$ with generators $iq_1,\ldots,iq_n,\partial_{q_1},\ldots,\partial_{q_n}$ satisfying the commutation relations $[q_j,q_k]=0$ and $[\partial_{q_j},q_k]=\delta_{jk}$.

Denote by $\mathcal{F}$ a suitable function space (e.g. the space of polynomials, or smooth funtions). The *symplectic Dirac operator* on $(\mathbb{R}^{2n},\omega_0)$ is the first-order (in the base variables x and y) differential operator acting on a symplectic spinor-valued functions space $\mathcal{F}\otimes\mathbb{S}^\infty$ given by $D_s=\sum_{j=1}^n(iq_j\partial_{y_j}-\partial_{q_j}\partial_{x_j})$. With respect to the symplectic Fischer inner product (see [4]), we obtain the dual operator $X_s=\sum_{j=1}^n(iq_jx_j+\partial_{q_j}y_j)$. These operators satisfy the relations:

$$\begin{aligned}[]
[\mathbb{E}+n, X_s] &= X_s\,,\\
[\mathbb{E}+n, D_s] &= -D_s\,,\\
[D_s, X_s] &= -i(\mathbb{E}+n)\,,
\end{aligned}$$

where $\mathbb{E}=\sum_{j=1}^n(x_j\partial_{x_j}+y_j\partial_{y_j})$ is the *Euler operator*. In other words, the three operators give rise to a copy of the Lie algebra $\mathfrak{sl}(2)$.

3 The interaction with a complex structure

3.1 Definition of the twisted symplectic Dirac operators

We will now introduce a complex structure $\mathbb{J}$ on the symplectic manifold $(\mathbb{R}^{2n},\omega_0)$ which is compatible with the symplectic form ω_0. This means that $\omega_0(x,\mathbb{J}y)$ defines a Riemannian metric g. Otherwise said, we will be working with the canonical Kähler manifold $(\mathbb{R}^{2n},\omega_0,g,\mathbb{J})$. By Darboux's theorem, we obtain, with respect to the canonical symplectic basis $\{e_j\}_{j=1}^{2n}$ the following complex structure $\mathbb{J}=\left(\begin{smallmatrix}0 & -I_n\\ I_n & 0\end{smallmatrix}\right)$. This means that the action of the complex structure $\mathbb{J}$ on $\mathbb{R}^{2n}$ is given by

$$(x_1,\ldots,x_n,y_1,\ldots,y_n)\mapsto(y_1,\ldots,y_n,-x_1,\ldots,-x_n).$$

The new differential operators acting on symplectic spinor-valued functions

$$\tilde{D}_s=\sum_{j=1}^n iq_j\partial_{x_j}+\partial_{y_j}\partial_{q_j}\,,\qquad \tilde{X}_s=\sum_{j=1}^n x_j\partial_{q_j}-iy_jq_j\,,\qquad \mathbb{E}=\sum_{j=1}^n x_j\partial_{x_j}+y_j\partial_{y_j}\,,$$

also give rise to a copy of the Lie algebra $\mathfrak{sl}(2)$. We call these first two operators the *twists* of D_s and X_s. Both sets of operators, i.e. (D_s,X_s) and $(\tilde{D}_s,\tilde{X}_s)$, are symplectic invariant, albeit under the following two *different* realisations of the symplectic Lie algebra given by:

$$\begin{cases}
X_{jk}=x_j\partial_{x_k}-y_k\partial_{y_j}-(q_k\partial_{q_j}+\frac{1}{2}\delta_{jk}),\\
Y_{jk}=x_j\partial_{y_k}+x_k\partial_{y_j}+i\partial_{q_j}\partial_{q_k}\,,\\
Z_{jk}=y_j\partial_{x_k}+y_k\partial_{x_j}+iq_jq_k\,,\\
Y_{jj}=x_j\partial_{y_j}+\frac{i}{2}\partial_{q_j}^2\,,\\
Z_{jj}=y_j\partial_{x_j}+\frac{i}{2}q_j^2\,,
\end{cases}
\quad\text{and}\quad
\begin{cases}
\tilde{X}_{jk}=x_j\partial_{x_k}-y_k\partial_{y_j}+q_k\partial_{q_j}+\frac{1}{2}\delta_{jk}\,, & 1\le j\le k\le n\,,\\
\tilde{Y}_{jk}=x_j\partial_{y_k}+x_k\partial_{y_j}-iq_jq_k\,, & j<k=1,\ldots,n\,,\\
\tilde{Z}_{jk}=y_j\partial_{x_k}+y_k\partial_{x_j}-i\partial_{q_j}\partial_{q_k}\,, & j<k=1,\ldots,n\,,\\
\tilde{Y}_{jj}=x_j\partial_{y_j}-\frac{i}{2}q_j^2\,, & j=1,\ldots,n\,,\\
\tilde{Z}_{jj}=y_j\partial_{x_j}-\frac{i}{2}\partial_{q_j}^2 & j=1,\ldots,n\,.
\end{cases}$$

Of course, it is not very useful that D_s and $\widetilde{D}_s$ are invariant under different (yet isomorphic) $\mathfrak{sp}(2n,\mathbb{R})$-realisations. Therefore, we will perform a symmetry reduction so that both operators become invariant under *one and the same* Lie algebra. To that end, we need to find the symplectic matrices which commute with the complex structure. We claim that

$$\mathsf{Sp}_{\mathbb{J}}(2n,\mathbb{R}) := \{M \in \mathsf{Sp}(2n,\mathbb{R}) \mid M\mathbb{J} = \mathbb{J}M\},$$

defines a realisation for the unitary Lie group. In order to see this, assume that M is of the block-form: $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C and D are $(n \times n)$—matrices. The condition that M is symplectic is equivalent to the one of the following conditions: the matrices $A^T C$ and $B^T D$ are symmetric and $A^T D - C^T B = I$. So, in order to determine $\mathsf{Sp}_{\mathbb{J}}(2n,\mathbb{R})$ we need to determine the symplectic matrices M which commute with the complex structure $\mathbb{J}$. The latter conditions means that

$$\begin{aligned} M\mathbb{J} = \mathbb{J}M &\iff \mathbb{J}^{-1}M\mathbb{J} = M \\ &\iff \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &\iff \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \end{aligned}$$

This implies $A = D$ and $B = -C$. In other words the matrix M is of the form: $M = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. Next, we still have the condition that M is symplectic, i.e.

$$\begin{aligned} M^T\Omega M = \Omega &\iff \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &\iff \begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -C^TA + A^TC & -C^TB + A^TD \\ -D^TA + B^TC & -D^TB + B^TD \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &\iff \begin{cases} A^TC = C^TA, \\ A^TD - C^TB = I, \\ B^TC - D^TA = -I, \\ B^TD = D^TB. \end{cases} \end{aligned}$$

This means that $A^T C$ and $B^T D$ should be symmetric matrices and $A^T D - C^T B = I$. But now, due to the first condition this reduces to $B^T A = A^T B$ and $A^T A + B^T B = I$. In other words, the matrices we are looking for must be of the form $M = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$ with $B^T A = A^T B$ and $A^T A + B^T B = I$, i.e.

$$\begin{pmatrix} A^T & -B^T \\ B^T & A^T \end{pmatrix}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^TA + B^TB & A^TB - B^TA \\ B^TA - A^TB & B^TB + A^TA \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Which is exactly the condition for a unitary matrix. The map

$$\Phi : \mathsf{Sp}_{\mathbb{J}}(2n,\mathbb{R}) \to \mathsf{U}(n) : M \mapsto A + iB,$$

gives the wanted isomorphism.

3.2 Unitary invariant symplectic Dirac operators

One can now check that the symplectic Dirac operator and its twist, are unitary invariant differential operators. This can be done by verifying that the operators commute with the following realisation of unitary Lie algebra $\mathfrak{u}(n)$:

$$\begin{cases} A_{jk} = y_j\partial_{x_k} + y_k\partial_{x_j} - x_j\partial_{y_k} - x_k\partial_{y_j} + i(q_jq_k - \partial_{q_j}\partial_{q_k}), & 1 \le j < k \le n, \\ B_{jj} = y_j\partial_{x_j} - x_j\partial_{y_j} + \frac{i}{2}\left(q_j^2 - \partial_{q_j}^2\right), & 1 \le j \le n, \\ C_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j} + y_j\partial_{y_k} - y_k\partial_{y_j} + q_j\partial_{q_k} - q_k\partial_{q_j}, & 1 \le j < k \le n. \end{cases}$$

This means that we can refine the $\mathfrak{sp}(2n)$-invariant PDE $D_sf = 0$ into two $\mathfrak{u}(n)$-invariant PDEs given by $D_sf = 0$ and $\widetilde{D}_sf = 0$, for a symplectic spinor valued polynomial $f \in \mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$. In analogy with the orthogonal case, we call the solutions *hermitian symplectic monogenics* (or h-symplectic monogenics in short).

3.3 Symplectic Dolbeault operators

Moreover, there is a second way of introducing the twist of the symplectic Dirac operator. Let us define the following operators which are known in the literature as the *symplectic Dolbeault operators* [3] defined by means of $D_z = \frac{D_s + i\widetilde{D}_s}{2}$ and $D_z^\dagger := \frac{D_s - i\widetilde{D}_s}{2}$. One easily verifies that

$$\frac{1}{2}(D_s + i\widetilde{D}_s) = -\sum_{j=1}^{n} \mathfrak{F}_j\partial_{z_j}, \quad \text{and} \quad \frac{1}{2}(D_s - i\widetilde{D}_s) = \sum_{j=1}^{n} \mathfrak{F}_j^\dagger\partial_{\overline{z}_j},$$

where we have introduced the symbols $\mathfrak{F}_j = (q_j + \partial_{q_j})$, $\mathfrak{F}_j^\dagger = (q_j - \partial_{q_j})$ and $\partial_{\overline{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ is the Cauchy-Riemann operator in the relevant variable with conjugate $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$. The structure of the operators D_z and $D_z^\dagger$ is similar to the orthogonal case. However, the raising/lowering operators $\mathfrak{F}_j$ and $\mathfrak{F}_j^\dagger$ are used instead of isotropic Witt vectors $\mathfrak{f}_j$ and $\mathfrak{f}_j^\dagger$ (see for instance [5] and the references therein).

3.4 Class of simultaneous solutions of D_s and $\widetilde{D}_s$

We will now describe a wide class of examples of h-symplectic monogenics, by making the link with holomorphic functions in several variables. Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ be a complex-valued function in several complex variables which is of the class $\mathcal{C}^1(\Omega)$ (i.e. continuously differentiable). We say that f is holomorphic (in several variables) if $\partial_{\overline{z}_j} f(z_1, \ldots, z_n) = 0$ for all $1 \le j \le n$. Moreover, we denote the set of holomorphic functions in Ω by $\mathrm{Hol}(\Omega)$.

In order not to overload notations, we use the summation convention. Suppose that we have a function of the form $F(x, y, q) = e^{-\frac{1}{2}|q|^2}H(x, y)$. Letting the symplectic Dirac operator act on F gives:

$$\begin{aligned} D_s\left(e^{-\frac{1}{2}|q|^2}H(x,y)\right) &= (iq_k\partial_{y_k} - \partial_{x_k}\partial_{q_k})\left(e^{-\frac{1}{2}|q|^2}H(x,y)\right) \\ &= iq_k e^{-\frac{1}{2}|q|^2}\partial_{y_k}H(x,y) + e^{-\frac{1}{2}|q|^2}q_k\partial_{x_k}H(x,y) \\ &= e^{-\frac{1}{2}|q|^2}q_k(\partial_{x_k} + i\partial_{y_k})H(x,y). \end{aligned}$$

We note that this equals zero if $(\partial_{x_k} + i\partial_{y_k})H(x, y) = 0$ for all $k = 1, \ldots, n$, i.e. if $H(x, y)$ is a holomorphic function in several variables. Completely similar,

$$\begin{aligned} \widetilde{D}_s\left(e^{-\frac{1}{2}|q|^2}H(x,y)\right) &= (iq_k\partial_{x_k} + \partial_{y_k}\partial_{q_k})\left(e^{-\frac{1}{2}|q|^2}H(x,y)\right) \\ &= e^{-\frac{1}{2}|q|^2}q_k(i\partial_{x_k} - \partial_{y_k})H(x,y) \\ &= ie^{-\frac{1}{2}|q|^2}q_k(\partial_{x_k} + i\partial_{y_k})H(x,y), \end{aligned}$$

which is zero for holomorphic $H(x, y)$.

This means that every function of the form $e^{-\frac{1}{2}(q_1^2+\cdots+q_n^2)}H(x,y)$ with H an holomorphic function in several variables is a solution of both D_s and $\widetilde{D}_s$. This observation generalises the class of solutions obtained by Habermann in [2] for $n=1$. It turned out that there are much *more solutions* than the ones of this form. In order to describe these systematically, we will need the notion of Howe dualities and corresponding Fischer decompositions. This will be done in full detail in our upcoming paper [6]. In the following section we will reveal the algebraic structures required for this approach.

4 A unitary Howe duality associated with D_s and $\widetilde{D}_s$

Recall that the Lie algebra $\mathfrak{su}(1,2)$ is a quasi-split real form of the complex Lie algebra $\mathfrak{sl}(3)$ and is defined in terms of matrices as

$$\mathfrak{su}(1,2)=\left\{\begin{pmatrix}\alpha & \beta & ic\\ \gamma & \overline{\alpha}-\alpha & -\overline{\beta}\\ id & -\overline{\gamma} & -\overline{\alpha}.\end{pmatrix} \mid c,d\in\mathbb{R}\ \&\ \alpha,\beta,\gamma\in\mathbb{C}\right\}.$$

A basis of the Lie algebra $\mathfrak{su}(1,2)$ is given by the the following eight matrices:

$$H_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad H_2=\begin{pmatrix}i&0&0\\0&-2i&0\\0&0&i\end{pmatrix},$$

$$X_1=\begin{pmatrix}0&1&0\\0&0&-1\\0&0&0\end{pmatrix},\quad X_2=\begin{pmatrix}0&i&0\\0&0&i\\0&0&0\end{pmatrix},\quad X_3=\begin{pmatrix}0&0&i\\0&0&0\\0&0&0\end{pmatrix},$$

$$Y_1=\begin{pmatrix}0&0&0\\1&0&0\\0&-1&0\end{pmatrix},\quad Y_2=\begin{pmatrix}0&0&0\\i&0&0\\0&i&0\end{pmatrix},\quad Y_3=\begin{pmatrix}0&0&0\\0&0&0\\i&0&0\end{pmatrix}.$$

The commutation relations of these matrices give rise to the following table:

$[\cdot,\cdot]$	H_1	H_2	X_1	X_2	X_3	Y_1	Y_2	Y_3
H_1	0	0	X_1	X_2	$2X_3$	$-Y_1$	$-Y_2$	$-2Y_3$
H_2		0	$3X_2$	$-3X_1$	0	$-3Y_2$	$3Y_1$	0
X_1			0	$2X_3$	0	H_1	H_2	$-Y_2$
X_2				0	0	H_2	$-H_1$	$-Y_1$
X_3					0	$-X_2$	$-X_1$	$-H_1$
Y_1						0	$-2Y_3$	0
Y_2							0	0
Y_3								0

The Lie algebra generated by the symplectic Dirac operators $D_s,\widetilde{D}_s$ and their duals $X_s,\widetilde{X}_s$ gives rise to a copy of the Lie algebra $\mathfrak{su}(1,2)$. In order to close the algebra, we introduce the following differential operators:

$$\mathcal{O}:=\sum_{j=1}^n i(x_j\partial_{y_j}-y_j\partial_{x_j})+\partial_{q_j}^2-q_j^2,\quad \Delta:=\sum_{j=1}^n \partial_{x_j}^2+\partial_{y_j}^2,\quad\text{and}\quad r^2:=\sum_{j=1}^n {x_j}^2+{y_j}^2.$$

It follows from straightforward calculations that the eight operators can be identified with the matrices from above: $H_1\longleftrightarrow\mathbb{E},H_2\longleftrightarrow\mathcal{O},X_1\longleftrightarrow X_t,X_2\longleftrightarrow X_s,X_3\longleftrightarrow r^2,Y_1\longleftrightarrow\widetilde{D}_s,Y_2\longleftrightarrow D_s$ and $Y_3\longleftrightarrow\Delta$.

1. We have two copies of the Heisenberg algebra: $\mathrm{Alg}\{D_s, \tilde{D}_s, \Delta\} \cong \mathrm{Alg}\{X_s, \widetilde{X}_s, r^2\} \cong \mathfrak{h}_3$.
2. We have three copies of $\mathfrak{sl}(2)$: $\mathrm{Alg}\{D, D^\dagger, \mathbb{E}\} \cong \mathrm{Alg}\{\tilde{D}, \tilde{D}^\dagger, \mathbb{E}\} \cong \mathrm{Alg}\{\Delta, r^2, \mathbb{E}\} \cong \mathfrak{sl}(2)$.

This means that there is a canonical $\mathfrak{su}(1,2)$-action on the space of spinor valued polynomials $\mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ where restricting to the subalgebra $\mathrm{Alg}(D_s, X_s)$ corresponds to $\mathfrak{sl}(2)$-copy obtained in the Howe duality for symplectic Clifford analysis (see [4] for more details). Now, taking into account the symplectic Dirac operators and its twists, we obtain the dual pair $\mathsf{U}(n) \times \mathfrak{su}(1,2)$ (i.e. the underlying group of invariance, together with the algebra generated by the operators and their duals).

We now focus on the reduction of the symplectic spinor space. In the orthogonal case, the spinor space $\mathbb{S}$ decomposes under the action of the unitary group $\mathsf{U}(n)$ as $\mathbb{S} = \bigoplus_r \mathbb{S}_{(r)}$, with $\mathbb{S}_{(r)}$ inequivalent irreducible pieces, which are eigenspaces of the *fermionic* quantum oscillator (also called spin-Euler operator, see for instance [5]). In the symplectic case, the relevant operator for decomposing the infinite dimensional spinor space is the *bosonic* quantum oscillator. The Hamiltonian of the quantum oscillator, the so-called Hermite operator, is given by

$$\mathcal{H} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \qquad f(q) \mapsto \frac{1}{2} \sum_{j=1}^{n} \left(\partial_{q_j}^2 - q_j^2 \right) f(q).$$

Note that we can write $\mathcal{O} = \sum_{j=1}^n i(x_j \partial_{y_j} - y_j \partial_{x_j}) + 2\mathcal{H}$, so that the Hermite operator is in fact the spinor-valued part of the operator $\mathcal{O}$, i.e. the differential operator in ∂_{q_j} and the variables q_j. Moreover, the eigenspaces can be identified with the irreducible decomposition of $\mathbb{S}^\infty$ into $\mathfrak{u}(n)$-irreducible representations. This means that the symplectic spinor space $\mathbb{S}^\infty$ decomposes into $\mathfrak{u}(n)$-irreducible representations $\widetilde{S}^\infty_{(k)}$ of dimension $\binom{n+k-1}{k}$ which can be thought of as k-homogeneous polynomials or the eigenspaces of the Hermite operator $\mathcal{H}$.

Moreover, the solutions of the corresponding Dirac operators, called monogenics, can be introduced from a purely representation theoretical viewpoint. In general, this boils down to determining the decomposition (this is called a *Fischer decomposition*) $\mathcal{P}_k(\mathbb{R}^m, \mathbb{C}) \otimes \mathbf{S}$ where $\mathbf{S}$ is the spinor space, which is $\mathbb{S}$ in the orthogonal case and $\mathbb{S}^\infty$ in the symplectic case, where we take $m = 2n$ in particular. We denote by $\mathcal{M}_k$ the k-homogeneous solutions of the Dirac operator $\partial_x = \sum_{j=1}^m e_j \partial_{x_j}$, these are called monogenics. They can be defined as follows:

$$\mathcal{M}_k \longleftrightarrow (k, 0, \dots, 0) \boxtimes \mathbb{S} = (k) \boxtimes \left(\frac{1}{2}, \dots, \pm\frac{1}{2} \right) \cong \left(k + \frac{1}{2}, \dots, \pm\frac{1}{2} \right),$$

where $\boxtimes$ denotes the Cartan product of the $\mathfrak{so}(m)$-representations and $(k, 0, \dots, 0)$ is the space of k-homogeneous harmonics (solutions of the Laplacian). In the symplectic case, we have:

$$\begin{aligned} \mathcal{M}_k^s \longleftrightarrow (k, 0, \dots, 0)_s \boxtimes \mathbb{S}^\infty &= (k)_s \boxtimes \left(\left(-\frac{1}{2}, \dots, -\frac{1}{2} \right) \oplus \left(-\frac{1}{2}, \dots, -\frac{3}{2} \right) \right) \\ &\cong \left(k - \frac{1}{2}, \dots, -\frac{1}{2} \right) \oplus \left(k - \frac{1}{2}, \dots, -\frac{3}{2} \right), \end{aligned}$$

where $(k, 0, \dots, 0)_s$ is the space of k-homogeneous polynomials. In order to obtain an algebraic characterisation of the space of h-symplectic monogenics, one proceeds as follows. First of all, we note that we need to consider the symplectic spinors $\mathbb{S}^\infty$ from an unitary viewpoint. We saw that $\mathbb{S}^\infty$ decomposes as an infinite direct sum of finite dimensional $\mathfrak{u}(n)$-modules $\widetilde{S}^\infty_{(k)}$ which are in fact eigenspaces of the Hermite operator. We denote the branched spinor space (which is in fact a direct sum of $\mathfrak{u}(n)$-irreps) by $\widetilde{\mathbb{S}^\infty}$. However, the space of k-homogeneous polynomials is *not* irreducible as a $\mathfrak{u}(n)$-module and we denote the branched module by $\widetilde{(k)}$. This means that the Cartan product $\mathcal{M}_k^{hs} \longleftrightarrow \widetilde{(k)} \boxtimes \widetilde{\mathbb{S}^\infty}$ would be a well-educated guess as a representation theoretical definition of the h-symplectic monogenics. Recall that these are the symplectic spinor-valued polynomial functions $f \in \mathcal{P}(\mathbb{R}^{2n}, \mathbb{C}) \otimes \mathcal{S}(\mathbb{R}^n)$ such that $D_s f = \widetilde{D}_s f = 0$. The explicit calculation of the Cartan product (and more generally the tensor product) will be done in [6]. Moreover, as an application we will prove a Fischer decomposition for the Howe dual pair we obtained in this paper.

5 Conclusion

In this paper we investigated a Howe dual pair occurring in symplectic Clifford analysis by allowing a compatible complex structure. This Howe duality is of the form $(G, \mathfrak{g}')$ where G is the underlying invariance group for which the relevant Dirac operators are invariant and $\mathfrak{g}'$ is the algebra generated by the Dirac operators and their duals. Depending on the orthogonal or symplectic framework, we have the following 'types' of Clifford analysis and *refinements*:

1. Orthogonal geometry (giving rise to a Clifford algebra)
   (a) Clifford analysis: $\mathsf{SO}(n) \times \mathfrak{osp}(1|2)$
   (b) Hermitian Clifford analysis $\mathsf{U}(n) \times \mathfrak{osp}(2|2)$
   (c) Quaternionic Clifford analysis $\mathsf{USp}(n) \times \mathfrak{osp}(4|2)$
2. Symplectic geometry (giving rise to a Weyl algebra)
   (a) Symplectic Clifford analysis: $\mathsf{Sp}(2n) \times \mathfrak{sl}(2)$
   (b) Hermitian symplectic Clifford analysis: $\mathsf{U}(n) \times \mathfrak{su}(1,2)$
   (c) Quaternionic symplectic Clifford analysis: $\mathsf{USp}(n) \times ?$

Thus far, we extended the framework of hermitian Clifford analysis in the presence of a symplectic structure in the case of the (flat) Kähler manifold $\mathbb{R}^{2n}$. It is an interesting question to further reduce the symmetry to the compact symplectic group $\mathsf{USp}(\mathsf{n})$ so that we have the chain $\mathsf{Sp}(2n) \supset \mathsf{U}(n) \supset \mathsf{USp}(n)$. In our furture work [6], we will describe the Fischer decomposition accompanying this new Howe dual pair.

6 Acknowledgements

Funding information The author is supported by the FWO-EoS project 'Symplectic Techniques in Differential Geometry' G0H4518N.

References

[1] D. Eelbode and G. Muarem, *The orthogonal branching problem for symplectic monogenics*, Adv. Appl. Clifford Algebras **32**, 32 (2022), doi:10.1007/s00006-022-01215-1.

[2] K. Habermann and L. Habermann, *Introduction to symplectic Dirac operators*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540334200 (2006), doi:10.1007/b138212.

[3] E. O. Korman, *Symplectic Dolbeault operators on Kähler manifolds*, Ann. Glob. Anal. Geom. **44**, 339 (2013), doi:10.1007/s10455-013-9369-x.

[4] H. De Bie, P. Somberg and V. Souček, *The metaplectic Howe duality and polynomial solutions for the symplectic Dirac operator*, J. Geom. Phys. **75**, 120 (2014), doi:10.1016/j.geomphys.2013.09.005.

[5] D. Eelbode, *Stirling numbers and spin-Euler polynomials*, Exp. Math. **16**, 55 (2007), doi:10.1080/10586458.2007.10128982.

[6] D. Eelbode, G. Muarem, *A unitary refinement of the symplectic Fischer decomposition*, *In preparation*.

SciPost Phys. Proc. 14, 039 (2023)

Wigner function analysis of finite matter-radiation systems

Eduardo Nahmad-Achar⋆, Ramón López-Peña,
Sergio Cordero and Octavio Castaños Garza

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Apartado Postal 70-543, México 04510 CDMX

⋆ nahmad@nucleares.unam.mx

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We show that the behaviour in phase space of the Wigner function associated to the electromagnetic modes carries the information of both, the entanglement properties between matter and field, and the regions in parameter space where quantum phase transitions take place. A finer classification for the continuous phase transitions is obtained through the computation of the surface of minimum fidelity.

Received 06-10-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.039

1 Introduction

Quantum phase transitions (QPT) are studied in nuclear, molecular, quantum optics, and condensed matter physics, and have potential applications in the design of quantum technologies [1]. The Wigner function gives a complete description of a quantum system in phase space; it allows for the calculation of all the quantities that the usual wave function gives, and negative values in the function appear as a consequence of interference between distant points in phase space. In a generalized Dicke model of 3-level atoms interacting with 2 electromagnetic modes, it may be used to analyse the behaviour in phase space of the two radiation modes of light across the finite phase diagram of the quantum ground state, and supply further evidence of the quantum phase transitions revealed by the fidelity criterion.

When the linear entropy for all the subsystems is calculated and compared with the behaviour of the Wigner function, we see that the entanglement between the substates responds to how the bulk of the ground state changes from a subset of the basis with a major contribution from one kind of photons, to a subset with a major contribution of the other one.

2 The generalised Dicke model

The multipolar Hamiltonian for the dipole interaction between a 2-mode radiation field and a 3-level atomic system in the long wave approximation ($\hbar = 1$) is

$$\mathbf{H} = \mathbf{H}_D + \mathbf{H}_{int} \,,$$

with

$$\mathbf{H}_D = \sum_{j<k}^{3} \Omega_{jk} \, \mathbf{a}^\dagger_{jk} \, \mathbf{a}_{jk} + \sum_{j=1}^{3} \omega_j \, \mathbf{A}_{jj} \,,$$

and

$$\mathbf{H}_{int} = -\frac{1}{\sqrt{N_a}} \sum_{j<k}^{3} \mu_{jk} \left(\mathbf{A}_{jk} + \mathbf{A}_{kj}\right)\left(\mathbf{a}_{jk} + \mathbf{a}^\dagger_{jk}\right) .$$

Here, N_a denotes the number of particles, $\mathbf{a}^\dagger_{jk}$, $\mathbf{a}_{jk}$ are creation and annihilation photon operators, Ω_{jk} is the frequency of the mode which promotes transitions between the atomic levels ω_j and ω_k, $\mathbf{A}_{ij}$ are the matter operators obeying the $U(3)$ algebra, with $\sum_{k=1}^{3} \mathbf{A}_{kk} = N_a \, \mathbf{I}_{\text{matter}}$, and μ_{jk} is the coupling parameter between atomic levels ω_j and ω_k.

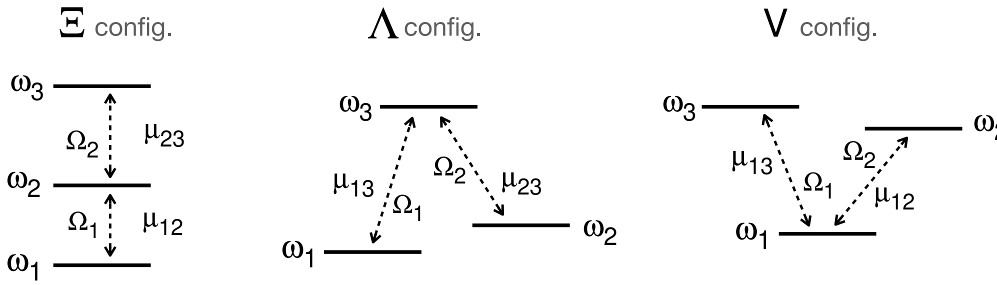


Figure 1: Atomic configurations for 3-level systems, showing the possible transitions and coupling strengths μ_{ij}.

We have the atomic configurations shown in Figure 1, customarily labelled by Ξ, Λ, and V, due to their shape resembling these letters, and where we label the atomic energy levels following $\omega_1 \leq \omega_2 \leq \omega_3$ and for simplicity fix $\omega_1 = 0$ and $\omega_3 = 1$; therefore, all energies are measured in terms of $\hbar\omega_3$. Note that particular atomic configurations are obtained by making an appropriate dipolar strength μ_{ij} vanish.

2.1 Variational study

A variational study involving coherent states for both matter and field provides a good approximation of the ground state energy surface per particle [2, 3]. Figure 2 shows the phase diagrams from a variational study using coherent test states, for the different atomic configurations Ξ, Λ, and V (from left to right), as well as the order of the transitions according to the Ehrenfest classification. We distinguish a *normal* region (N, in medium grey) where the atoms decay individually, and *collective* regions S_{ij} where the decay is proportional to $N_a(N_a + 1)$ and in which only one kind of photon contributes to the ground state. Continuous black lines denote the separatrices dividing these regions.

It is important to note that the signature of the phase diagram remains when the symmetries of the Hamiltonian are restored in the variational solution and the thermodynamic limit $N_a \to \infty$ is taken.

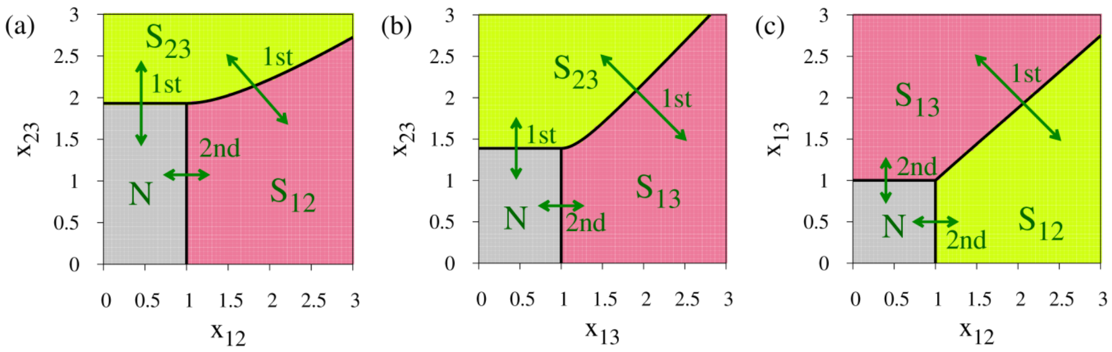


Figure 2: Phase diagrams from a variational study using coherent test states, for the different atomic configurations Ξ, Λ, and V (from left to right). The order of the transitions according to the Ehrenfest classification is shown. The parameters used are: Ξ-configuration: $\omega_2/\omega_3 = 1/3$; Λ- configuration: $\omega_2/\omega_3 = 1/10$; V-configuration: $\omega_2/\omega_3 = 8/10$. Here, $x_{ij} = \mu_{ij}/\mu_c$ is the dimensionless dipolar coupling strength, where μ_c stands for its critical value in the two-level system $\{ij\}$ in the limit $N_a \to \infty$.

2.2 Numerical quantum solution

The exact calculation of the ground state involves a numerical diagonalisation of the Hamiltonian matrix. The Hamiltonian is invariant under parity transformations of the form

$$\Pi_1 = e^{i\pi \mathbf{K}_1}, \quad \Pi_2 = e^{i\pi \mathbf{K}_2},$$

where $\mathbf{K}_s$, $s = 1,2$, are constants of motion when the rotating wave approximation (RWA) is taken [7]. Accordingly, the Hilbert space $\mathcal{H}$ divides naturally into four subspaces

$$\mathcal{H} = \mathcal{H}_{ee} \oplus \mathcal{H}_{eo} \oplus \mathcal{H}_{oe} \oplus \mathcal{H}_{oo},$$

where subscripts $\sigma = \{ee, eo, oe, oo\}$ denote the even e or odd o parity of Π_1 and Π_2, respectively.

We use basis states labeled by $|\nu_{12}, \nu_{13}, \nu_{23}\rangle \otimes |n_1, n_2, n_3\rangle$, with $n_1 + n_2 + n_3 = N_a$ and $\nu_{jk} = 0, 1, \cdots, \infty$, which denote Fock states.

Since the dimension of the Hilbert space is $\dim(\mathcal{H}) = \infty$, we need to use a truncation criterion. For the set of eigenvalues of $\mathbf{K}_1$, $\mathbf{K}_2$, we take this criterion as follows [8]: choose values $k_{i\max}$ to satisfy

$$1 - \mathcal{F}(k_{1\max}, k_{2\max}) \leq 10^{-10},$$

where $\mathcal{F}(k_1, k_2) = |\langle \psi(k_1, k_2)|\psi(k_1+2, k_2+2)\rangle|^2$ is the fidelity between the state $|\psi(k_1, k_2)\rangle$ containing all eigenvalues up to k_1 and k_2, and the state $|\psi(k_1+2, k_2+2)\rangle$. This ensures that the energy calculated remains without variation to one part in 10^{-8}. Other criteria may be used, of course, depending on the problem in question.

3 Fidelity as signature of QPT in *finite* systems

Quantum phase transitions are determined by singularities in the wave function of the ground state, and these may be studied by the method of Ginzburg-Landau, or using catastrophe theory, in the thermodynamic limit [4]. Another criterion is by the loci where the fidelity between neighbouring states $|\Psi_g(\xi_1)\rangle$, $|\Psi_g(\xi_2)\rangle$ along parametric lines $\xi(t)$ in parameter space

$$\mathcal{F}(\rho_{\xi(t)}, \rho_{\xi(t+\delta)}) = |\langle \Psi_g(\xi(t))|\Psi_g(\xi(t+\delta))\rangle|^2,$$

presents a minimum (see e.g. [5, 6] and references therein). We have followed this method for *finite* systems to find the separatrices in parameter space. We call these *quantum phase transitions*, in contrast to other terminology that appears in the literature, since the constitution of the ground state changes significantly as one crosses a separatrix. The *surface of minimum fidelity* is calculated by considering neighbouring points in directions parallel to the axes ($x_{jk} = 0$), along identity lines, and along their orthogonal directions, thereby finding the local minima. Here, $x_{jk} = \mu_{jk}/\mu_c$, where

$$\mu_c = \frac{1}{2}\sqrt{\Omega_{jk}(\omega_k - \omega_j)},$$

stands for the critical coupling value in a two-level $\{jk\}$ system, in the limit $N_a \to \infty$. Thus, x_{jk} is the dimensionless dipolar coupling.

In the case of the generalised *quantum* Rabi model, the quantum separatrices for a single 3-level atom interacting dipolarly with two modes of electromagnetic field are given in Figure 3, for the three atomic configurations, Ξ, Λ, and V (from left to right), when in resonance with the field modes [7]. The parity of the Hilbert subspace in which the ground state lives is marked by colours and by the letters $\{ee, eo, oe, oo\}$, and we see that a much richer structure appears in contrast with the limit $N_a \to \infty$ shown in Fig. 2.

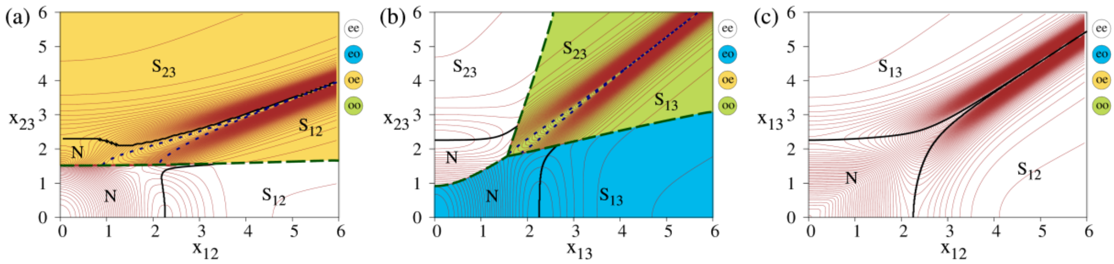


Figure 3: Quantum phase diagrams for the three atomic configurations, Ξ, Λ, and V (from left to right), for one atom when in resonance with the field modes. Different types of transitions are shown (see text). For the Ξ-configuration we have used $\Omega_{12} = 1/4$, $\Omega_{23} = 3/4$ and $\omega_2 = 1/4$; for the Λ-configuration $\Omega_{13} = 1$, $\Omega_{23} = 9/10$, and $\omega_2 = 1/10$; and for the V-configuration $\Omega_{12} = 4/5$, $\Omega_{13} = 1$ and $\omega_2 = 4/5$.

Quantum phase transitions for a finite system appear where the ground state changes abruptly, and this may be determined by calculating the fidelity or the fidelity susceptibility between neighbouring states. We can distinguish three types of loci of points where this takes place (cf. Figure 3):

1. Dashed lines: *discontinuous transitions*, the fidelity between neighbouring states falls to zero, and the separatrix in this case borders along orthogonal Hilbert subspaces of different parity;

2. Continuous lines: *stable continuous transitions*, $F(\xi) \neq 0$ and it remains different from zero as N_a increases;

3. Dotted lines: *unstable continuous transitions*, $F(\xi) \neq 0$ but reaches zero in the large N_a limit.

This classification is further corroborated through the behaviour of the Wigner function for each field mode, as we shall see in the next section. Note that stable and unstable continuous transitions can also be distinguished by means of the Bures distance, which measures the difference of two probability densities of the quantum system; for the stable continuous transition the value of the Bures distance will be smaller than for the unstable continuous transition.

4 Wigner function in the Λ-configuration

First order quantum phase transitions, according to the Ehrenfest classification, can be always associated to zero fidelity values, i.e., discontinuous transitions, and the corresponding eigenstates are orthogonal.

A finer classification of the continuous transitions is more evident through the study of the *Wigner function*, since this classification is based on whether the bulk of the ground state remains in a sub-basis of the total basis or not. Here we shall focus on the Λ-configuration, which appears to have a richer structure.

We may use the parity operators for the Λ-configuration

$$\begin{aligned}\mathbf{K}_1 &= \boldsymbol{\nu}_{13} + \boldsymbol{\nu}_{23} + \mathbf{A}_{33}\,,\\ \mathbf{K}_2 &= \boldsymbol{\nu}_{23} + \mathbf{A}_{11} + \mathbf{A}_{33}\,,\end{aligned}$$

to replace the electromagnetic quanta oscillation numbers

$$\nu_{13} = k_1 - k_2 + n_1\,, \qquad \nu_{23} = k_2 - n_1 - n_3\,,$$

and thus denote the ground state of the system as

$$|\psi_{\rm gs}\rangle = \sum_{k_1,k_2}\sum_{n_1,n_3}^{N_a} C_{k_1,k_2,n_1,n_3} \times |k_1 - k_2 + n_1, k_2 - n_1 - n_3,\, n_1,\, N_a - n_1 - n_3,\, n_3\rangle\,,$$

from which we calculate the reduced density matrices ϱ_{jk} $(j<k)$ for modes ν_{jk}.

Notice that for the case of a single atom, for maximum values of $x_{jk} = 6$ and for the desired precision of 10^{-10} established in Sec. 2.2, the ground state function lives in a Hilbert space of dimension $\dim(\mathcal{H}) = 1395$, while for a precision of 10^{-15} the dimension must at least be $\dim(\mathcal{H}) = 2079$ [8].

Thus, the Wigner functions for the reduced density matrices are

$$\begin{aligned}W_{13}(q,\,p) &= \sum_{k_1,k_2,k_1'}\sum_{n_1,n_3} C_{k_1,k_2,n_1,n_3}\, C^*_{k_1',k_2,n_1,n_3}\, W_{|k_1-k_2+n_1\rangle\langle k_1'-k_2+n_1|}(q,p)\,,\\ W_{23}(q,\,p) &= \sum_{k_1,k_2,k_2'}\sum_{n_1,n_3} C_{k_1,k_2,n_1,n_3}\, C^*_{k_1,k_2',n_1,n_3}\, W_{|k_2-n_1-n_3\rangle\langle k_2'-n_1-n_3|}(q,p)\,,\end{aligned}$$

where $W_{|n\rangle\langle m|}(q,p)$ is the Weyl symbol for the operator $\rho_{nm} = |n\rangle\langle m|$ [9, 10].

We may plot these Wigner functions as functions of the field quadratures $(q,\,p)$ at various points at either side of a separatrix, to see their behaviour as the system undergoes a phase transition [7].

Figure 4 shows the behaviour of W_{13} as the system goes through a stable-continuous transition (red dot along a continuous grey evaluation trajectory). The elongation presenting a bimodal distribution is a consequence of photon contribution ν_{13} becoming significant. Regions where the Wigner function W_{13} is negative (black) appear as we move away from the normal region and cross the separatrix, because the number of photons in mode ν_{13} grows from zero: we now have a superposition of states with different values of ν_{13}.

Figure 5 shows the behaviour of both, W_{13} and W_{23}, as the system goes through an unstable-continuous transition (red dot along a continuous grey evaluation trajectory): close to the separatrix in dotted lines both photon contributions are significant. We note that both Wigner functions present elongated (bimodal) distributions. Above the separatrix the contribution of photons ν_{23} dominates and W_{23} has major regions with negative values; when the transition occurs, the field mode contributions to the ground state change their roles.

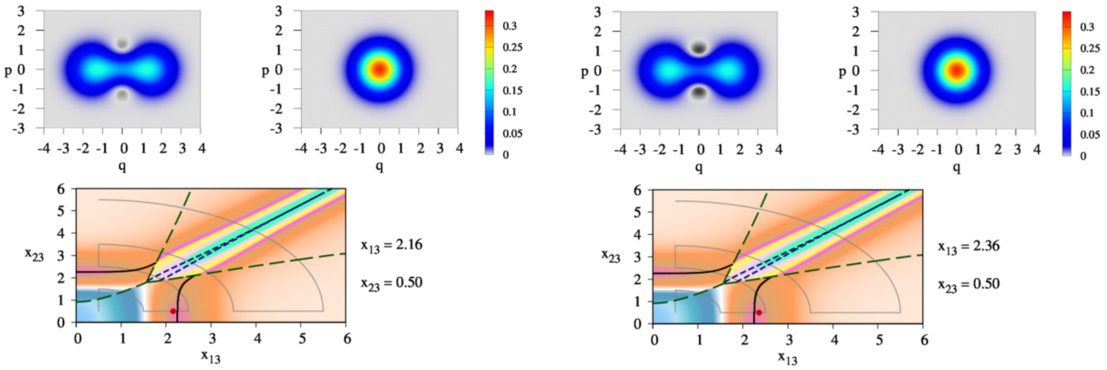


Figure 4: Behaviour of the Wigner function W_{13} and W_{23}, as the system goes through a stable-continuous transition. Regions where it becomes negative (black) reflect the existence of a superposition of states with different values of ν_{13}. (In each case, the continuous dim grey line is the evaluation trajectory, the red dot indicates the evaluation point in parameter space.) Note that, through this transition, W_{23} does not change.

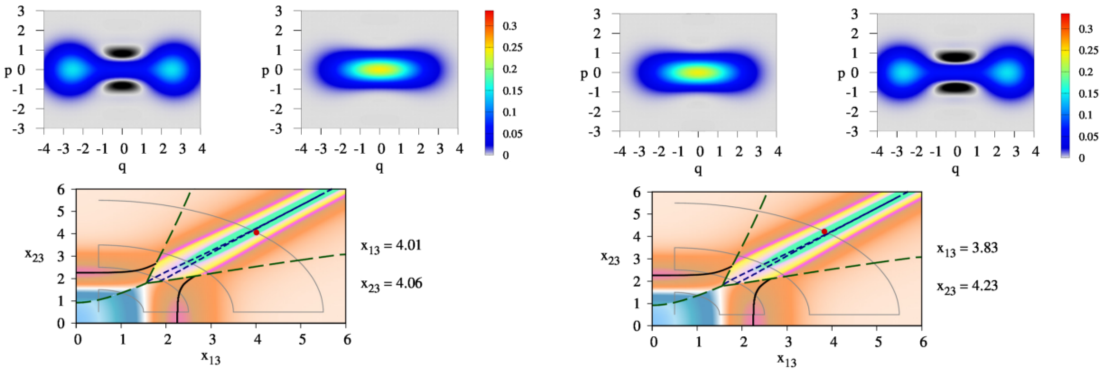


Figure 5: Behaviour of W_{13} and W_{23} as the system goes through an unstable-continuous transition. Across the transition the field mode contributions to the ground state change their roles $S_{13} \rightleftharpoons S_{23}$. (In each case, the continuous dim grey line is the evaluation trajectory, the red dot indicates the evaluation point in parameter space.)

We see that the Wigner function characterises completely the phase diagram. In the normal region the Wigner function describes a classical behaviour of the field (W takes positive values) and at least one photon mode remains in the vacuum, while the collective region is characterised by a Wigner function in which the quantumness of the photon modes is clearly shown; it divides itself into two regions, in each of which a single radiation mode dominates.

Videos showing the behaviour of the Wigner function for each mode, along the full trajectory shown in Figure 5, may be found for all the atomic configurations in the website of IOP *Physica Scripta*: Ξ-configuration; Λ-configuration; V-configuration.

4.1 Correlation between Wigner function and entanglement

Bimodality and negativity of Wigner function reflect which field mode dominates in the superradiant region, and not the parity of the state. This is evident when we compare it with an entanglement measure (e.g., the linear entropy) [7]. We define

$$S_{\nu_1} = 1 - \mathrm{Tr}\left(\rho^2_{\nu_1}\right),$$
$$S_{\nu_2} = 1 - \mathrm{Tr}\left(\rho^2_{\nu_2}\right),$$
$$S_{\nu-m} = 1 - \mathrm{Tr}\left(\rho^2_{\nu_1\nu_2}\right),$$

to be, respectively, the linear entropy measuring correlation between field mode 1 and the rest of the system (matter + field mode 2), the linear entropy measuring correlation between field mode 2 and the rest of the system (matter + field mode 1), and the linear entropy measuring correlation between matter and field modes 1 and 2.

Figure 6 shows their plots along a trajectory which crosses all detected transitions in parameter space. When the ground state is dominated by the vacuum state of the field (small values of the coupling parameters inside the Normal region), the correlation between one mode of the field, say i, and the rest of the system (matter + field mode j with $i \neq j$), is null $S_{L_i} = 0$ and the Wigner function is unimodal. This field-mode i vs. matter + field-mode j entanglement reaches its maximum as soon as we cross into the super-radiant region, the Wigner function showing negative values at a vicinity of the origin of quadrature q and small non-zero values of quadrature p. It then falls rapidly to zero as soon as we enter the region where field mode j dominates, even if a parity change is not had.

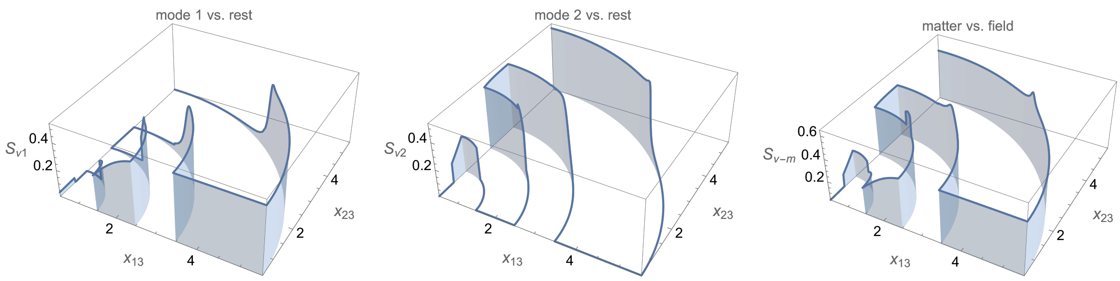


Figure 6: Plots of the different linear entropies $S_{\nu1}$, $S_{\nu2}$, and $S_{\nu-m}$, along a trajectory which crosses all detected transitions in parameter space.

5 Conclusion

We have shown the results of the characteristics of the ground state for a single three-level atom interacting dipolarly with a two-mode electromagnetic field. The symmetries of the system allow for the division the quantum state space into subspaces which have a well-defined parity. We have used a fidelity criterion to determine the quantum phase transitions for the three three-level configurations.

We calculated the Wigner function for each of the electromagnetic modes Ω_{13} and Ω_{23}, and showed the behaviour of these in various regions of the parameter space, which supplies further evidence of the quantum phase transitions revealed by the fidelity criterion; the regions where it takes negative values (the system exhibiting non-classical behaviour) were determined. Besides providing the phase transitions and a finer classification of them, it is interesting to note that the Wigner function can be and has been measured experimentally [11,12].

The linear entropy for all the subsystems was calculated and compared with the behaviour of the Wigner function; we see that the entanglement between the substates responds to how the bulk of the ground state changes from a subset of the basis with a major contribution from one kind of photons, to a subset with a major contribution of the other one, and not to the state parity even for large values of the coupling parameters.

6 Acknowledgements

Funding information This work was partially supported by DGAPA-UNAM (under projects IN112520, and IN100323).

References

[1] S. Sachdev, *Quantum phase transitions*, Cambridge University Press, Cambridge, UK, ISBN 9780521514682 (2011), doi:10.1017/CBO9780511973765.

[2] S. Cordero, E. Nahmad-Achar, R. López-Peña and O. Castaños, *Polychromatic phase diagram for n-level atoms interacting with ℓ modes of an electromagnetic field*, Phys. Rev. A **92**, 053843 (2015), doi:10.1103/PhysRevA.92.053843.

[3] S. Cordero, E. Nahmad-Achar, O. Castaños and R. López-Peña, *A general system of n levels interacting with ℓ electromagnetic modes*, Phys. Scr. **92**, 044004 (2017), doi:10.1088/1402-4896/aa6363.

[4] R. Gilmore, *Catastrophe theory for scientists and engineers*, Dover Publications, Mineola, USA, ISBN 9780486675398 (1993).

[5] P. Zanardi and N. Paunković, *Ground state overlap and quantum phase transitions*, Phys. Rev. E **74**, 031123 (2006), doi:10.1103/PhysRevE.74.031123.

[6] S.-J. Gu, *Fidelity approach to quantum phase transitions*, Int. J. Mod. Phys. B **24**, 4371 (2010), doi:10.1142/S0217979210056335.

[7] R. López-Peña, S. Cordero, E. Nahmad-Achar and O. Castaños, *Quantum phase diagrams of matter-field Hamiltonians II: Wigner function analysis*, Phys. Scr. **96**, 035103 (2021), doi:10.1088/1402-4896/abd654.

[8] S. Cordero, O. Castaños, R. López-Peña and E. Nahmad-Achar, *Reduced bases for a three-level atom interacting with a two-mode radiation field*, Phys. Rev. A **99**, 033811 (2019), doi:10.1103/PhysRevA.99.033811.

[9] V. V. Dodonov and V. I. Man'ko, *Evolution of multidimensional systems. Magnetic properties of ideal gases of charged particles*, in *Invariants and the evolution of nonstationary quantum systems*, Nova Science Publishers, Hauppauge, USA, ISBN 9780941743495 (1989).

[10] O. Castaños, S. Cordero, R. López-Peña and E. Nahmad-Achar, *Phase space properties of light within the generalised Dicke model*, Phys. Scr. **93**, 085102 (2018), doi:10.1088/1402-4896/aacd43.

[11] G. Nogues, A. Rauschenbeutel, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, S. Haroche, L. G. Lutterbach and L. Davidovich, *Measurement of a negative value for the Wigner function of radiation*, Phys. Rev. A **62**, 054101 (2000), doi:10.1103/PhysRevA.62.054101.

[12] K. Banaszek, C. Radzewicz, K. Wódkiewicz and J. S. Krasiński, *Direct measurement of the Wigner function by photon counting*, Phys. Rev. A **60**, 674 (1999), doi:10.1103/PhysRevA.60.674.

SciPost Phys. Proc. 14, 040 (2023)

Group theoretical derivation of consistent particle theories

Giuseppe Nisticò★

Università della Calabria, Italy
INFN, gr. collegato di Cosenza, Italy

★ giuseppe.nistico@unical.it

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Current quantum theories of an elementary free particle assume unitary space inversion and anti-unitary time reversal operators. In so doing robust classes of possible theories are discarded. The present work shows that consistent theories can be derived through a strictly deductive development from the principle of relativistic invariance and position covariance, also with anti-unitary space inversion and unitary time reversal operators. In doing so the class of possible consistent theories is extended for positive but also zero mass particles. In particular, consistent theories for a Klein-Gordon particle are derived and the non-localizability theorem for a non zero helicity massless particle is extended.

Received 16-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.040

1 Introduction

Relativistic quantum theories of single free particle can be deductively derived from the principles of *relativistic invariance* and *covariance* [1] - [4]; the first principle implies that the Hilbert space of the quantum theory of a free particle must admit a *transformer triplet* $(U, \mathcal{S}, \mathcal{T})$ formed by a unitary representation U of the universal covering group $\tilde{\mathcal{P}}_+^\uparrow$ of the proper orthochronus Poincaré group $\mathcal{P}_+^\uparrow$ and by the operators $\mathcal{S}$ and $\mathcal{T}$, which realize the quantum transformations implied by the transformations of $\mathcal{P}_+^\uparrow$, by space inversion $\mathfrak{s}$ and by time reversal $\mathfrak{t}$, respectively. Yet the literature, except some works [5] [6] with specific aims different from the present one, excludes transformer triplets with $\mathcal{S}$ anti-unitary or with $\mathcal{T}$ unitary, from the pionering works of Wigner, Bargmann [1] - [3], to subsequent investigations [4] - [9] . In so doing robust classes of triplets, and hence of possible theories, are lost. For instance, there is no such a triplet for a consistent theory of Klein-Gordon particles.[1]

The motivation for the exclusion of $\mathcal{T}$ unitary or $\mathcal{S}$ anti-unitary was their implication of negative spectral values for the hamiltonian operator P_0, values deemed inconsistent because

[1] Klein-Gordon theory, indeed, was obtained through canonical quantization [10], [11], but it predicts inconsistencies, such as negative probabilities [12].

P_0 was identified with the *positive* relativistic kinetic energy operator $E_{kin} = \mu(1-\dot{\mathbf{Q}}^2)^{-1/2}$, where $\dot{\mathbf{Q}}$ is the "velocity" operator. But remark 3.1 shall show that the hamiltonian operator P_0 does not always coincide with E_{kin}, so that a unitary T or an anti-unitary S can be consistent.

In the present article we show how a strictly deductive development of consistent quantum theories of elementary free particle can be successfully carried out without *apriori* preclusions about the unitary or anti-unitary character of S or T. As results, classes of consistent possible theories for a positive mass particle are expicitly identified, which meaningfully extend the class of the current theories; in particular, consistent theories of Klein-Gordon particle are derived. Also in the case of a massless particle the approach extends the class of possible theories. Furthermore, the non-localizability theorem for non zero helicity massless particles is extended to the new theories with T unitary or S anti-unitary.

Section 2 shows how the relativistic invariance principle implies that every theory of elementary free particle admits a transformer triplet. In section 3 the class of possible consistent theories for a positive mass particle is identified; this class contains consistent theories with S anti-unitary, e.g. consistent theories of Klein-Gordon particle. Section 4 identifies the class of consistent theories for a zero mass elementary free particle; once again, besides the current theories, it contains theories with S anti-unitary or T unitary. A more accurate and more general argument is presented, which denies localizability of non zero helicity mass zero particles

2 General implications of Poincaré invariance

2.1 Prerequisites and notation

First of all, it is worth to fix the notaion for any quantum theory based on a Hilbert space $\mathcal{H}$:

- $\Omega(\mathcal{H})$ denotes the set of all self-adjoint operators representing observables;

- $\mathcal{S}(\mathcal{H})$ denotes the set of all density operators ρ identified with quantum states;

- $\mathcal{U}(\mathcal{H})$ denotes the group of all unitary unitary operators;

- $\mathcal{V}(\mathcal{H})$ is the larger group of all unitary or anti-unitary operators.

The Poincaré group $\mathcal{P}$ is a very important mathematical structure for the present work, because it is the group of symmetry transformations for a free particle. $\mathcal{P}$ is the group generated by $\mathcal{P}_+^\uparrow \cup \{\mathfrak{t}, \mathfrak{s}\}$, where $\mathcal{P}_+^\uparrow$ is the proper orthochronus Poincaré group, $\mathfrak{t}$ and $\mathfrak{s}$ are the time reversal and space inversion transformations. The proper orthochronus group $\mathcal{P}_+^\uparrow$ is a connected group generated by 10 one-parameter subgroups, namely the subgroup $\mathcal{T}_0$ of time translations, the three subgroups $\mathcal{T}_j$ ($j = 1,2,3$) of spatial translations, the three subgroups $\mathcal{R}_j$ of spatial rotations, the three subgroups $\mathcal{B}_j$ of Lorentz boosts, relative to the three spatial axes x_j. Time reversal $\mathfrak{t}$ and space inversion $\mathfrak{s}$ are not connected with the identity transformation $e \in \mathcal{P}$. Given any vector $\underline{x} = (x_0, \mathbf{x}) \in \mathbb{R}^4$, where x_0 is called the *time component* of $\underline{x}$ and $\mathbf{x} = (x_1, x_2, x_3)$ is called the *spatial component* of $\underline{x}$, time reversal $\mathfrak{t}$ transforms $\underline{x} = (x_0, \mathbf{x})$ into $(-x_0, \mathbf{x})$ and space inversion $\mathfrak{s}$ transforms $\underline{x} = (x_0, \mathbf{x})$ into $(x_0, -\mathbf{x})$.

The universal covering group of $\mathcal{P}_+^\uparrow$ is the semidirect product $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \circledS SL(2,\mathbb{C})$ of the time-space translation group $\mathbb{R}^4$ and the group $SL(2,\mathbb{C}) = \{\underline{\Lambda} \in GL(2,\mathbb{C}) \mid \det \underline{\Lambda} = 1\}$. Accordingly, $\tilde{\mathcal{P}}_+^\uparrow$ is simply connected and there is a canonical homomorphism $\mathsf{h} : \tilde{\mathcal{P}}_+^\uparrow \to \mathcal{P}_+^\uparrow$, $\tilde{g} \to \mathsf{h}(\tilde{g}) \in \mathcal{P}_+^\uparrow$, which restricts to an isomorphism within a small enough neighborhood of the identity $(0, \mathbb{I}_{\mathbb{C}^2})$ of $\tilde{\mathcal{P}}_+^\uparrow$. By $\tilde{\mathcal{T}}_0$, $\tilde{\mathcal{T}}_j, \tilde{\mathcal{R}}_j, \tilde{\mathcal{B}}_j, \tilde{\mathcal{L}}_+^\uparrow$ we denote the subgroups of $\tilde{\mathcal{P}}_+^\uparrow$ which correspond to the subgoroups $\mathcal{T}_0$, $\mathcal{T}_j, \mathcal{R}_j$, $\mathcal{B}_j$, $\mathcal{L}_+^\uparrow$ of $\mathcal{P}_+^\uparrow$, through the homomorphism h.

2.2 Quantum theoretical implications for an elementary free particle

Since a free particle is a particular kind of isolated system, we begin by showing the derivation of the general structure of the quantum theory of an isolated system. By $\mathcal{F}$ we denote the class of the (inertial) reference frames that move uniformly with respect to each other. A physical system is an *isolated system* if the following *invariance principle* holds.

$\mathcal{IP}$ *The theory of an isolated system is invariant with respect to changes of frames within $\mathcal{F}$.*

If Σ belongs to $\mathcal{F}$, then Σ_g denotes the frame related to Σ by such g, for every $g \in \mathcal{P}$. Given an observable $\mathcal{A}$ represented by the operator $A \in \Omega(\mathcal{H})$, let $\mathcal{M}_A$ be a procedure to measure $\mathcal{A}$; then the invariance principle implies that another measuring procedure $\mathcal{M}'_A$ must exist, which is with respect to Σ_g identical to what is $\mathcal{M}_A$ with respect to Σ, otherwise the principle $\mathcal{IP}$ would be violated. Hence, $\mathcal{IP}$ implies the existence [12] of the so called *quantum transformation associated to* g, i.e., of a mapping

$$S_g : \Omega(\mathcal{H}) \to \Omega(\mathcal{H}), \quad A \to S_g[A],$$

where $S_g[A]$ is the self-adjoint operator that represents the observable measured by $\mathcal{M}'_A$.

To every element $\tilde{g}$ of the covering group $\tilde{\mathcal{P}}_+^\uparrow$ we can associate the quantum transformation $S_{\mathsf{h}(\tilde{g})} \equiv S_{\tilde{g}}$ through the canonical homomorphism h. In [12] it is proved that the properties of quantum transformations, under a continuity condition for $\tilde{g} \to S_{\tilde{g}}$, imply that

Imp.1. a continuous unitary representation U of $\tilde{\mathcal{P}}_+^\uparrow$ exists such that $S_{\tilde{g}}[A] = U_{\tilde{g}} A U_{\tilde{g}}^{-1}$, and

Imp.2. two operators $\mathrm{S}, \mathrm{T} \in \mathcal{V}(\mathcal{H})$ exist such that $S_{\mathsf{s}}[A] = \mathrm{S} A \mathrm{S}^{-1}$ and $S_{\mathsf{t}}[A] = \mathrm{T} A \mathrm{T}^{-1}$.

Thus, the principle $\mathcal{IP}$ has the following fundamental implication.

(FI) *The quantum theory of an isolated system admits a transformer triplet $(U, \mathrm{S}, \mathrm{T})$ such that implications Imp.1 and Imp.2 hold.*

Given a transformer triplet $(U, \mathrm{S}, \mathrm{T})$, let $P_0, P_j, J_j, K_j \in \Omega(\mathcal{H})$ be the *selfadjoint generators* of U; so [12], if $\tilde{g} \in \tilde{\mathcal{T}}_0$ (resp., $\tilde{\mathcal{T}}_j$, $\tilde{\mathcal{R}}_j$, $\tilde{\mathcal{B}}_j$) is identified by the parameter t (resp., a, θ, u), then

$$U_{\tilde{g}} = e^{iP_0 t}, \quad (\text{resp., } U_{\tilde{g}} = e^{iP_j a},\ U_{\tilde{g}} = e^{J_j \theta},\ U_{\tilde{g}} = e^{iK_j \frac{1}{2} \ln \frac{1+u}{1-u}}). \tag{1}$$

The generator P_0 relative to time translations is the *hamiltonian operator*, so that

$$(i) \quad \frac{d}{dt} A_t \equiv \dot{A}_t = i[P_0, A_t], \qquad (ii) \quad \frac{d}{dt} \rho_t \equiv \dot{\rho}_t = -i[P_0, \rho_t]. \tag{2}$$

By *"elementary" free particle* we mean an isolated system whose quantum theory has a *unique* three-operator $\mathbf{Q} \equiv (Q_1, Q_2, Q_3)$ with $Q_j \in \Omega(\mathcal{H})$, called *position* operator, such that $(U(\tilde{\mathcal{P}}_+^\uparrow), \mathrm{S}, \mathrm{T}; \mathbf{Q})$ is an *irreducible* system of operators, and satisfying the following conditions.

(Q.1) $[Q_j, Q_k] = \mathbb{O}$, for all $j, k = 1, 2, 3$; this condition establishes that a measurement of position yields all three values of the coordinates of the same specimen of the system.

(Q.2) For every $g \in \mathcal{P}$, the position operator $\mathbf{Q}$ and the transformed position operator $S_g[\mathbf{Q}]$ satisfy the transformation properties of position with respect to g.

As proved in [12], the transformer triplet $(U, \mathrm{S}, \mathrm{T})$ of the quantum theory of an elementary free particle must be *irreducible*. Thus, the identification of all possible theories of an elementary free particle can be carried out in two steps: first by identifying all irreducible transformer triplets $(U, \underleftarrow{\mathrm{S}}, \overleftarrow{\mathrm{T}})$, and then selecting those triplets for which a unique position operator $\mathbf{Q}$ exists.

The mathematical group structural properties of $\mathcal{P}$ imply [12], [16] that each irreducible triplet $(U, \underleftarrow{\mathrm{S}}, \overleftarrow{\mathrm{T}})$ is characterized by a number $\mu \in \mathbb{C}$, called *mass*, with $\mu^2 \in \mathbb{R}$, such that $P_0^2 - \mathbf{P}^2 = \mu^2 \mathbb{I}$.

3 Quantum theories of positive mass elementary free particle

To identify the positive mass possible theories, we shall identify the irreducible triplets with $\mu > 0$; then, the triplets admitting a three-operator $\mathbf{Q}$ satisfying $(Q.1)$, $(Q.2)$ are singled out.

3.1 Positive mass irreducible triplets

Following [12], for any pair (μ, s), where $\mu > 0$ and s is an integral or half-integral number $s \in \frac{1}{2}\mathbb{N}$ called *spin*, there is at least one irreducible triplet. Conversely, every irreducible triplet is characterized by one such a pair. The following theorem yields a first classification.

Theorem 3.1. *If $(U, \underleftarrow{\mathrm{S}}, \overleftarrow{\mathrm{T}})$ is an irreducible triplet with non-negative mass $\mu \geq 0$, then*
i) $\sigma(P_0) = (-\infty, -\mu]$ or $\sigma(P_0) = [\mu, \infty)$ or $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$, where $\sigma(P_0)$ is the spectrum of P_0.

Moreover, $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$ if and only if $\overleftarrow{\mathrm{T}}$ is unitary or $\underleftarrow{\mathrm{S}}$ is anti-unitary.
ii) Each class $\mathcal{I}(\mu, s)$ of all irreducible triplets with positive mass $\mu > 0$ decomposes as

$$\mathcal{I}(\mu, s) = \mathcal{I}^-(\mu, s) \cup \mathcal{I}^+(\mu, s) \cup \mathcal{I}^{-+}(\mu, s), \tag{3}$$

where $\mathcal{I}^-(\mu, s)$, $\mathcal{I}^+(\mu, s)$ and $\mathcal{I}^{-+}(\mu, s)$ are respectively the classes of irreducible triplets with $\sigma(P_0) = (-\infty, -\mu]$, $\sigma(P_0) = [\mu, \infty)$ and $\sigma(P_0) = (-\infty, -\mu] \cup [\mu, \infty)$.

The representation U of a triplet in $\mathcal{I}^+(\mu, s)$ or $\mathcal{I}^-(\mu, s)$ can be irreducible or not. We refer to [12] for a complete identification of the irreducible triplets of $\mathcal{I}^{\pm}(\mu, s)$ with U irreducible. Therein also instances of triplet in $\mathcal{I}^+(\mu, s)$ and $\mathcal{I}^+(\mu, s)$ with U reducible are explicitly shown.

The representation U of a triplet in $\mathcal{I}^{-+}(\mu, s)$ is always reducible [12], namely $U = U^+ \oplus U^-$ where $U^{\pm}$ belongs to a triplet in $\mathcal{I}^{\pm}(\mu, s)$. Moreover, U^+ is reducible if and only if U^- is reducible.

The class of all irreducible triplets of $\mathcal{I}^{-+}(\mu, s)$ with U^+ irreducible can be found in [12], where also triplets of $\mathcal{I}^{-+}(\mu, s)$ with U^+ reducible are concretely shown.

3.2 Theories of elementary free particle with positive mass

To determine the possible theories of positive mass elemetary free particle, we have to select irreducible triplets of $\mathcal{I}(\mu, s)$ identified in [12] for which a position $\mathbf{Q}$ satisfying $(Q.1)$ and $(Q.2)$ exists. Condition $(Q.2)$ can be only partially imposed. In fact, while the covariance properties with respect to translations, rotations, time reversal and space inversion are known and explicitly expressed by the following relations [12]

$$(i)\ \ [Q_j, P_k] = i\delta_{jk}, \quad (ii)\ \ [J_j, Q_k] = i\epsilon_{jkl} Q_l, \quad (iii)\ \ \overleftarrow{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\overleftarrow{\mathrm{T}}, \quad (iv)\ \ \underleftarrow{\mathrm{S}}\mathbf{Q} = -\mathbf{Q}\overleftarrow{\mathrm{T}}, \tag{4}$$

the explicit relations that establish the transformation properties of position with respect to boosts are not available, yet [12]. However, conditions (4) are sufficient to uniquely identify $\mathbf{Q}$ for some subclasses of irreducible triplets, according to the following theorem [12].

Theorem 3.2. *Given a triplet in $\mathcal{I}^+(\mu,0)$ with U irreducible there is a unique three-operator* $\mathbf{Q}$*, satisfying (Q.1) and (4). Modulo unitary isomorphism, the resulting theory has Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$, where $d\nu(\mathbf{p}) = \frac{dp_1 dp_2 dp_3}{p_0}$ with $p_0 = \sqrt{\mu^2 + \mathbf{p}^2}$,*

- *generators defined by*

$$(P_j\psi)(\mathbf{p}) = p_j\psi(\mathbf{p}), \quad (P_0\psi)(\underline{p}) = p_0\psi(\underline{p}), \quad J_k = \mathcal{J}_k^{(0)}, \quad K_j = \mathcal{K}_j^{(0)}, \tag{5}$$

where $\mathcal{J}_k^{(0)} = -i\left(p_l \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_l}\right), \quad \mathcal{K}_j^{(0)} = ip_0 \frac{\partial}{\partial p_j};$

- $\mathcal{S} = \Upsilon$, $\mathcal{T} = \mathcal{K}\Upsilon$*, where $\mathcal{K}$ and Υ are defined by $\mathcal{K}\psi(\mathbf{p}) = \overline{\psi(\mathbf{p})}$, $(\Upsilon\psi)(\mathbf{p}) = \psi(-\mathbf{p})$.*

- *The position operator is $\mathbf{Q} = \mathbf{F}$, where $\mathbf{F}$ is the Newton-Wigner [13] operator defined by*

$$F_j = i\frac{\partial}{\partial p_j} - \frac{i}{2p_0^2}p_j\,. \tag{6}$$

Analogously, there is only one theory based on a triplet in $\mathcal{I}^-(\mu,0)$ with U irreducible. It differs from that in $\mathcal{I}^+(\mu,0)$ by $P_0 = -p_0$ and $K_j = -\mathcal{K}_j^{(0)}$. There are only two theories based on triplets of $\mathcal{I}^{-+}(\mu,0)$ with U^+ irreducible. They share the Hilbert space and generators:[2] $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu) \oplus L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$

$$P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix}, \quad P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad J_k = \begin{bmatrix} \mathcal{J}_k^{(0)} & 0 \\ 0 & \mathcal{J}_k^{(0)} \end{bmatrix}, \quad K_j = \begin{bmatrix} \mathcal{K}_j^{(0)} & 0 \\ 0 & -\mathcal{K}_j^{(0)} \end{bmatrix}. \tag{7}$$

The two theories differ for the different pairs $(\mathcal{S}_1, \mathcal{T}_1)$, $(\mathcal{S}_2, \mathcal{T}_2)$ of space inversion and time reversal operators; indeed $\mathcal{S}_1 = \mathcal{S}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathcal{K}$ while $\mathcal{T}_1 = \mathcal{K}\Upsilon\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathcal{T}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

For both theories the position operator is $\mathbf{Q} = \begin{bmatrix} \mathbf{F} & 0 \\ 0 & \mathbf{F} \end{bmatrix}$.

For all triplets with $s > 0$ (Q.1) and (4) are not sufficient [12] to completely identify $\mathbf{Q}$.

Remark 3.1. In both theories based on $\mathcal{I}^{-+}(\mu,0)$ the hamiltonian operator P_0 has also negative spectral values. But since the "velocity" is $\dot{\mathbf{Q}} = \frac{d}{dt}\mathbf{Q} = i[P_0, \mathbf{Q}] = \begin{bmatrix} \frac{\mathbf{p}}{p_0} & 0 \\ 0 & -\frac{\mathbf{p}}{p_0} \end{bmatrix}$, we compute that $E_{kin} = \mu(1 - \dot{\mathbf{Q}}^2)^{-1/2} = p_0 > \mathbb{O}$, i.e. the theories are consistent.

3.3 Conclusions for the positive mass case

According to section 3.2, four classes of possible consistent theories are completely determined by following the present approach, with U or U^+ irreducible. However, the class of theories based on $\mathcal{I}^{\pm}(\mu,0)$ with U reducible and the class of theories based on $\mathcal{I}^{-+}(\mu,0)$ with U^+ reducible are not empty; concrete examples are given in [12]. They are *new species theories*, i.e. they correspond to none of the known theories. Hence, our approach extends the class of consistent theories of positive mass elementary spin 0 free particle.

Moreover, it provides consistent theories for Klein-Gordon particles. Indeed, by means of a unitary transformation, operated by the operator $Z = Z_1 Z_2$, where $Z_2 = \frac{1}{\sqrt{p_0}}\mathbb{I}$ and Z_1 is the

[2] If $\psi \in L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu) \oplus L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$, we write $\psi \equiv \psi_1 \oplus \psi_2 \equiv \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$, $\psi_1, \psi_2 \in L_2(\mathbb{R}^3, \mathbb{C}^{2s+1}, d\nu)$).

inverse of the *Fourier-Plancherel* operator, the theories based on $\mathcal{I}^{-+}(\mu,0)$ turn out to be equivalent to two theories with Hilbert space $\hat{\mathcal{H}} = Z\left(L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu)\right) \equiv L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)$, with the self-adjoint generators

$$\hat{P}_j = \begin{bmatrix} -i\frac{\partial}{\partial x_j} & 0 \\ 0 & -i\frac{\partial}{\partial x_j} \end{bmatrix}, \quad \hat{P}_0 = \sqrt{\mu^2 - \nabla^2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\hat{J}_j = -i\left(x_k\frac{\partial}{\partial x_l} - x_l\frac{\partial}{\partial x_k}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{K}_j = \frac{1}{2}\left(x_j\sqrt{\mu^2-\nabla^2} + \sqrt{\mu^2-\nabla^2}x_j\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

while $\hat{\mathrm{T}}_1 = \mathcal{K}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{\mathrm{S}}_1 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\hat{\mathrm{T}}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{\mathrm{S}}_2 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
The position operator is $\hat{Q}_j = \begin{bmatrix} x_j & 0 \\ 0 & x_j \end{bmatrix}$.

These $\hat{P}_j$, $\hat{J}_j$, $\hat{K}_j$, $\hat{\mathcal{H}}$ are generators and Hilbert space of Klein-Gordon theory of spin-0 particle [10] [11]. However, since the position operator is the multiplication operator, the position probability density must be $\rho(t,\mathbf{x}) = |\hat{\psi}_1(t,\mathbf{x})|^2 + |\hat{\psi}_2(t,\mathbf{x})|^2$, hence non-negative. Thus, our extended class includes consistent theories for Klein-Gordon particle free from the inconsistent negative probabilities of the early theory.

It turns out [12] that in all triplets with non zero spin, the position operator $\mathbf{Q}$ is not uniquely determined by (Q.1) and (4). On the other hand, the transformation properties of position with respect to boosts, expressed for instance by a relation for $[K_j, Q_k]$, are not available in order to better identify $\mathbf{Q}$ by imposing them.

To each solution $\mathbf{Q}$ of (Q.1) and (4) there correspond a different $[K_j, Q_k]$, in general. For instance, Dirac theory for spin 1/2 particle [14] [15] is *completely* characterized by the relation $[K_j, Q_k] = -\frac{i}{2}(Q_j\dot{Q}_k + \dot{Q}_kQ_j)$ satisfied by the posistion operator of Dirac theory; however, other solutions $\mathbf{Q}$ yielding other relations for $[K_j, Q_k]$ are theoretically consistent too.

4 Quantum theories of zero mass elementary free particle

Analogously to the positive mass case, the possible quantum theories of zero mass particle are determined first by identifying the class $\mathcal{I}_0$ of the irreducible transformer triplets with $\mu = 0$, and then by selecting those triplets that admit a unique position operator. According to theorem 3.1.i the class $\mathcal{I}_0$ decomposes as $\mathcal{I}_0^+ = \mathcal{I}_0^+ \cup \mathcal{I}_0^- \cup \mathcal{I}_0^{-+}$, where $\mathcal{I}_0^-$ (resp., $\mathcal{I}_0^+$, $\mathcal{I}_0^{-+}$) denotes the class of irreducible triplets with $\sigma(P_0) = (-\infty, 0]$ (resp., $\sigma(P_0) = [0,\infty)$, $\sigma(P_0) = \mathbb{R}$).

4.1 Zero mass irreducible triplets

In [16] the irreducible triplets of $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$ with U irreducible, and of $\mathcal{I}_0^{-+}$ with U^+ irreducible are completely identified. The results are collected by the following statement.

Theorem 4.1. *Modulo unitary isomorphisms, there is only one triplet $(U, \mathrm{S}, \mathrm{T})$ in $\mathcal{I}_0^+$ and in $\mathcal{I}_0^-$ with U irreducible, whose Hilbert space is $\mathcal{H} = L_2(\mathbb{R}^3, d\nu)$, and*

$$(P_j\psi)(\mathbf{p}) = p_j\psi(\mathbf{p}), \quad P_0\psi(\mathbf{p}) = \pm p_0\psi(\mathbf{p}), \quad J_j = J_j^{(0)}, \quad K_j = \pm K_j^{(0)}, \quad \mathrm{S} = \Upsilon, \quad \mathrm{T} = \mathcal{K}\Upsilon.$$

If $(U, \mathrm{S}, \mathrm{T})$ is an irreducible triplet of $\mathcal{I}_0^{-+}$ with U^+ irreducible, then $m \in \mathbb{Z}$ exists such *that* $\mathcal{H} = L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu)$ *and*

$$P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 \\ 0 & p_j \end{bmatrix},$$

$$J_j = \begin{bmatrix} J_j^{(0)} + j_j & 0 \\ 0 & J_j^{(0)} - j_j \end{bmatrix}, \quad K_j = \begin{bmatrix} K_j^{(0)} + k_j & 0 \\ 0 & -K_j^{(0)} + k_j \end{bmatrix},$$

where $j_1 = \frac{m}{2}\frac{p_1 p_0}{p_1^2+p_2^2}$, $j_2 = \frac{m}{2}\frac{p_2 p_0}{p_1^2+p_2^2}$, $j_3 = 0$, $k_1 = -\frac{m}{2}\frac{p_2 p_3}{p_1^2+p_2^2}$, $k_2 = \frac{m}{2}\frac{p_3 p_1}{p_1^2+p_2^2}$, $k_3 = 0$.

With $m = 0$ *there are six triplets, each characterized by a different pair* $(\mathrm{T}_n, \mathrm{S}_n)$, $n = 1, 2, ..., 6$.

For every $m \neq 0$ *in* $\mathcal{I}_0^{-+}$ *there are two triplets with different pairs* $(\mathrm{T}_a, \mathrm{S}_a)$ *and* $(\mathrm{T}_b, \mathrm{S}_b)$.

Remark 4.1. *For the zero mass case the helicity operator* $\hat{\lambda} = \frac{\mathbf{J}\cdot\mathbf{P}}{p_0}$ *plays an important role. Theorem 4.1 and (5) imply [16] that* $\hat{\lambda} = 0$ *for the triplets in* $\mathcal{I}_0^{\pm}$.

Using theorem 4.1 we see that $\hat{\lambda} = -\frac{m}{2}$ *for every triplet of* $\mathcal{I}_0^{-+}$ *with* U^+ *irreducible.*

4.2 The theories of elementary free particle with zero mass

The possible theories of elementary free particle with zero mass can now be identified by selecting the triplets that admit a unique position operator. One conclusion shared by the past approaches states that no position operator exists for massless particles with non-zero helicity. Yet, the theoretical structures where such non-existence is proven [7] [9] are triplets where S is unitary and T is anti-unitary. The present approach highlights that this is a serious shortcoming, because according to theorem 3.1 these structures must be triplets in $\mathcal{I}_0^+$ or $\mathcal{I}_0^-$. But according to section 4.1 irreducible triplets with non-zero helicity can exist only in $\mathcal{I}_0^{-+}$. Therefore, these proofs do not apply.

In fact our approach proves the following theorems [16].

Theorem 4.2. *If* $\hat{\lambda} \neq 0$, *then in every triplet of* $\mathcal{I}_0$ *there is no three-operator satisfying* (Q.1) *and (4.i), (4.ii).*

Theorem 4.3. *For the triplet of* $\mathcal{I}_0^+$ *or of* $\mathcal{I}_0^-$, *with* U *irreducible, there is only one three-operator satisfying* (Q.1) *and* (4), *namely Newton-Wigner operator* $\mathbf{Q} = \mathbf{F}$.

Since the search for a position operator must be restricted to triplets with $\hat{\lambda} = 0$, in $\mathcal{I}_0^{-+}$ only triplets with $m = 0$ have to be checked.

Theorem 4.4. *The triplets of* $\mathcal{I}_0^{-+}$ *with a three-operator satisfying* (Q.1) *and* (4) *are three of the six triplets with* $m = 0$ *of theorem 4.1, characterized by* $\mathrm{T}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathrm{S}_1 = \mathcal{K}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *by* $\mathrm{T}_2 = \mathrm{T}_1$, $\mathrm{S}_2 = \Upsilon\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *and by* $\mathrm{T}_3 = \mathcal{K}\Upsilon\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\mathrm{S}_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathcal{K}$.

In all the three theories $\mathbf{Q} = \begin{bmatrix} \mathbf{F} & 0 \\ 0 & \mathbf{F} \end{bmatrix}$.

4.3 Conclusions for the zero mass case

The current literature in fact restricts the search for theories of massless elementary free particle to triplets with T anti-unitary and S unitary, i.e. to triplets of $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$. Our approach proves that consistent theories can be developed also if T is unitary or S is anti-unitary. As a consequence, the class of possible theories extends to include a subclass of $\mathcal{I}_0^{-+}$.

Furthermore, the non-existence proofs of a position operator for non zero helicity massless particles extends to the larger class of possible theories, because the operators $\overleftarrow{\mathrm{T}}$ and $\underleftarrow{\mathrm{S}}$ play no role in the new theorem 4.2.

References

[1] E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Nucl. Phys. B Proc. Suppl. **6**, 9 (1989), doi:10.1016/0920-5632(89)90402-7.

[2] V. Bargmann and E. P. Wigner, *Group theoretical discussion of relativistic wave equations*, Proc. Natl. Acad. Sci. **34**, 211 (1948), doi:10.1073/pnas.34.5.211.

[3] V. Bargmann, *On unitary ray representations of continuous groups*, Ann. Math. **59**, 1 (1954), doi:10.2307/1969831.

[4] A. S. Wightman, *On the localizability of quantum mechanical systems*, Rev. Mod. Phys. **34**, 845 (1962), doi:10.1103/RevModPhys.34.845.

[5] N. Debergh, J.-P. Petit and G. D'Agostini, *On evidence for negative energies and masses in the Dirac equation through a unitary time-reversal operator*, J. Phys. Commun. **2**, 115012 (2018), doi:10.1088/2399-6528/aaedcc.

[6] V. V. Varlamov, *Spinor structure and internal symmetries*, Int. J. Theor. Phys. **54**, 3533 (2015), doi:10.1007/s10773-015-2596-0.

[7] S. Weinberg, *Feynman rules for any spin. II. Massless particles*, Phys. Rev. **134**, B882 (1964), doi:10.1103/PhysRev.134.B882.

[8] L. L. Foldy, *Synthesis of covariant particle equations*, Phys. Rev. **102**, 568 (1956), doi:10.1103/PhysRev.102.568.

[9] T. F. Jordan, *Simple derivation of the Newton-Wigner position operator*, J. Math. Phys. **21**, 2028 (1980), doi:10.1063/1.524712.

[10] O. Klein, *Quantentheorie und fünfdimensionale Relativittstheorie*, Z. Phys. **37**, 895 (1926), doi:10.1007/BF01397481.

[11] W. Gordon, *Der Comptoneffekt nach der Schrödingerschen Theorie*, Z. Phys. **40**, 117 (1926), doi:10.1007/BF01390840.

[12] G. Nisticò, *Group theoretical derivation of consistent free particle theories*, Found. Phys. **50**, 977 (2020), doi:10.1007/s10701-020-00364-2.

[13] T. D. Newton and E. P. Wigner, *Localized states for elementary systems*, Rev. Mod. Phys. **21**, 400 (1949), doi:10.1103/RevModPhys.21.400.

[14] P. A. M. Dirac, *The quantum theory of the electron*, Proc. R. Soc. Lond., Contain. Pap. Math. Phys. Character **117**, 610 (1928), doi:10.1098/rspa.1928.0023.

[15] G. Nisticò, *Consistent theories of free Dirac particle without singular predictions*, Particles **6**, 1 (2022), doi:10.3390/particles6010001.

[16] G. Nisticò, *Group theoretical derivation of consistent massless particle theories*, Found. Phys. **51**, 112 (2021), doi:10.1007/s10701-021-00494-1.

On Möbius gyrogroup and Möbius gyrovector space

Kurosh Mavaddat Nezhaad* **and Ali Reza Ashrafi**

Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Kashan, Kashan 87317-53153, I. R. Iran

* kuroshmavaddat@gmail.com

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Gyrogroups are new algebraic structures that appeared in 1988 in the study of Einstein's velocity addition in the special relativity theory. These new algebraic structures were studied intensively by Abraham Ungar. The first gyrogroup that was considered into account is the unit ball of Euclidean space $\mathbb{R}^3$ endowed with Einstein's velocity addition. The second geometric example of a gyrogroup is the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. To construct a gyrogroup structure on $\mathbb{D}$, we choose two elements $z_1, z_2 \in \mathbb{D}$ and define the Möbius addition by $z_1 \oplus z_2 = \frac{z_1+z_2}{1+\bar{z_1}z_2}$. Then $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup. If we define $r \odot x = \frac{(1+|x|)^r-(1-|x|)^r}{(1+|x|)^r+(1-|x|)^r} \frac{x}{|x|}$, where $x \in \mathbb{D}$ and $r \in \mathbb{R}$, then $(\mathbb{D}, \oplus, \odot)$ will be a real gyrovector space. This paper aims to survey the main properties of these Möbius gyrogroup and Möbius gyrovector space.

Received 18-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.041

1 Introduction

Throughout this paper, $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ denotes the extended complex plane and $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the complex open unit disk. We refer the interested readers to consult [1,2] for more information on this topic.

If G is a non-empty set and "+" is a binary operation, then $(G, +)$ is called a groupoid. A permutation $f : G \to G$ with this property that $f(x + y) = f(x) + f(y)$, $x, y \in G$, is said to be an automorphism of G. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$. A gyrogroup [3] is a groupoid $(G, +)$ satisfying the following axioms:

1. G has a left identity under the binary operation, "+".

2. Each element of G has a left inverse.

3. There exists a mapping gyr : $G \times G \to \mathrm{Aut}(G)$ which satisfies the following conditions:
   (a) For each three elements, a, b, and c of G, $a + (b + c) = (a + b) + \mathrm{gyr}[a, b]c$. This property is named the left gyroassociativity of G.
   (b) For every two elements $a, b \in G$, $\mathrm{gyr}[a + b, a] = \mathrm{gyr}[a, b]$. This equality is called the left loop property.

It is custom to wite $\mathrm{gyr}[a, b]$ as $\mathrm{gyr}(a, b)$. This algebraic structure share interesting analogous theorems with classical group theory.

If $(G, \oplus)$ is a gyrogroup, then the following properties of G are important in the context of gyrogroup theory [4]:

1. If $a \oplus b = a \oplus c$ then $b = c$; (general left cancelation law)
2. For each $a \in G$, $\mathrm{gyr}[0, a] = I$;
3. If x is a left inverse of a, then $\mathrm{gyr}[x, a] = I$;
4. $\mathrm{gyr}[a, a] = I$;
5. Every left identity is a right identity;
6. The left identity is unique;
7. Every left inverse is a right inverse;
8. The left inverse of each element is unique;
9. For arbitrary elements $a, b \in G$, $\ominus a \oplus (a \oplus b) = b$; (left cancelation law)
10. For arbitrary elements $a, b, x \in G$, $\mathrm{gyr}[a, b]x = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x))$; (gyrator identity)
11. For all elements $a, b \in G$, $\mathrm{gyr}[a, 0] = \mathrm{gyr}[0, b] = I$ and $\mathrm{gyr}[a, b]0 = 0$;
12. For arbitrary elements $a, b, x \in G$, $\mathrm{gyr}[a, b] \ominus x = \ominus \mathrm{gyr}[a, b]x$.

The Möbius transformations are defined as the linear fractional transformations $f(z) = \frac{az+b}{cz+d}$ of $\hat{\mathbb{C}}$, where a, b, c and d are complex numbers satisfying $ad - bc \neq 0$. These transformations constitute a group under composition of functions. This group is isomorphic to the group $PGL(2, \mathbb{C}) = \frac{GL(2,\mathbb{C})}{Z(GL(2,\mathbb{C}))}$. By restricting conditions to 2×2 matrices with $\det A = 1$, we will have the special linear group $\mathbf{SL}(2, \mathbb{R})$, which is well-known that it maps upper-half plane to itself [2].

The Möbius transformations of complex open disk $\mathbb{D}$ are defined as the mappings given by $z \mapsto e^{i\theta} \frac{a+z}{1+\bar{a}z} = e^{i\theta}(a \oplus z)$, where $a, z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. By seeing this transformation as an addition $\oplus$, we will have Möbius addition of complex open unit disk which is given by $x \oplus y = \frac{x+y}{1+\bar{x}y}$, where $x, y \in \mathbb{D}$, and $\bar{x}$ denotes the conjugate of x. It is easy to see that Möbius addition is neither associative nor commutative. For the sake of fixing the lack of associativity and commutativity, Abraham Ungar introduced the gyrations of these algebraic structures as

$$\mathrm{gyr}[x, y]z = \frac{x \oplus y}{y \oplus x} z = \frac{1 + x\bar{y}}{1 + \bar{x}y} z,$$

where $x, y, z \in \mathbb{D}$ [5,6]. This definition will make a gyrogroup structure on $\mathbb{D}$. It is well-known that the gyrators of this gyrogroup is not closed under composition of functions. We refer to [7] for applications of gyrogroup theory in non-Euclidean geometry.

2 Möbius structures

Recall that the Möbius addition of two elements x and y in $\mathbb{D}$ is defined as $x \oplus y = \frac{x+y}{1+\bar{x}y}$. It is easy to see that for each element $z \in \mathbb{D}$, $\ominus z = -z$. The gyrator gyr$[a,b]$ is defined as $\mathrm{gyr}[a,b](x) = \frac{a\oplus b}{b\oplus a}x = \frac{1+a\bar{b}}{1+\bar{a}b}x$. This proves that $\mathrm{gyr}[a,b](c \oplus d) = \mathrm{gyr}[a,b](c) \oplus \mathrm{gyr}[a,b](d)$. It is well-known that in each gyrogroup $\mathrm{gyr}^{-1}[a,b] = \mathrm{gyr}[b,a]$ and by definition of a gyrogroup, all gyrations are automorphisms. By definition of a gyrator, $\mathrm{gyr}[a,b](b \oplus a) = a \oplus b$ and so $(\mathbb{D}, \oplus)$ is gyrocommutative.

Following Ungar [6], by some calculations one can see that $\mathrm{gyr}[u,v](w) = w + 2\frac{Au+Bv}{D}$, where $A = -u.w \parallel v \parallel^2 + v.w + 2(u.v)(v.w)$, $B = -v.w \parallel u \parallel^2 - u.w$, $D = 1 + 2u.v + \parallel u \parallel^2 \parallel v \parallel^2$ and $u, v, w \in \mathbb{V}_s$, where $\mathbb{V}_s$ is generalization of Möbius disk gyrogroup, $(\mathbb{D}, \oplus)$ to s-ball of $\mathbb{V}$, $\mathbb{V}_s = \{\mathbb{V}_s \in \mathbb{V} : \parallel v \parallel < s\}$, and its inner product and norm, . and $\parallel . \parallel$, are inherited from its space $\mathbb{V}$ and $+$ denotes the addition of vectors in $\mathbb{V}$. Note that the Möbius addition and Möbius scalar multiplication in $\mathbb{V}_s$ reduce to the vector addition and scalar multiplication, respectively, as s tends to infinity.

Ferreira and Ren [8], studied the algebraic structure of Möbius gyrogroups by a Clifford algebra approach. They started from an arbitrary real inner product space of dimension n and then construct a paravector space from which it is possible to study the Möbius gyrogroups. The most important result of Ferreira and Ren is giving a characterization of the Möbius sub-gyrogroups of $\mathbb{D}$.

A triple $(P, \oplus, \odot)$ consisting a non-empty set P together with two binary operations $\oplus$ and $\odot$ are called a real gyrovector space, if for all real numbers r, r_1, r_2 and all elements $x, y, z \in P$ the following are satisfied: (i) $(G, \oplus)$ is a gyrogroup; (ii) $(r_1 + r_2) \odot x = r_1 \odot x + r_2 \odot x$; (iii) $r_1 r_2 \odot x = r_1 \odot (r_2 \odot x)$; $1 \odot x = x$; $\mathrm{gyr}[x,y](r \odot z) = r \odot \mathrm{gyr}[x,y](z)$; $\mathrm{gyr}[r_1 \odot x, r_2 \odot x] = I$. The gyrovector space is the main object of Ungar's theory of analytic hyperbolic geometry [4]. Kinyon and Ungar [9], applied the relationshops between the geometric and algebraic properties of the Möbius gyrovector space to obtain an interesting geometric picture of these objects.

Suppose $\mathbb{I} = (-1,1)$, $x, y \in \mathbb{I}$ and $r \in \mathbb{R}$. Define $x \oplus y = \frac{x+y}{1+xy}$ and $r \odot x = \frac{(1+x)^r - (1-x)^r}{(1+x)^r + (1-x)^r}$. Then it is easy to see that $\mathbb{I}$ is a real vector space. Kinyon and Ungar [9] presented an interesting discussion to show that this real verctor space can be generalized to the real gyrovector space $(\mathbb{D}, \oplus, \odot)$ in which $x \oplus y = \frac{x+y}{1+\bar{x}y}$ and $r \odot x = \frac{(1+|x|)^r - (1-|x|)^r}{(1+|x|)^r + (1-|x|)^r} \frac{x}{|x|}$, $x, y \in \mathbb{D}$ and $r \in \mathbb{R}$. The first example, $\mathbb{I}$, is an interesting algebraic example for the Euclidean geometry, but the second one, $(\mathbb{D}, \oplus)$, is an important algebraic example for hyperbolic geometry.

Watanabe [10] introduced the notion of gyrolinear indipendence of vectors in $\mathbb{D}$. To define, we assume that a finite subset $A = \{a_1, a_2, \ldots, a_n\}$ of vectors in $\mathbb{D}$ is given. If for each permutation $\sigma \in S_n$, $r_{\sigma(1)} \odot a_{\sigma(1)} \oplus r_{\sigma(2)} \odot a_{\sigma(2)} \oplus \ldots r_{\sigma(n)} \odot a_{\sigma(n)} = 0$ implies that $r_1 = r_2 = \ldots = r_n = 0$, then A is called a gyrolinear independent set of $\mathbb{D}$. By [10, Lemma 8], every vector of a finite gyrolinearly independent set A is non-zero, and every subset of A is also gyrolinearly independent.

Demirel [11] gave an important sharp inequality between members of $\mathbb{D}$. He proved that if $x_1, x_2, \ldots, x_n$ are non-zero elements of $\mathbb{D}$, then $| \oplus_{j=1}^{n} x_j| \le \oplus_{j=1}^{n} |x_j|$, where $\oplus_{j=1}^{n} |x_j| = |x_1| + |x_2| + \ldots |x_n|$, $|x_j| = |x_j \ominus 0|$, and

$$\oplus_{j=1}^{n} x_j = (\ldots((x_1 \oplus x_2) \oplus x_3) \oplus \ldots \oplus x_{n-1}) \oplus x_n\,.$$

Abe and Watanabe [12] proved that every finitely generated gyrovector subspace in the Möbius gyrovector space is the intersection of the vector subspace generated by the same generators and the Möbius ball. They applied this result to present a notion of orthogonal gyrodecomposition and determined its relationship with the orthogonal decomposition. The most important result of this paper is related to the following two questions in the Möbius gyrovector space $(\mathbb{D}, \oplus, \odot)$.

1. $\{r_1 \odot a_1 \oplus r_2 \odot a_2 \mid r_1, r_2 \in \mathbb{R}\} = \{s_2 \odot a_2 \oplus s_1 \odot a_1 \mid s_1, s_2 \in \mathbb{R}\}$?
2. $r \odot (s_1 \odot a_1 \oplus s_2 \odot a_2) \in \{r_1 \odot a_1 \oplus r_2 \odot a_2 \mid r_1, r_2 \in \mathbb{R}\}$?

They gave an affirmative answer to both Questions 1 and 2.

3 Concluding remark

In this paper we survey most important result in literature on the Möbius gyrogroup $(\mathbb{D}, \oplus)$ and Möbius gyrovector space $(\mathbb{D}, \oplus, \odot)$. Ferreira and Ren [8] characterized all subgyrogroups of the Möbius gyrogroup $(\mathbb{D}, \oplus)$. We end this paper with the following question:

Question 3.1. Is there any characterization of subgyrovector spaces of $(\mathbb{D}, \oplus, \odot)$?

References

[1] M. V. Lawson, *The Möbius inverse monoid*, J. Algebra **200**, 428 (1998), doi:10.1006/jabr.1997.7242.

[2] V. V. Kisil, *Geometry of Möbius transformations*, Imperial College Press, London, UK, ISBN 9781848168589 (2012), doi:10.1142/p835.

[3] A. A. Ungar, *Thomas rotation and the parametrization of the Lorentz transformation group*, Found. Phys. Lett. **1**, 57 (1988), doi:10.1007/BF00661317.

[4] A. A. Ungar, *Analytic hyperbolic geometry and Albert Einstein's special theory of relativity*, World Scientific, Singapore, ISBN 9789812772299 (2008), doi:10.1142/6625.

[5] A. A. Ungar, *The holomorphic automorphism group of the complex disk*, Aequat. Math. **47**, 240 (1994), doi:10.1007/BF01832962.

[6] A. A. Ungar, *From Möbius to gyrogroups*, Amer. Math. Month. **115**, 138 (2008), doi:10.1080/00029890.2008.11920506.

[7] A. A. Ungar, *The hyperbolic Pythagorean theorem in the Poincaré disc model of hyperbolic geometry*, Amer. Math. Month. **106**, 759 (1999), doi:10.2307/2589022.

[8] M. Ferreira and G. Ren, *From Möbius to gyrogroups: A Clifford algebra approach*, J. Algebra **328**, 230 (2011), doi:10.1016/j.jalgebra.2010.05.014.

[9] M. K. Kinyon and A. A. Ungar, *The gyro-structure of the complex unit disk*, Math. Mag. **73**, 273 (2000), doi:10.2307/2690974.

[10] K. Watanabe, *Orthogonal gyroexpansion in Möbius gyrovector spaces*, J. Funct. Spaces 1518254 (2017), doi:10.1155/2017/1518254.

[11] O. Demirel, *The first sharp gyrotriangle inequality in Möbius gyrovector space* $(\mathbb{D}, \oplus, \otimes)$, Forum Geom. **17**, 439 (2017).

[12] T. Abe and K. Watanabe, *Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space*, J. Math. Anal. Appl. **449**, 77 (2017), doi:10.1016/j.jmaa.2016.11.039.

SciPost Phys. Proc. 14, 042 (2023)

Clocking mechanism from a minimal spinning particle model

Tobiasz Pietrzak[1⋆] and Łukasz Bratek[2]

1 H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences,
ul. Eljasza-Radzikowskiego 152, PL 31342 Kraków, Poland
2 Institute of Physics, Cracow University of Technology,
ul. Podchorażych 1, PL-30084 Kraków, Poland

⋆ tobiasz.pietrzak@ifj.edu.pl

Group ICGTMP

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

The clock hypothesis plays an important role in the theory of relativity. To test this hypothesis, a mechanical model of an ideal clock is needed. Such a model should have the phase of its intrinsic periodic motion increasing linearly with the affine parameter of the clock's center of mass worldline. A class of relativistic rotators introduced by Staruszkiewicz in the context of an ideal clock is studied. A singularity in the inverse Legendre transform leading from the Hamiltonian to the Lagrangian leads to new possible Lagrangians characterized by fixed values of mass and spin. In free motion the rotators exhibit intrinsic motion with the speed of light and fixed frequency.

Received 17-12-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.042

---

1 Introduction

By definition an ideal clock always measures its proper time. The equality of time measured by natural clocks and that of ideal clock has been verified to a high degree of precision [1], however it is not known whether this equality always holds true. Discrepancies could occur for extreme accelerations of order c^2/L where L is a length scale characterising a given system (e.g. $10^{29}\frac{m}{s^2}$ for electron's Zitterbewegung frequency). Accelerations that high are not yet experimentally attainable. Nevertheless, an attempt can be made to theoretically test the clock hypothesis (which refers to classical concepts) within the same framework one uses to describe real mechanical systems. In this respect a classical model of the ideal clock must be devised.[1]

[1]A spatially extended quantum field-theoretical model of a clock devised in the clock hypothesis context [2] goes beyond this conceptual limitation. The authors concluded that *no device built according to the rules of quantum field theory can measure proper time along its path*. It is also known that for any timelike worldline in any spacetime, there is a sufficiently small light clock that accurately measures the proper time [3], however this kind of clock is not a mechanical system.

As a purely mathematical construct unrelated to any material mechanism, an ideal clock would be a simple non-quantum device. The mechanism of such a clock could be designed in the following way. In the momentum rest frame, the image of the spatial direction of the Pauli-Lubański four-vector could be identified with the equator on the Riemann sphere of null directions and used as the clock's face. On the other hand, the image of a null direction (carrying the spinning degrees of freedom) would be a point moving about the equator, counting the number of times the phase has been increased by 2π, and thus represent the clock's hand.

Such a model has been proposed by Staruszkiewicz [4]. It is based on the concept of a relativistic rotator – a dynamical system described by position, single null direction (thus with 5 degrees of freedom) and, additionally, two-dimensional parameters – mass m and length l used to set the values of Casimir invariants, respectively, to m^2 and $-\frac{1}{4}m^4 l^2$. It seems that the model provides the simplest mechanical system whose clocking frequency could be fixed this way. Among the entirety of Lagrangians possible for the family of relativistic rotators considered in [4], there are only two which satisfy the last requirement above. As later shown at the Lagrangian level [5] the unique Lagrangians are defective when interpreted as dynamical systems with 5 degrees of freedom (the Hessian rank is 4, not 5). This explained the observation [6] that in free motion of the clock the phase, and hence the clocking frequency, remained indeterminate as functions of the proper time of the center of momentum frame, contrary to the original motivation.

A possible way to find the required Lagrangian and stabilise the clocking frequency leads through the inverse Legendre transformation (from the Hamiltonian to a Lagrangian). As observed for the rotator in [7], a singularity in this transformation distinguishes intrinsic motion with the speed of light. This changes the analytic form of the required Lagrangian.[2]

2 Staruszkiewicz class of relativistic rotators.

A class of relativistic rotators is defined by the following Hamiltonian action introduced by Staruszkiewicz [4]

$$S = -m \int \mathrm{d}\lambda \sqrt{\dot{x}\dot{x}}\, f(\xi), \qquad \xi \equiv -l^2 \frac{\dot{k}\dot{k}}{(k\dot{x})^2}, \quad f'(\xi) \not\equiv 0. \tag{1}$$

Here,[3] the dot denotes differentiation with respect to λ – an arbitrary parameter along the worldline, and f can be arbitrary non-constant and positive function of a reparametrization invariant argument ξ depending on the spinning degrees of freedom through a null direction k (the latter property means that ξ must be a Poincaré scalar, independent of arbitrary scale of null vector k).

Representations of the Poincaré group are enumerated by the eigenvalues of two Casimir operators (for the case of massive representations). These operators are the square of the momentum four-vector $C_1 = p^\mu p_\mu$ and the square of the Pauli-Lubański four-vector $C_2 = W^\mu W_\mu$, where:

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} p_\nu M_{\alpha\beta}, \qquad M_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha + \Sigma_{\alpha\beta}.$$

The expression $\Sigma_{\alpha\beta}$ represents the internal angular momentum (spin). To find suitable Lagrangians in the considered class of rotators one can proceed as follows. The conserved quantities p_α and $M_{\alpha\beta}$ are determined from the action (1) (with $\Sigma_{\alpha\beta} = k_\alpha \pi_\beta - k_\beta \pi_\alpha$), where the

[2]These results can be considered new as they are based on yet unpublished paper [7].

[3]Throughout this paper x^μ denotes the position vector, k^μ is the single null direction carrying the spinning degrees of freedom. The scalar product is denoted by $xy \equiv \eta_{\alpha\beta}x^\alpha y^\beta = x^\alpha y_\alpha$ (Einstein's summation convention is used), where $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$, and $\epsilon^{0123} = 1$ for the Levi-Civita completely anti-symmetric pseudotensor. Greek indices run over $0, 1, 2, 3$ and 0 stands for the time component.

momenta canonically conjugated to x^μ and k^μ read, respectively,

$$p_\mu = m\left[f(\xi)\frac{\dot{x}_\mu}{\sqrt{\dot{x}\dot{x}}} - 2\xi f'(\xi)\frac{\sqrt{\dot{x}\dot{x}}}{k\dot{x}}k_\mu\right], \qquad \text{and} \qquad \pi_\mu = 2m\frac{\sqrt{\dot{x}\dot{x}}}{\dot{k}\dot{k}}\xi f'(\xi)\dot{k}_\mu .$$

The corresponding Casimir invariants can now be calculated

$$C_1 = m^2[f^2(\xi) - 4\xi f(\xi)f'(\xi)], \qquad C_2 = -4m^4 l^2 \xi f^2(\xi)[f'(\xi)]^2 .$$

By requiring that $C_1 \equiv m^2$ and $C_2 \equiv -\frac{1}{4}m^4 l^2$ (identically), one gets two first-order differential equations that, remarkably, have a common solution of the form $f(\xi) = \sqrt{1 \pm \sqrt{\xi}}$. The Hamiltonian action describing these rotators takes on the form

$$S = -m\int \mathrm{d}\lambda\,\sqrt{\dot{x}\dot{x}}\sqrt{1 \pm \sqrt{-l^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}}} + \int \mathrm{d}\lambda\,\Lambda\, kk\,, \tag{2}$$

with Λ being a Lagrange multiplier. As will be explained below, the dynamical system defined by the action (2) is not suitable as a clock. However, it is equivalent to a geometric model of a spinning particle introduced earlier in a different context by Lyakhovich, Segal, and Sharapov [8] and as such can be used with success.

3 Hessian rank deficiency for subluminal intrinsic motion

In the Lagrangian form of dynamics, there are s Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, 2, \ldots, s,$$

for a dynamical system with s (physical) degrees of freedom. In this form the Lagrangian L is assumed to be a function of s generalised coordinates $q^i = Q^i(\lambda)$ and the corresponding velocities $v^i = \dot{Q}^i(\lambda)$ that altogether characterise the physical state of the system. Differentiating the Lagrange equations with respect to the independent parameterization λ, one gets a system of second-order equations

$$H_{ij}a^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial v^i \partial q^j}v^j - \frac{\partial^2 L}{\partial \lambda \partial v^i}, \qquad H_{ij} \equiv \frac{\partial^2 L}{\partial v^i \partial v^j} .$$

Provided that $\det[H_{ij}] \not\equiv 0$ for this system, one can express accelerations $a^i = \ddot{Q}^i(\lambda)$ as independent functions of positions and velocities. When the Hessian determinant $\det(H_{ij})$ is non-vanishing the Lagrangian is called regular, otherwise it is called singular. For a singular Lagrangian, there is an infinite number of accelerations available from which a dynamical system can choose at any stage of its motion. The regularity (or singularity) is a qualitative feature, independent of the particular coordinates in which the Lagrangian has been expressed.

Note, that the discussion just above assumes that the Lagrangian has been expressed in terms of the physical degrees of freedom only. In a more general situation, the notion of a Lagrangian regularity or singularity becomes context-dependent. The reason for this is that, in describing a dynamical system, one can use a Lagrangian involving only s physical degrees of freedom or a Lagrangian in an extended configuration space involving additional r, non-dynamical degrees of freedom. The Hessian square matrix has dimension s in the first case and dimension $s + r$ in the extended case. In both cases, however, the Hessian rank must not be lower than s. The use of a Lagrangian description involving spurious or auxiliary degrees of freedom often makes the description more transparent or easier to tackle with, for example, fully covariant. In order not to come into confusion, instead of referring to singularity/regularity of a Lagrangian it would be better to refer to the rank of the Hessian matrix (denoted with $\mathrm{Rk}(H)$) as it is not changed when additional gauge degrees of freedom are introduced and, accordingly, the Lagrangian is rewritten in an extended configuration space.

A good example is provided by the ordinary point particle. Its Lagrangian in the covariant form $L = -m\sqrt{\dot{x}\dot{x}}$ is singular – it involves a spurious gauge degree of freedom. In the gauge $\lambda = x^0$ one gets a regular Lagrangian $L = -m\sqrt{1-\dot{\boldsymbol{x}}\dot{\boldsymbol{x}}}$, where $\dot{\boldsymbol{x}}$ is the spatial velocity vector. In both cases, the Hessian rank is 3 and equals the number of degrees of freedom considered physical in the context of a point particle. Similarly, when it comes to a relativistic rotator, the Lagrangian recast in a form involving only the 5 physical degrees of freedom characteristic of a genuine rotator should be regular, which means that the determinant of the corresponding 5×5 Hessian matrix must be non-vanishing. This implies that the rank of the full 8×8 Hessian matrix of the original singular Lagrangian (1) involving also non-dynamical degrees of freedom should be 5 too.

One can verify the condition $\mathrm{Rk}(H) = 5$ for all members of the considered family of relativistic rotators (1) regarded as dynamical systems with 5 physical degrees of freedom. Following the calculation presented in [5], one can start with Cartesian coordinates (x, y, z) and spherical angles (φ, θ) describing the position and the null direction in a reference system of some inertial observer. The arbitrary parameter λ can be set to be proportional to the time of that observer, $\lambda = l^{-1}t$. Then, in terms of the vector matrices $V = [\dot{x}, \dot{y}, \dot{z}]^T$, $N = [\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]^T$ and $W = \left[\dot{\theta}, \dot{\varphi}\sin\theta\right]^T$, the Lagrangian form (1) gets reduced to

$$L = -m\sqrt{1-V^TV}f(\xi), \quad \text{with} \quad \xi = \frac{W^TW}{(1-N^TV)^2} \quad \text{and} \quad f'(\xi) \not\equiv 0. \tag{3}$$

The Hessian determinant can be found by taking components of vectors V and W as independent velocity variables (linearly related to the original set of velocities) and using some identities for determinants of block matrices. As shown in [5], the resulting determinant reads

$$\det[H_{ij}] \propto f^3(\xi)\left[f'(\xi)\right]^2\left(1 + 2\xi\left(\frac{f'(\xi)}{f(\xi)} + \frac{f''(\xi)}{f'(\xi)}\right)\right),$$

where the proportionality factor (not shown) is independent of f. Hence, only with f satisfying the differential equation $\big(f(\xi) + 2\xi f'(\xi)\big)f'(\xi) + 2\xi f(\xi)f''(\xi) = 0$ the Lagrangian (3) will be singular. This equation has only one solution such that $f'(\xi) \not\equiv 0$, namely

$$f(\xi) = a\sqrt{1 \pm b\sqrt{\xi}},$$

with a and b being positive integration constants to be set by the Casimir parameters.

Now it becomes clear that the only Lagrangian with deficient rank in the investigated family of relativistic rotators (1) is that defined by the action (2) (its Hessian rank is 4, not 5). In consequence of this the phase of the clocking mechanism has the nature of a gauge variable [5, 9], which is the reason why the dynamical system (2) cannot be interpreted as a clock.

4 Singularities in the inverse Legendre transformation. Zitterbewegung with the speed of light.

According to Dirac's method [10], the Hamiltonian for a (reparametrization invariant) relativistic system is a linear combination of first-class constraints (whose Poisson bracket with all other constraints is vanishing). The coefficients of this combination are arbitrary functions of the independent parameter. There are four such constraints for the Lagrangian (2): the first two follow from the requirement imposed on both Casimir invariants: $C_1 \equiv pp \simeq m^2$ and $C_2 \equiv -\det \mathrm{Gram}(p,k,\pi) \simeq -\frac{1}{4}m^4 l^2$; the other two constraints concern the particular realisation of the spinning degrees of freedom described by a null direction k (with the corresponding conjugate momentum π): $kk \simeq 0$ and $k\pi \simeq 0$ – the latter ensures that the physical state is independent of the arbitrary scale of k. All of these constraints are first-class. Remembering that one can use any equivalent combination of constraints, it immediately follows that the total Hamiltonian, as implied by the original Lagrangian form (2), can be taken as [9]

$$\mathcal{H} = \frac{u_1}{2m}\left[pp - m^2\right] + \frac{u_2}{2m}\left[pp + \frac{4}{l^2 m^2}(kp)^2 \pi\pi\right] + u_3 k\pi + u_4 kk\,, \tag{4}$$

with u_i's being independent arbitrary functions.[4] Now the Hamiltonian constraints follow from the equations $\partial_{u_i}\mathcal{H} = 0$ while the velocities are defined through the Hamiltonian equations:

$$\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{u_1 + u_2}{m} p^\mu + u_2 \frac{4\,(kp)(\pi\pi)}{l^2 m^3} k^\mu\,, \qquad \dot{k}^\mu = \frac{\partial \mathcal{H}}{\partial \pi_\mu} = u_2 \frac{4\,(kp)^2}{l^2 m^3}\pi^\mu + u_3 k^\mu\,. \tag{5}$$

Now, the Hamiltonian form (4) can be assumed as a starting point. All Lagrangians corresponding to the Hamiltonian (4) can be obtained by applying the inverse Legendre transformation. The form of the resulting Lagrangian $L \equiv p\dot{x} + \pi\dot{k} - \mathcal{H}$, when expressed in terms of the velocities, is subject to the invertibility of the map (5) restricted to the submanifold defined by the Hamiltonian constraints. On this submanifold induced is a corresponding map between two sets of scalar variables $\{u_1, u_2, u_3, kp, p\pi\}$ and $\{\dot{k}\dot{k}, \dot{k}\dot{x}, \dot{x}\dot{x}, k\dot{x}, k\dot{k}\}$ which is easier to investigate:

$$\begin{aligned} &\dot{x}\dot{x} = u_1^2 - u_2^2\,, \qquad k\dot{x} = (u_1+u_2)\frac{kp}{m}\,, \qquad \dot{k}\dot{k} = -\frac{4(kp)^2}{l^2m^2}u_2^2\,, \\ &\dot{k}\dot{x} = (u_1+u_2)\left[\frac{4(kp)(p\pi)}{m^3 l^2}u_2 + u_3\right]\frac{kp}{m}\,, \qquad k\dot{k} = 0\,. \end{aligned} \tag{6}$$

The number of new constraints for velocities depends on the rank of the Jacobi matrix of the above mapping. It can be shown that this rank depends only on the variables u_1, u_2, and equals 4 for $u_1^2 \neq u_2^2 \neq 0$, 3 for $u_1 = u_2 \neq 0$, and 2 for $u_1 = -u_2 \neq 0$.

In passing from the Hamiltonian to the Lagrangian, one may first assume that $u_1 + u_2 \neq 0$ and $u_2 \neq 0$. Then the momenta expressed as functions of velocities and u_i's read

$$p^\mu = \frac{m}{u_1+u_2}\dot{x}^\mu - \frac{l^2 m (u_1+u_2)^2(\dot{k}\dot{k} - 2u_3 k\dot{k})}{4(k\dot{x})^2 u_2}\frac{k^\mu}{k\dot{x}}\,, \qquad \pi^\mu = \frac{l^2 m (u_1+u_2)^2}{4(k\dot{x})^2 u_2}(\dot{k}^\mu - u_3 k^\mu)\,.$$

From the constraint equations $pp - m^2 = 0$ and $pp + \frac{4}{l^2m^2}(kp)^2(\pi\pi) = 0$ two conditions for u_1 and u_2 follow:

$$\frac{\dot{x}\dot{x}}{(u_1+u_2)^2} + \frac{u_1+u_2}{2u_2}\xi = 1\,, \quad \text{and} \quad \frac{(u_1+u_2)^2}{4u_2^2}\xi = 1\,. \tag{7}$$

The resulting u_1, u_2 can be expressed as independent functions of the velocities, provided that the Jacobian determinant of the transformation (7) — regarded as one leading from variables

[4] The Hamiltonian formulation of the whole class of relativistic rotators defined by the general Lagrangian (1) was presented in [9]. This formulation uses the minimal phase space in terms of four-vectors. There is also possible a description of dynamical systems in extended phase spaces that upon reduction should recover the minimal Hamiltonian form. In the case of the particular Lagrangian (2) such an approach was presented by Das and Ghosh [11] who also obtained the Hamiltonian (4). They started with a counterpart of Lagrangian (2) written in an extended space exploiting a trick, introduced by Lukierski Stichel and Zakrzewski [12], in which additional auxiliary variables allow one to make the time derivative structure of the original Lagrangian easier to tackle with.

$(\dot{x}\dot{x}, \xi)$ to variables (u_1, u_2) which, up to a constant factor, is equal to $\frac{\xi \dot{x}\dot{x}}{u_2^3(u_1+u_2)}$ — is non-zero. In this case the resulting Lagrangian overlaps with that in the action integral (2). However, assuming that the condition $\dot{x}\dot{x} \neq 0$ is not satisfied, two other Lagrangians are possible.

In the first case $u_1 = u_2$, and the corresponding new velocity constraints follow:

$$\frac{\dot{x}\dot{x}}{k\dot{x}} = 0\,, \qquad l^2 \frac{\dot{k}\dot{k}}{k\dot{x}} + k\dot{x} = 0\,.$$

Then, from (6), $u_1 = \chi$, $u_2 = \chi$, $u_3 = \nu$, $kp = \frac{m}{2\chi} k\dot{x}$ and $p\pi = \frac{l^2 m^2}{2k\dot{x}}\left[\frac{\dot{k}\dot{x}}{k\dot{x}} - \nu\right]$ with χ and ν being arbitrary functions. After discarding a total derivative involving $k\dot{k}$ and the higher order terms in the velocity constraints, the resulting Lagrangian can be cast in the following form linear in these constraints

$$L = \frac{m\kappa}{2}\frac{\dot{x}\dot{x}}{k\dot{x}} + \frac{m}{4\kappa}\left[l^2 \frac{\dot{k}\dot{k}}{k\dot{x}} + k\dot{x}\right] + \Lambda\, kk\,. \tag{8}$$

Here, $\kappa(\chi) \equiv \frac{kp}{m}$ is a new variable independent of velocities while Λ is a Lagrange multiplier.

In the second case, for $u_1 = -u_2$, a restricted Legendre transformation should be considered with p^μ left (for a while) unaltered. Using equations (5) and (6), one can find that $\pi = \mp \frac{lm^2}{2} \frac{\dot{\boldsymbol{k}} - u_3 \boldsymbol{k}}{kp\sqrt{-\dot{k}\dot{k}}}$ and $u_2 = \mp \frac{lm}{2kp}\sqrt{-\dot{k}\dot{k}}$. Now, integrating away the term linear in $k\dot{k}$, another Lagrangian is obtained in the form

$$L = p\dot{x} \pm \frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{kp} + \Lambda\, kk\,. \tag{9}$$

Inferred from equations (5) and (6) the result $\dot{x}^\mu = \pm \frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{(kp)^2} k^\mu$ can be re-obtained by performing arbitrary variations of the Lagrangian with respect to p^μ, hence $e\dot{x} = \pm \frac{lm^2}{2}\frac{\sqrt{-\dot{k}\dot{k}}}{2(kp)^2} ek$ for any vector e^μ, and this fact can be used to eliminate p^μ from (9). Accordingly, the alternative form of the above Lagrangian can be taken to be

$$L = m\left[\frac{-4l^2 \dot{k}\dot{k}}{(ek)^2 (e\dot{x})^2}\right]^{1/4} e\dot{x} + \Lambda\, kk\,,$$

which involves arbitrary (timelike) e^μ (then the condition $ek \neq 0$ is satisfied) playing the role of the initial momentum p.

Unlike the Lagrangian (2), the new Lagrangians (8) and (9) have analytic structure compatible with the constraint $\dot{x}\dot{x} = 0$. They describe intrinsic motion with the speed of light (see Appendix).

5 Conclusion

In this paper, the present status of Staruszkiewicz's relativistic rotators in free motion was discussed. The original motivation behind introducing the rotators was the idea of devising a model of an ideal clock that could be used to test the clock hypothesis [4]. However, the constraints imposed on the Casimir invariants for the purpose of realising the quantum irreducibility idea on the classical level, lead to Lagrangians with deficient Hessian rank (which is 4 instead of 5) when subluminal intrinsic motion is assumed from the start. In consequence of this the clocking rate remains arbitrary function of the proper time in the momentum rest frame.

However, at the level of constrained Hamiltonians, one makes no a priori assumptions about the velocities. Constraints on velocities may appear when passing from the Hamiltonian to the Lagrangian. With this method one recovers the original Lagrangian with subluminal motion when the rank of the inverse Legendre transformation is maximal. For a lower rank (when this transformation becomes singular) one obtains two new Lagrangians (8) and (9) with intrinsic motion with the speed of light (the motion of the momentum rest frame is still subluminal). The solutions are presented in Appendix.

The dynamical systems described by the new Lagrangians exhibit behaviour that can viewed as a counterpart of Zitterbewegung known for two states of Dirac's free electron (see the interesting and original discussion by Breit [13]). The existence of the two systems conforms with the distinguished role of the constraint $\dot{x}\dot{x} = 0$. It remains to investigate how these systems would behave when appropriately coupled with the electromagnetic or gravitational field.

A Appendix

For both Lagrangians, the momentum $P \equiv \partial_{\dot{x}}$ is conserved, hence $P = me$, where e is a constant unit future-oriented timelike four-vector and m is a mass parameter. The equation $\partial_\Lambda L = 0$ implies $kk = 0$. The arbitrary parameterization λ and the arbitrary scale of k can be chosen so that $e\dot{x} = 1$ and $ke = 1$. Furthermore, the spatial vector n defined by $k = e + n$ is unit and orthogonal to e: $nn = -1$ and $ne = 0$.

For the Lagrangian (9), the equation $\partial_p L = 0$ implies $\dot{x} = (l/2)\Omega(e+n)$ with $\Omega \equiv \sqrt{-\dot{n}\dot{n}}$, which in turn gives $\Omega = 2/l$ from the previous condition $e\dot{x} = 1$ which is now seen to identify λ with the time t in the momentum frame (in which the time axis is directed along e). Finally, $\dot{x} = e + n$. The momentum $\Pi \equiv \partial_{\dot{k}} L$ reduces to $\Pi = -(ml^2/4)\dot{n}$. Since $\partial_k L = -me + 2\Lambda(e+n)$, the respective Lagrangian equation reduces to the equation for large circles on a unit sphere, $\ddot{n} + (2/l)^2 n = 0$, where the Lagrange multiplier $\Lambda = m/2$ was earlier determined upon taking the scalar product $n(\dot{\Pi} - \partial_k L) = 0$ and using the identity $\dot{n}\dot{n} + n\ddot{n} = 0$ satisfied by any vector with constant product nn. The solution reads $n = a\cos\phi + b\sin\phi$, where $\phi = (2/l)t$ is the phase, a and b are constant vectors such that $aa = -1 = bb$, $ab = 0$, $ae = 0 = be$. Substituting this in the other equation for x and integrating, one obtains $x = et + (l/2)(a\sin\phi - b\cos\phi)$. The phase $\phi = (2/l)t$ is a unique function of the proper time t in the momentum frame and $\dot{x}\dot{x} = 0$.

For the Lagrangian (8), the conserved momentum $P \equiv \partial_{\dot{x}} L = me$ implies

$$\frac{\kappa}{k\dot{x}}\dot{x} = e - \frac{1}{4\kappa}\left(1 - \frac{l^2\dot{k}\dot{k} + 2\kappa^2\dot{x}\dot{x}}{(k\dot{x})^2}\right)k, \quad \text{hence} \quad \kappa = ek\,.$$

Now, taking scalar products of the above equation with e and $\dot{x}$, and with itself, one gets three equations from which one finds that $k\dot{x} = 2(ke)(e\dot{x})$, $l^2\dot{k}\dot{k} + (k\dot{x})^2 = 0$, and $\dot{x}\dot{x} = 0$. This in turn implies $\dot{x}/(e\dot{x}) = 2e - k/(ke)$. By applying the gauge $e\dot{x} = 1$ and $ke = 1$ as in the previous case, and then the decomposition $k = e + n$, one finally obtains $\dot{n}\dot{n} = -4/l^2$ and $\dot{x} = e - n$ (note the sign difference with the previous case). Then one finds in an analogous way as before, that $\ddot{n} + (2/l)^2 n = 0$, however with $\Lambda = -m/4$. This leads to a solution $x = et - (l/2)(a\sin\phi - b\cos\phi)$ with $\phi = (2/l)t$.

References

[1] J. Bailey et al., *Measurements of relativistic time dilatation for positive and negative muons in a circular orbit*, Nature **268**, 301 (1977), doi:10.1038/268301a0.

[2] K. Lorek, J. Louko and A. Dragan, *Ideal clocks – A convenient fiction*, Class. Quantum Gravity **32**, 175003 (2015), doi:10.1088/0264-9381/32/17/175003.

[3] S. C. Fletcher, *Light clocks and the clock hypothesis*, Found. Phys. **43**, 1369 (2013), doi:10.1007/s10701-013-9751-3.

[4] A. Staruszkiewicz, *Fundamental relativistic rotator*, Acta Phys. Pol. B Proc. Suppl. **1**, 109 (2008).

[5] Ł. Bratek, *Fundamental relativistic rotator: Hessian singularity and the issue of the minimal interaction with electromagnetic field*, J. Phys. A: Math. Theor. **44**, 195204 (2011), doi:10.1088/1751-8113/44/19/195204.

[6] V. Kassandrov, N. Markova, G. Schäfer and A. Wipf, *On a model of a classical relativistic particle of constant and universal mass and spin*, J. Phys. A: Math. Theor. **42**, 315204 (2009), doi:10.1088/1751-8113/42/31/315204.

[7] Ł. Bratek, *Relativistic ideal clock*, (arXiv preprint) doi:10.48550/arXiv.1511.00419.

[8] S. M. Kuzenko, S. L. Lyakhovich and A. Y. Segal, *A geometric model of the arbitrary spin massive particle*, Int. J. Mod. Phys. A **10**, 1529 (1995), doi:10.1142/S0217751X95000735.

[9] Ł. Bratek, *Can rapidity become a gauge variable? Dirac Hamiltonian method and relativistic rotators*, J. Phys. A: Math. Theor. **45**, 055204 (2012), doi:10.1088/1751-8113/45/5/055204.

[10] P. A. M. Dirac, *Generalized Hamiltonian dynamics*, Can. J. Math. **2**, 129 (1950), doi:10.4153/CJM-1950-012-1.

[11] S. Das and S. Ghosh, *Relativistic spinning particle in a noncommutative extended spacetime*, Phys. Rev. D **80**, 085009 (2009), doi:10.1103/PhysRevD.80.085009.

[12] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, *Acceleration-extended Galilean symmetries with central charges and their dynamical realizations*, Phys. Lett. B **650**, 203 (2007), doi:10.1016/j.physletb.2007.04.058.

[13] G. Breit, *An Interpretation of Dirac's theory of the electron*, Proc. Natl. Acad. Sci. **14**, 553 (1928), doi:10.1073/pnas.14.7.553.

Segal's contractions, AdS and conformal groups

Daniel Sternheimer⋆∘

Department of Mathematics, Rikkyo University, Tokyo, Japan
Institut de Mathématiques de Bourgogne, Dijon, France

⋆ Daniel.Sternheimer@u-bourgogne.fr

∘ Honorary Professor, St.Petersburg State University, and
Board of Governors Member, Ben Gurion University of the Negev, Israel

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Symmetries and their applications always played an important role in I.E. Segal's work. I shall exemplify this, starting with his correct proof (at the Lie group level) of what physicists call the "O'Raifeartaigh theorem", continuing with his incidental introduction in 1951 of the (1953) Inönü–Wigner contractions, of which the passage from AdS (SO(2,3)) to Poincaré is an important example, and with his interest in conformal groups in the latter part of last century. Since the 60s Flato and I had many fruitful interactions with him around these topics. In a last section I succintly relate these interests in symmetries with several of ours, especially elementary particles symmetries and deformation quantization, and with an ongoing program combining both.

Received 17-11-2022
Accepted 28-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.043

Contents

1 Prologue **2**

2 Some history, anecdotes and background material **2**
2.1 First interactions with Segal 2
2.2 ICM 1966 and around 3
2.3 Later interactions. 4

3 Contractions, conformal group & covariant equations **5**
3.1 A tachyonic survey of contractions and related notions 5
3.2 Conformal groups and conformally covariant equations 6
3.3 Relativistically Covariant Equations 7

4 AdS, AdS/CFT, deformation quantization & perspectives **8**
4.1 Classical limit, deformation quantization and avatars 8
4.2 AdS, AdS/CFT and particle physics 8

5 Conclusion **9**

References **10**

1 Prologue

In July 2018 a special session dedicated to Irving Ezra Segal (13 September 1918 – 30 August, 1998) was organized during the first day of the 32nd International Colloquium on Group Theoretical Methods in Physics (Group32) that was held at Czech Technical University in Prague, Czech Republic, from Monday 9th July until Friday 13th July 2018. It was meant to be a homage to this immense scientist on the occasion of the centenary of his birth and became a commemoration. In the Notices of the American Mathematical Society [33] were published contributions concerning the life and work of I. E. Segal by a number of leading scientists, part of whom are/were not with us anymore [Baez, John C.; Beschler, Edwin F.; Gross, Leonard; Kostant, Bertram; Nelson, Edward; Vergne, Michèle; Wightman, Arthur S.]. Among these I will only quote what Edward Nelson (1932 - 2014) said of him (p. 661): *It is rare for a mathematician to produce a life work that at the time can be fully and confidently evaluated by no one, but the full impact of the work of Irving Ezra Segal will become known only to future generations.*

The text of my invited talk in that special session was sent by me in December 2018 to the organizers, and sent by them to IOP in early 2019, together with all other contributions. In April 2019 the editors informed me that the Proceedings of Group32 were published, with a link that recently changed to https://iopscience.iop.org/issue/1742-6596/1194/1 (IOP publishes a very large number of conference proceedings, mostly in physics.) In December 2021, looking for a more precise reference, I was surprised not to find there my contribution. Apparently someone at IOP "forgot" to include my contribution, *without informing me nor the organizers of the fact*. The organizers of Group34 (in Strasbourg) very kindly agreed to include my text, which as the reader can see deals indeed with "Group Theoretical Methods in Physics", in the Proceedings of Group34. The sections of the following text are essentially my original (December 2018) contribution to Group32.

2 Some history, anecdotes and background material

2.1 First interactions with Segal

The first interactions we (Moshe Flato and I) had with I.E. Segal were probably on the occasion of the controversy that arose in 1965 around what physicists still call "the O'Raifeartaigh theorem". Indeed in 1965 Moshe and I submitted to the Physical Review Letters (PRL) a contribution [19] criticizing that of Lochlainn O'Raifeartaigh, published there the same year [26]. In the latter paper was "proved" that the so-called "internal" (unitary) and external (Poincaré) symmetries of elementary particles can be combined only by direct product. In our rebuttal Moshe insisted that we write that the proof (of O'Raifeartaigh, who by the way became a good friend after we met) was "lacking mathematical rigor," a qualification which incidentally (especially at that time) many physicists might consider as a compliment. Our formulation was deliberately provocative, because Moshe felt that we were criticizing a "result" which, for a variety of reasons, many in the "main stream" wanted to be true.

Remark. The "theorem" of O'Raifeartaigh was formulated at the Lie algebra level, where the proof is not correct because it implicitly assumes that there is a common domain of analytic vectors for all the generators of an algebra containing both symmetries. In fact, as it was formulated, the result is even wrong, as we exemplified later with counterexamples. The result was proved shortly afterward by Res Jost and, independently, by Irving Segal [30] but only in the more limited context of unitary representations of Lie groups. In those days "elementary particle spectroscopy" was performed mimicking what had been done in atomic and molecular spectroscopy, where one uses a unitary group of symmetries of the (known) forces. As a student of Racah, Moshe mastered these techniques. The latter approach was extended somehow to nuclear physics, then to particle physics. That is how, to distinguish between neutrons and protons, Heisenberg introduced in 1932 "isospin," with SU(2) symmetry. When "strange" particles were discovered in the 50s, it became natural to try and use as "internal" symmetry a rank-2 compact Lie group. In early 1961 Fronsdal and Ben Lee, with Behrends and Dreitlein, all present then at UPenn, studied all of these. At the same time Salam asked his PhD student Ne'eman to study only SU(3), in what was then coined "the Eight-Fold way" by Gell'Mann because its eight-dimensional adjoint representation could be associated with mesons of spin 0 and 1, and baryons of spin $\frac{1}{2}$. Since spin is a property associated with the "external" Poincaré group, it was simpler to assume that the two are related by direct product. Hence the interest in the "O'Raifeartaigh theorem." For this and much more see e.g. Section 2 in [35] and references therein. The Editors of PRL objected to our formulation. In line with the famous Einstein quote ("The important thing is not to stop questioning, curiosity has its reason for existing.") Moshe insisted on keeping it "as is." The matter went up to the President of the American Physical Society, who at that time was Felix Bloch, who consulted his close friend Isidor Rabi. [In short, Rabi discovered NMR, which is at the base of MRI, due to Bloch.] Rabi naturally asked who is insisting that much. When he learned that it was Moshe, who a few years earlier, when Rabi was giving a trimester course in Jerusalem, kept asking hard questions which he often could not answer, he said: "If he insists he must have good reasons for it. Do as he wishes." The Editors of PRL followed his advice.

2.2 ICM 1966 and around

The following year (in April 1966), at a conference in Gif-sur-Yvette on "the extension of the Poincaré group to the internal symmetries of elementary particles" which Moshe (then 29) naturally co-organized, Christian Fronsdal told Moshe: "You wrote that impolite paper." This was the beginning of a long friendship, which lasts to this day and is at the origin of important scientific works, many of which deal with applications of group theory in physics, and related issues on quantization.

Shortly afterward (16–26 August 1966) an important mathematical event happened: the International Congress of Mathematicians (where the acronym ICM came into wide usage). Until then the scientific exchanges between the USSR and "the West" had been very limited. A record number of mathematicians attended (4,282 according to official statistics), of which 1,479 came from the USSR, 672 from other "Socialist countries" in Europe, while over 1,200 came from "Western countries", including 280 from France: Moshe (then still only citizen of Israel but working in France since October 1963) and I were among the latter. Irving Segal came from the US. I remember that we and many from the French delegation traveled to Moscow on a Tupolev plane, organized as in a train with compartments seating eight. In Moscow, we were accommodated, together with many "ordinary" participants (some VIPs, among them Segal if I remember correctly, were accommodated in "smaller" hotels closer to the Kremlin), in the huge hotel Ukraina (opened in 1957, the largest hotel in Europe), one of the seven Stalinist skyscrapers in Moscow with a height of 206 meters (including the spire, 73 meters long) and total floor area ca. 88,000m^2, a small city in itself. On every floor there were "etazhniks" supposed to help but in fact checking on the guests. (We encountered the same system 10 years later in Taipei...).

The opening ceremony of ICM 1966, as it is now known, was held in the Kremlin. During the long party which followed we met, and instantly became friends with, many leading Soviet scientists like Nikolay Nikolayevich Bogolyubov (who invited us to Dubna after the Congress), Ludwig Dmitrievich Faddeev and Israel Moiseevich Gelfand. There were also a number of cultural events, both official and optional. I remember that, near a centrally located hotel, we (Moshe and I) and Segal were looking for a taxi in order to get to one such event. Segal entered in a random run, trying to get one, to no avail. Moshe, who spoke fluent Russian (though he could not read nor write it), calmly managed with the doorman of the hotel to get one for the three of us.

At the end of 1966 we made our first visit to the US, starting with Princeton at the invitation of E.P. Wigner with whom Moshe (being a student of Racah) had established connection. We also went to Brookhaven National Laboratory (where my cousin Rudolph Sternheimer spent most of his career, and where the editor of PRL, Sam Goudsmith, was located). Very generously Segal invited us to MIT, then and during our following visits (in 1969 and later) and accommodated us in the Sheraton Commander near Harvard square.

We had many subjects of common interest, mostly around group theory in relation with physics (for some, see below). On the anecdotical side, at some point during our second visit to US (in 1969, with Jacques Simon) the discussion came around Fock space [22]. Segal insisted that it should be called "Fock – Cook space" because his student Joseph M.Cook made it more precise in his 1951 Thesis in Chicago (ProQuest Dissertations Publishing, 1951. T-01196). Not surprisingly that unusual terminology did not catch. [F.J. Dyson wrote in Mathematical Reviews, about the announcement in PNAS: "The author has set up a mathematically precise and rigorous formulation of the theory of a linear quantized field, avoiding the use of singular functions. The formalism is equivalent to the usual one, only it is more carefully constructed, so that every operator is a well-defined Hilbert space operator and every equation has an unambiguous meaning. Fields obeying either Fermi or Bose statistics are included."]

2.3 Later interactions.

An anecdotical event, among our later interactions, also related to the Soviet Union (of the latter days) is the following. Our first visit to Leningrad (also technically the last one, because our subsequent visits were to St. Petersburg ...) occured in the Spring of 1989. We (Moshe and I) were accommodated in a recent Finnish-built hotel, not very fancy and at the entrance of the city when coming from the airport, but functional. Alain Connes, who visited at the same time, had a nice room in a top floor of Hotel Evropeiskaya (now back to its original splendor and named Grand Hotel Europe); however at the time the hotel was rather run down and e.g. water reached his room only a few hours per day!

If I remember correctly it was then that we met in the USSR another visitor, Irving Segal, who introduced us to his second wife Martha Fox, whom he had married in 1985. [His first wife, Osa Skotting, had left him at some point (some said for another woman) and not long afterward remarried in 1986 with an old flame of her, Saunders Mac Lane, whom we met on many occasions in the University of Chicago, always dressed in tartan trousers (the MacLean tartan, of course).]

Interestingly we met both Irving and Martha shortly thereafter at a workshop in Varna, where we were all accommodated in a nice "rest house" for the "nomenklatura" of the Bulgarian Communist party. There we met also for the first time Vladimir Drinfeld and a number of other "Eastern bloc" mathematicians. We were warned by our Bulgarian friend Ivan Todorov (who, as physicist and Academician, had access to information that was not widely publicized) to be careful when drinking wine, because after the Chernobyl disaster in April 1986, many agricultural products (including mushrooms and especially wines) produced in the following months and years were contaminated. [At that time French authorities claimed that the Chernobyl radioactive cloud did not cross the Rhine, which of course nobody believed.] In any case there were still enough older wines in Bulgaria for us to enjoy in the evenings, and we did, including with Martha who liked the company of these (then younger) scientists. One evening Irving spent some time in scientific discussions with a distinguished colleague after which, seeing us in the lobby, he said: "Martha you are tired, please come". Martha denied being tired but then Irving insisted:"Martha you are tired, and besides you have some duties to perform." At this point Joe Wolf quipped: "Not here I hope!" She had to leave. I told the story later to some friends in the US, and one of them remarked: "It must have worked because recently at MIT Segal has been distributing cigars on the occasion of the birth of their daughter Miriam."

3 Contractions, conformal group & covariant equations

3.1 A tachyonic survey of contractions and related notions

In 1951, in a side remark at the end of an article [28], Segal introduced the notion of contractions of Lie algebras, that was "introduced in physics" in a more explicit form two years later [25] by Eugene Wigner and Erdal Inönü. [The latter was the son of Ismet Inönü who in 1938 succeeded Ataturk as president of Turkey. Eventually Erdal (1926 – 2007) had a political career, becoming himself interim Prime Minister in 1993.] The notion has been studied and generalized by a number of people. For an informative more recent paper, see e.g. [36]. (I had something to do with its publication.)

The 1951 paper by Segal was analyzed in Mathematical Reviews by Roger Godememt. It included some nasty remarks (something rare in the Reviews but not infrequent with Godement), in particular, after giving a number of simpler proofs of a few results, Godement wrote: "Tout cela est très facile. L'article se termine par quelques exemples inspirés de problèmes physiques, à propos desquels l'auteur émet des opinions et suggestions dont la discussion demanderait des connaissances cosmologiques et métaphysiques que le rapporteur n'a malheureusement pas eu le temps d'acquérir." [All this is very easy. The paper ends with some examples inspired by physical problems; in connection with these the author expresses opinions and suggestions, the discussion of which would require cosmological and metaphysical knowlegde which the reviewer unfortunately did not have the time to obtain.] These examples include the notion of contractions of Lie algebras, and more!

In a nutshell, a typical example of contraction consists in multiplying part of the generators of some linear basis of a Lie algebra by a parameter ϵ which then is made 0. In particular, when multiplying the Lorentz boosts of the Lie algebra $\mathfrak{so}(3,1)$ by $\epsilon \to 0$, one obtains the Lie algebra of the Euclidean group $E(3)$ ($\mathfrak{so}(3) \cdot \mathbb{R}^3$).

The notion of contractions of Lie algebras is a kind of inverse of the more precise notion that became known, ten years later, after the seminal paper by Murray Gerstenhaber [23], as deformations of (Lie) algebras. Immediately thereafter it became clear to many, especially in France where Moshe Flato had arrived in 1963, that the symmetry of special relativity (the Poincaré Lie algebra $\mathfrak{so}(3,1) \cdot \mathbb{R}^4$) is a deformation of that of Newtonian mechanics (the Galilean Lie algebra, semi-direct product of $E(3)$ in which the $\mathbb{R}^3$ are velocity translations, and of space-time translations). Or, conversely, that Newtonian mechanics is a contraction (in the sense of Segal), of special relativity.

These notions were extensively discussed at the above-mentioned April 1966 conference in Gif. In other words, special relativity can be viewed, from the symmetries viewpoint, as a deformation. A natural question, which already then arose in the mind of Moshe, was then to ask whether quantum mechanics, the other major physical discovery of the first half of last century, can also be viewed as a deformation. It was more or less felt, because of the notion of "classical limit" and though in this case we deal with an infinite dimensional Lie algebra, that classical mechanics is a kind of contraction (when $\hbar \to 0$) of the more elaborate notion of quantum mechanics. But the inverse operation is far from obvious, if only because in quantum mechanics the bracket is the commutator of operators on some Hilbert space while in classical mechanics we deal with the Poisson bracket of classical observables, functions on some phase space.

At the same time, in 1963/64, I participated in the Cartan–Schwartz seminar at IHP (Institut Henri Poincaré) on the (proof of the) seminal theorem of Atiyah and Singer on the index of elliptic operators, which had just been announced without a proof. My share (2 talks), on the Schwartz side, in the Spring of 1964, was the multiplicative property of the analytic index, crucial for achieving dimensional reduction that was an important ingredient of the proof. Moshe followed that important seminar. But it was only more than a dozen years later, after we developed what became known as deformation quantization, that we realized that the composition of symbols of differential operators was a deformation of the commutative product of functions (what we called a star-product), or conversely that the commutative product is a contraction of our star-product.

3.2 Conformal groups and conformally covariant equations

The conformal group (here, the Lie group $SO(4,2)$ or a covering of it) was introduced ten years before Segal was born as a symmetry of Maxwell equations by Harry Bateman (in 1908 and 1910) [3, 4] and Ebenezer Cunningham (in 1910) [9]. In 1936 (a year after an article, also in Ann. Math., in which he studied the extension of the electron wave equation to de Sitter spaces) Dirac [11] made this fact more precise. Yet not many realized the fact, possibly because in addition to the Poincaré group (a group of linear transformations of space-time) there were 4 generators of "inversions", nonlinear transformations. For a long time many (including textbooks authors and some colleagues physicists of Moshe in Dijon) were convinced that the Poincaré group is the most general group of invariance of special relativity.

We were introduced to this group in 1965 by Roger Penrose during a visit to David Bohm at Birbeck College of the university of London. We were impressed by Roger Penrose. When we told that to André Lichnerowicz he remarked: "Half of what he says is true." That half proved to be seminal and worth a Nobel prize. Inasmuch as the conformal group is concerned we made immediate use of it in a number of papers in a variety of contexts [7, 20].

In December 1969 Moshe was visiting KTH (the Royal Institute of Technology) in Stockholm, at the time when Gell'mann gave there the traditional scientific lecture on the occasion of his Nobel prize. For an unknown reason, he chose to center his talk on the conformal group, which by then we knew very well, in particular because we had studied in detail the conformal covariance of field equations [17]. Moshe (then 32) did not hesitate to interrupt him a few times, asking from the back of the auditorium (im)pertinent questions to which Gell'Mann's only answer was: "Good question." Eventually Moshe said that he did not ask for marks for his questions, but would like answers. At that point Gell'Mann, who is known to be very fast, remarked: "I didn't know there would be specialists in the audience." Then Moshe, who was even faster, replied: "Until now you insulted only me, now you are insulting the Nobel Committee, who is sitting here [in the first row]." That is not a good way to make friends. After the lecture half of the Committee members, instead of joining Gell'Mann for a lunch at the US Embassy, joined Moshe for a (better) lunch at the French Embassy, in honor of Samuel Beckett who that year was the Nobel laureate in Literature (but had sent his publisher to collect the Prize, prefering to remain in sunny Tunisia with his young companions).

For some more information on Moshe Flato (who coincidentally was born September 17, 4 days and 19 years after Segal, and, like Segal, died in 1998, almost 3 months after Segal) see e.g. [14, 15].

In addition to the above mentioned papers, we published a few other papers around the conformal group. Moshe had discussed the issue with Segal, who at first didn't seem interested in the idea. But the question apparently remained in his mind and not long afterward he dealt with the conformal group from a different point of view. In his first of many publications on the subject [31] (reviewed by Victor Guillemin) and [32] appears the universal covering space of the conformal compactification of Minkowski space, in connection with a simple explanation for the "red shift" observed by astronomers in studying quasars. At the time, though he had many more important contributions (albeit mostly of mathematical nature) Segal was very proud of his explanation, in spite of the fact that many astronomers were critical, because while his explanation worked well for some galaxies, it did not work so well for others, which Segal did not consider. That is one more example of what Sir Michael Atiyah said at the International Congress of Mathematical Physics in London in 2000 (his contribution there was published in [2]): "Mathematicians and physicists are two communities separated by a common language." This refers to a famous saying which is often attributed to George Bernard Shaw but seems to date back to Oscar Wilde (in "The Canterville Ghost", 1887): "We have really everything in common with America nowadays, except, of course, language."

3.3 Relativistically Covariant Equations

In 1960 Segal published an important paper [29] in the first volume of the Journal of Mathematical Physics, extensively (often with personal remarks, e.g. in relation with QED) analyzed in Mathematical Reviews by Arthur Wightman, in a review much longer than the abstract of the paper. Among many interesting ideas, that were further developed in subsequent papers, appears there a Poisson bracket on the (infinite dimensional) space of initial conditions for the Klein Gordon equation.

Our interest in that structure was triggered by the approach we made, from the 70s to the 90s and in parallel with deformation quantization (see below), of many nonlinear evolution equations of physics, as covariant under a kind of deformation of the symmetry of linear (free) part. [That approach has not yet attracted enough attention from specialists of these PDEs and ODEs, possibly because the tools used, involving e.g. group representations and their cohomologies, which serve as a basis for a careful analytical study of such equations, are foreign to PDEs specialists.] It culminated in the "tour de force" extensive [18] study of CED (classical electrodynamics), namely "Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations" (see also references therein and a few later developments), where it is explained in detail.

Very appropriately it is dedicated to the memory of Julian Schwinger, "the chief creator of QED" (certainly in its analytical form). Indeed a rigorous passage from CED to QED, from the point of view of deformation quantization, will require a Hamiltonian structure on the space of initial conditions for CED, of the kind introduced by Segal, the quantized fields being considered as functionals on that space. That is one more example of how our works were, and should be in the future, intimately intertwined with those of Segal.

4 AdS, AdS/CFT, deformation quantization & perspectives

In this section I succintly present my ongoing research on the convergence of topics which we were concerned with in the 60s, around symmetries of elementary particles, with later works (from the 70s) on the essence of quantization and around conformal groups. The former are closely related to our above mentioned first interaction with Segal and the latter to applications in physics of mathematical tools he developed, in relation with both quantization and conformal groups. An early presentation can be found e.g. in [35].

4.1 Classical limit, deformation quantization and avatars

Since that part is developed in numerous reviews, by many, I shall here give only an ultra-short presentation, starting with the connection with Segal's contractions. Indeed, the fact that classical mechanics is, in a sense, a "contraction" of quantum mechanics, was essentially known to many, one of the first being Dirac [10], and has been expressed precisely e.g. by Hepp in [24]. Quite naturally the idea that quantization should be some kind of a deformation was "in the back of the mind" of many, but how to express that precisely was far from obvious. After [5, 6] appeared one of them demanded from André Lichnerowicz to be quoted for the idea, but André did not know how we could include "the back of the mind" of that person in our list of references! [Incidentally our 1977 UCLA preprint of [5, 6] was sent to Annals of Physics by Schwinger, who had published in 1960 [27] a short paper which turned out to be related to it.]

That it should be possible to formulate such an idea in a mathematically precise way was implicitly felt by Dirac in [12], where he went on by developing his approach to quantization of constrained systems (in geometrical language, coupled second class constraints reduces $\mathbb{R}^{2n}$ phase space to a symplectic submanifold, and first class constraints reduce it further to what we called a Poisson manifold):

...One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines...

That is the path we followed in our foundational papers [5, 6] that are extensively quoted, directly and even more implicitly. The notion became a classic, and constitutes an item in the Mathematics Subject Classification. For a detailed review see e.g. [13]. The even more developed notions of quantum groups and of noncommutative geometry, which had different origins, appeared essentially shortly afterward and may be considered as avatars. For this and more, see e.g. [34, 35].

4.2 AdS, AdS/CFT and particle physics

As is well known, the Anti de Sitter group AdS_4, $SO(2,3)$ (or a covering of it), can be viewed either as the conformal group of a 2+1 dimensional flat space-time, or a deformation (with negative curvature) of the Poincaré group of usual Minkowski (3+1) dimensional space-time, the latter being a "Segal" contraction of AdS_4. That has many physical consequences, including for particle physics, which have been studied by many authors, especially since the 70s. See e.g. [1] (which has been quoted by Witten as an early instance of the AdS/CFT correspondence) and references therein, where e.g. is shown how AdS representations contract to the Poincaré group, and many later papers by us and others.

The two massless representations of the Poincaré group in 2+1 dimensions have a unique extension to representations of its conformal group AdS_4 (that feature exists in any higher dimension). The latter were discovered in 1963 by Dirac and called by him "singletons". We called them "Rac" for the scalar perticle (because it has only one component, "Rac" means only in Hebrew) and "Di" for the helicity $\frac{1}{2}$ one (which has 2 components), on the pattern of Dirac's "bra" and "ket".

Among the many applications to particle physics, including with conformal symmetry, one should mention that the photon can be considered as *dynamically* composed of two Racs, in a way compatible with QED [16], and that the leptons can also be considered as composites of singletons [21], in a way generalizing the electroweak unification theory. Thus, in the same way as (special) relativity and quantum mechanics can be considered as deformations, deforming Minkowski space to Anti de Sitter (with a tiny negative curvature) can explain photons and leptons as composites. A natural question is how to extend that to the heavier hadrons.

The approach I am advocating ([35] and work in progress, in particular a Springer Brief in Mathematical Physics with Milen Yakimov), based on the strong belief that one passes from one level of physical theories to another by a deformation in some category, is to deform the symmetry one step further, to some quantized Anti de Sitter (qAdS), possibly with multiple parameters (commuting so far, since one does not know yet how to do treat deformations with noncommutative parameters, e.g. quaternions), and even at roots of unity since the Hopf algebra of quantum groups at roots of unity is finite dimensional. Maybe one could then find the "internal symmetries" as symmetries of the deformation "parameters", putting on solid ground that "colossus with clay feet" called the Standard Model. Vast programme, as could have said de Gaulle. Interestingly the development of the required mathematics (of independent interest) would be related to a number of Segal's works.

5 Conclusion

The above scientific and anecdotical samples show how, in spite of being a generation apart, our lives have been "intertwined" for over 30 years with that of I.E. Segal. The use of symmetries in physics have been a kind of watermark throughout our works, beyond their apparent diversity. A special mention is due to the conformal group (of Minkowski space-time) which has played an important role throughout the works of I.E. Segal, in particular in his late cosmological applications. In this century, very modestly, I have been trying to develop (unconventional in a different way) consequences in that direction.

Acknowledgements

First I want to thank the organizers of Group32, in particular Patrick Moylan who managed the special session dedicated to I.E. Segal, for inviting me to it. Very special thanks are due to the organizers of 34$^{\text{th}}$ International Colloquium on Group Theoretical Methods in Physics (Group34), in particular to the Chair of its Organizing Committee Michel Rausch de Traubenberg, for kindly giving "hospitality" to my contribution in honor of I.E. Segal, among those of

Group34, which are being published with SciPost. [They also invited me to attend the meeting in Strasbourg, which I could not accept, inter alia, because of the last wave of Covid-19.] And to Rutwig Campoamor-Stursberg, a distinguished member of the Group34 Organizing Committee, for the “TeXploit” of adapting my plain LaTeX contribution to the (unfriendly for me) requirements of the SciPost style, including concerning the bibliography.

References

[1] E. Angelopoulos, M. Flato, C. Fronsdal and D. Sternheimer, *Massless particles, conformal group, and de Sitter universe*, Phys. Rev. D **23**, 1278 (1981), doi:10.1103/PhysRevD.23.1278.

[2] M. Atiyah, *On the unreasonable effectiveness of physics in mathematics*, in *Highlights of mathematical physics*, American Mathematical Society, Providence, USA, ISBN 9780821832233 (2000).

[3] H. Bateman, *The conformal transformations of a space of four dimensions and their applications to geometrical optics*, Proc. Lond. Math. Soc. **7**, 70 (1909), doi:10.1112/plms/s2-7.1.70.

[4] H. Bateman, *The transformation of the electrodynamical equations*, Proc. Lond. Math. Soc. **8**, 223 (1910), doi:10.1112/plms/s2-8.1.223.

[5] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Phys. **111**, 61 (1978), doi:10.1016/0003-4916(78)90224-5.

[6] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. II. Physical applications*, Ann. Phys. **111**, 111 (1978), doi:10.1016/0003-4916(78)90225-7.

[7] D. Bohm, M. Flato, D. Sternheimer and J. P. Vigier, *Conformal-group symmetry of elementary particles*, Nuovo Cimento **38**, 1941 (1965), doi:10.1007/BF02750118.

[8] J. M. Cook, *The mathematics of second quantization*, Proc. Natl. Acad. Sci. **37**, 417 (1951), doi:10.1073/pnas.37.7.417.

[9] E. Cunningham, *The principle of relativity in electrodynamics and an extension thereof*, Proc. Lond. Math. Soc. **8**, 77 (1910), doi:10.1112/plms/s2-8.1.77.

[10] P. A. M. Dirac, *The Lagrangian in quantum mechanics*, Phys. Z. Sowjetunion **3**, 64 (1933).

[11] P. A. M. Dirac, *Wave equations in conformal space*, Ann. Math. **37**, 429 (1936), doi:10.2307/1968455.

[12] P. A. M. Dirac, *The relation of classical to quantum mechanics*, in *Proceedings of the second Canadian mathematical congress*, Vancouver, Canada (1951).

[13] G. Dito and D. Sternheimer, *Deformation quantization: Genesis, developments and metamorphoses*, in *Deformation quantization*, de Gruyter, Berlin, Germany, ISBN 9783110172478 (2002), doi:10.1515/9783110866223.9.

[14] *Moshé Flato, the man and the scientist*, in G. Dito and D. Sternheimer (eds.), *Conférence Moshé Flato 1999*, Vol. 1, p. 3-54, Kluwer Academic Publishing, Dordrecht, Netherlands, ISBN 9780792365402 (2000).

[15] *Moshé Flato — Personal recollections*, Lett. Math. Phys. **48**, 5 (1999), doi:10.1023/A:1017293905798.

[16] M. Flato and C. Fronsdal, *Composite electrodynamics*, J. Geom. Phys. **5**, 37 (1988), doi:10.1016/0393-0440(88)90013-7.

[17] M. Flato, J. Simon and D. Sternheimer, *Conformal covariance of field equations*, Ann. Phys. **61**, 78 (1970), doi:10.1016/0003-4916(70)90377-5.

[18] M. Flato, J. C. H. Simon and E. Taflin, *Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations*, Mem. Amer. Math. Soc. **127**, 606 (1997), doi:10.1090/memo/0606.

[19] M. Flato and D. Sternheimer, *Remarks on the connection between external and internal symmetries*, Phys. Rev. Lett. **15**, 934 (1965), doi:10.1103/PhysRevLett.15.934.

[20] M. Flato and D. Sternheimer, *Remarques sur les automorphismes causals de l'espace-temps*, C. R. Acad. Sci. Paris Sér. A-B **263**, A953 (1966).

[21] C. Frønsdal, *Singletons and neutrinos*, Lett. Math. Phys. **52**, 51 (2000), doi:10.1023/A:1007693518414.

[22] V. Fock, *Konfigurationsraum und zweite Quantelung*, Z. Phys. **75**, 622 (1932), doi:10.1007/BF01344458.

[23] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. **79**, 59 (1964), doi:10.2307/1970484.

[24] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Commun. Math. Phys. **35**, 265 (1974), doi:10.1007/BF01646348.

[25] E. Inonu and E. P. Wigner, *On the contraction of groups and their representations*, Proc. Natl. Acad. Sci. **39**, 510 (1953), doi:10.1073/pnas.39.6.510.

[26] L. O'Raifeartaigh, *Mass differences and Lie algebras of finite order*, Phys. Rev. Lett. **14**, 575 (1965), doi:10.1103/PhysRevLett.14.575.

[27] J. Schwinger, *The special canonical group*, Proc. Natl. Acad. Sci. **46**, 1401 (1960), doi:10.1073/pnas.46.10.1401.

[28] I. E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. **18**, 221 (1951), doi:10.1215/S0012-7094-51-01817-0.

[29] I. E. Segal, *Quantization of nonlinear systems*, J. Math. Phys. **1**, 468 (1960), doi:10.1063/1.1703683.

[30] I. Segal, *An extension of a theorem of L. O'Raifeartaigh*, J. Funct. Anal. **1**, 1 (1967), doi:10.1016/0022-1236(67)90023-7.

[31] I. Segal, *Causally oriented manifolds and groups*, Bull. Amer. Math. Soc. **77**, 958 (1971), doi:10.1090/S0002-9904-1971-12815-X.

[32] I. E. Segal, *Theoretical foundations of the chronometric cosmology*, Proc. Natl. Acad. Sci. **73**, 669 (1976), doi:10.1073/pnas.73.3.669.

[33] J. C. Baez, E. F. Beschler, L. Gross, B. Kostant, E. Nelson, M. Vergne, and A. S. Wightman, *Irving Ezra Segal*, Not. Am. Math. Soc. **46**, 659 (1999), https://www.ams.org/notices/199906/mem-segal.pdf.

[34] D. Sternheimer, *The deformation philosophy, quantization and noncommutative space-time structures*, in *Higher structures in geometry and physics*, Birkhäuser, Boston, USA, ISBN 9780817647346 (2010), doi:10.1007/978-0-8176-4735-3_3.

[35] D. Sternheimer, *"The important thing is not to stop questioning", including the symmetries on which is based the Standard Model*, in *Geometric methods in physics*, Birkhäuser, Cham, Switzerland, ISBN 9783319062471 (2014), doi:10.1007/978-3-319-06248-8_2.

[36] E. Weimar-Woods, *Contractions, generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. **12**, 1505 (2000), doi:10.1142/S0129055X00000605.

SciPost Phys. Proc. 14, 044 (2023)

Towards higher super-σ-model categories

Rafał R. Suszek⋆

Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw

⋆ suszek@fuw.edu.pl

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

A simplicial framework for the gerbe-theoretic modelling of supercharged-loop dynamics in the presence of worldsheet defects is discussed whose equivariantisation with respect to global supersymmetries of the bulk theory and subsequent orbit decomposition lead to a natural stratification of and cohomological superselection rules for target-space supergeometry, expected to encode essential information on the quantised 2*d* field theory. A physically relevant example is analysed in considerable detail.

Received 03-01-2023
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.044

---

1 Introduction

Models of geometrodynamics of extended distributions of (super)charge, known as nonlinear σ-models with topological Wess–Zumino terms, remain an active field of physical and mathematical research, not least because of their applicability in the description of a wide range of dynamical systems, from the critical field theory of collective excitations of quantum spin chains all the way to classical superstring theory, and because of the great wealth of the underlying mathematics. The hierarchical higher-geometric structures – a.k.a. p-gerbes and their morphisms – associated with background $(p+2)$-form fields in these models, which couple to the worldvolume (super)charge current through distinguished Cheeger–Simons differential characters generalising the standard line holonomy (as derived for $p=1$ in [1–4]), have long been known not only to give rise to a canonical prequantisation of the models, as in [1, 5], and to provide a natural cohomological classification of the models themselves, their boundary conditions and generic defects, but also – to lead to a categorification of their prequantisable group-theoretic symmetries, including the gauged ones, and more general dualities (*e.g.*, T-duality). These correspondences can oftentimes be implemented by certain defect networks embedded in the σ-model spacetime, to which sheaf-theoretic data of $k>0$-cells of the weak $(p+1)$-categories with p-gerbes as 0-cells are pulled back along the σ-model field in order to render the Dirac–Feynman amplitude of the σ-model with defects well-defined, *cp.* [4–7]. In particular, the distinguished *topological* defects, studied at some length in [3–6, 8], enter the generalised world-sheet orbifold construction of [4,8] of the σ-model on the space M/G_σ of orbits of an action of a rigid-symmetry group G_σ on the target space M of the original σ-model, in which they serve to model the so-called G_σ-twisted sector of the orbifold theory, instrumental in the gauging of G_σ. Finally, in some highly symmetric settings, such as, *e.g.*, that of the Wess–Zumino–Witten (WZW) σ-model, there are defects realising the *generating* symmetries of the σ-model [3,4,6,9] which encode highly nontrivial information on the ensuing quantum field theory, such as, *e.g.*, its fusion rules and Moore–Seiberg data. Therefore, it is clear that understanding the structure of the fully fledged higher categories behind the topological couplings of poly-phase σ-models, and constructing concrete examples of their k-cells, is a goal of fundamental relevance to the study of these important field theories.

This note gives a concise account of a proposal, advanced in [9] on the basis of the earlier studies [4–7], for an effective structurisation of the stratified target spaces and the higher-geometric objects over them defining the 2d σ-model in the presence of defects compatible with configurational symmetries of the bulk field theory. The proposal is formulated in the most general (target-space) $\mathbb{Z}/2\mathbb{Z}$-graded setting of the Green–Schwarz-type super-σ-model and uses mixed group-theoretic, simplicial and cohomological tools. It paves the way to a systematic construction of maximally supersymmetric defects in the flat $\mathbb{Z}/2\mathbb{Z}$-graded target geometry, and leads to interesting novel predictions for a higher-geometric target-space realisation of the non-perturbative data of the bulk theory in the highly (super)symmetric setting of the WZW(-type) models with Lie-(super)group targets.

2 A bicategory for the super-σ-model with defects

The 2d super-σ-model is a lagrangean theory of (super)fields from the mapping supermanifold $[\Sigma, M] = \underline{\mathrm{Hom}}_{\mathbf{sMan}}(\Sigma, M)$ defined for a closed oriented (Graßmann-)even manifold Σ (the 'spacetime' of the model) and a supermanifold M. For M even, $[\Sigma, M]$ is represented by $C^\infty(\Sigma, M)$ and we obtain a theory of embeddings shaped by an interplay between forces sourced by two tensor fields on the target space M: a metric tensor g and a closed 3-form Kalb–Ramond field χ with periods from $2\pi\mathbb{Z}$. For M properly graded, the inner-Hom functor $[\Sigma, M]$ is to be evaluated on the nested family of superpoints $\mathbb{R}^{0|N}$, $N \in \mathbb{N}^\times$, both tensors are even, and the 'metric' g typically degenerates in the Graßmann-odd directions. In either setting, the 3-form geometrises, in the sense of [10], as a gerbe $\mathcal{G}$ with connective structure of curvature χ. The gerbe trivialises upon pullback to the 2d worldsheet Σ, thereby defining – for $\partial\Sigma = \emptyset$ – the topological Wess–Zumino (WZ) coupling of χ to the charged-loop current in M, given by the surface holonomy [1] of $\mathcal{G}$ along the image of Σ in M.

In the boundary σ-model, with $\partial\Sigma \neq \emptyset$, specifying the gerbe $\mathcal{G}$ alone is *not* sufficient to obtain a consistent field theory as the topological term becomes ill-defined. Instead, a distinguished 1-cell from the bicategory $\mathfrak{BGrb}_\nabla(M)$ of (1-)gerbes over M is required to exist over a submanifold $\iota_D : D \hookrightarrow M$ into which $\partial\Sigma$ is constrained to map, to wit, a trivialisation $\mathcal{T}_D : \iota_D^*\mathcal{G} \cong \mathcal{I}_\omega$ in terms of a trivial gerbe $\mathcal{I}_\omega$ associated with a *global* de Rham primitive $\omega \in \Omega^2(D)$ of χ, *cp.* [2]. The trivialisation is to be pulled back to $\partial\Sigma$ by the σ-model field.

The boundary σ-model can be viewed as a two-phase field theory in which the bulk phase, defined by the triple $(M, \mathrm{g}, \mathcal{G})$ abuts onto the empty phase $(\mathbb{R}^{0|0}, 0, \mathcal{I}_0)$ across the boundary domain wall (or boundary *defect*) $\partial\Sigma$. In this picture, the target space is the *stratified* supermanifold $\widetilde{M} \equiv M \sqcup \mathbb{R}^{0|0}$, and the two limiting field configurations at the defect are modelled by the two mappings $\iota_1 \equiv \iota_D : D \longrightarrow M \subset \widetilde{M}$ and $\iota_2 : D \dashrightarrow \mathbb{R}^{0|0} \subset \widetilde{M}$. This is the point of departure for a far-reaching generalisation, contemplated in [3,4], in which an arbitrary number of phases coexist over 2d patches within Σ, separated by *defect lines* which, in turn, intersect at a *discretuum* of *defect junctions*, graded by their valence. The physical prototype of this situation is the decomposition of a demagnetised ferromagnetic medium into Weiss domains of uniform magnetisation which jumps across domain walls between them, and the most natural application of defects in the field-theoretic setting under consideration is the modelling of the *twisted sector* in the theory with the target space given by the orbispace of an action (not necessarily free or proper) of a symmetry group $\mathrm{G}_\sigma \subset \mathrm{Isom}(M, \mathrm{g})$ of the σ-model for $[\Sigma, M]$ – this orbifold σ-model can be *defined* in terms of (G_σ-classes of) patchwise continuous field configurations in M with G_σ-jump discontinuities localised at an arbitrarily fine mesh of defect lines carrying data of a G_σ-equivariant structure on $\mathcal{G}$ [4,7,11,12].

The above considerations set the stage for a precise definition of the poly-phase super-σ-model: It starts with a splitting of $\Sigma \in \partial^{-1}\emptyset$ by an embedded defect graph $\Gamma \subset \Sigma$ into a family $\{D_i\}_{i\in I}$ of (topologically) closed $2d$ domains D_i which compose the extended worldsheet $\widehat{\Sigma} = \sqcup_{i\in I} D_i$ and intersect at a family $\{L_{(i,j)}\}_{(i,j)\in I_\Gamma \subset I^{\times 2}}$ of closed oriented defect lines $L_{(i,j)}$. The latter form the extended defect graph $\widehat{\Gamma} = \sqcup_{(i,j)\in I_\Gamma} L_{i,j} \subset \widehat{\Sigma} \times_\Sigma \widehat{\Sigma} \equiv \widehat{\Sigma}^{[2]}$ and join transversally at defect junctions whose ensemble $V = \sqcup_{\nu\geq 3} V_\nu$ decomposes into subsets $V_\nu \subset \widehat{\Sigma}^{[\nu]}$ of junctions of a fixed valence ν, further split as $V_\nu = \sqcup_{\varepsilon_\nu \in (\mathbb{Z}/2\mathbb{Z})^{\times\nu}} V_{\varepsilon_\nu}$, $\varepsilon_\nu = (\varepsilon^{(\nu)}_{k,k+1})$, $k \in \overline{1,n}$ into subcomponents V_{ε_ν} with distinct cyclic ($(\nu, \nu+1) \equiv (\nu, 1)$) sequences of *in*-coming ($\varepsilon^{(\nu)}_{k,k+1} = +1$) and *out*-going ($\varepsilon^{(\nu)}_{k,k+1} = -1$) defect lines, *cp.* [4]. There are canonical projections $p_1, p_2 : \widehat{\Gamma} \longrightarrow \widehat{\Sigma}$ (assigning to a given defect line the corresponding components of the boundaries of the two domains separated by it) and $p^{\varepsilon_\nu}_{k,k+1} : V_{\varepsilon_\nu} \longrightarrow \widehat{\Gamma}$ (assigning to a given defect junction the endpoints of the defect lines converging at it, in anti-clockwise cyclic order), satisfying the obvious identities of order 2, *e.g.*, $p_2 \circ p^{(+++)}_{1,2} = p_1 \circ p^{(+++)}_{2,3}$ for two defect lines $L_{1,2}$ and $L_{2,3}$ *in*-coming at $v \in V_{(+++)}$. To the manifolds (with boundary) $\widehat{\Sigma}, \widehat{\Gamma}$ and V_ν, we associate the respective strata M_0, M_1 and $M_{\nu-1} = \sqcup_{\varepsilon_\nu\in(\mathbb{Z}/2\mathbb{Z})^{\times\nu}} T_{\varepsilon_\nu}$ of the target supermanifold (possibly further stratified, as in the boundary example above), which determine a natural decomposition $\mathcal{F}_\sigma \equiv [\widehat{\Sigma}, M_0] \sqcup [\widehat{\Gamma}, M_1] \sqcup \sqcup_{\nu\geq 3} \sqcup_{\varepsilon_\nu\in(\mathbb{Z}/2\mathbb{Z})^{\times\nu}} [V_{\varepsilon_\nu}, T_{\varepsilon_\nu}]$ of the space of σ-model fields. These target strata are endowed with smooth structure maps $\iota_1, \iota_2 : M_1 \longrightarrow M_0$ and $\pi^{\varepsilon_\nu}_{k,k+1} : T_{\varepsilon_\nu} \longrightarrow M_1$ subject to relations of order 2 mirroring those satisfied by their worldsheet counterparts, requisite for the consistency of the field-theoretic framework. The structure maps give rise to a family of pullback operators: $\Delta = \iota_2^* - \iota_1^*$ and $\Delta_{\varepsilon_\nu} = \sum_{k=1}^{\nu} \varepsilon^{(\nu)}_{k,k+1}\, \pi^{\varepsilon_\nu *}_{k,k+1}$ which obey the identities $\Delta_{\varepsilon_\nu} \circ \Delta = 0$ and thus establish a relative-cohomological structure on $M \equiv M_0 \sqcup M_1 \sqcup \sqcup_{\nu\geq 3} \sqcup_{\varepsilon_\nu\in(\mathbb{Z}/2\mathbb{Z})^{\times\nu}} T_{\varepsilon_\nu}$ in which the form $\Omega \equiv (\chi, \omega, 0) \in \Omega^3(M_0) \oplus \Omega^2(M_1) \oplus \Omega^1(M_{\nu\geq 3})$ acquires the status of a relative de Rham 3-cocycle, *cp.* [7, Sec. 7.2], and the defect strata M_1 and T_{ε_ν} play the rôle of correspondence spaces: The former supports a trivialisation of the Δ-image of the 'bulk' gerbe $\mathcal{G}$ on M_0, whereas the latter carries a secondary trivialisation of the Δ_{ε_ν}-image of that primary trivialisation. In order to motivate this result of the in-depth analysis reported in [4], let us consider the special case in which $\iota_{D^\times} : D^\times \equiv (\iota_1, \iota_2)(M_1) \hookrightarrow M_0^{\times 2}$ is an embedding, and the connected component $\mathbb{S}^1 \subset \Gamma$ mapped to M_1 separates diffeomorphic domains D_1 and D_2. Invoking the Wong–Affleck 'folding trick' [13], we may then regard the defect line as a boundary defect in the σ-model on D_1 (onto which D_2 has been 'folded') with the target supermanifold $M_0^{\times 2}$ and the gerbe $\mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G}^*$ (the dualisation of $\mathrm{pr}_2^*\mathcal{G}$ reflects the flip of the orientation accompanying the 'folding', *cp.* [14]), and with the boundary $\mathbb{S}^1$ sent to $D^\times$. The reasoning of [2], referred to previously, now calls for a trivialisation $\mathrm{pr}_1^*\mathcal{G} \otimes \mathrm{pr}_2^*\mathcal{G}^* \cong \mathcal{I}_\omega$, or, equivalently, a so-called gerbe bi-module $\Phi : \iota_1^*\mathcal{G} \cong \iota_2^*\mathcal{G} \otimes \mathcal{I}_\omega$ over M_1. This turns out to be the right structure for an *arbitrary* choice of the correspondence space $(M_1, \iota_\cdot)$, *cp.* [4]. Note in the passing that there is always the distinguished *identity defect*, mapping to the component $M_0 \subset M_1$ with $\iota_A \restriction_{M_0} \equiv \mathrm{id}_{M_0}$ and $\Phi\restriction_{M_0} \equiv \mathrm{id}_{\mathcal{G}}$. There is no equally straightforward and general argument elucidating the gerbe-theoretic structure to be pulled back to the V_{ε_ν}, but we may give a heuristic reasoning which emphasises the main idea behind the rigorous construction while circumnavigating its technicalities, given *ibidem*. In it, we assume all lines to be *in*-coming at υ for the sake of simplicity and drop the sign labels. Thus, whenever $I_{(\nu)} \equiv (\pi_{1,2}, \pi_{2,3}, \ldots, \pi_{\nu-1,\nu}, \pi_{\nu,1})(T_{(++\cdots+)}) \subset M_1^{\times\nu}$ is a submanifold, *and* the value of the action functional of the σ-model is invariant under homotopy moves of the defect within Σ, *i.e.*, when we are dealing with a *topological* defect in a conformally invariant σ-model, we may – upon cutting out a disc centred on a given $\upsilon \in V_{(++\cdots+)}$ whose boundary intersects each line emanating from υ once, and subsequently cutting out from it another disc of radius $r \approx 0$ with the same properties – deform the defect lines $L_{k,k+1}$ on the ensuing annulus in such a way that the angular distance between the nearest neighbours $(L_{k,k+1}, L_{k+1,k+2})$ is $\varepsilon \approx 0 (\approx \nu\varepsilon)$. Now we should be able to read off the sought-after vertex structure from the limit $\varepsilon \searrow 0$ succeeded by $r \searrow 0$. The former leaves us with a single defect line mapped into $M_1^{(\nu)} = \{(q_1, q_2, \ldots, q_\nu) \in M_1^{\times\nu} \mid \iota_2(q_k) = \iota_1(q_{k+1}),\ k \in \overline{1, \nu-1}\}$ and carrying the data of the composite 1-isomorphism $(\Phi_\nu \otimes \mathrm{id}_{\mathcal{I}_{\omega_1+\omega_2+\cdots+\omega_{\nu-1}}}) \circ \cdots \circ (\Phi_3 \otimes \mathrm{id}_{\mathcal{I}_{\omega_1+\omega_2}}) \circ (\Phi_2 \otimes \mathrm{id}_{\mathcal{I}_{\omega_1}}) \circ \Phi_1$ with $(\Phi_k, \omega_k) = \mathrm{pr}_k^*(\Phi, \omega)$, representing the *fusion* of the ν gerbe bi-modules. The latter reproduces the endpoint $\widetilde{\upsilon}$ of the fused defect, or, equivalently, a junction between it and the identity defect stretching from $\widetilde{\upsilon}$ to the boundary of the big disc (and mapping to $M_0 \subset M_1^{(\nu)}$ in a natural manner). In order to understand what ought to be put at the junction, we fold the disc along the fused defect *and* its identity extension, whereupon we obtain a half-disc worldsheet with a *piecewise* boundary condition at the fold (the other part of its boundary has a different status, and so we do not consider it here). At this stage, we may apply a dimensionally reduced variant of the folding trick to the boundary (Chan–Paton) degrees of freedom at the defect junction $\widetilde{\upsilon}$ mapped to $T_{(++\cdots+)}$, whereby it transpires that $\widetilde{\upsilon}$ should carry a trivialisation of $(\Phi_{\nu,1} \otimes \mathrm{id}_{\omega_{1,2}+\omega_{2,3}+\cdots+\omega_{\nu-1,\nu}}) \circ \cdots \circ (\Phi_{3,4} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}+\omega_{2,3}}}) \circ (\Phi_{2,3} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}}}) \circ \Phi_{1,2}$ for $(\Phi_{k,k+1}, \omega_{k,k+1}) = \pi_{k,k+1}^*(\Phi, \omega)$ (this makes sense as Φ is represented by a principal $\mathbb{C}^\times$-bundle). The heuristic argument does not fix the (global) connection of the trivialisation, it is only the detailed computation of [4] which shows that it should be null. Altogether, we end up with the superstring background $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ composed of: the bulk target $\mathcal{M} = (M_0, \mathrm{g}, \mathcal{G})$ in which (M_0, g) is a quasi-metric supermanifold with a gerbe $\mathcal{G}$ of curvature χ over it; the $\mathcal{G}$-*bi-brane* $\mathcal{B} = (M_1, \iota_\cdot, \omega, \Phi)$ with the bimodule $\Phi : \iota_1^*\mathcal{G} \cong \iota_2^*\mathcal{G} \otimes \mathcal{I}_\omega$; and the $(\mathcal{G}, \mathcal{B})$-*inter-bi-brane* $\mathcal{J} = \sqcup_{\nu \geq 3} \sqcup_{\varepsilon_\nu \in (\mathbb{Z}/2\mathbb{Z})^{\times\nu}} (T_{\varepsilon_\nu}, \pi_{\cdot,\cdot}^{\varepsilon_\nu}, \varphi_{\varepsilon_\nu})$ with the component fusion 2-isomorphisms

$$\varphi_{\varepsilon_\nu} \ : \ \left(\Phi_{\nu,1}^{(\nu)} \otimes \mathrm{id}_{\mathcal{I}_{\Delta_{\varepsilon_\nu}\omega - \omega_{\nu,1}^{(\nu)}}}\right) \circ \cdots \circ \left(\Phi_{3,4}^{(\nu)} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}^{(\nu)}+\omega_{2,3}^{(\nu)}}}\right) \circ \left(\Phi_{2,3}^{(\nu)} \otimes \mathrm{id}_{\mathcal{I}_{\omega_{1,2}^{(\nu)}}}\right) \circ \Phi_{1,2}^{(\nu)} \cong \mathrm{id}_{\pi_1^{\varepsilon_\nu *}\mathcal{G}} \,,$$

where $(\Phi_{k,k+1}^{(\nu)}, \omega_{k,k+1}^{(\nu)}) \equiv \pi_{k,k+1}^{\varepsilon_\nu *}(\Phi^{\varepsilon_{k,k+1}^{(\nu)}}, \varepsilon_{k,k+1}^{(\nu)}\,\omega)$, and where $\pi_1^{\varepsilon_\nu} = \iota_1 \circ \pi_{1,2}^{\varepsilon_\nu}$ if $\varepsilon_{1,2}^{(\nu)} = +1$, and $= \iota_2 \circ \pi_{1,2}^{(\varepsilon_\nu)}$ if $\varepsilon_{1,2}^{(\nu)} = -1$. The superfield theory is determined by the Dirac–Feynman amplitude $\mathcal{A}_{\mathrm{DF}}[\xi] = \exp(\mathrm{i}\, S_\sigma[\xi])$ on $\mathcal{F}_\sigma$ in which the action 'functional' splits $S_\sigma[\xi] = S_{\mathrm{metr}}[\xi] + S_{\mathrm{WZ}}[\xi]$ into the 'metric' term $S_{\mathrm{metr}}[\xi] = \int_\Sigma \mathrm{Vol}(\Sigma, \xi^*\mathrm{g})$ and the 'topological' WZ term given by the decorated-surface holonomy $\exp(\mathrm{i}\, S_{\mathrm{WZ}}[\xi]) = \mathrm{Hol}_{(\mathcal{G},\Phi,(\varphi_\cdot))}(\xi|\Gamma)$ of [4].

The higher-supergeometric elements $\mathcal{G}, \Phi$ and $\varphi_{\varepsilon_\nu}$ of $\mathfrak{B}$ are distinguished 0-, 1- and 2-cells, respectively, of $\mathfrak{BGrb}_\nabla(M)$, *cp.* [15, 16]. The fundamental property of and the main rationale for physical interest in these objects, discussed at length in, *i.a.*, [1, 2, 5], is that they canonically determine – *via* cohomological transgression, originally proposed in [1] – a prequantisation of the above superfield theory and encode a lot of non-trivial information on its (nonperturbative) structure, *cp.* [17] for a recent review.

3 The trinity: Simpliciality, symmetry and semisimplicity

The construction of the poly-phase super-σ-model from the previous section features submanifolds of Segal's nerve of the Σ-fibred pair groupoid $\mathrm{Pair}_\Sigma(\widehat{\Sigma})$ of the extended worldsheet. The nerve is a canonical example of a *simplicial manifold*, and the identities of order 2 satisfied by the structure maps $p^{\varepsilon_\nu}_{k,k+1}$ and p_A are readily seen to *follow* from the elementary simplicial identities obeyed by the face maps of $\mathrm{Pair}_\Sigma(\widehat{\Sigma})$. The order-2 relations between the structure maps $\pi^{\varepsilon_\nu}_{k,k+1}$ and ι_A of the background $\mathfrak{B}$ are a target-space realisation of their worldsheet counterparts. They give rise to subsets in another simplicial supermanifold, to wit, the nerve of the pair groupoid $\mathrm{Pair}(M_0)$ of the bulk target space (*cp.*, $D^\times$ and $I_{(\nu)}$). While there is no *a priori* reason to expect that components of the stratified target M form a simplicial supermanifold, there are important circumstances in which they do: This happens, *e.g.*, in σ-models with topological defects with *induction*, studied in [4] in the context of orbifolding, in which defect junctions of valence $\nu > 3$ can be obtained from binary trees of defect junctions of valence 3 in a limiting procedure in which the lengths of all *internal* edges are sent to 0, at no cost in the value of the action functional (owing to the topological nature of the defects). As a result, fusion 2-isomorphisms for the junctions of higher valence are induced, through (vertical) composition, from the elementary ones for trivalent junctions which define the binary-tree resolutions, *cp.* [4] and our heuresis in the previous section. Such an intrinsically *semi*-simplicial structure is promoted to a fully fledged simplicial one through adjunction of the identity defect, encountered previously, whose gerbe-theoretic data $\mathrm{id}_{\mathcal{G}}$ are provided by the bulk gerbe itself [16]. This defect can be drawn anywhere in the worldsheet (in particular, it can be attached to any defect junction, whereby the valence of the junction is increased by 1). Its existence suggests the incorporation of the degeneracy maps of a simplicial target.

Our hitherto considerations lead us to the definition of a *simplicial superstring background* with the target given by (a submanifold in) a stratified simplicial supermanifold $(M_\bullet, d^{(\bullet)}_\cdot, s^{(\bullet)}_\cdot)$ with face maps $d^{(n+1)}_i : M_{n+1} \longrightarrow M_n$ and degeneracy maps $s^{(n)}_i : M_n \longrightarrow M_{n+1}$ defined for $i \in \overline{0, n+1}$ and for all $n \in \mathbb{N}$, and subject to the standard simplicial identities. The $d^{(n+1)}_i$ reproduce the previously considered structure maps (for all but the ν-th line *in*-coming) *uniquely* as $(\iota_1, \iota_2) = (d^{(1)}_1, d^{(1)}_0)$ and $(\pi^{(3)}_{1,2}, \pi^{(3)}_{2,3}, \pi^{(3)}_{1,3}) = (d^{(2)}_2, d^{(2)}_0, d^{(2)}_1)$, and – for $\nu > 3$ – also $\pi^{(\nu)}_{1,\nu} = d^{(2)}_1 \circ d^{(3)}_1 \circ \cdots \circ d^{(\nu-1)}_1$ and $\pi^{(\nu)}_{k,k+1} = d^{(2)}_2 \circ d^{(3)}_2 \circ \cdots \circ d^{(\nu-k)}_2 \circ d^{(\nu-k+1)}_0 \circ d^{(\nu-k+2)}_0 \circ d^{(\nu-1)}_0$ for $k \in \overline{1, \nu-1}$, consistently with the identities of order 2 mentioned earlier. The $s^{(n)}_i$ account for the existence of the flat identity (sub-)bi-brane $s^{(0)*}_0 \Phi \equiv \mathrm{id}_{\mathcal{G}} : s^{(0)*}_0 d^{(1)*}_1 \mathcal{G} = \mathcal{G} \cong \mathcal{G} = s^{(0)*}_0 d^{(1)*}_0 \mathcal{G} \otimes \mathcal{I}_{s^{(0)*}_0 \omega}$ and allow to write down fusion 2-isomorphisms for defect junctions with identity defect lines attached.

As simpliciality seems to be favoured by topological defects, which are transmissive to the Virasoro currents of a *conformal* σ-model, the Segal–Sugawara realisation of the Virasoro algebra within the universal enveloping algebra of a Kač–Moody algebra of a simple Lie algebra, known, *e.g.*, from the study of the WZW σ-model, suggests a natural direction of enhancement of our construction: Focusing on σ-models with a rich configurational symmetry (*e.g.*, those with targets given by homogeneous spaces of Lie supergroups), we may combine simpliciality with symmetry to further constrain the geometry of the supertarget and the simplicial gerbe over it. The point of departure is the identification of the bicategorial realisation of rigid symmetries of the σ-model, which is readily achieved for symmetries induced by isometries of the metric bulk target (M_0, g). Thus, we consider a Lie supergroup G_σ together with actions $M_n\lambda : \mathrm{G}_\sigma \times M_n \longrightarrow M_n$, $n \in \mathbb{N}$, and so also with the fundamental vector fields $\mathcal{K}^n_X = -(X \otimes \mathrm{id}_{\mathcal{O}_{M_n}}) \circ M_n\lambda^*$ over the $M_n \equiv (|M_n|, \mathcal{O}_{M_n})$, labelled by elements of the tangent Lie superalgebra $\mathfrak{g}_\sigma \ni X$ of G_σ. The structure maps $\pi^{(\nu)}_{k,k+1}$ and ι_A are assumed G_σ-equivariant to ensure a natural alignment of the bulk and defect-quiver variations of the action functional engendered by the $\mathcal{K}_X^n$, *cp.* [4, 18]. We then say that G_σ is a Lie supergroup of prequantisable rigid (configurational) symmetries of the super-σ-model if (i) Ω is G_σ-invariant; (ii) the action admits a generalised relative (co)momentum $\kappa_\cdot : \mathfrak{g}_\sigma \longrightarrow \Omega^1(M_0) \oplus \Omega^0(M_1)$ such that $\mathrm{D}_{M_\cdot}\kappa_X = -\iota_{\mathcal{K}_X}\Omega$, where $\mathrm{D}_{M_\cdot}$ is the relative de Rham differential [18, Prop. 2.8]; and (iii) it lifts to the higher-geometric components of $\mathfrak{B}$ as a G_σ-indexed[1] family of 1-isomorphisms $\Lambda_g : M_0\lambda_g^*\mathcal{G} \cong \mathcal{G}$ and 2-isomorphisms $\lambda_g : (\iota_2^*\Lambda_g \otimes \mathrm{id}_{\mathcal{I}_\omega}) \circ M_1\lambda_g^*\Phi \cong \Phi \circ \iota_1^*\Lambda_g$ satisfying the coherence identities $M_{\nu-1}\lambda_g^*\varphi_{\varepsilon_\nu} = (\varphi_{\varepsilon_\nu}\circ\mathrm{id})\bullet(\mathrm{id}\circ\lambda^{(\nu)}_{g\,1,2})\bullet(\mathrm{id}\circ\lambda^{(\nu)}_{g\,2,3}\circ\mathrm{id})\bullet\cdots\bullet(\mathrm{id}\circ\lambda^{(\nu)}_{g\,\nu-1,\nu}\circ\mathrm{id})\bullet(\lambda^{(\nu)}_{g\,\nu,1}\circ\mathrm{id})$ (in which some *canonical* 2-isomorphisms have been dropped for brevity). The existence of such a coherent lift is a necessary and sufficient condition for the invariance of the decorated-surface holonomy under the symmetry transformations $\xi \mapsto M_\cdot\lambda_g \circ \xi$, and can be read off from [18, Thm. 4.4]. The marriage between simpliciality and symmetry thus defined can be neatly established by declaring the structure maps of $(M_\bullet, d_\cdot^{(\bullet)}, s_\cdot^{(\bullet)})$ G_σ-equivariant and, accordingly, by propagating a *given* bulk symmetry $M_0\lambda$ over $M_\bullet$ to engender a simplicial G_σ-space with an action $M_\bullet\lambda : \mathrm{G}_\sigma \times M_\bullet \longrightarrow M_\bullet$. This yields *maximally (super)symmetric defects*.

Drawing further motivation from the WZW σ-model, with its generating bi-chiral loop-group symmetries and the corresponding maximally symmetric bi-branes of [3, 4, 6] which are supported over *orbits* of the bulk-group action, we come to the final stage of the rather natural structurisation of the super-σ-model. It boils down to imposing the requirement of *semisimplicity* upon the simplicial target G_σ-supermanifold, by which we mean a (disjoint-sum) decomposition of $(M_\bullet, d_\cdot^{(\bullet)}, s_\cdot^{(\bullet)}, M_\bullet\lambda)$ into orbits of the simplicial G_σ-action. The demand that these support – as a token of quantum-mechanical consistency of the construction – a simplicial gerbe described previously is then anticipated to give rise to cohomological superselection rules for the *admissible* orbits in the decomposition, which are the only ones that we choose to keep. For $M_\bullet$ topologically nontrivial, this is bound to yield powerful constraints on the ensuing target geometry, coming from the standard Dirac-quantisation argument. Compactness of the target geometry should then result in a rationalisation of the (super)background.

4 The maximally supersymmetric simplicial Lie superbackgrounds

A prime example of a super-σ-model to which the above principles may be applied *constructively*, and in which their consequences may be explored, is the WZW(-type) σ-model, with the bulk target given by a (Kostant–)Lie supergroup G. The latter is endowed with a canonical bi-invariant Cartan 3-cocycle $\chi_{\mathrm{C}}^q = q\,\mathrm{tr}_{\mathfrak{g}}(\theta_{\mathrm{L}} \wedge [\theta_{\mathrm{L}} \overset{\wedge}{,} \theta_{\mathrm{L}}])$, $\mathfrak{g} \equiv \mathrm{sLie}\mathrm{G}$ which geometrises, in physically interesting cases (*e.g.*, for G even compact and connected, and on the super-Minkowski group) and for a suitable choice of the loop charge $q \in \mathbb{R}$, as a supersymmetric gerbe $\mathcal{G}_{\mathrm{C}}$. The target is to be seen as an orbit of an action $\mathrm{G}_0\lambda$ of a subgroup of the group $\mathrm{G} \times \mathrm{G}$ of left and right translations on G. The corresponding maximally supersymmetric defects implement the *right* regular action $\wp$ of G on itself, *cp.* [3,4,6], and so here the stratified target $M_\bullet$ embeds in the nerve $\mathrm{N}(\mathrm{G} \rtimes_\wp \mathrm{G})$ of the right action groupoid $\mathrm{G} \rtimes_\wp \mathrm{G}$. The morphism composition of the latter groupoid represents supergroup multiplication $\mathrm{m}_{\mathrm{G}} : \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G}$, which admits a gerbe-theoretic realisation in the form of a (generalised) supersymmetric *multiplicative structure*, instrumental in the construction of the bi-brane for the said multiplicative defect. The structure is a simplicial gerbe over Segal's model of the classifying space $B\mathrm{G}$ of G (containing $\mathrm{N}(\mathrm{G} \rtimes_\wp \mathrm{G})$ as a simplicial sub-manifold), and as such comprises a distinguished 1-isomorphism $\mathcal{M} : \mathrm{pr}_1^*\mathcal{G}_{\mathrm{C}} \otimes \mathrm{pr}_2^*\mathcal{G}_{\mathrm{C}} \cong \mathrm{m}_{\mathrm{G}}^*\mathcal{G}_{\mathrm{C}} \otimes \mathcal{I}_{\varrho_{\mathrm{PW}}}$ over $\mathrm{G}^{\times 2}$, written in terms of a Polyakov–Wiegmann 2-form ϱ_{PW}, and a coherent (quasi-)associator 2-isomorphism α over

[1]We put our discussion in the so-called $\mathcal{S}$-point picture for the sake of simplicity.

$\mathrm{G}^{\times 3}$, *cp.* [9, 19] for details. As the defect of interest maps to $\mathrm{G}^{\times 2}$ and has structure maps $(\iota_1, \iota_2) = (\mathrm{pr}_1, \wp \equiv \mathrm{m}_\mathrm{G})$, we may invoke the existence of $\mathcal{M}$ in the Wong–Affleck argument of Sec. 2 to conclude that $\mathcal{G}_\mathrm{C}$ must trivialise over the second cartesian factor $\iota_D : D \hookrightarrow \mathrm{G}$ in the relevant bi-brane geometry $\widetilde{\iota}_D : \mathrm{G} \times D \hookrightarrow \mathrm{G}^{\times 2}$. This, however, implies that the bi-brane arises as a product $\mathcal{B}_{\mathrm{maxym}} \equiv (\mathrm{G} \times D, \mathrm{pr}_1, \wp, \widetilde{\iota}_D^* \varrho_{\mathrm{PW}} - \mathrm{pr}_2^* \omega_D, \Phi)$ of another kind of *fusion*: $\Phi = (\widetilde{\iota}_D^* \mathcal{M} \otimes \mathrm{id}_{\mathcal{I}_{-\mathrm{pr}_2^* \omega_D}}) \circ (\mathrm{id}_{\mathrm{pr}_1^* \mathcal{G}_\mathrm{C}} \otimes \mathrm{pr}_2^* \mathcal{T}_D^{-1} \otimes \mathrm{id}_{\mathcal{I}_{-\mathrm{pr}_2^* \omega_D}})$ between $\widetilde{\iota}_D^* \mathcal{M}$ and the *boundary* bi-brane $\mathcal{B}_\partial \equiv (D, \iota_D, *, \omega_D, \mathcal{T}_D)$ introduced before. By a similar argument [9], the existence of α turns the problem of constructing $\varphi_{\varepsilon_{\nu+1}}$ into a search – over (the second factor in) a disjoint union of $\mathrm{G}_\nu\lambda$-orbits within $\mathrm{G} \times \widetilde{D}_\nu \equiv \mathrm{G} \times (D^{\times \nu} \cap \mathrm{m}_{1\_\nu}^{-1}(D))$ with $\mathrm{m}_{1\_\nu} \equiv \mathrm{m}_\mathrm{G} \circ (\mathrm{m}_\mathrm{G} \times \mathrm{id}_\mathrm{G}) \circ \cdots \circ (\mathrm{m}_\mathrm{G} \times \mathrm{id}_{\mathrm{G}^{\times \nu-2}})$ – for *boundary* fusion 2-isomorphisms $\varphi_{\nu+1}^{(\partial)} : \otimes_{i=1}^\nu \mathrm{pr}_i^* \mathcal{T}_D \cong (\mathrm{m}_{1\_\nu}^* \mathcal{T}_D \otimes \mathrm{id}) \circ \mathcal{M}_{1\_\nu-1,\nu} \restriction_{\widetilde{D}_\nu}$, defined in terms of the 1-isomorphism $\mathcal{M}_{1\_\nu-1,\nu} = \mathcal{M}_{12\cdots\nu-1,\nu} \circ \cdots \circ (\mathcal{M}_{12,3} \otimes \mathrm{id}) \circ (\mathcal{M}_{1,2} \otimes \mathrm{id})$ in which $\mathcal{M}_{12\cdots k-1,k} = ((\mathrm{m}_{1\_k-1} \times \mathrm{id}_\mathrm{G}) \circ (\mathrm{pr}_1, \mathrm{pr}_2, \ldots, \mathrm{pr}_k))^* \mathcal{M}$. (Incidentally, this gives a physical meaning to the seemingly meaningless concept of 'fusion of branes' discussed in the literature [20].)

In the quantisation of the σ-model determined by the gerbe, states are represented by disc Dirac–Feynman amplitudes [21, Sec. 4.1], and so it stands to reason that the boundary bi-brane $\mathcal{B}_\partial$ provides a geometric realisation of the spectrum of the quantum theory. Accordingly, we may anticipate that fusion 2-isomorphisms carry information on its Verlinde fusion ring. We conclude this note with a review of evidence which corroborates these expectations and a recapitulation of conjectures based firmly thereon in the setting of the bosonic WZW σ-model. For further details, as well as novel bicategorial constructions for the Green–Schwarz super-σ-model with the super-Minkowskian target, we refer the Reader to the extensive study [9].

The bosonic WZW defect & another link to the CS theory. The bulk WZW σ-model for the compact 1-connected Lie group G is defined by the Cartan–Killing metric $\mathrm{g} = -6q\, \mathrm{tr}_\mathfrak{g}(\theta_\mathrm{L} \otimes \theta_\mathrm{L})$ and the Cartan 3-form χ_C^q with $24\pi q \equiv \mathrm{k} \in \mathbb{N}^\times$, co-normalised in a manner which ensures non-anomalous conformality of the (quantised) field theory. Integrality of the *level* k implies the existence of a unique (isoclass of) gerbe geometrising χ_C^q – the k-th tensor power of the Gawędzki–Hitchin–Meinrenken basic gerbe $\mathcal{G}_\mathrm{C}$ over G. The rigid symmetries of the bulk theory make up $\mathrm{G}_\sigma = \mathrm{G} \times \mathrm{G}$ and lift to $\mathsf{N}_\bullet(\mathrm{G} \rtimes_\wp \mathrm{G})$ as $\mathrm{G}_n\lambda : \mathrm{G}_\sigma \times \mathrm{G}^{\times n+1} \ni ((x,y), g, h_i) \longmapsto (x \cdot g \cdot y^{-1}, \mathrm{Ad}_y(h_i)) \in \mathrm{G}^{\times n+1}$. Cohomological arguments localise $\mathcal{B}_\partial$ over the disjoint union $D = \sqcup_{\lambda \in \mathrm{P}_+^\mathrm{k}} \mathcal{C}_\lambda$ of the conjugacy classes $\mathcal{C}_\lambda = \mathrm{Ad}_\mathrm{G}(\mathrm{e}^{2\pi \mathrm{i}\, \lambda/\mathrm{k}})$ labelled by weights λ from the fundamental affine Weyl alcove P_+^k at level k, with $\omega_\partial \restriction_{\mathcal{C}_\lambda}$ fixed *uniquely* by the bi-chiral loop-group extension of $\mathrm{G}_1\lambda$, *cp.* [2,9]. The existence of $(\mathcal{M}, \alpha)$ is unobstructed, and so $\mathcal{B}_\partial$ induces the non-boundary maximally symmetric bi-brane as discussed above. Thus, the connected components $\mathrm{G} \times \mathcal{C}_\lambda$ of the defect-line target are in a 1-1 correspondence with chiral sectors of the bulk Hilbert space, furnishing *integrable* highest-weight representations $\mathcal{V}_{\lambda,\mathrm{k}}$ of the Kač–Moody algebra $\widehat{\mathfrak{g}}_\mathrm{k}$.

The localisation of $\mathcal{B}_{\mathrm{maxym}}$ conforms with the 'triple-S' argument of Sec. 3, and the last remark strengthens the expectation that the associated inter-bi-brane carries *geometric* information on the Verlinde fusion ring of the WZW σ-model. Recall that the ring structure is encoded by the multiplicity spaces in the decomposition, into the $\mathcal{V}_{\lambda_3,\mathrm{k}}$, of the Hilbert space of the *boundary* WZW σ-model on a strip with boundaries carrying the data of $\mathcal{B}_\partial$, or, equivalently, by the spaces $\mathscr{C}(\mathcal{V}_{\lambda_1,\mathrm{k}} \otimes \mathcal{V}_{\lambda_2,\mathrm{k}}, \mathcal{V}_{\lambda_3,\mathrm{k}})$ of rank-3 conformal blocks of the (chiral) bulk WZW theory, *cp.* [22]. Each such space is the Hilbert space of the $3d$ Chern–Simons (CS) theory on the time cylinder $\mathbb{R} \times \mathbb{C}\mathrm{P}^1$ over $\mathbb{C}\mathrm{P}^1$, coupled to vertical Wilson lines $\mathbb{R} \times \{\sigma_i\}$, $i \in \{1,2,3\}$ with holonomies along the respective non-contractible loops, encircling simply the punctures σ_i, valued in the $\mathcal{C}_{\lambda_i}$, *cp.* [21]. We may now look for imprints of these structures in the *boundary* component T_{++-}^∂ of the elementary inter-bi-brane geometry $T_{++-} = \mathrm{G} \times T_{++-}^\partial$ (*e.g.*), which the 'triple-S' argument predicts to be (a subset in) the disjoint union of Ad_G-orbits in $T_{\lambda_1,\lambda_2}^{\lambda_3} \equiv (\mathcal{C}_{\lambda_1} \times \mathcal{C}_{\lambda_2}) \cap \mathrm{m}_\mathrm{G}^{-1}(\mathcal{C}_{\lambda_3})$ for any λ_1, λ_2 and λ_3. And, remarkably, we find them! Indeed, the necessary condition for the existence of $\varphi_3^{(\partial)}$ on $T^{\lambda_3}_{\lambda_1,\lambda_2}$ is the vanishing of $\Delta_{++-}\omega$ (which restricts to T^{∂}_{++-}), and the latter 2-form turns out to be... the partially symplectically reduced presymplectic form on the state space of the CS theory described above. Its partial reduction, due to Alekseev and Malkin [23], with respect to the 'pointed' gauge group associated with a homological decomposition of $\mathbb{C}P_{(3)} \equiv \mathbb{C}P \setminus \{\sigma_1, \sigma_2, \sigma_3\}$ relative to a point $\sigma_* \in \mathbb{C}P_{(3)}$, leaves us with the tangent of the residual gauge group $[\sigma_*, \mathrm{G}] \equiv \mathrm{G}$ as the characteristic distribution of $\Delta_{++-}\omega$, which is *just* the symmetry group G of the Ad_{G}-orbits in T^{∂}_{++-}, whose disjoint union composes the *classical* state space of the CS theory in the parametrisation of [23]. In the light of the interpretation of the conformal blocks as intertwiners $\mathrm{Hom}_{\widehat{\mathfrak{g}}_{\mathrm{k}}}(\mathcal{V}_{\lambda_1,\mathrm{k}} \otimes \mathcal{V}_{\lambda_2,\mathrm{k}}, \mathcal{V}_{\lambda_3,\mathrm{k}})$ between current-symmetry sectors of the chiral bulk theory,[2] this result is also in keeping with the identification of the *transgressed* fusion 2-isomorphisms transmissive to rigid bulk symmetries (which the WZW ones are, *cp.* [18]) as intertwiners of the symmetry representations on the (twisted) state spaces fusing at the defect junction, *cp.* [5, 7].

The structure of the simplicial WZW target is ultimately determined by the requirement of existence of the 2-isomorphisms $\varphi_\nu^{(\partial)}$, expected to distinguish a subfamily within the disjoint union of Ad_{G}-orbits in the boundary factors $T^{\partial}_{\varepsilon_\nu}$ of the T_{ε_ν}. Taking into account the above highly nontrivial result (which generalises to arbitrary ν), we are led to the following conjectures:

1. The fusion 2-isomorphisms $\varphi_\nu^{(\partial)}$ exist only over manifolds $\times_{i=1}^{\nu-1} \mathcal{C}_{\lambda_i} \cap \mathrm{m}^{-1}_{1\_\nu-1}(\mathcal{C}_{\lambda_\nu})$ with non-vanishing Verlinde numbers $\dim \mathscr{C}(\otimes_{i=1}^{\nu-1} \mathcal{V}_{\lambda_i,\mathrm{k}}, \mathcal{V}_{\lambda_\nu,\mathrm{k}})$ (for conformal blocks of arbitrary rank).

2. Given such a manifold, the number of those Ad_{G}-orbits in its decomposition which support $\varphi_\nu^{(\partial)}$ is given by the corresponding Verlinde number.

One may also anticipate that the fusion 2-isomorphisms of valence $\nu > 3$ are induced from the elementary ones with $\nu = 3$ due to simpliciality of the WZW background, and that the inter-bi-brane fusing matrices defined, as described in detail in [9], by the associator move of [4] relating inequivalent such induction schemes for $\varphi_4^{(\partial)}$, are intimately related to the standard fusing matrices of the bulk WZW σ-model. The first conjecture was corroborated for the special case of $\mathrm{G} = \mathrm{SU}(2)$ in [6], and the last expectation hinges on the highly nontrivial cohomological evidence from the simple-current sector gathered in [4] as well as on simple considerations of the topologicality of the maximally symmetric defect. Various geometric, algebraic and field-theoretic arguments in favour of the second conjecture were given in [9].

References

[1] K. Gawędzki, *Nonperturbative quantum field theory*, Springer, Boston, USA, ISBN 9781461280538 (1988), doi:10.1007/978-1-4613-0729-7.

[2] K. Gawędzki and N. Reis, *WZW branes and gerbes*, Rev. Math. Phys. **14**, 1281 (2002), doi:10.1142/S0129055X02001557.

[3] J. Fuchs, C. Schweigert and K. Waldorf, *Bi-branes: Target space geometry for world sheet topological defects*, J. Geom. Phys. **58**, 576 (2008), doi:10.1016/j.geomphys.2007.12.009.

[4] I. Runkel and R. R. Suszek, *Gerbe-holonomy for surfaces with defect networks*, Adv. Theor. Math. Phys. **13**, 1137 (2009), doi:10.4310/ATMP.2009.v13.n4.a5.

[2] More accurately, we should speak of quantum intertwiners between $U_{q(\mathrm{k})}(\mathfrak{g})$-modules, *cp.* [22].

[5] R. R. Suszek, *Defects, dualities and the geometry of strings via gerbes. I. Dualities and state fusion through defects*, (arXiv preprint) doi:10.48550/arXiv.1101.1126.

[6] I. Runkel and R. R. Suszek, *Affine su(2) fusion rules from gerbe 2-isomorphisms*, J. Geom. Phys. **61**, 1527 (2011), doi:10.1016/j.geomphys.2011.03.008.

[7] R. R. Suszek, *Defects, dualities and the geometry of strings via gerbes II. Generalised geometries with a twist, the gauge anomaly and the gauge-symmetry defect*, (arXiv preprint) doi:10.48550/arXiv.1209.2334.

[8] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B **763**, 354 (2007), doi:10.1016/j.nuclphysb.2006.11.017.

[9] R. R. Suszek, *On symmetric simplicial (super)string backgrounds, (super-)WZW defect fusion and the Chern-Simons theory*, (arXiv preprint) doi:10.48550/arXiv.2208.05312.

[10] M. K. Murray, *Bundle gerbes*, J. Lond. Math. Soc. **54**, 403 (1996), doi:10.1112/jlms/54.2.403.

[11] J.-H. Jureit and T. Krajewski, *Quasi-quantum groups from strings*, J. Phys.: Conf. Ser. **103**, 012005 (2008), doi:10.1088/1742-6596/103/1/012005.

[12] J. Fröhlich, J. Fuchs, I. RunkeL and C. Schweigert, *Defect lines, dualities and generalised orbifolds*, in *XVIth international congress on mathematical physics*, World Scientific, Singapore, ISBN 9789814304627 (2010), doi:10.1142/9789814304634_0056.

[13] E. Wong and I. Affleck, *Tunneling in quantum wires: A boundary conformal field theory approach*, Nucl. Phys. B **417**, 403 (1994), doi:10.1016/0550-3213(94)90479-0.

[14] K. Gawędzki, R. R. Suszek and K. Waldorf, *Bundle gerbes for orientifold sigma models*, Adv. Theor. Math. Phys. **15**, 621 (2011), doi:10.4310/ATMP.2011.v15.n3.a1.

[15] D. Stevenson, *The geometry of bundle gerbes*, PhD thesis, The University of Adelaide, Adelaide, Australia (2000).

[16] K. Waldorf, *More morphisms between bundle gerbes*, Theory Appl. Categories **18**, 240 (2007).

[17] R. R. Suszek, *Higher supergeometry for the super-σ-model*, Rev. Roum. Math. Pures Appl. **66**, 347 (2021).

[18] K. Gawędzki, R. R. Suszek and K. Waldorf, *The gauging of two-dimensional bosonic sigma models on world-sheets with defects*, Rev. Math. Phys. **25**, 1350010 (2013), doi:10.1142/S0129055X13500104.

[19] A. L. Carey, S. Johnson, M. K. Murray, D. Stevenson and B.-L. Wang, *Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories*, Commun. Math. Phys. **259**, 577 (2005), doi:10.1007/s00220-005-1376-8.

[20] A. L. Carey and B.-L. Wang, *Fusion of symmetric D-branes and Verlinde rings*, Commun. Math. Phys. **277**, 577 (2007), doi:10.1007/s00220-007-0399-8.

[21] Y. Nutku, *Conformal field theory*, CRC Press, Boca Raton, USA, ISBN 9780429502873 (2018), doi:10.1201/9780429502873.

[22] K. Gawędzki, I. T. Todorov and P. Tran-Ngoc-Bich, *Canonical quantization of the boundary Wess-Zumino-Witten model*, Commun. Math. Phys. **248**, 217 (2004), doi:10.1007/s00220-004-1107-6.

[23] A. Y. Alekseev and A. Z. Malkin, *Symplectic structure of the moduli space of flat connection on a Riemann surface*, Commun. Math. Phys. **169**, 99 (1995), doi:10.1007/BF02101598.

Construction of matryoshka nested indecomposable N-replications of Kac-modules of quasi-reductive Lie superalgebras, including the sl(m/n) and osp(2/2n) series

Jean Thierry-Mieg[1⋆], Peter D. Jarvis[2,3†] and Jerome Germoni[4‡]
with an appendix by Maria Gorelik[5∘]

1 NCBI, National Library of Medicine, National Institute of Health,
8600 Rockville Pike, Bethesda MD20894, U.S.A.
2 School of Natural Sciences (Mathematics and Physics), University of Tasmania,
Private Bag 37, Hobart, Tasmania 7001, Australia.
3 Alexander von Humboldt Fellow.
4 Université Claude Bernard Lyon 1, CNRS UMR 5208,
Institut Camille Jordan, F-69622 Villeurbanne, France.
5 Department of Mathematics, the Weizmann Institute of Science, Rehovot, Israel.

⋆ mieg@ncbi.nlm.nih.gov , † peter.jarvis@utas.edu.au ,
‡ germoni@math.univ-lyon1.fr , ∘ maria.gorelik@weizmann.ac.il

Group ICGTMP
34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We construct a new class of finite dimensional indecomposable representations of simple superalgebras which may explain, in a natural way, the existence of the heavier elementary particles. In type I Lie superalgebras sl(m/n) and osp(2/2n) , one of the Dynkin weights labeling the finite dimensional irreducible representations is continuous. Taking the derivative, we show how to construct indecomposable representations recursively embedding N copies of the original irreducible representation, coupled by generalized Cabibbo angles, as observed among the three generations of leptons and quarks of the standard model. The construction is then generalized in the appendix to quasi-reductive Lie superalgebras.

Received 18-10-2022
Accepted 21-07-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.045

1 Introduction

In Kac's complete classification of the simple Lie superalgebras [1, 2], two families contain an even generator y commuting with the even subalgebra, namely the $A(m-1,n-1) = sl(m/n), m \neq n$ and the $C(n+1) = osp(2/2n)$ superalgebras. They admit a single Dynkin diagram with a single odd positive simple root β [3].

The even subalgebra, in the corresponding Chevalley basis, has the structure:

$$\begin{aligned} [h_i,h_j] = 0\,, \quad [h_i,e_j] = C_{ij} e_j\,, \quad [h_i,f_j] = -C_{ij} f_j\,, \\ [y,h_i] = [y,e_i] = [y,f_i] = 0\,, \quad i,j = 1,2,\ldots,r\,, \end{aligned} \tag{1}$$

where r, h_i, e_i, f_i and C_{ij} denote respectively the rank, the Cartan commuting generators, the raising and the lowering generators associated to the simple roots, and the Cartan matrix of the semisimple even Lie subalgebra $sl(m) \oplus sl(n)$, respectively $sp(2n)$, with rank $r = m+n-2$, respectively $r = n$. The remaining raising (respectively lowering) generators of the even semisimple subalgebra are generated by the iterated commutators of the e (respectively f) generators limited by the Serre rule $ad(e_i)(e_j)^{-C_{ij}+1} = 0$. Finally, the additional even generator y, that physicists often call the hypercharge, centralizes the even subalgebra. Even in finite dimensional representations, y is not quantized, and as shown below, this is the cornerstone of our new construction of nested indecomposable N-replications of an arbitrary Kac module which we propose to call matryoshka representations.

In its odd sector, the superalgebra has P odd raising generators u_i corresponding to the P positive odd roots β_i and P odd lowering generators v_i corresponding to the $-\beta_i$, with $P = mn$ for $sl(m/n)$, or $P = 2n$ for $osp(2/2n)$. In both cases, the u_i sit in the irreducible fundamental representation of the even subalgebra. We call u_1 the lowest weight vector of the u_i representation; u_1 corresponds to the simple positive odd root $\beta = \beta_1$. Reciprocally, we call v_1 the highest weight vector of the v_i. For our following analysis, the important relations are

$$\begin{aligned} [y,u_i] = u_i\,, \quad [y,v_i] = -v_i\,, \\ \{u_i,u_j\} = \{v_i,v_j\} = 0\,, \\ \{u_i,v_j\} = d_{ij}^a \mu_a + ky\,\delta_{ij}\,, \end{aligned} \tag{2}$$

where d_{ij}^a and k are constants ($k \neq 0$) and the μ_a span the even generators of type (h,e,f). That is: the hypercharge y grades the superalgebra, with eigenvalues $(0,\pm1)$. The u_i anticommute with each other. So do the v_i. Finally and most important, the anticommutator of the odd raising operator u_i with the odd lowering operator v_i corresponding to the opposite odd root depends linearly on the hypercharge y. In particular, $\{u_1,v_1\} = h_\beta = d_{11}^a h_a + ky$, where k is non zero and h_β is the Cartan generator associated to the odd simple root β. See for example the works of Kac [1,4] or the dictionary on superalgebras by Frappat, Sciarrino and Sorba [5] for details.

2 Construction of the Kac modules

Following Kac [4], choose a highest weight vector Λ defined as an eigenstate of the Cartan generators (h_i, y), and annihilated by all the raising generators (e_i, u_j). The eigenvalues a_i of the Cartan operators h_i are called the even Dynkin labels. The eigenvalue b of the Cartan operator h_β corresponding to the odd simple root is called the odd Dynkin label:

$$h_i\Lambda = a_i\Lambda\,, \quad \{u_1,v_1\}\,\Lambda = h_\beta\Lambda = b\Lambda\,. \tag{3}$$

Construct the corresponding Verma module using the free action on Λ of the lowering generators (f,v) modulo the commutation relations of the superalgebra. Since the v anticommute, the polynomials in (f,v) acting on Λ are at most of degree P in v, and hence the Verma module is graded by the hypercharge y and contains exactly P layers.

Consider the antisymmetrized product w^- of all the odd lowering generators $(v_i, i = 1,2,...,P)$. The state $\overline{\Lambda} = w^-\Lambda$ is a highest weight with respect to the even subalgebra $e_i\overline{\Lambda} = 0$. Indeed e_i annihilates Λ and each term in the Leibniz development of $[e_i, w^-]$ contains a repetition of one of the v generators, and hence vanishes.

Let ρ be the half supersum of the even and odd positive roots

$$\rho = \rho_0 - \rho_1 = \tfrac{1}{2}(\sum \alpha^+ - \sum \beta^+)\,. \tag{4}$$

Let w^+ be the antisymmetrized product of all the u generators. As shown by Kac [4], we have

$$w^+\overline{\Lambda} = w^+ w^- \Lambda = \pm \prod_i < \Lambda + \rho | \beta_i > \Lambda\,, \tag{5}$$

where the product iterates over the P positive odd roots β_i, the sign depends on the relative ordering of w^+ and w^- and the bilinear form $< | >$ is a symmetrized version of the Cartan metric. If this product is non-zero, the Verma module is called typical. Λ belongs to the orbit of the $\overline{\Lambda}$ and *vice-versa*, hence they both belong to the same irreducible submodule. If the scalar product $< \Lambda + \rho | \beta_i >$ vanishes for one or more odd positive root β_i, the Verma module is no longer irreducible but only indecomposable since Λ is not in the orbit of $\overline{\Lambda}$. It is then called atypical of type i and there exists a state ω_i with Cartan eigenvalues $\Lambda_i - \beta_i$ which is a sub highest weight annihilated by all the even and odd raising operators (e, u). In the present study, we do not quotient out by this submodule but preserve the indecomposable Verma module construction because we want to preserve the continuity in b. Notice that in the A and C superalgebras that we are studying the odd roots are on the light-cone of the Cartan root space: $< \beta_i | \beta_i >= 0$. Therefore, if Λ is atypical i, the secondary highest weight $\Lambda - \beta_i$ is also atypical i.

As in the Lie algebra case, this Verma module is infinite dimensional, because of the acceptable iterated action of the even lowering generators f. But as we just discussed, the iterated action of the anticommuting odd lowering operators v saturates at layer P.

Let us now recall for completeness the usual procedure to extract a finite dimensional irreducible module from a Lie algebra Verma module. All the states with negative even Dynkin labels which are annihilated by the even raising generators can be quotiented. For example, given a Chevalley basis (h, e, f) for the Lie algebra $sl(2)$ and a Verma module with highest weight Λ, we have

$$\begin{gathered} [h, e] = e\,, \quad [h, f] = -f\,, \quad [e, f] = 2h\,, \\ h\Lambda = a\Lambda\,, \quad e\Lambda = 0\,, \end{gathered} \tag{6}$$

hence

$$hf^n\Lambda = (a - 2n) f^n\Lambda\,, \quad ef^n\Lambda = n(a - n + 1) f^{n-1}\Lambda\,. \tag{7}$$

If a is a positive integer, the Verma module can be quotiented by the orbit of the state $f^{a+1}\Lambda$, and the equivalence classes form an irreducible module of finite dimension $a + 1$.

Generalizing to a superalgebra, all the even Dynkin labels a_i associated to the Cartan operators $h_i, i = 1, 2, ..., r$ are restricted to non negative integers. We pass to the quotient in each even submodule and define the Kac module as the resulting finite dimensional quotient space. The crucial observation is that the identification of the even sub highest weights ω requests to solve a set of equations involving the even Dynkin labels a_i, but independent of the odd Dynkin weight b, which remains non-quantized. For example, in $sl(2/1)$, the state $\omega = (a f v - (a + 1) v f)\Lambda$ is an even highest weight [6]. But please remember that we do not quotient out the atypical submodules.

Note that this procedure does not extend to the type II Lie-Kac superalgebras $B(m, n)$, $D(m, n)$, $F(4)$ and $G(3)$, because these algebras contain even generators with hypercharge $y = \pm 2$. For example the generator associated to the lowest weight of the adjoint representation. Indeed, the supplementary root of the (affine) extended Dynkin diagram is even. Thus, the Kac module is finite dimensional if and only if its hidden extended Dynkin label is also a non negative integer. This integrality constraint involves b. So the representations of the type II superalgebras are finite dimensional only for quantized values of b, see Kac [4] for the original proofs and [7, 8] for examples.

The remaining even highest weights $\Lambda^p_{(...)}$ are spread over the P layers with hypercharge decreasing from y down to $y-P$. On the zeroth layer, we have $\Lambda^0 = \Lambda$, on the first layer we have the P weights $\Lambda^1_i = \Lambda - \beta_i$, on the second layer we have the $P(P-1)/2$ weights $\Lambda^2_{ij} = \Lambda - \beta_i - \beta_j$, $i \neq j$, down to the P^{th} layer $\Lambda^P_{12...P} = \overline{\Lambda}$, each time excluding the even highest weight vectors with negative even Dynkin labels, since they have been quotiented out. For an explicit construction of the matrices of the indecomposable representations of $sl(2/1)$, we refer the reader to our study [6] and references therein.

To conclude, if the Kac module with highest weight Λ is typical, it is irreducible. If it is atypical, it is indecomposable. In both cases, its even highest weights are the $\Lambda^p_{(...)}$ and the whole module is given by the even orbits of the $\Lambda^p_{(...)}$ with non negative even Dynkin labels and hypercharge $y-p$.

3 On the derivative of the odd raising generators

Consider a finite D dimensional Kac module with highest weight Λ, typical or atypical, as described in the previous section. Call a_i the even Dynkin labels and b the odd Dynkin label and y the eigenvalue of the hypercharge Y acting on the highest weight state. Notice that y is a linear combination of b and the even Dynkin weights a_i. As shown above, in our Chevalley basis, the matrices representing the (e,f,v) generators in the Verma module are by construction independent of b, and the matrix representing the hypercharge generator Y can be written as $Y = yI + \alpha^i h_i$, where I is the identity, h_i the even Cartan generators of the semi simple even subalgebra (i.e. excluding Y) and the α^i are constants independent of y. This remains true in the Kac module because the quotient operations needed to pass to the finite dimensional submodule does not involve b. Finally, the matrices representing the odd raising generators u are linear in b, i.e. in y, because, when we push an odd raising generator u acting from the left through an element of the Kac module, i.e. through a polynomial in (f,v) acting on Λ, we must contract u with one of the v generators before u touches Λ.

Now consider the derivatives u'_i of the odd raising u_i matrices

$$u'_i(a) = \partial_y u_i(a,y)\,. \tag{8}$$

Using (2), we derive the anticommutation relations

$$\{u'_i, v_j\} = \partial_y \{u_i, v_j\} = \partial_y (d^a_{ij}\mu_a + ky\,\delta_{ij}) = k\delta_{ij}\,, \tag{9}$$

where the μ matrices span the even generators (h,e,f), and where $\mu(a)$ and $v(a)$ are independent of y. Another way of seeing the same results is to compute the $\{u_i(a,y), v_j(a)\}$ anticommutator, divide by y and take the limit when y goes to infinity. Since the matrix elements of the even generators are all bounded when y diverges, except the hypercharge Y with spectrum $y, y-1, ..., y-P$, we arrive at the same conclusion: the $\{u', v\}$ anticommutator is proportional to the identity on the whole Kac module. Many explicit examples of the matrices $u(a,y), u'(a), v(a)$ can be found in our extensive study of $sl(2/1)$ [6].

This result holds for the Verma modules, for the typical-irreducible Kac modules and for the atypical-indecomposable Kac modules of type I superalgebras, but does not hold for the type II superalgebras or for the irreducible atypical modules of the type I superalgebras because we need continuity in b. Indeed, we proved in [9] by a cohomology argument that the fundamental atypical triplet of $sl(2/1)$ cannot be doubled.

The procedure does not hold for the simple superalgebras $psl(n/n)$. It works for $sl(n/n)$, but this superalgebra is not simple because if $m = n$ the (m/n) identity operator Y is supertraceless and generates an invariant 1-dimensional subalgebra that can be quotiented out.

The resulting simple superalgebra $psl(n/n)$ corresponds to the quantized case $y=0$ and we cannot take the derivative.

4 Construction of an indecomposable N-replication of a Kac module

Given a finite D dimensional Kac module, typical or atypical-indecomposable, represented by $D\times D$ matrices (μ,y,u,v), constructed as above and where μ collectively denotes the even matrices of type (h,e,f), consider the doubled matrices of dimension $2D\times 2D$:

$$M=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix},\quad Y=\begin{pmatrix}y & I\\ 0 & y\end{pmatrix},\quad U=\begin{pmatrix}u & u'\\ 0 & u\end{pmatrix},\quad V=\begin{pmatrix}v & 0\\ 0 & v\end{pmatrix},\tag{10}$$

where we used the $D\times D$ matrices u' constructed in the previous section. By inspection, the matrices (M,Y,U,V) have the same super-commutation relations as the matrices (μ,y,u,v) and therefore form an indecomposable representation of the same superalgebra of doubled dimension $2D$. This representation cannot be diagonalized since the matrix Y representing the hypercharge cannot be diagonalized because of its block Jordan structure.

The block u' can be rescaled via a change of variables

$$\begin{aligned}Q&=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix},\quad Q^{-1}=\begin{pmatrix}1/\lambda & 0\\ 0 & 1\end{pmatrix},\\ QUQ^{-1}&=\begin{pmatrix}u & \lambda u'\\ 0 & u\end{pmatrix},\quad QYQ^{-1}=\begin{pmatrix}y & \lambda I\\ 0 & y\end{pmatrix}.\end{aligned}\tag{11}$$

Furthermore, we can construct a module of dimension ND, for any positive integer N by iterating the previous construction. By changing variables we can then introduce a complex parameter λ at each level. For example, for $N=3$, we can construct

$$\begin{aligned}M&=\begin{pmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{pmatrix},\quad Y=\begin{pmatrix}y & I & 0\\ 0 & y & I\\ 0 & 0 & y\end{pmatrix},\quad \tilde{Q}Y\tilde{Q}^{-1}=\begin{pmatrix}y & \lambda_1 I & 0\\ 0 & y & \lambda_2 I\\ 0 & 0 & y\end{pmatrix},\\ U&=\begin{pmatrix}u & u' & 0\\ 0 & u & u'\\ 0 & 0 & u\end{pmatrix},\quad V=\begin{pmatrix}v & 0 & 0\\ 0 & v & 0\\ 0 & 0 & v\end{pmatrix},\quad \tilde{Q}U\tilde{Q}^{-1}=\begin{pmatrix}u & \lambda_1 u' & 0\\ 0 & u & \lambda_2 u'\\ 0 & 0 & u\end{pmatrix}.\end{aligned}\tag{12}$$

Matryoshka theorem: Given any finite dimensional, typical or atypical, Kac module of a type I simple superalgebra, $A(m/n)$, $m\neq n$ or $C(n)$, using the derivative u' of the odd raising generators with respect to the hypercharge y which centralizes the even subalgebra, we can construct an indecomposable representation recursively embedding N replications of the original module.

We propose the name matryoshka because this nested structure strongly resemble the famous Russian dolls.

5 Conclusion

Representation theory of Lie algebras and superalgebras involves three increasingly difficult steps: classification, characters and construction. In Lie algebra theory, we can rely on three major results: all finite dimensional representations of the semisimple Lie algebras are completely reducible, their irreducible components are classified by the Dynkin labels of their highest weight state, their characters are given by the Weyl formula. Nevertheless, the actual construction of the matrices, although known in principle, remains challenging. We only know the matrices in closed analytic form in the case of $sl(2)$.

Finite dimensional simple Lie superalgebras have been classified by Kac [1]. In the present study, we only consider the simple basic classical superalgebras of type 1, $sl(m/n)$, $m \neq n$ and $osp(2, 2n)$ which are characterized by the existence of an even generator, the hypercharge y, commuting with the even subalgebra. As for Lie algebras, their irreducible modules can be classified by the Dynkin labels of their highest weight and Kac [4] has discovered in 1977 an elegant generalization of the Weyl formula.

But there are two additional difficulties. First, as found by Kac, the hypercharge y of the finite dimensional modules is not quantized, but for certain discrete values, the Kac module ceases to be irreducible but becomes indecomposable. One can quotient out one or several invariant submodules and the Weyl-Kac formula of the irreducible quotient module is not known in general [10, 11]. Furthermore, there is a rich zoology of finite dimensional indecomposable modules which were progressively discovered by Kac [4], Scheunert [12], Marcu [13, 14], Su [15], and others, culminating in the classification of Germoni [16, 17]. See [6] for an explicit description of the indecomposable $sl(2/1)$ modules.

A particular class, first described by Marcu [14], is of great interest in physics because it has implications for the standard model of leptons and quarks. These particles are well described by $sl(2/1)$ irreducible modules graded by chirality [18–23]. However, experimentally, they appear as a hierarchy of three quasi identical families, for example the muon and the tau behave as heavy electrons. This hierarchical structure has no clear explanation in Lie algebra theory. Furthermore, the three families leak into each other in a subtle way first described by Cabibbo (C) for the strange quarks and generalized to all three families by Kobayashi and Maskawa (KM). In a certain technical sense, the axis of the electroweak interactions is not orthogonal to the axes of the strong interactions, but tilted by small angles, called the CKM angles. As a result the weak interactions are not truly universal because the heavier quarks leak into the lighter quarks. Again, this experimental phenomenon has no explanation in Lie algebra theory precisely because all representations are completely reducible.

Marcu found in 1980 [14] that the fundamental $sl(2/1)$ quartet can be duplicated and triplicated in an indecomposable way. Coquereaux, Haussling, Scheck and coworkers [24–26] have proposed in the 90's to interpret these representations as a description of the CKM mechanism. This raises several questions: is the construction of Marcu limited to three generations, as observed experimentally in the case of the quarks and leptons, or does there exist indecomposable modules involving more layers? Is this property specific of $sl(2/1)$, or is it applicable to other simple Lie-Kac superalgebras?

We have previously partially answered these questions. In [16, 17] the existence of multi generations indecomposable modules is indicated. In [9], we proved, using cohomology, that any Kac module of a type I superalgebra can be duplicated. But these were just proofs of existence.

In the present study, using the derivative of the odd generators relative to the hypercharge, we have shown that any Kac module of a type I Lie-Kac superalgebra $sl(m/n)$, $m \neq n$ and $osp(2/2n)$ can be replicated any desired number of times in an indecomposable way. We have also shown that atypical representations cannot be replicated. We can therefore, in this framework predict the existence of three species of sterile right neutrinos from the observation of the non-zero PMNS (leptonic CKM) mixing angles.

In the appendix presented below (A), we further show that these results are valid for any Kac module $K(L)$ over a quasi-reductive Lie superalgebra $\mathfrak{g}$ of type I. As the reader will notice, the style of this appendix contributed by M.G. is more general and more abstract. We hope that this split/joint presentation will appeal to the wide audience of the G34 conference, equally composed of mathematicians and physicists. The main result is that the “matryoshka N-replication” of the Kac module $K(L)$ has the structure of a module over a Heisenberg superalgebra.

These results are interesting for physics, surprising relative to Lie algebra theory, and very specific as we actually construct the matrices of these “matryoshka” Russian dolls indecomposable modules, in terms of the matrices of the original Kac module, rather than limit our analysis to their existence, classification, or the calculation of their characters.

Acknowledgments

The authors heartily thank the organizers of the G34 conference

Author contributions JTM, PDJ and JG developed the work on the type 1 superalgebras, MG contributed the generalization to quasi-reductive superalgebras presented in the appendix.

Funding information JTM was supported by the Intramural Research Program of the National Library of Medicine, National Institute of Health. MG was supported by ISF 1957/21 grant.

A Appendix: Generalization to quasi-reductive Lie superalgebras

Contributed by Maria Gorelik.

A finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is *quasi-reductive* if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a semisimple $\mathfrak{g}_0$-module. Quasi-reductive Lie superalgebras were introduced in [27, 28]. Simple quasi-reductive Lie superalgebras are classical Lie superalgebras in Kac's classification (see [2]). For other examples and partial classification of quasi-reductive Lie superalgebras, see [27], [28], and [29].

As above, the base field is $\mathbb{C}$ and $\mathfrak{g}$ is a quasi-reductive Lie superalgebra with an even Cartan subalgebra $\mathfrak{h}$. This means that $\mathfrak{h}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ and that $\mathfrak{g}^{\mathfrak{h}} = \mathfrak{h}$. The only simple quasi-reductive Lie superalgebras which do not satisfy this assumption are Q-type superalgebras. We denote by $\mathfrak{h}'$ the center of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$; one has

$$\mathfrak{h} = \mathfrak{h}' \times \mathfrak{h}'', \quad \text{where } \mathfrak{h}'' := [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \cap \mathfrak{h}\,.$$

We identify $(\mathfrak{h}')^*$ with the subspace $(\mathfrak{h}'')^{\perp} = \{\nu \in \mathfrak{h}^* |\ \nu(h) = 0 \quad \text{for any } h \in \mathfrak{h}''\}$. One has

$$\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \times \mathfrak{h}'\,.$$

For an $\mathfrak{h}$-module N we denote by N_ν the generalized weight space corresponding to $\nu \in \mathfrak{h}^*$:

$$N_\nu = \{v \in N |\ \forall h \in \mathfrak{h}\,,\ (h - \nu(h))^s v = 0\,,\ \text{ for } s >> 0\}\,.$$

All modules in this section are assumed to be *locally finite over* $\mathfrak{h}$ *with generalized finite-dimensional weight spaces*: this means that $N = \oplus_\nu N_\nu$ and $\dim N_\nu < \infty$ for all $\nu \in \mathfrak{h}^*$. We set

$$\operatorname{ch} N := \sum_\nu \dim N_\nu e^\nu\,.$$

A quasi-reductive Lie superalgebra is of *type I* if $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-graded superalgebra. In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ is a reductive Lie algebra, $\mathfrak{g}_{\pm 1}$ are odd commutative subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ acts diagonally on $\mathfrak{g}_{\pm 1}$. Examples of quasi-reductive Lie superalgebra of type I include $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2|2n)$, $\mathfrak{p}_n$ and others.

A.1 Self-extensions of highest weight modules

Let M be a module with the highest weight λ (i.e. M is a quotient of $M(\lambda)$), and let $v_\lambda \in M$ be the highest weight vector, i.e. the image of the canonical generator of $M(\lambda)$. Introduce the natural map

$$\Upsilon_M : \mathrm{Ext}^1_{\mathfrak{g}}(M, M) \to \mathfrak{h}^*,$$

as follows. Let $0 \to M \to^{\phi_1} N \to^{\phi_2} M \to 0$ be an exact sequence. Let $v := \phi_1(v_\lambda)$ and fix $v' \in N_\lambda$ such that $\phi_2(v') = v_\lambda$. Observe that v, v' is a basis of N_λ and so there exists $\mu \in \mathfrak{h}^*$ such that for any $h \in \mathfrak{h}$ one has $h(v') = \lambda(h)v' + \mu(h)v$ (i.e., the representation $\mathfrak{h} \to \mathrm{End}(N_\lambda)$ is $h \mapsto \begin{pmatrix} \lambda(h) & \mu(h) \\ 0 & \lambda(h) \end{pmatrix}$). The map Υ_M assigns μ to the exact sequence. It is easy to see that $\Upsilon_M : \mathrm{Ext}^1(M, M) \to \mathfrak{h}^*$ is injective.

Notice that if $0 \to M \to N_1 \to M \to 0$ and $0 \to M \to N_2 \to M \to 0$ are two exact sequences then

$$N_1 \cong N_2 \iff \Upsilon_M(N_1) = c\Upsilon_M(N_2), \quad \text{for some } c \in \mathbb{C} \setminus \{0\}. \tag{A.1}$$

If N is an extension of M by M (i.e., $N/M \cong M$) we denote by $\Upsilon_M(N)$ the corresponding one-dimensional subspace of $\mathfrak{h}^*$, i.e. $\Upsilon_M(N) = \mathbb{C}\mu$, where μ is the image of the exact sequence $0 \to M \to N \to M \to 0$.

A.1.1

Let M be a finite-dimensional highest weight module. Since $\mathfrak{g}_0$ is reductive, the algebra $\mathfrak{h}''$ acts diagonally on any finite-dimensional $\mathfrak{g}$-module. Therefore the image of Υ_M annihilates $\mathfrak{h}''$, so lies in $(\mathfrak{h}')^*$. In particular, for the image of Υ_M is zero and $\mathrm{Ext}^1(M, M) = 0$ if $\mathfrak{h}' = 0$. In other words, the finite-dimensional highest weight modules do not admit non-splitting self-extensions if $\mathfrak{g}_{\bar{0}}$ is semisimple (for instance, if $\mathfrak{g}$ is a basic classical Lie superalgebra of type II).

A.2 Kac modules

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a quasi-reductive superalgebra of type I.

The following useful construction appears in several papers including [30]. For a given $\mathfrak{g}_0$-module M, we may extend M trivially to a $\mathfrak{g}_0 + \mathfrak{g}_1$-module and introduce the Kac module

$$K(M) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0 + \mathfrak{g}_1} M.$$

This defines an exact functor $K : \mathfrak{g}_0 - Mod \to \mathfrak{g} - Mod$ which is called *Kac functor*. It is easy to see that $K(M)$ is indecomposable if and only if M is indecomposable.

As $\mathfrak{g}_0$-module we have $K(M) \cong M \otimes \Lambda\mathfrak{g}_{-1}$. Since $\Lambda\mathfrak{g}_{-1}$ is a finite-dimensional module with a diagonal action of $\mathfrak{h}$, M is finite-dimensional (resp., diagonal $\mathfrak{h}$-module) if and only if $K(M)$ is finite-dimensional (resp., diagonal $\mathfrak{h}$-module) . Moreover, M is a locally finite $\mathfrak{h}$-module with generalized finite-dimensional weight spaces if and only if $K(M)$ is such a module.

A.2.1 Self extensions of Kac modules

Take $\nu \in (\mathfrak{h}')^*$ and let $J_n(\nu)$ be the $(n|0)$-dimensional indecomposable $\mathfrak{h}$-module spanned by $v_1, \dots, v_n$ with the action $hv_i = \nu(h)v_{i+1}$), $hv_n = 0$. (h acts on $V_2(\nu)$ $\begin{pmatrix} 0 & 0 \\ \nu(h) & 0 \end{pmatrix}$). Observe that $hJ_n(\nu) = 0$ for all $h \in \mathfrak{h}''$. We view $J_n(\nu)$ as $\mathfrak{g}_0$-module with the zero action of $[\mathfrak{g}_0, \mathfrak{g}_0]$.

For any $\mathfrak{g}_0$-module L the product $J_n(\nu) \otimes L$ is an indecomposable $\mathfrak{g}_0$-module which admits a filtration of length n with the factors isomorphic to L. Thus $K(L \otimes J_n(\nu))$ is an indecomposable $\mathfrak{g}$-module which admits a filtration of length n with the factors isomorphic to the Kac module $K(L)$; we denote this module by $K(L; n; \nu)$. In particular, $K(L; 2; \nu)$ is a self-extension of the Kac module $K(L)$.

A.2.2

Let L be a finite-dimensional highest weight $\mathfrak{g}_0$-module. Then $K(L)$ is a finite-dimensional highest weight $\mathfrak{g}$-module By A.1.1, the image of $\Upsilon_{K(L)}$ lies in $(\mathfrak{h}')^*$. Using the above construction we obtain that the the image of Υ_M is equal to $(\mathfrak{h}')^*$.

A.2.3

Lemma. *If L is a $\mathfrak{g}_0$-module, where $\mathfrak{h}$ acts locally finitely with finite-dimensional generalized weight spaces, then $K(L; n; \nu) \cong K(L; n; \mu)$ if and only if $\nu \in \mathbb{C}^* \mu$.*

Proof. If $\nu \in \mathbb{C}^* \mu$, then $J_n(\nu) \cong J_n(\mu)$ and thus $K(L; n; \nu) \cong K(L; n; \mu)$.

Conversely, assume that $K(L; n; \nu) \cong K(L; n; \mu)$. As $\mathfrak{g}_0$-modules

$$K(L; n; \nu) \cong L \otimes J_n(\nu) \otimes \Lambda \mathfrak{g}_{-1} \cong K(L) \otimes J_n(\nu) .$$

Thus for any $\lambda \in \mathfrak{h}^*$ we have $K(L; n; \nu)_\lambda \cong K(L)_\lambda \otimes J_n(\nu)$. Take λ such that $K(L)_\lambda \neq 0$. Then $K(L; n; \nu) \cong K(L; n; \mu)$ implies

$$K(L)_\lambda \otimes J_n(\nu) \cong K(L)_\lambda \otimes J_n(\mu) , \quad \text{as } \mathfrak{h}\text{-modules.} \tag{A.2}$$

We will use the following fact: if V_1, V_2 are two modules over a one-dimensional Lie algebra $\mathbb{C}x$ with the minimal polynomials of x on V_i equal to $(x - c_i)_i^k$, then the minimal polynomial of x on $V_1 \otimes V_2$ equals to $(x - (c_1 + c_2))^{k_1+k_2}$.

Assume that $\nu \notin \mathbb{C}^* \mu$. Then there exists $h \in \mathfrak{h}$ such that $\nu(h) = 0 \neq \mu(h)$. Recall that $h - \lambda(h)$ acts nilpotently on $K(L)_\lambda$, so the minimal polynomial of h on $K(L)_\lambda$ takes the form $(h - \lambda(h))^k$. By above, the minimal polynomial of h on $K(L)_\lambda \otimes J_n(\nu)$ (resp., on $K(L)_\lambda \otimes J_n(\mu)$) is $(h - \lambda(h))^k$ (resp., $(h - \lambda(h))^{k+n}$). Hence (A.2) does not hold: a contradiction. □

A.3 Action of the Heisenberg superalgebra

Let $\mathfrak{g}$ be a quasi-reductive superalgebra of type I. Retain notation of A.2.1. Let $\iota' : \mathfrak{g}_0 \to \mathfrak{h}'$ be the projection along the decomposition $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \times \mathfrak{h}'$. We endow the superspace $H = \mathfrak{g}_{-1} \oplus \mathfrak{h}' \oplus \mathfrak{g}_1$ by the structure of Lie superalgebra with the bracket $[-,-]_n$ given by

$$[\mathfrak{g}_1, \mathfrak{g}_1]_n := [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]_n := 0 , \quad [a_-, a_+]_n := \iota'([a_-, a_+]) , \quad [h, a_\pm]_n := 0 ,$$

for all $a_\pm \in \mathfrak{g}_{\pm 1}$ and $h \in \mathfrak{h}'$. Observe that H is a quasi-reductive superalgebra of type I (in fact, H is the direct product an odd Heisenberg superalgebra and a commutative superalgebra). For an $\mathfrak{h}'$-module M we denote by $K_H(M)$ the Kac module for H constructed as in A.2.1.

Let L be a $\mathfrak{g}_0$-module. The construction (10) defines an action of H on a self-extension of a Kac module $K(L)$; we describe this action below.

A.3.1

Fix $\mu \in (\mathfrak{h}')^*$. Consider a one-parameter family of n-dimensional $\mathfrak{g}_0$-modules $V_n(t\nu)$ constructed in A.2.1 (for $t \in \mathbb{R}$). Then $K(L;n;t\nu)$ is a one-parameter family of self-extensions of $K(L)$: this self-extension is splitting if $t=0$; by (A.1), for $t \neq 0$ all these modules are isomorphic. As a vector space $K(L;n;t\nu))$ is canonically isomorphic to

$$V := L \otimes J_n(\nu) \otimes \Lambda\mathfrak{g}_{-1}\,.$$

We let $\rho_t : \mathfrak{g} \to \mathrm{End}(V)$ be the representation corresponding to $K(L;n;t\nu)$.

It is easy to see that $\rho_t(u) = \rho_0(u)$ for $u \in [\mathfrak{g}_0, \mathfrak{g}_0] + \mathfrak{g}_{-1}$ and that for $u \in \mathfrak{h}' + \mathfrak{g}_1$ one has $\rho_t(u) = \rho_0(u) + t\rho'_t(u)$ for some $\rho'_t(u) \in \mathrm{End}(V)$.

A.3.2

Lemma. *Define a linear map $\phi : H \to \mathfrak{gl}(V)$ by*

$$\phi(a_-) := \rho_0(a_-)\,, \quad \phi(a_+) := \rho'_t(a_+)\,, \quad \phi(h) := \rho'_t(h)\,,$$

for $a_\pm \in \mathfrak{g}_\pm$ and $h \in \mathfrak{h}'$. Then ϕ is the H-representation isomorphic to the Kac module $K_H(L', n, \nu)$ where $L \cong L'$ as a superspace and the action of $H_0 = \mathfrak{h}'$ on L' is trivial.

Proof. Let us check that ϕ is a homomorphism of Lie superalgebras. For $a_-, b_- \in \mathfrak{g}_-$ one has $[a_-, b_-]_n = [a_-, b_-] = 0$ and

$$[\phi(a_-), \phi(b_-)] = \rho_0([a_-, b_-]) = 0 = \phi([a_-, b_-]_n)\,.$$

By above, $\frac{\partial^2 \rho_t(a)}{\partial^2 t} = 0$ for any $a \in \mathfrak{g}$. Therefore for any $a, b \in \mathfrak{g}$ one has

$$0 = \frac{\partial^2 \rho_t([a,b])}{\partial^2 t} = [\rho'_t(a), \rho'_t(b)]\,.$$

Taking $a, b \in \mathfrak{g}_1 + \mathfrak{h}'$ we get $0 = [\phi(a), \phi(b)] = \phi([a,b]_n)$.

One has

$$[\rho_t(a), \rho'_t(b)] + [\rho'_t(a), \rho_t(b)] = \rho'_t([a,b])\,.$$

Using this formula for $a = a_-$ and $b = a_+$ we obtain

$$[\phi(a_-), \phi(a_+)] = [\rho_t(a_-), \rho'_t(a_+)] = \rho'_t([a_-, a_+]) = \rho'_t(\iota'[a_-, a_+]) = \phi([a_-, a_+]_n)\,.$$

Finally, taking $h \in \mathfrak{h}'$ and $a_- \in \mathfrak{g}_-$ we get

$$[\phi(h), \phi(a_-)] = [\rho'_t(h), \rho_t(a_-)] = \rho'_t([h, a_-]) = 0\,,$$

so $[\phi(h), \phi(a_-)] = \phi([h, a_-]_n)$. Hence ϕ is a homomorphism, so ϕ defines a representation of H. Denote the corresponding H-module by N. Let us check that the linear isomorphism $L \xrightarrow{\sim} L'$ induces the H-module isomorphism $N \xrightarrow{\sim} K_H(L', n, \nu)$.

Since $K(L, n, t\nu)$ is a free $\mathfrak{g}_{-1}$-module generated by the subspace $L \otimes V_n(t\nu)$, N is a free $\mathfrak{g}_{-1}$-module generated by the subspace $L' \otimes V_n(t\nu)$. For any $v \in L`\otimes V_n(t\nu)$ one has $\rho_t(a_+)(v) = 0$, so $\phi(a_+)(v) = 0$. Take $h \in \mathfrak{h}'$. Let $v_1, \ldots, v_n$ be the standard basis of $J_n(\nu)$ (see A.2.1). For $w \in L$ we have

$$\rho_t(h)(w \otimes v_i) = \rho_0(h)(w) \otimes v_i + t\nu(h) w \otimes v_{i+1}\,,$$

so $\phi(h)(w \otimes v_i) = \nu(h) w \otimes v_{i+1}$. Hence $N = K_H(L', n, \nu)$ as required. □

A.4 Another construction of the action of the Heisenberg superalgebra

A natural question is to find a more "natural" construction for Heisenberg superalgebra and its action. This can be done as follows.

A.4.1

Let $\mathfrak{t}$ be any Lie superalgebra. We introduce the increasing filtration by $\mathcal{F}^0(\mathfrak{t}) = 0$, $\mathcal{F}^1(\mathfrak{t}) := \mathfrak{t}_{\bar{1}}$ and $\mathcal{F}^2(\mathfrak{t}) := \mathfrak{t}$.

The associated graded Lie superalgebra $\tilde{H} := \mathrm{gr}_{\mathcal{F}}(\mathfrak{t})$ is naturally isomorphic to $\mathfrak{t}$ as a vector superspace; denoting this linear isomorphism by $\iota : \mathfrak{t} \to \tilde{H}$ we obtain the following formulae for the bracket on $\tilde{H}$:

$$[\tilde{H}_{\bar{0}}, \tilde{H}] = 0\,, \quad [a,b] := \iota([\iota^{-1}(a), \iota^{-1}(b)])\,, \quad \text{if } a,b \in \tilde{H}_{\bar{1}}\,.$$

If $\dim \mathfrak{t} < \infty$, then $\tilde{H}$ is quasi-reductive. If $\mathfrak{t}$ is $\mathbb{Z}$-graded and finite-dimensional, then $\tilde{H}$ is quasi-reductive of type I ($\tilde{H}$ is the direct product an odd Heisenberg superalgebra and a commutative superalgebra).

If N is a $\mathfrak{t}$-module generated by a subspace N' we can define a compatible increasing filtration on N by setting $\mathcal{F}^0(N) = N'$ and $\mathcal{F}^i(N) = \mathcal{F}^i(\mathcal{U}(\mathfrak{t}))N'$. The associated graded module $\mathrm{gr}_{\mathcal{F}}(N)$ has a structure of $\tilde{H}$-module.

A.4.2 Application to A.3

Retain notation of A.3. Let $\mathfrak{t} := \mathfrak{g}$ be quasi-reductive of type I. Identify $\tilde{H}_{\bar{0}}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$. Since $\tilde{H}_{\bar{0}}$ lies in the center of $\tilde{H}$, any subspace of $\mathfrak{g}_0$ is an ideal of $\tilde{H}$. It is easy to see that $\tilde{H}/[\mathfrak{g}_0, \mathfrak{g}_0]$ is isomorphic to H constructed in A.3.

Set $N := K(L,n,\nu)$. Fix $h \in \mathfrak{h}'$ such that $\nu(h) = 1$. Recall that the h acts on $J_n(\nu)$ by a Jordan cell: $J_n(\nu)$ is spanned by $v_1, hv_2, h^2v_1, \ldots, h^{n-1}v$. $L \otimes J_n(\nu)$ is spanned by $L \otimes v_1, h(L \otimes v_1), \ldots, h^{n-1}(L \otimes v_1)$, so $N' := L \otimes v_1$ generates N over $\mathfrak{g}$. Define the increasing filtration on N as above. The associated graded module $\mathrm{gr}_{\mathcal{F}}(N)$ is a $\tilde{H}$-module. We have $[\mathfrak{g}_0, \mathfrak{g}_0](\mathcal{F}^i(N)) = \mathcal{F}^i(N)$ for each i, so $[\mathfrak{g}_0, \mathfrak{g}_0]$ annihilates $\mathrm{gr}_{\mathcal{F}}(N)$. Hence $\mathrm{gr}_{\mathcal{F}}(N)$ is an H-module. It is not hard to see that this module is isomorphic to the H-module constructed in Lemma A.3.2.

Conclusion: the matryoshka N-replication of the Kac module $K(L)$ has the structure of a module over a Heisenberg superalgebra.

References

[1] V. G. Kac, *Classification of simple Lie superalgebras*, Funct. Anal. Appl. **9**, 263 (1975), doi:10.1007/BF01075610.

[2] V. G. Kac, *Lie superalgebras*, Adv. Math. **26**, 8 (1977), doi:10.1016/0001-8708(77)90017-2.

[3] L. Frappat, A. Sciarrino and P. Sorba, *Structure of basic Lie superalgebras and of their affine extensions*, Commun. Math. Phys. **121**, 457 (1989), doi:10.1007/BF01217734.

[4] V. G. Kac, *Characters of typical representations of classical Lie superalgebras*, Commun. Algebra **5**, 889 (1977), doi:10.1080/00927877708822201.

[5] L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie superalgebras*, (arXiv preprint) doi:10.48550/arXiv.hep-th/9607161.

[6] J. Thierry-Mieg, P. D. Jarvis and J. Germoni, *Explicit construction of the finite dimensional indecomposable representations of the simple Lie-Kac $SU(2/1)$ superalgebra and their low level non diagonal super Casimir operators*, (arXiv preprint) doi:10.48550/arXiv.2207.06545.

[7] J. Thierry-Mieg, *Irreducible representations of the basic classical Lie superalgebras OSp(m/2n), SU(m/n), SSU(n/n), D(2/1, α), G(3), F(4)*, in *20th conference on group theoretical methods in physics*, Trieste, Italy (1983).

[8] J. Thierry-Mieg, *Table of irreducible representations of the basic classical Lie superalgebras OSp(m/2n), SU(m/n), SSU(n/n), D(2/1, α), G(3), F(4)*, Preprints Observatoire de Paris-Meudon and LBL Berkeley (1985), https://ftp.ncbi.nlm.nih.gov/repository/acedb/1985.characters.pdf.

[9] P. D. Jarvis and J. Thierry-Mieg, *Indecomposable doubling for representations of the type I Lie superalgebras sl(m/n) and osp(2/2n)*, J. Phys. A: Math. Theor. **55**, 475206 (2022), doi:10.1088/1751-8121/aca22d.

[10] J. Van der Jeugt, J. W. B. Hughes, R. C. King and J. Thierry-Mieg, *A character formula for singly atypical modules of the Lie superalgebra sl(m/n)*, Commun. Algebra **18**, 3453 (1990), doi:10.1080/00927879008824086.

[11] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, in *Lie theory and geometry*, Birkhäuser, Boston, USA, ISBN 9781461266853 (1994), doi:10.1007/978-1-4612-0261-5_15.

[12] M. Scheunert, W. Nahm and V. Rittenberg, *Irreducible representations of the osp(2,1) and spl(2,1) graded Lie algebras*, J. Math. Phys. **18**, 155 (1977), doi:10.1063/1.523149.

[13] M. Marcu, *The tensor product of two irreducible representations of the spl(2,1) superalgebra*, J. Math. Phys. **21**, 1284 (1980), doi:10.1063/1.524577.

[14] M. Marcu, *The representations of spl(2,1)-an example of representations of basic superalgebras*, J. Math. Phys. **21**, 1277 (1980), doi:10.1063/1.524576.

[15] Y. Su, *Classification of finite dimensional modules of the Lie superalgebra sl(2/1)*, Commun. Algebra **20**, 3259 (1992), doi:10.1080/00927879208824514.

[16] J. Germoni, *Représentations indécomposables des superalgèbres de Lie spéciales linéaires*, C. R. Acad. Sci. Math. **324**, 1221 (1997), doi:10.1016/S0764-4442(99)80403-0.

[17] J. Germoni, *Indecomposable representations of special linear Lie superalgebras*, J. Algebra **209**, 367 (1998), doi:10.1006/JABR.1998.7520.

[18] Y. Ne'eman, *Irreducible gauge theory of a consolidated Salam-Weinberg model*, Phys. Lett. B **81**, 190 (1979), doi:10.1016/0370-2693(79)90521-5.

[19] D. B. Fairlie, *Higgs fields and the determination of the Weinberg angle*, Phys. Lett. B **82**, 97 (1979), doi:10.1016/0370-2693(79)90434-9.

[20] P. H. Dondi and P. D. Jarvis, *A supersymmetric Weinberg-Salam model*, Phys. Lett. B **84**, 75 (1979), doi:10.1016/0370-2693(79)90652-X.

[21] Y. Ne'eman and J. Thierry-Mieg, *Geometrical gauge theory of ghost and Goldstone fields and of ghost symmetries*, Proc. Natl. Acad. Sci. **77**, 720 (1980), doi:10.1073/PNAS.77.2.720.

[22] J. Thierry-Mieg and Y. Ne'eman, *Exterior gauging of an internal supersymmetry and SU(2/1) quantum asthenodynamics*, Proc. Natl. Acad. Sci. **79**, 7068 (1982), doi:10.1073/PNAS.79.22.7068.

[23] J. Thierry-Mieg and P. D. Jarvis. *SU(2/1) superchiral self-duality: A new quantum, algebraic and geometric paradigm to describe the electroweak interactions*, J. High Energy Phys. **04**, 001 (2021), doi:10.1007/JHEP04(2021)001.

[24] R. Coquereaux, *Elementary fermions and SU(2|1) representations*, Phys. Lett. B **261**, 449 (1991), doi:10.1016/0370-2693(91)90455-Y.

[25] R. Häussling and F. Scheck, *Triangular mass matrices of quarks and Cabibbo-Kobayashi-Maskawa mixing*, Phys. Rev. D **57**, 6656 (1998), doi:10.1103/PhysRevD.57.6656.

[26] R. Häussling, M. Paschke and F. Scheck, *Leptonic generation mixing, noncommutative geometry and solar neutrino fluxes*, Phys. Lett. B **417**, 312 (1998), doi:10.1016/S0370-2693(97)01407-X.

[27] V. Serganova *Quasireductive supergroups*, in *New developments in Lie theory and its applications*, American Mathematical Society, Providence, USA, ISBN 9780821852590 (2011).

[28] V. Mazorchuk, *Parabolic category O for classical Lie superalgebras*, in *Advances in Lie superalgebras*, Springer, Cham, Switzerland, ISBN 9783319029511 (2014), doi:10.1007/978-3-319-02952-8_9.

[29] N. I. Ivanova and A. L. Onishchik, *Parabolic subalgebras and gradings of reductive Lie superalgebras*, J. Math. Sci. **152**, 1 (2008), doi:10.1007/S10958-008-9049-8.

[30] C.-W. Chen, V. Mazorchuk, *Simple modules over Lie superalgebras*, Trans. Amer. Math. Soc. **374**, 899 (2021), doi:10.1090/tran/8303.

The parastatistics of braided Majorana fermions

Francesco Toppan⋆

Centro Brasileiro de Pesquisas Físicas CBPF, Rua Dr. Xavier Sigaud 150,
Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil.

⋆ toppan@cbpf.br

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

This paper presents the parastatistics of braided Majorana fermions obtained in the framework of a graded Hopf algebra endowed with a braided tensor product. The braiding property is encoded in a t-dependent 4×4 braiding matrix B_t related to the Alexander-Conway polynomial. The nonvanishing complex parameter t defines the braided parastatistics. At $t = 1$ ordinary fermions are recovered. The values of t at roots of unity are organized into levels which specify the maximal number of braided Majorana fermions in a multiparticle sector. Generic values of t and the $t = -1$ root of unity mimick the behaviour of ordinary bosons.

Received 18-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.046

1 Introduction

Braided Majorana fermions have been intensively investigated since the [1] Kitaev's proposal that they can be used to encode the logical operations of a topological quantum computer which offers protection from decoherence (see also [2–4]). In this talk I present consequences and open questions about the parastatistics of $\mathbb{Z}_2$-graded braided Majorana qubits derived from the results of [5]; this paper applied to $\mathbb{Z}_2$-graded qubits the [6] framework of a graded Hopf algebra endowed with a braided tensor product. A nonvanishing complex braiding parameter t controls the spectra of multiparticle Majorana fermions. Inequivalent physics is derived for the set of t roots of unity which are organized into different levels $(L_2, L_3, \ldots, L_\infty)$. The levels interpolate between ordinary fermions (L_2 for $t = 1$) and the spectrum of bosons ("L_∞" recovered at $t = -1$). The intermediate levels L_k for $k = 3, 4, 5, \ldots$ implement a special type of parafermionic statistics (see [7–9]) which allows at most $k - 1$ braided Majorana excited states in any given multiparticle sector.

The paper is structured as follows. In Section 2 the braiding of $\mathbb{Z}_2$-graded qubits is illustrated. In Section 3 the truncations of the spectra at roots of unity are discussed. The consequences for the parastatistics are presented in Section 4.

2 Braiding $\mathbb{Z}_2$-graded qubits

We present the main ingredients of the construction. A single Majorana fermion can be described as a $\mathbb{Z}_2$-graded qubit which defines a bosonic vacuum state $|0\rangle$ and a fermionic excited state $|1\rangle$:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

The operators acting on the $\mathbb{Z}_2$-graded qubit close the $\mathfrak{gl}(1|1)$ superalgebra. In a convenient presentation they can be defined as

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Their (anti)commutators are

$$\begin{aligned} &[\alpha,\beta] = \beta\,, \qquad [\alpha,\gamma] = -\gamma\,, \qquad [\alpha,\delta] = 0\,, \qquad [\delta,\beta] = -\beta\,, \qquad [\delta,\gamma] = \gamma\,, \\ &\{\beta,\beta\} = \{\gamma,\gamma\} = 0\,, \qquad\qquad \{\beta,\gamma\} = \alpha + \delta\,. \end{aligned} \tag{3}$$

The diagonal operators α, β are even, while β, γ are odd, with γ the fermionic creation operator.

The excited state is a Majorana since it is a fermion which coincides with its own antiparticle. This is a consequence of the fact that the (2) matrices span the Clifford algebra $Cl(2,1)$ which, see [10,11], is of real type (implying that the charge conjugation operator is the identity).

The construction of multiparticle $\mathbb{Z}_2$-graded qubits is obtained via the coproduct Δ of the graded Hopf algebra $\mathcal{U}(\mathfrak{gl}(1|1))$, the Universal Enveloping Algebra of $\mathfrak{gl}(1|1)$.

The braiding of the graded qubits is realized by introducing a braided tensor product $\otimes_{br}$ such that, for the operators a, b ($\mathbb{I}$ is the identity) one can write

$$(\mathbb{I} \otimes_{br} a) \cdot (b \otimes_{br} \mathbb{I}) = \Psi(a,b)\,, \tag{4}$$

where the right hand side operator $\Psi(a,b)$ satisfies braided compatibility conditions.

For the purpose of braiding $\mathbb{Z}_2$-graded qubits it is only necessary to specify the braiding property of the creation operator γ:

$$(\mathbb{I} \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}) = \Psi(\gamma,\gamma)\,. \tag{5}$$

A consistent choice for the right hand side is to set

$$\Psi(\gamma,\gamma) = B_t \cdot (\gamma \otimes \gamma)\,, \tag{6}$$

where B_t is a 4×4 constant matrix which depends on the complex parameter $t \neq 0$. The dot in the right hand side denotes the standard matrix multiplication.

The braiding compatibility condition is guaranteed by assuming B_t to be given by

$$B_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}, \tag{7}$$

since B_t satisfies

$$(B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) = (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t)\,. \tag{8}$$

The matrix B_t is the R-matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra [12] and is related, see [13], to the Burau representation of the braid group.

3 Truncations at roots of unity

The requirement that

$$B_t^n = \mathbb{I}_4\,, \tag{9}$$

for some $n = 2, 3, \ldots$ finds solution for the $n-1$ roots of the polynomial $b_n(t)$. This set of polynomials is defined as

$$b_{n+1}(t) = \sum_{j=0}^{n} (-t)^j\,,$$

so that

$$\begin{aligned} b_1(t) &= 1\,,\\ b_2(t) &= 1-t\,,\\ b_3(t) &= 1-t+t^2\,,\\ b_4(t) &= 1-t+t^2-t^3\,,\\ b_5(t) &= 1-t+t^2-t^3+t^4\,,\\ \ldots &= \ldots \end{aligned}$$

The set of $b_k(t)$ polynomials enters the construction of multiparticle states. The n-particle vacuum $|0\rangle_n$ is given by the tensor product of n single-particle vacua:

$$|0\rangle_n = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \qquad (n \;\text{ times})\,. \tag{10}$$

The fermionic excited states are created by applying powers of tensor products involving the single-particle creation operator γ. For $n = 2, 3$ one has, e.g., that the first excited state is created by

$$\begin{aligned} \gamma_{(2)} &= \mathbb{I}_2 \otimes_{br} \gamma + \gamma \otimes_{br} \mathbb{I}_2\,,\\ \gamma_{(3)} &= \mathbb{I}_2 \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma + \mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2 + \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \mathbb{I}_2\,. \end{aligned} \tag{11}$$

By taking into account the braided tensor product one obtains, for the second and third excited states,

$$\begin{aligned} \gamma_{(2)}^2 &= (1-t)\cdot(\gamma \otimes_{br} \gamma)\,,\\ \gamma_{(3)}^2 &= (1-t)\cdot(\mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \gamma + \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma + \gamma \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2)\,,\\ \gamma_{(3)}^3 &= (1-t)(1-t+t^2)\cdot(\gamma \otimes_{br} \gamma \otimes_{br} \gamma)\,. \end{aligned}$$

This construction works in general. The $b_k(t) = 0$ roots of the polynomials produce truncations at the higher order excited states and the corresponding spectrum of the theory.

4 The levels and the associated parastatistics

The single-particle Hamiltonian H can be identified with the operator δ in (2). It follows that the single-particle excited state has energy level $E = 1$.This is also true (due to the property of the Hopf algebra coproduct) for the first excited state in the multiparticle sector. Each creation operator produces a quantum of energy.

In the n-particle sector the energy spectrum of the theory depends on whether t produces a truncated or untruncated spectrum. The notion of truncation level acquires importance.

A "level-k" root of unity, for $k = 2, 3, 4, \ldots$, is a a solution t_k of the $b_k(t_k) = 0$ equation such that, for any $k' < k$, $b_{k'}(t_k) \neq 0$.

The physical significance of a level-k root of unity is that the corresponding braided multiparticle Hilbert space can accommodate at most $k-1$ Majorana spinors.

The special point $t = 1$, being the solution of the $b_2(t) \equiv 1-t = 0$ equation, is a level-2 root of unity. It gives the ordinary total antisymmetrization of the fermionic wavefunctions. The $t = 1$ level-2 root of unity encodes the Pauli exclusion principle of ordinary fermions.

With an abuse of language, the $t = -1$ root of unity, which does not solve any $b_k(t) = 0$ equation, can be called a root of unity of ∞ level.

The physics does not depend on the specific value of t, but only on the root of unity level. A generic t which does not coincide with a root of unity produces the same untruncated spectrum of the $t = -1$ "L_∞" level.

The following energy spectra are derived.

Case a, truncated L_k level: the n-particle energy eigenvalues E are

$$\begin{aligned} E &= 0, 1, \ldots, n\,, && \text{for} \quad n < k\,, \\ E &= 0, 1, \ldots, k-1\,, && \text{for} \quad n \geq k\,; \end{aligned}$$

a plateau is reached for the maximal energy level $k-1$; this is the maximal number of braided Majorana fermions that can be accommodated in a multiparticle Hilbert space;

Case b, untruncated ($t = -1$) L_∞ level: the n-particle energy eigenvalues E are

$$E = 0, 1, \ldots, n\,, \qquad \text{for any} \quad n\,;$$

there is no plateau in this case. The energy eigenvalues grow linearly with N.

We can associate the roots of unity levels to fractions.

Let $t = e^{i\theta} = e^{if\pi}$ with $f \in [0, 2[$. The following fractions correspond to the roots of unity levels:

$$\begin{aligned} L_\infty &= 1\,, \\ L_2 &= 0\,, \\ L_3 &= \frac{1}{3}, \frac{5}{3}\,, \\ L_4 &= \frac{1}{2}, \frac{3}{2}\,, \\ L_5 &= \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}\,, \\ L_6 &= \frac{2}{3}, \frac{4}{3}\,, \\ L_7 &= \frac{1}{7}, \frac{3}{7}, \frac{5}{7}, \frac{9}{7}, \frac{11}{7}, \frac{13}{7}\,, \\ L_8 &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\,, \\ \ldots &= \ldots \end{aligned}$$

As an example, the 5 roots of $b_6(t) = 1-t+t^2-t^3+t^4-t^5$ are classified, for $t = exp(i\theta)$, into:

level-2 root, $\theta = 0$,
level-3 roots $\theta = \pi/3$ and $5\pi/3$,
level-6 roots $\theta = 2\pi/3$ and $4\pi/3$.

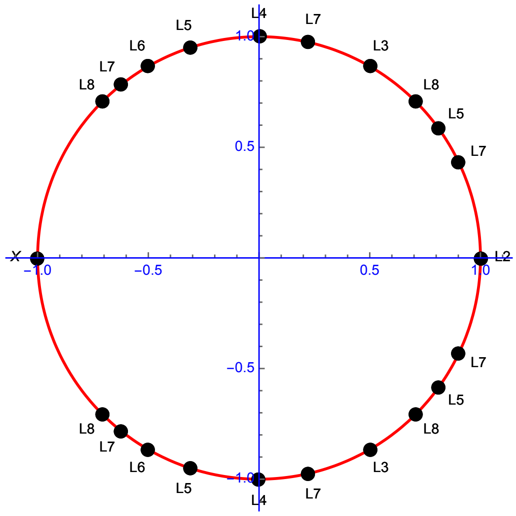


Figure 1: Roots of unity up to level 8.

The above figure shows how the roots of unity are accommodated up to level 8.

The level k root accommodates at most k inequivalent energy levels in the multiparticle states.

5 Conclusion

The [5] braided multiparticle quantization of Majorana fermions produces truncations of the spectra at certain values of t roots of unity. This feature points towards a relation between the braided tensor product framework here discussed and the representations of quantum groups at roots of unity where similar truncations, see [14, 15], are observed. The precise connection of the two approaches is on the other hand not yet known and still an open question. The representations of the quantum group $\mathcal{U}_q(\mathfrak{gl}(1|1)$ at roots of unity have been classified and presented in [16] (see also [17]). A possibility to investigate the connection seems to be offered by the scheme of [18] which shows how a quasitriangular Hopf algebra can be converted into a braided group.

On a separate issue it should be mentioned that a forthcoming paper will present, with the help of intertwining operators, the construction of the braided tensor product $\otimes_{br}$ in terms of an ordinary tensor product $\otimes$. This construction relates the observed parastatistics of Majorana fermions to the "mixed brackets" (which interpolate ordinary commutators and anticommutators) that have been introduced in [19] in defining the Volichenko algebras.

Acknowledgements

Funding information This work was supported by CNPq (PQ grant 308846/2021-4).

References

[1] A. Y. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303**, 2 (2003), doi:10.1016/S0003-4916(02)00018-0.

[2] S. B. Bravyi and A. Y. Kitaev, *Fermionic quantum computation*, Ann. Phys. **298**, 210 (2002), doi:10.1006/aphy.2002.6254.

[3] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, *Non-Abelian anyons and topological quantum computation*, Rev. Mod. Phys. **80**, 1083 (2008), doi:10.1103/RevModPhys.80.1083.

[4] L. H. Kauffman, *Knot logic and topological quantum computing with Majorana fermions*, in *Logic and algebraic structures in quantum computing*, Cambridge University Press, Cambridge, UK, ISBN 9781107033399 (2016), doi:10.1017/CBO9781139519687.

[5] F. Toppan, *First quantization of braided Majorana fermions*, Nucl. Phys. B **980**, 115834 (2022), doi:10.1016/j.nuclphysb.2022.115834.

[6] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, UK, ISBN 9780511613104 (1995), doi:10.1017/CBO9780511613104.

[7] G. Gentile, *Osservazioni sopra le statistiche intermedie*, Nuovo Cimento **17**, 493 (1940), doi:10.1007/BF02960187.

[8] H. S. Green, *A generalized method of field quantization*, Phys. Rev. **90**, 270 (1953), doi:10.1103/PhysRev.90.270.

[9] O. W. Greenberg and A. M. L. Messiah, *Selection rules for parafields and the absence of para particles in nature*, Phys. Rev. **138**, B1155 (1965), doi:10.1103/PhysRev.138.B1155.

[10] S. Okubo, *Real representations of finite Clifford algebras. I. Classification*, J. Math. Phys. **32**, 1657 (1991), doi:10.1063/1.529277.

[11] H. L. Carrion, M. Rojas and F. Toppan, *Quaternionic and octonionic spinors. A classification*, J. High Energy Phys. **04**, 040 (2003), doi:10.1088/1126-6708/2003/04/040.

[12] L. H. Kauffman and H. Saleur, *Free fermions and the Alexander-Conway polynomial*, Commun. Math. Phys. **141**, 293 (1991), doi:10.1007/BF02101508.

[13] N. Reshetikhin, C. Stroppel and B. Webster, *Schur-Weyl-type duality for quantized $gl(1|1)$, the burau representation of braid groups, and invariant of tangled graphs*, in *Perspective in analysis, geometry, and topology*, Birkhäuser, Boston, USA, ISBN 9780817682767 (2011), doi:10.1007/978-0-8176-8277-4_16.

[14] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35**, 89 (1990), doi:10.1007/BF00147341.

[15] C. de Concini and V. G. Kac, *Representations of quantum groups at roots of 1*, in *Operator algebras, unitary representations, enveloping algebras, and invariant theory*, Birkhäuser, Boston, USA, ISBN 9783764334895 (2000).

[16] O. Viro, *Quantum relatives of the Alexander polynomial*, St. Petersburg Math. J. **18**, 391 (2007), doi:10.1090/S1061-0022-07-00956-9.

[17] A. Sartori, *The Alexander polynomial as quantum invariant of links*, Ark. Mat. **53**, 177 (2015), doi:10.1007/s11512-014-0196-5.

[18] D. Gurevich and S. Majid, *Braided groups of Hopf algebras obtained by twisting*, Pacific J. Math. **162**, 27 (1994), doi:10.2140/pjm.1994.162.27.

[19] D. Leites and V. Serganova, *Metasymmetry and Volichenko algebras*, Phys. Lett. B **252**, 91 (1990), doi:10.1016/0370-2693(90)91086-Q.

SciPost Phys. Proc. 14, 047 (2023)

On the spin content of the classical massless Rarita–Schwinger system

Mauricio Valenzuela⋆ and Jorge Zanelli

Centro de Estudios Científicos (CECs), Arturo Prat 514, Valdivia, Chile
Facultad de Ingeniería, Arquitectura y Diseño, Universidad San Sebastián, Valdivia, Chile

⋆ mauricio.valenzuela@uss.cl

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We analyze the Rarita–Schwinger massless theory in the Lagrangian and Hamiltonian approaches. At the Lagrangian level, the standard gamma-trace gauge fixing constraint leaves a spin–$\frac{1}{2}$ and a spin–$\frac{3}{2}$ propagating Poincaré group helicities. At the Hamiltonian level, the result depends on whether the Dirac conjecture is assumed or not. In the affirmative case, a secondary first class constraint is added to the total Hamiltonian and a corresponding gauge fixing condition must be imposed, completely removing the spin–$\frac{1}{2}$ sector. In the opposite case, the spin–$\frac{1}{2}$ field propagates and the Hamilton field equations match the Euler-Lagrange equations.

Received 24-11-2022
Accepted 11-08-2023
Published 24-11-2023
Check for updates
doi:10.21468/SciPostPhysProc.14.047

1 Introduction

In 1939, Markus Fierz and Wolfgang Pauli discussed the obstacles in the attempt to quantize fields of arbitrary spin ≥ 1 interacting with photons [1]. Two years later, William Rarita and Julian Schwinger simplified the Fierz-Pauli treatment, writing down a set of field equations describing fermions of arbitrary spin $\geq 3/2$ [2]. The Rarita-Schwinger system (**RS**) describes a field of spin $k+1/2$ as a tensor-spinor of rank k, $\psi^{\alpha}_{\mu_1\cdots\mu_k}$, symmetric in its tensor indices $\mu_1\cdots\mu_k$, satisfying a Dirac-like field equation with mass,

$$(\partial\!\!\!/ + M)\psi_{\mu_1\cdots\mu_k} = 0\,, \qquad \gamma^{\mu}\psi_{\mu\mu_2\cdots\mu_k} = 0\,. \tag{1}$$

The subsidiary conditions $\partial^{\mu}\psi_{\mu\mu_2\cdots\mu_k} = 0$ (transverse) and $\psi^{\mu}{}_{\mu\mu_2\cdots\mu_k} = 0$ (traceless), appear as consequence of (1) for $M \neq 0$. In the spin–$\frac{3}{2}$ case, of a vector-spinor ψ^{α}_{μ}, Rarita and Schwinger also noted that there is a class of Lagrangians parametrized by the mass (M) and a dimensionless coefficient (A) that gives rise to the equations (1) (see *e.g.* [3–5]). Then they chose some A “which permits a relatively simple expression of the equations of motion in the presence of electromagnetic fields”.

The description of spin–$\frac{3}{2}$ particles adopted in supergravity, however, traditionally referred to as the Rarita–Schwinger Lagrangian [6,7] (see also [8] and references therein) corresponds to a different choice of A which in the massless limit gives the Lagrangian[1]

$$\mathcal{L} := -\frac{i}{2}\bar{\psi}_\mu \gamma^{\mu\nu\lambda}\partial_\nu \psi_\lambda \,, \tag{2}$$

whose corresponding field equations are (see [9, 10] for the chiral spinor version of these equations)

$$\gamma^{\mu\nu\lambda}\partial_\nu \psi_\lambda = 0\,. \tag{3}$$

We emphasize that (2) is not equivalent to massless limit of the original RS action since, as shown below, the γ-trace condition arises as a gauge choice and not as consequence of the field equations.

The action changes by a boundary term and (3) is invariant under the gauge transformation

$$\delta\psi_\mu = \partial_\mu \epsilon\,. \tag{4}$$

Eq. (3) can also be written as

$$\not{\partial}\psi_\mu - \partial_\mu \gamma\cdot\psi = 0\,, \qquad \partial\cdot\psi - \not{\partial}\gamma\cdot\psi = 0\,, \tag{5}$$

and, in the $\gamma^\mu\psi_\mu = 0$ gauge, as

$$\not{\partial}\psi_\mu = 0\,, \qquad \partial^\mu\psi_\mu = 0\,, \qquad \gamma^\mu\psi_\mu \overset{gf}{=} 0\,. \tag{6}$$

The third equation corresponds to the gauge choice that fixes the freedom (4), where the symbol $\overset{gf}{=}$ reflects this. In the massive RS system, $\partial^\mu\psi_\mu = 0$ is a consistency condition of the field equations, hence (6) can be obtained from the massless limit of the original RS equations,[2] however (6) can not be obtained by direct variation of the massless action.

The RS field ψ_μ belongs to the reducible representation $\frac{3}{2}\oplus\frac{1}{2}$ of the Lorentz group and it can be split into its irreducible parts as $\psi_\mu := \rho_\mu + \gamma_\mu\kappa$, where the spin-3/2 part ρ_μ is gamma-traceless, $\gamma^\mu\rho_\mu = 0$, and κ represents the spin-1/2. Then the Euler-Lagrange equations (5) read

$$(D-1)\not{\partial}\kappa - \partial^\mu\rho_\mu = 0\,, \qquad \not{\partial}\rho_\mu - \gamma_\mu\not{\partial}\kappa - (D-2)\partial_\mu\kappa = 0\,. \tag{7}$$

Using the gauge freedom (4) it is possible to make ρ_μ, not only gamma-traceless but also divergence-free. Hence, the first equation reduces to the massless Dirac equation for the spin–$\frac{1}{2}$ field κ, while the second becomes a Dirac equation for the massless spin–$\frac{3}{2}$ with source $\partial_\mu\kappa$. This shows that at least in the gauge $\partial^\mu\rho_\mu = 0$ both spin sectors seem to propagate.

Another way to see that the RS may propagate is the fact that in the vacuum of the spin–$\frac{3}{2}$ field $\rho_\mu = 0$ the RS action produces the Dirac action,

$$\mathcal{L} := i\frac{(D-1)(D-2)}{2}\,\bar{\kappa}\not{\partial}\kappa\,, \tag{8}$$

where D is the number of spacetime dimensions.

[1] Here γ_μ, $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$, are Dirac matrices, $\eta_{\mu\nu} = \mathtt{diag}(-1,1,\dots)$ and $\gamma_{\mu\cdots\nu} = \gamma_{[\mu}\cdots\gamma_{\nu]}$ are completely antisymmetric products. We assume the Majorana reality condition $\psi^\dagger = \psi$, $\bar{\psi} = \psi^t C$, were $C^t = -C$.

[2] This is analogous to the transverse condition $\partial^\mu A_\mu = 0$ that is required by consistency of the Proca equations but is only a gauge option in Maxwell's theory.

Though this result might seem straightforward, it should be unexpected if the spin–$\frac{1}{2}$ is pure gauge. One would expect a trivial action principle, as it happens in standard gauge theories. In this paper we look for an explanation to this problem.

By excellence, Hamiltonian analysis is the standard framework for elucidating what are the degrees of freedom of gauge systems. We shall see that either result can be obtained depending on whether or not the validness of the Dirac conjecture—which says that all first class constraints are gauge generators—is assumed. This is a technical observation: if the Dirac conjecture is not assumed, and a gauge fixing condition is impossed only for the primary first class constraint, then the spin–$\frac{1}{2}$ field propagates. Otherwise, the sum of the secondary first class constraint to the extended Hamiltonian introduce a new arbitrary function of time (the corresponding Lagrange multiplier) which, in order to produce a deterministic system of equations, requires an additional gauge fixing condition, which removes the spin–$\frac{1}{2}$ field, in agreement with [8, 11–14].

We shall see exactly in which step of the Dirac algorithm the two branches are generated: the one in which the spin–$\frac{1}{2}$ sector remains and the other where it is removed.

2 Space and time splitting

In terms of Poincaré group the vector-spinor ψ_μ^α can be decomposed in irreducible representation of spins $(1 \oplus 0) \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$ [8]. Hence the RS action should be regarded has a spin–$\frac{1}{2}$ and spin–$\frac{3}{2}$ particle systems. The Poincaré spin split can be achieved explicitly in terms of spin-block projectors [8, 15, 16], which involve nonlocal operators. These projectors reveals the gauge invariant components of the RS action and it produces a decoupled system of spin–$\frac{3}{2}$ and spin–$\frac{1}{2}$ governed by Dirac kinetic terms, in agreement with the discussion following (7) (for further details see [17]). The second spin–$\frac{1}{2}$ mode is the pure gauge mode, as it belongs to the kernel of the RS kinetic operator off-shell. The analogous treatment of the Maxwell theory would introduce the transverse (spin-one) and the longitudinal (spin-0) projectors, of which only the transverse mode would propagate.

Non-locality along time directions, however, may be problematic: they might be incompatible with the integration of the field equations under initial conditions on a Cauchy surface. Thus another method is necessary to analyze the problem. Spatial nonlocality, on the other hand, is compatible with the Cauchy data because it leaves the initial surface intact and therefore gauge transformations or field redefinitions involving nonlocal spatial operators such as $\not\nabla^{-1} := (\gamma^i \partial_i)^{-1}$, should not lead to inconsistencies.

We first split the vector-spinor ψ_μ into ψ_0 and ψ_i, which is in turn split into three more pieces: the spatial divergence $\partial^i \psi_i$, the γ–trace $\gamma^i \psi_i$, and a spatial γ–traceless and divergenceless vector-spinor ξ_i ($\gamma^i \xi_i = 0 = \partial^i \xi_i$). Thus we have one spin–$\frac{3}{2}$ field ξ_i, and three spin–$\frac{1}{2}$ representations of the spatial rotation group, ψ_0, $\partial^i \psi_i$ and $\gamma^i \psi_i$. We shall see that the γ–traceless and divergenceless conditions (6) remove one spin–$\frac{1}{2}$ representations of the rotation group each, whilst the third is a propagating spin–$\frac{1}{2}$ irreducible representation of the Poincaré group.

Consider the decomposition of the identity of vector-spinors in three orthogonal projectors, $\mathbb{1} = P^{\perp\!\perp} + P^N + P^L$,

$$(P^N)_{ij} := \frac{1}{D-2} N_i N_j \,, \qquad (P^L)_{ij} := L_i L_j \,, \qquad P^{\perp\!\perp} = \mathbb{1} - P^N - P^L \,, \tag{9}$$

where $N_i := \gamma_i - L_i$ and $L_i := \not\nabla^{-1} \partial_i$. These are space-like spin-block projectors that decompose the spatial vector-spinor as

$$\psi_i = \xi_i + N_i\zeta + L_i\lambda\,, \quad \text{where} \quad \xi_i = P^{\perp\perp}{}_i{}^j\psi_j\,, \quad \zeta = \frac{1}{D-2}N^i\psi_i\,, \quad \lambda = L^i\psi_i\,, \tag{10}$$

which can be verified with the help of the identities $N_i N^i = D-2$, $L_i L^i = 1$, $N_i L^i = 0$.

The gamma traceless relation is equivalent to

$$\psi_0 = -\gamma_0\gamma^i\psi_i = -\gamma_0((D-2)\zeta + \lambda)\,, \tag{11}$$

which, together with the decomposition (10), reduce (6) to

$$\partial\!\!\!/\,\xi_i = 0\,, \qquad \partial\!\!\!/\,\tilde{\lambda} = 0\,, \qquad \dot{\zeta} = 0\,, \qquad \nabla\!\!\!\!/\,\zeta = 0\,, \tag{12}$$

where $\tilde{\lambda} = \gamma^0\lambda$. Thus the explicit solution of (6) containing the spin–$\frac{1}{2}$ and spin–$\frac{3}{2}$ sector is,

$$\psi_0 = \gamma^0\lambda\,, \qquad \psi_i = \xi_i + \partial_i\nabla\!\!\!\!/^{-1}\lambda\,, \tag{13}$$

given in terms of standard solutions of the Dirac equations (12), restricted by double-transverse condition $\gamma^i\xi_i = \partial^i\xi_i = 0$ [13, 18]. It follows that fields ξ_i and λ propagate, whilst ζ is a constant spinor –and must therefore vanish on-shell–, and ψ_0 is not an independent field. Hence, the massless RS equations in the gauge $\gamma^\mu\psi_\mu = 0$ eliminates two spin–$\frac{1}{2}$ modes, ψ_0 and ζ, whilst one spin–$\frac{1}{2}$ and one spin–$\frac{3}{2}$ field propagate. Thus, there are no arbitrary functions of time left in the system: the dynamical equations (12) completely determine the evolution of the fields, provided initial data is given on a Cauchy surface.

The number of degrees of freedom in the system is defined by one half the number of functions in the set $\{\xi_i^\alpha(t_0,\vec{x}), \lambda^\alpha(t_0,\vec{x})\}$ necessary to specify the evolution. These functions are: $k\times(D-1)-2k$ components of ξ_i^α, $\alpha = 1,\ldots,k$, and $k = 2^{[D/2]}$ components in λ^α; in both cases the Dirac equation restricts half of them. In total, we are left with $k(D-2)/2$ degrees of freedom which are equivalent to two massless states of spin–$\frac{1}{2}$ and spin–$\frac{3}{2}$, respectively.

It is relevant now to discuss the picture in Dirac's time honored Hamiltonian analysis [19].

3 Hamiltonian analysis

Splitting ψ_μ as in (10) one obtains, up to boundary terms,

$$\mathcal{L} = -i\bar{\psi}_0\gamma^{0ij}\partial_i\psi_j + \frac{i}{2}\bar{\psi}_i\gamma^{0ij}\dot{\psi}_j - \frac{i}{2}\bar{\psi}_i\gamma^{ijk}\partial_j\psi_k\,. \tag{14}$$

The definition of momenta, $\pi^\mu := \partial\mathcal{L}/\partial\dot{\psi}_\mu$, yields the primary constraints

$$\pi^0 \approx 0\,, \tag{15}$$

$$\chi^i := \pi^i - \tfrac{i}{2}\mathcal{C}^{ij}\psi_j \approx 0\,, \tag{16}$$

where $\mathcal{C}^{ij}_{\alpha\beta} := -(C\gamma^{0ij})_{\alpha\beta} = \mathcal{C}^{ji}_{\beta\alpha}$ is invertible, $\mathcal{C}^{ij}_{\alpha\beta}(\mathcal{C}^{-1})^{\beta\kappa}_{jm} := \delta^i_m\delta^\kappa_\alpha$,

$$(\mathcal{C}^{-1})^{\alpha\beta}_{ij} = \Big(-\frac{1}{(D-2)}\gamma_i\gamma_j\gamma_0 C^{-1} + \delta_{ij}\gamma_0 C^{-1}\Big)^{\alpha\beta}\,. \tag{17}$$

The constraint (15) states that ψ_0 is a Lagrange multiplier and (16) is a consequence of the first order character of the system. The Hamiltonian, including a linear combination of the primary constraints is

$$H = \int d^{D-1}x\,\Big(i\bar{\psi}_0\gamma^{0ij}\partial_i\psi_j + \frac{i}{2}\bar{\psi}_i\gamma^{ijk}\partial_j\psi_k + \chi^i_\alpha\mu^\alpha_i + \pi^0_\alpha\mu^\alpha_0\Big)\,, \tag{18}$$

where μ_i and μ_0 are arbitrary spinorial Lagrange multipliers. Preservation in time of the primary constraint $\pi^0 \approx 0$ yields a secondary constraint,

$$\dot{\pi}^0 = \{\pi^0, H\} = -\frac{\delta H}{\delta \psi_0} = -iC\gamma^{0ij}\partial_i\psi_j \approx 0 \quad \Leftrightarrow \quad \varphi := -i\mathcal{C}^{ij}\partial_i\psi_j \approx 0\,, \tag{19}$$

which is equivalent to the equation of motion obtained varying (14) with respect to ψ_0. Preservation in time of the other primary constraints,

$$\dot{\chi}^i = -(C\gamma^i\gamma_0C^{-1})\varphi + i\mathcal{C}^{ij}\partial_j\psi_0^\beta + iC\partial^i\gamma^j\psi_j - iC\not{\nabla}\psi^i + i\mathcal{C}^{ij}\mu_j \approx 0\,, \tag{20}$$

and of the secondary one,

$$\dot{\varphi} = i\mathcal{C}^{ij}\partial_i\mu_j \approx 0\,, \tag{21}$$

yield conditions that determine the Lagrange multipliers μ_i in terms of the phase space fields, and there are no further constraints. It is easily checked that χ^i is second class, while π^0 and the linear combination of constraints

$$\tilde{\varphi} := \varphi + i\partial_i\chi^i \approx 0 \tag{22}$$

are first class. The second class constraint χ^i will be eventually dropped, leaving $\pi^0 \approx 0$ and $\varphi \approx 0$ as the only remaining first class constraints.

It should be noticed that the secondary constraint φ is not purely first class. In particular (21) fixes part of the Lagrange multipliers of the system, whilst $\tilde{\varphi}$ mixes primary and secondary constraints. This will be relevant for the discussion of the Dirac conjecture.

The system can be reduced to the surface of the second class constraints (16) by strongly setting $\chi^i = 0$ and replacing Poisson by Dirac brackets, $\{f, g\}_D := \{f, g\} - \{f, \chi^i_\alpha\}\mathcal{C}^{-1\alpha\beta}_{ij}\{\chi^j_\beta, g\}$, which in the variables ψ_0, π^0, ξ, ζ and λ, reads

$$\begin{aligned}\{f, g\}_D = (-1)^f \int d^{D-1}z \Bigg[&-i\frac{\delta f}{\delta \xi_i} P^{\perp\perp}_{ij}\gamma_0C^{-1}\frac{\delta g}{\delta \xi_j} - i\frac{D-3}{D-2}\frac{\delta f}{\delta\lambda}\gamma_0C^{-1}\frac{\delta g}{\delta\lambda} \\ &+ i\frac{1}{D-2}\left(\frac{\delta f}{\delta\lambda}\gamma_0C^{-1}\frac{\delta g}{\delta\zeta} + \frac{\delta f}{\delta\zeta}\gamma_0C^{-1}\frac{\delta g}{\delta\lambda}\right) + \left(\frac{\delta f}{\delta\psi_0^\alpha}\frac{\delta g}{\delta\pi^0_\alpha} + \frac{\delta f}{\delta\pi^0_\alpha}\frac{\delta g}{\delta\psi_0^\alpha}\right)\Bigg]\,,\end{aligned} \tag{23}$$

and the first class Hamiltonian (18) reduces to

$$H_1 = \int d^{D-1}x \left(i(D-2)\bar{\psi}_0\gamma^0\not{\nabla}\zeta - \frac{i(D-2)(D-3)}{2}\bar{\zeta}\not{\nabla}\zeta + \frac{i}{2}\bar{\xi}^i\not{\nabla}\xi_i + \pi^0_\alpha\mu_0^\alpha \right), \tag{24}$$

where the first class secondary constraint $\varphi \approx 0$ is equivalent to $\not{\nabla}\zeta \approx 0$.

The question now is whether one should add this secondary first class constraint to the Hamiltonian as an independent gauge generator. This is equivalent to asking whether the Dirac conjecture (**DC**) holds in this case, namely, whether all secondary first class constraints generate gauge transformations. If the conjecture is valid, the gauge transformations generated by φ would require gauge fixing; if that is not the case, φ does not generate gauge transformations, it should not be included in the Hamiltonian and no gauge fixing would be required.

One can examine the effect of adding φ to the Hamiltonian (24) with a Lagrange multiplier. The time evolution defined by $\dot{f} = \{f, H'\}$, with respect to the *extended Hamiltonian* $H' := H_1 + \tau^\alpha\varphi_\alpha$, is

$$\dot{\xi}_i = -\gamma_0\not{\nabla}\xi_i\,, \qquad \dot{\lambda} = -(D-3)\gamma_0\not{\nabla}\zeta + \not{\nabla}\psi_0 + \not{\nabla}\tau\,, \tag{25}$$

$$\dot{\psi}_0 = -\mu_0\,, \qquad \pi^0 = 0\,, \qquad \dot{\zeta} = 0, \qquad \not{\nabla}\zeta = 0\,. \tag{26}$$

The gauge symmetry generated by π^0 is fixed by specifying ψ_0, which can be chosen to implement the standard γ-traceless condition in (6) as $\psi_0 + \gamma_0\gamma^i\psi_i \approx 0$. This, together with $\pi^0 \approx 0$, form a pair of second class constraints that can be readily eliminated from the phase space. This gauge choice is accessible since π^0 generates arbitrary shifts in ψ_0 and, in particular, the shift $\delta\psi_0 = -(\psi_0 + \gamma_0\gamma^i\psi_i)$, renders $\psi'_\mu = \psi_\mu + \delta\psi_\mu$ γ-traceless. Thus in the phase space spanned by ξ_i, ζ, λ, using (11) reduces the system (25) to

$$\dot{\xi}_i = -\gamma_0 \nabla\!\!\!/ \xi_i\,, \qquad \dot{\lambda} = \gamma_0 \nabla\!\!\!/ \lambda + \nabla\!\!\!/ \tau\,, \qquad \dot{\zeta} = 0 = \nabla\!\!\!/ \zeta\,. \tag{27}$$

Assuming the DC as valid, would imply that the evolution of λ is indeterminate, from the presence of the arbitrary function of time τ, and therefore an external gauge condition in convolution with the constraint $\varphi \approx 0$ would be necessary. Choosing the gauge condition $\lambda \approx 0$, the stationary condition $\dot{\lambda} \approx 0$ determines the Lagrange multipliers, $\tau = 0$. This removes the spin–$\frac{1}{2}$ sector and the only propagating field is ξ_i. In this case, the Hamilton approach, with the Dirac conjecture assumed to be valid, do not match the Euler-Lagrange equations (12).

The framework that matches the Lagrangian approach is the one where the DC is not assumed. Then φ would not be regarded as a gauge generator, and it should not be added to H_1. This is equivalent to setting $\tau = 0$ in (27), and we reproduce the Euler-Lagrange equations (12) and (7) in the gauge $\partial^\mu\rho_\mu = 0$, whose solution is given by the spin–$\frac{1}{2}$ –spin–$\frac{3}{2}$ system (13).

The two scenarios presented above are consistent. Although in the first case the resulting Hamiltonian evolution is not equivalent to the Lagrangian dynamics, it yields a physical subsystem. There are Lagrangian models whose Hamiltonian formulation leads to secondary first class constraints that do not generate gauge transformations [20,21,23]. For those counterexamples to the DC it is still possible to *postulate* the validity of the conjecture without running into inconsistencies. Moreover it has been argued that not adopting the Dirac conjecture might lead to problems in the quantization, which supports the idea that it would be safer to assume the validity of the DC in general [23].

On the other hand, it seems unnecessary to postulate the DC in our case; the resulting system is still consistent and in agreement with the Lagrangian description, and the Dirac bracket (23) does not lead to quantization problems of the sort found in the counterexample of the DC in [23]: the Dirac field can be quantized. In addition, in Chapter 3 of [23] the DC is shown to follow from Dirac's constrained Hamiltonian analysis for dynamical systems in which first and second class constraints do not mix in the process. As noted above (22), this condition does not hold here since the secondary constraint $\{H, \pi^0\} = \varphi = \tilde{\varphi} - i\partial_i\chi^i$ is a linear combination of first class and second class constraints. Furthermore, the constraint $\tilde{\varphi}$ (22) is a mixture of a secondary first class constraint and a second class one. Since it mixes both types, it does not have the form required by the proof of the DC presented in [23]. For a critical discussion on the DC see [24,25].

If the DC is not valid because some secondary first class constraints do not generate gauge transformations, there is no need to provide a gauge condition for those constraints, and the standard formula for the counting of degrees of freedom [22,23,27] generalizes as

$$2\times\begin{bmatrix}\text{Number of}\\ \text{d.o.f.}\end{bmatrix} = \begin{bmatrix}\text{Dimension of}\\ \text{phase space}\end{bmatrix} - \begin{bmatrix}\text{2nd class}\\ \text{constraints}\end{bmatrix} - 2\times\begin{bmatrix}\text{1st class}\\ \text{gauge}\\ \text{generators}\end{bmatrix} - \begin{bmatrix}\text{1st class}\\ \text{non-gauge}\\ \text{generators}\end{bmatrix}.$$

Note that the last term on the right hand side could be odd, leading to a paradoxical (possibly inconsistent) quantum scenario. However in systems of spinors, first class constraints have an even number of components and therefore not necessarily inconsistent. For the RS system in 4 dimensions, this counting gives $(16 \times 2 - 12 - 2 \times 4 - 4)/2 = 4$ degrees of freedom, which correspond to two spin–$\frac{3}{2}$ helicities plus two spin–$\frac{1}{2}$ helicities. In references [8, 11–14], on the other hand, the DC is assumed to be valid, concluding that there are only 2 degrees of freedom, those of a massless spin–$\frac{3}{2}$ field.

4 Conclusions

The apparent presence of a propagating spin–$\frac{1}{2}$ mode in the RS system contradicts the expectation that the spin–$\frac{1}{2}$ field is a pure gauge mode. A dynamical spin–$\frac{1}{2}$ mode in RS sounds similar to the claim that there is a propagating spin-0 field in the Maxwell theory. However, in contrast to what happens in gauge theories like Maxwell, Yang-Mills or Chern-Simons when evaluated on a pure gauge configuration like $A_\mu = \Lambda^{-1}\partial_\mu \Lambda$, the RS action neither vanishes nor reduces to a boundary term when evaluated on $\psi_\mu = \gamma_\mu \zeta$ for a generic ζ. This means that configurations $\psi_\mu = \gamma_\mu \zeta$ are not zero-modes of the action, unlike what happens in gauge theories for pure gauge configurations. The reduction $\psi_\mu = \gamma_\mu \zeta$ is precisely what is done in unconventional supersymmetry [28–32], while in supergravity the complementary option is selected by imposing $\gamma^\mu \psi_\mu = 0$ [8].

As for quantization issues, the spin–$\frac{3}{2}$ sector of the massless RS field has been quantized in various approaches [11, 12, 14, 33]. In all of them, both spin–$\frac{1}{2}$ sectors of the Poincaré group decomposition are factored out. Following reference [33]—where it is shown that the massless RS field decomposes in a spin–$\frac{1}{2}$ (pure gauge) sector with 0-norm, and spin–$\frac{1}{2}$ and spin–$\frac{3}{2}$ sectors of positive norm—the massless RS can be quantized à la Gupta-Bleuler factoring out only the zero norm state.

So far we have assumed a flat spacetime, although the generalization to a curved background is straightforward. In the light of these results, it would be interesting to consider supergravity theories without enforcing the validity of the Dirac conjecture, which must contain a spin–$\frac{1}{2}$ excitation along with the gravitino. The spin–$\frac{1}{2}$ sector will inherit the gravity and gauge interactions of the vector spinor, which would generate new supergravity phenomenology.

Acknowledgements

We warmly thank discussions with L. Andrianopoli, G. Bossard, N. Boulanger, C. Bunster, A. Marrani, R. Matrecano, R. Noris, M. Rausch de Traubenberg, P. Sundell, M. Trigiante for their challenging questions, comments and suggestions.

Funding information This work was partially funded by grant FONDECYT 1220862 and USS-VRID project VRID-INTER22/10.

References

[1] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, Proc. R. Soc. Lond., A. Math. Phys. Sci. **173**, 211 (1939), doi:10.1098/rspa.1939.0140.

[2] W. Rarita and J. Schwinger, *On a theory of particles with half-integral spin*, Phys. Rev. **60**, 61 (1941), doi:10.1103/PhysRev.60.61.

[3] K. Johnson and E. C. G. Sudarshan, *Inconsistency of the local field theory of charged spin 3/2 particles*, Ann. Phys. **13**, 126 (1961), doi:10.1016/0003-4916(61)90030-6.

[4] L. M. Nath, B. Etemadi and J. D. Kimel, *Uniqueness of the interaction involving spin 3/2 particles*, Phys. Rev. D **3**, 2153 (1971), doi:10.1103/PhysRevD.3.2153.

[5] M. Benmerrouche, R. M. Davidson and N. C. Mukhopadhyay, *Problems of describing spin-3/2 baryon resonances in the effective Lagrangian theory*, Phys. Rev. C **39**, 2339 (1989), doi:10.1103/PhysRevC.39.2339.

[6] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress toward a theory of supergravity*, Phys. Rev. D **13**, 3214 (1976), doi:10.1103/PhysRevD.13.3214.

[7] S. Deser and B. Zumino, *Consistent supergravity*, Phys. Lett. B **62**, 335 (1976), doi:10.1016/0370-2693(76)90089-7.

[8] P. van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68**, 189 (1981), doi:10.1016/0370-1573(81)90157-5.

[9] R. Ducrocq, M. Rausch de Traubenberg and M. Valenzuela, *A pedagogical discussion of $N = 1$ four-dimensional supergravity in superspace*, Mod. Phys. Lett. A **36**, 2130015 (2021), doi:10.1142/S0217732321300159.

[10] M. Rausch de Traubenberg and M. Valenzuela, *A supergravity primer*, World Scientific, Singapore, ISBN 9789811210518 (2020), doi:10.1142/11557.

[11] G. Senjanović, *Hamiltonian formulation and quantization of the spin-3/2 field*, Phys. Rev. D **16**, 307 (1977), doi:10.1103/PhysRevD.16.307.

[12] M. Pilati, *The canonical formulation of supergravity*, Nucl. Phys. B **132**, 138 (1978), doi:10.1016/0550-3213(78)90262-6.

[13] S. Deser, J. H. Kay and K. S. Stelle, *Hamiltonian formulation of supergravity*, Phys. Rev. D **16**, 2448 (1977), doi:10.1103/PhysRevD.16.2448.

[14] E. S. Fradkin and M. A. Vasiliev, *Hamiltonian formalism, quantization and S matrix for supergravity*, Phys. Lett. B **72**, 70 (1977), doi:10.1016/0370-2693(77)90065-X.

[15] R. E. Behrends and C. Fronsdal, *Fermi decay of higher spin particles*, Phys. Rev. **106**, 345 (1957), doi:10.1103/PhysRev.106.345.

[16] V. I. Ogievetsky and E. S. Sokatchev, *Equations of motion for superfields*, in *4th international conference on nonlocal quantum field theory*, Alushta, USSR (1976).

[17] M. Valenzuela and J. Zanelli, *Massless Rarita-Schwinger equations: Half and three halves spin solution*, (arXiv preprint) doi:10.48550/arXiv.2305.00106.

[18] A. Das and D. Z. Freedman, *Gauge quantization for spin-3/2 fields*, Nucl. Phys. B **114**, 271 (1976), doi:10.1016/0550-3213(76)90589-7.

[19] P. A. M. Dirac, *Lectures on quantum mechanics*, Dover Publications, New York, USA, ISBN 9780486417134 (2001).

[20] R. Cawley, *Determination of the Hamiltonian in the presence of constraints*, Phys. Rev. Lett. **42**, 413 (1977), doi:10.1103/PhysRevLett.42.413.

[21] M. J. Gotay, *On the validity of Dirac's conjecture regarding first-class secondary constraints*, J. Phys. A Math. Gen. **16**, L141 (1999), doi:10.1088/0305-4470/16/5/003.

[22] M. Henneaux, C. Teitelboim, and J. Zanelli, *Gauge invariance and degree of freedom count*, Nucl. Phys. B **332**, 169 (1990), doi:10.1016/0550-3213(90)90034-B.

[23] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton, USA, ISBN 9780691037691 (1992).

[24] J. M. Pons, *On Dirac's incomplete analysis of gauge transformations*, Stud. Hist. Philos. Sci. B **36**, 491 (2005), doi:10.1016/j.shpsb.2005.04.004.

[25] J. Earman, *Tracking down gauge: An ode to the constrained Hamiltonian formalism*, in *Symmetries in physics: Philosophical reflections*, Cambridge University Press, Cambridge, UK, ISBN 9780521821377 (2003).

[26] M. Henneaux, C. Teitelboim and J. Zanelli, *Gauge invariance and degree of freedom count*, Nucl. Phys. B **332**, 169 (1990), doi:10.1016/0550-3213(90)90034-B.

[27] A. J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian systems*, Accademia Nazionale dei Lincei, Rome, Italy (1976).

[28] P. D. Alvarez, M. Valenzuela and J. Zanelli, *Supersymmetry of a different kind*, J. High Energy Phys. **04**, 058 (2012), doi:10.1007/JHEP04(2012)058.

[29] P. D. Alvarez, M. Valenzuela and J. Zanelli, *Role of gravity in particle physics: A unified approach*, Int. J. Mod. Phys. D **29**, 2041012 (2020), doi:10.1142/S0218271820410126.

[30] P. D. Alvarez, M. Valenzuela and J. Zanelli, *Chiral gauge theory and gravity from unconventional supersymmetry*, J. High Energy Phys. **07**, 205 (2020), doi:10.1007/JHEP07(2020)205.

[31] P. D. Alvarez, L. Delage, M. Valenzuela and J. Zanelli, *$\mathcal{N} = 2$ extended MacDowell-Mansouri supergravity*, J. High Energy Phys. **07**, 176 (2021), doi:10.1007/JHEP07(2021)176.

[32] P. D. Alvarez, L. Delage, M. Valenzuela and J. Zanelli, *Unconventional SUSY and conventional physics: A pedagogical review*, Symmetry **13**, 628 (2021), doi:10.3390/sym13040628.

[33] W. Heidenreich, *On solution spaces of massless field equations with arbitrary spin*, J. Math. Phys. **27**, 2154 (1986), doi:10.1063/1.527037.

Extensions of realisations for low-dimensional Lie algebras

Iryna Yehorchenko⋆†

Institute of Mathematics, National Academy of Sciences of Ukraine, Ukraine
Institute of Mathematics, Polish Academy of Sciences, Poland

⋆ iyegorch@imath.kiev.ua , † iyehorchenko@impan.pl

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We find extensions of realisations of some low-dimensional Lie algebras, in particular, for the Poincaré algebra for one space dimension. Using inequivalent extensions, we performed comprehensive classification of relative differential invariants for these Lie algebras. We show difference between classification of extensions of realisations, and classification of nonlinear realisations of Lie algebras.

Received 19-12-2022
Accepted 30-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.048

1 Introduction

Work on classification of extended realisations of Lie algebras was motivated by the paper by M. Fels and P. Olver [1] that presents importance and need to have a procedure to classify Lie algebras that are used in various fields (see e.g. references therein). Mathematical relevance of such problem is its importance for the classical invariance theory and theory of special functions, and relevance in physics and practical applications, beyond quantum mechanics and computer vision, is the need to provide symmetry classification of mathematical models for different processes. Relative differential invariants, RDIs (not only absolute invariants, ADIs) shall be needed for full group classification of PDEs invariant under some Lie algebras. Classification of RDIs is an interesting problem of the abstract algebra by itself, even without any relevance to differential equations and their symmetry. RDIs were already applied for characterisation of some ordinary differential equations [2], and I believe that it is possible to find a method for characterisation of partial differential equations using differential invariants of equivalence groups. Extensions of realisations may be used not only for classification of RDIs, but in any problems of symmetry classification of models when we increase the set of variables but want to preserve symmetry of the initial model. Another application of RDIs is study of singularities of curves and manifolds.

The methods for full description and classification of absolute differential invariants are well-known. However, it appeared more difficult to find a way to classify RDIs. For more information on the story behind this work see [3]. Some mathematicians (e.g. Oliver Glenn in [4]) claimed that they know how to classify the RDIs, but the reference [5] he cited does not contain the solution. The paper that inspired me to solve this problem [1] also did not contain a solution, and I presented my method of extended realisations of Lie algebras at the SPT-2007 conference [6]. Later I was informed by Pavel Winternitz that there exists an earlier alternative solution of this problem of classification of RDIs in their papers published in 1991 and 1993 [7] and [8] (these papers considered finding full sets of invariant differential equations, but their approach also completely solves a problem of classification of RDIs). However, I believe that the procedure proposed in [6] is much simpler. This procedure also allows to apply to RDIs all extensive theory for ADIs, e.g. ability to construct operators of invariant differentiation and fundamental bases of ADIs, enabling description of invariants of any order.

Here I present some new examples of classification of extended realisations that may be extended to higher-dimensional algebras. I also compare classification of extended realisations and classification of nonlinear realisations using an example of the Poincaré algebra for one space dimension, and show that these problems are different and produce different solutions.

Let us start with a realisation of a Lie algebra with the basis operators $L = \langle Q_m \rangle$,

$$Q_m = \zeta_{ms}(x_i)\partial_{x_s}\,, \tag{1}$$

where x_i are some variables that may be regarded as dependent or independent in construction of some equations or differential invariants, and ζ_{ms} are some functions of x_i. In the examples there dependent variables are designated specifically, we may use other letters. Hereinafter we will imply summation over the repeated indices, if not specially indicated otherwise.

We take additional variables R_k, and study extended action operators $\hat{Q}_m = Q_m + \lambda_{mjk} R_j \partial_{R_k}$ that form the same Lie algebra with the same structural constants. These additional variables provide classification of symmetries for mathematical models in spaces extended with these variables, but preserving symmetries of original models. In specific application of classification of RDIs, these additional variables will be excluded from functional bases of ADIs of extended realisations to get bases of RDIs for the original realisations. For a specific realisation of any Lie algebra L, we can classify all inequivalent extended action realisations for a finite number of additional variables.

We consider construction of extensions for realisations of low-dimensional Lie algebras that were listed and classified in [9, 10]. Here we will deal only with one- and two-dimensional algebras, and the three-dimensional Poincaré algebra for one space variable with the aim to present main ideas. The problem of classification of the extended realisations (not only linear) is interesting by itself, but for classification of relative invariants of Lie algebras we need specifically only linear extensions with nonzero coefficients at $R\partial_R$.

Definition 1. A function Θ depending on x, u and on partial derivatives u of order up to l (Θ may designate a set of functions $(\Theta_1, ..., \Theta_N)$) is called an RDI for the Lie algebra $L = \langle Q_m \rangle$, if it is an invariant of the l-th Lie prolongation of this algebra:

$$\overset{l}{Q}_m \Theta(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = \lambda_m(x, u, \underset{1}{u}, \ldots, \underset{l}{u})\Theta\,,$$

where λ_m are some functions; if $\lambda_m = 0$, Θ is an absolute differential invariant (ADI) of the algebra L; if $\lambda_m \neq 0$, it is a proper RDI.

Definition 2. A maximal set of functionally independent invariants of the order $r \leq l$ of a Lie algebra L is a functional basis of differential invariants of the order l for the algebra L.

Note that we cannot treat a set of independent RDIs the same way as we would treat a set of ADIs – a function of RDIs may be not invariant. E.g. linear combinations of RDIs generally speaking will not be RDIs. In the case when the task is to construct general extension operators for operators (1), we look for the extensions in the form

$$\hat{Q}_m = Q_m + a_m(x_i, R)\partial_R \,. \tag{2}$$

Definition 3. Extension of a realisation of the Lie algebra L with the basis operators of the form (1) is constructed of the same operators with more variables added (2).

To find scalar RDIs, we need to add just one new variable, and to find only linear extensions.

$$\hat{Q}_m = Q_m + a_m(x_i)R\partial_R \,. \tag{3}$$

I used the lists of non-equivalent realisations of two-dimensional Lie algebras in [9, 10]. Please see the additional references there. We must note that the idea of equivalence/non-equivalence for the extended realisation accounts for classification of the RDIs - the difference with classification just with respect to the local transformation will be seen on the example of the translation operators. Our classification of the linear extensions has in mind the following definition of equivalence of RDIs.

Definition 4. Two RDIs of the algebra L are called equivalent, if they can be transformed one into another by some transformation from the equivalence group of the relative invariance conditions.

$$\overset{l}{Q}_m R(x, u, \underset{1}{u}, \dots, \underset{l}{u}) = \lambda_m(x, u, \underset{1}{u}, \dots, \underset{l}{u})R \,.$$

We can also consider equivalence of pairs (R, λ) of RDIs with their respective multiplicators. It may be useful for practical purposes of description of invariant differential equations, as a linear combination of RDIs with the same multiplicator will also be an RDI and may be used to construct an invariant differential equation.

The procedure for description of RDIs proposed in [6]: 1) Construct Lie prolongations of the operators Q_m; 2) write operators of extended action; 3) classify realisations of the extended action up to transformations from equivalence group of the invariance conditions; 4) find a functional basis of ADIs for the inequivalent realisations; 5) construct RDIs and ADIs of the algebra L from absolute invariants of operators of extended action by elimination of ancillary variables. Note that ancillary variables R may enter the ADIs of the operators of the extended realisation as multipliers of the form $R^K, K \neq 0$ are some integers - the ADIs having the form FR^K, where F are some functions of dependent and independent variables, and derivatives of the dependent variables of the relevant order, will produce the RDIs of the form F.

Let us remind some properties of RDIs - a product of RDIs is also an RDI, an RDI in a non-zero degree will also be an RDI.

2 Extensions of algebras of translation operators

We start from a seemingly very easy case, that is a one-dimensional Lie algebra. It should be considered anyway for the purpose of comprehensive presentation. It is a Lie algebra whose basis consists of one operator. Any one separate first-order differential operator obviously forms a Lie algebra, and is locally equivalent to a translation operator.

Let us first construct a linear extension for the translation operator ∂_x. We will look for it in the form $\hat{Q} = \partial_x + R(x)F\partial_F$. It is easy to check that an arbitrary $R(x)$ will satisfy the commutator criterion for this algebra.

As to the standard classification up to equivalence with respect to the local transformations, we may use the Lie theorem on straightening out of vector fields as it was done e.g. in [11] (see [12]). However, though the operator $\hat{Q} = \partial_x + a(x)R\partial_R$ is certainly locally equivalent to $Q = \partial_x$, for our purpose of the RDI classification we need to consider operators with a non-vanishing coefficient at ∂_R. It is easy to see that a non-extended $Q = \partial_x$ does not produce any RDIs, but the extended operator gave us an RDI $R = \exp x$. This RDI, as well as other exponential RDIs of the translation operators, is not useful at all to describe invariant equations, but it should be listed if we aim at obtaining a comprehensive classification of RDIs. So, the procedure of classification of RDIs requires classification of linear extended algebras with nonzero coefficients at $R\partial_R$.

Proper classification procedures are described e.g. in [13] and in references therein, as well as in the papers on classification of realisations of low-dimensional Lie algebras of the first-order differential operators [9, 10]. Such classification in the case of operators considered here shall find algebras or single operators whose actions are not equivalent under local transformations of the following form: $\tilde{x} = \kappa(x, R)$, $\tilde{R} = \phi(x, R)$, $\tilde{Q}_m = \tilde{a}_m(\tilde{x})\partial_{\tilde{x}} + \tilde{b}_m(\tilde{x})\tilde{R}\partial_{\tilde{R}}$. Similar criteria would be also relevant for algebras involving more variables. However, for the purposes of this paper we can easily check equivalence or non-equivalence using invariants of the relevant algebras or operators.

3 Two-dimensional Lie algebras

Using the procedure, proposed in [6], we classify extended realisations of the following two-dimensional Lie algebras

$$(a)\ \partial_x\,,\quad x\partial_x\,,\qquad (b)\ \partial_x\,,\quad y\partial_x\,,\qquad (c)\ \partial_x\,,\quad x\partial_x + \partial_y\,, \tag{4}$$

excluding from our consideration here the algebras that consist only of the translation operators. The commutator of operators (4 (a)) is ∂_x. We consider the extension $\partial_x + a(x, y)R\partial_R$, $x\partial_x + b(x, y)R\partial_R$. From the commutation relations $b_x - xa_x = a$, we get determining conditions for the coefficients $-xa = \phi(y)$; a is arbitrary. We get $b = xa(x, y) + \phi(y)$, and the general form of the extended operators $\partial_x + a(x, y)R\partial_R$, $x\partial_x + (xa(x, y) + \phi(y))R\partial_R$. Similarly we construct extensions for algebras (b) and (c).

Table 1

	Basis Operators	General Extended Basis Operators	Inequivalent Extended Basis Operators
1	$\partial_x, x\partial_x,$	$\partial_x + a(x, y)R\partial_R,$ $x\partial_x + (xa(x, y) + \phi(y))R\partial_R,$	$\partial_x + R\partial_R,$ $x(\partial_x + R\partial_R) + \epsilon R\partial_R,$
2	$\partial_x,\ y\partial_x,$	$\partial_x + a(x, y)R\partial_R,$ $y\partial_x + (ya(x, y) + \phi(y))R\partial_R,$	$\partial_x + R\partial_R,$ $y\partial_x + (y + \epsilon)R\partial_R,$
3	$\partial_x,\ x\partial_x + \partial_y,$	$\partial_x + \Phi_x(x, y)R\partial_R,$ $x\partial_x + (\Phi_y - x\Phi_x)R\partial_R,$	$\partial_x + R\partial_R, \quad x\partial_x + R\partial_R.$

$a(x, y)$, $\phi(y)$, $\Phi(x, y)$ are arbitrary sufficiently smooth functions; ϵ is equal to 0 or 1.

Operators listed in Table 1 would allow calculation of zero-order RDIs. Finding higher-order RDIs would require finding extensions of the prolongations of operators of the initial realisation to the relevant order. We would like to point out that here we need extensions of the relevant prolongations of the operators being considered - not prolongations of extensions.

Here we will look for the **functional bases** of the RDIs up to the second order of derivatives. We find invariants for two independent variables x and y, and one dependent variable u. Let us point out that finding differential invariants depends essentially from choice and assignment of dependent and independent variables.

Table 2

	Extended Second Prolongations	Functional Bases of ADIs	Functional Bases of RDIs
1	$\partial_x + R\partial_R$, $x\partial_x - u_x\partial_{u_x} - u_{xx}\partial_{u_{xx}} - u_{xy}\partial_{u_{xy}} + R\partial_R$;	y, u, u_y, u_{yy}, u_xR, $u_{xx}R, u_{xy}R$;	y, u, u_y, u_{yy}, u_x, u_{xx}, u_{xy},
2	$\partial_x + R\partial_R$, $y\partial_x - u_x\partial_{u_y} - u_{xx}\partial_{u_{xy}} - u_{xy}\partial_{u_{yy}} + R\partial_R$,	y, u, u_x, u_{xx}, $\exp\frac{u_y}{u_x}R$, $u_xu_{xy} - u_yu_{xx}$, $u_{xy}^2 - 2u_{xx}u_{yy}$,	y, u, u_x, u_{xx}, $\exp u_y$, $u_xu_{xy} - u_yu_{xx}$, $u_{xy}^2 - 2u_{xx}u_{yy}$,
3	$\partial_x + R\partial_R$, $x\partial_x + \partial_y - u_x\partial_{u_x} - u_{xx}\partial_{u_{xx}} - u_{xy}\partial_{u_{xy}} + R\partial_R$,	$\frac{\exp y}{R}$, u, u_y, u_{yy}, u_xR, $u_{xx}R$, $u_{xy}R$,	$\exp y$, u, u_y, u_{yy}, u_x, u_{xx}, u_{xy}.

4 Poincaré algebra for one space dimension

Many extensions were studied for many famous algebras of the mathematical physics without limitations of linearity and without any relation to finding RDIs (see e.g. [14] for the Poincaré algebra $P(1,2)$, and [11] for $P(1,1)$). These nonlinear realisations were used to find their differential invariants and whence new equations invariant under these algebras.

We will illustrate difference in the problem of classification of the general extensions for Lie algebras and of the extensions with the purpose of the RDI classification on the example of the Poincaré algebra with two independent variables t and x and one dependent variable u:

$$\partial_t\,, \qquad \partial_x\,, \qquad J = t\partial_x + x\partial_t\,. \tag{5}$$

A functional basis of the second-order ADIs can be written as follows [11]:

$$I_1 = u\,, \qquad I_2 = u_t^2 - u_x^2\,, \qquad I_3 = u_{tt} - u_{xx}\,, \tag{6}$$
$$I_4 = (u_t - u_x)^2(u_{tt} + 2u_{tx} + u_{xx})\,, \qquad I_5 = (u_t + u_x)^2(u_{tt} - 2u_{tx} + u_{xx})\,.$$

Setting I_4, I_5 to zero, the authors actually obtained expressions that are RDIs

$$AR_1 = u_t - u_x\,, \quad AR_2 = u_t + u_x\,, \quad AR_3 = u_{tt} + 2u_{tx} + u_{xx}\,, \quad AR_4 = u_{tt} - 2u_{tx} + u_{xx}\,, \tag{7}$$

by means of listing invariant equations of the type $AR_i = 0$, but did not mention the concept of a relative differential invariant, and did not give any statements on full classification of such invariants.

Let us look at the extension of the standard realisation of the Poincaré algebra (5) constructed with the aim of classification of RDIs.

$$\partial_t + a(t,x)R\partial_R\,, \qquad \partial_x + b(t,x)R\partial_R\,, \qquad J = t\partial_x + x\partial_t + c(t,x)R\partial_R\,. \tag{8}$$

From the commutation relations $[P_t, P_x] = 0$, $[P_t, J] = P_x$, $[P_x, J] = P_t$ we get conditions on the functions $a(t,x)$, $b(t,x)$, $c(t,x)$: $a_x = b_t$, $c_t - ta_x - xa_t = b$, $c_x - tb_x - xb_t = a$, whence $a(t,x) = \Phi_t$, $b(t,x) = \Phi_x$, $c(t,x) = t\Phi_x + x\Phi_t + C$, where $\Phi = \Phi(t,x)$ is an arbitrary sufficiently smooth function of its arguments, and $C = const$.

Up to local equivalence and on condition of non-zero coefficients at $R\partial_R$, we obtain the following realisation

$$\partial_t + R\partial_R\,, \qquad \partial_x + R\partial_R\,, \qquad J = t(\partial_x + R\partial_R) + x(\partial_t + R\partial_R) + \epsilon R\partial_R\,, \tag{9}$$

where ϵ is equal to 0 or 1.

Operators (9) with $\epsilon = 0$ give from its functional basis of ADIs $R^{-1}\exp t$, $R^{-1}\exp x$ a set of RDIs $\exp t$, $\exp x$.

We can find an extended prolongation of realisation (5) using the commutation relations similarly, and obtain:

$$\begin{aligned}&\partial_t + R\partial_R\,, \qquad \partial_x + R\partial_R\,,\\ &J = t(\partial_x + R\partial_R) + x(\partial_t + R\partial_R) - u_t\partial_{u_x} - u_x\partial_{u_t} - u_{tt}\partial_{u_{xt}} - 2u_{xt}(\partial_{u_{xx}} + \partial_{u_{tt}}) - u_{xx}\partial_{u_{xt}} + \epsilon R\partial_R\,.\end{aligned} \tag{10}$$

Operators (10) with $\epsilon = 1$ give from its functional basis of first-order ADIs

$$(u_t + u_x)R^{-1}\,, \qquad (u_t - u_x)R\,, \tag{11}$$

first-order RDIs for the algebra (5) $u_t + u_x$, $u_t - u_x$.

The determining equation for the second-order ADIs of the form $F = F(u_{tt}, u_{xt}, u_{xt})$ of (5), will look as follows: $2u_{xt}(F_{u_{xx}} + F_{u_{tt}}) + (u_{xx} + u_{tt})F_{u_{xt}} + RF_R = 0$. The resulting second-order ADIs are

$$u_{tt} - u_{xx}\,, \qquad (u_{tt} + 2u_{xt} + u_{xx})R^{-2}\,, \qquad (u_{tt} - 2u_{xt} + u_{xx})R^2\,. \tag{12}$$

In the same way we obtain two inequivalent proper second-order RDIs: $u_{tt} + 2u_{xt} + u_{xx}$, $u_{tt} - 2u_{xt} + u_{xx}$.

An invariant $u_{tt} - u_{xx}$ is an ADI of (5), so we do not include it into the list of proper RDIs.

Products of invariants in the relevant degrees from the lists (11), (12) to eliminate ancillary variables R will give absolute invariants from the list (6). So to describe all non-equivalent RDIs up to the second order of the realisation being considered it is sufficient to take only zero-and first-order RDIs in addition to the list of ADIs.

The general extensions of the realisation (5) were studied in [11] , and a new extended realisation was found:

$$\partial_t\,, \qquad \partial_x\,, \qquad J = t\partial_x + x\partial_t + u\partial_u\,, \tag{13}$$

that is not locally equivalent to (5), as well as ADIs of (13) up to the second order.

$$\begin{gathered}A_1 = u_t + u_x\,, \qquad A_2 = (u_t - u_x)u^{-2}\,, \qquad A_3 = (u_{tt} - u_{xx})u^{-1}\,,\\ A_4 = (u_{tt} + 2u_{tx} + u_{xx})u\,, \qquad A_5 = (u_{tt} - 2u_{tx} + u_{xx})u^{-3}\,.\end{gathered}$$

To obtain a comprehensive classification of RDIs for this extended realisation, we need to extend it further and to consider the realisation

$$P_t = \partial_t + a(t,x,u)R\partial_R\,, \qquad P_x = \partial_x + b(t,x,u)R\partial_R\,, \qquad J = t\partial_x + x\partial_t + u\partial_u + c(t,x,u)R\partial_R\,.$$

From the commutation relations $[P_t, P_x] = 0$, $[P_t, J] = P_x$, $[P_x, J] = P_t$ we get conditions on the functions $a(t,x,u)$, $b(t,x,u)$, $c(t,x,u)$:

$$a_x = b_t\,, \qquad c_t - ta_x - xa_t - ua_u = b\,, \qquad c_x - tb_x - xb_t - ub_u = a\,,$$

whence

$$a(t,x,u) = \Phi_t\,, \qquad b(t,x,u) = \Phi_x\,, \qquad c(t,x,u) = t\Phi_x + x\Phi_t + u\Phi_u + C\,,$$

where $\Phi = \Phi(t,x,u)$ is an arbitrary sufficiently smooth function, and $C = const$.

Up to local equivalence and on condition of non-zero coefficients at $R\partial_R$, we obtain

$$\partial_t + R\partial_R\,, \qquad \partial_x + R\partial_R\,, \qquad J = t(\partial_x + R\partial_R) + x(\partial_t + R\partial_R) + u\partial_u + \epsilon R\partial_R\,, \tag{14}$$

where ϵ is equal to 0 or 1.

Operators (14) with $\epsilon = 0$ give from its functional basis of ADIs $R^{-1}\exp t$, $R^{-1}\exp x$ a set of RDIs $\exp t$, $\exp x$.

We can find the extended prolongation of realisation (5) using the commutation relations similarly, and obtain:

$$\begin{aligned}
&\partial_t + R\partial_R\,, \qquad \partial_x + R\partial_R\,,\\
&J = t(\partial_x + R\partial_R) + x(\partial_t + R\partial_R) + u\partial_u + u_x\partial_{u_x} + u_t\partial_{u_t} + u_{xx}\partial_{u_{xx}} + 2u_{xt}\partial_{u_{xt}} + u_{tt}\partial_{u_t}\\
&\quad - u_t\partial_{u_x} - u_x\partial_{u_t} - u_{tt}\partial_{u_{xt}} - 2u_{xt}(\partial_{u_{xx}} + \partial_{u_{tt}}) - u_{xx}\partial_{u_{xt}} + \epsilon R\partial_R\,. \qquad (15)
\end{aligned}$$

We can take relative differential invariants as follows:

$$IR_1 = u\,, \qquad IR_2 = u_t + u_x\,, \qquad IR_3 = u_t - u_x\,, \qquad IR_4 = u_{tt} - u_{xx}\,,$$
$$IR_5 = u_{tt} + 2u_{tx} + u_{xx}\,, \qquad IR_6 = u_{tt} - 2u_{tx} + u_{xx}\,.$$

5 Conclusion

We classified extended realisations for two-dimensional Lie algebras and for the Poincaré algebras for one space dimension, and found functional bases of absolute differential invariants for there new inequivalent realisations. These results allowed to classify RDIs for these realisations. It would be interesting to study RDIs for more algebras, and to look for new nonlinear realisations of such algebras.

Acknowledgments

I would like to thank the Institute of Mathematics of the Polish Academy of Sciences for their hospitality and grant support, to the National Academy of Sciences of the USA and the National Centre of Science of Poland for their grant support.

I would like also to mention my good memories and appreciation of my visit to the Centre de recherches mathématiques in Montreal by invitation of late Jiří Patera where I had fruitful discussions with Jiří Patera and Pavel Winternitz, and I also was able to study literature on differential invariants, and, in particular, find papers by Oliver Glenn.

Funding information The research and conference participation of the author were funded by Narodowe Centrum Nauki (Poland), Grant No.2017/26/A/ST1/00189.

References

[1] M. Fels and P. J. Olver, *On relative invariants*, Math. Ann. **308**, 701 (1997), doi:10.1007/s002080050097.

[2] J. Hietarinta and V. Dryuma, *Is my ODE a Painlevé equation in disguise?*, J. Nonlin. Math. Phys. **9**, 67 (2002), doi:10.2991/jnmp.2002.9.s1.6.

[3] I. Yehorchenko, *Extension of realisations for low-dimensional Lie algebras and relative differential invariants*, (arXiv preprint) doi:10.48550/arXiv.2212.05657.

[4] O. E. Glenn, *Complete systems of differential invariants*, Proc. Lond. Math. Soc. **s2**, 72 (1928), doi:10.1112/plms/s2-27.1.72.

[5] O. E. Glenn, *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations*, Am. J. Math. **46**, 37 (1924), doi:10.2307/2370660.

[6] I. Yehorchenko, *Relative invariants for Lie algebras: Construction and applications*, in *Symmetry and perturbation theory*, World Scientific, Singapore, ISBN 9789812776167 (2007), doi:10.1142/9789812776174_0051.

[7] J. F. Cariñena, M. A. del Olmo and P. Winternitz, *Cohomology and symmetry of differential equations*, in *Group theoretical methods in physics*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540540403 (1991), doi:10.1007/3-540-54040-7_118.

[8] J. F. Cariñena, M. A. del Olmo and P. Winternitz, *On the relation between weak and strong invariance of differential equations*, Lett. Math. Phys. **29**, 151 (1993), doi:10.1007/BF00749730.

[9] M. Nesterenko, *Transformation groups on real plane and their differential invariants*, Int. J. Math. Math. Sci. 017410 (2006), doi:10.1155/IJMMS/2006/17410.

[10] R. O. Popovych, V. M. Boyko, M. O. Nesterenko and M. W. Lutfullin, *Realizations of real low-dimensional Lie algebras*, J. Phys. A: Math. Gen. **36**, 7337 (2003), doi:10.1088/0305-4470/36/26/309.

[11] G. Rideau and P. Winternitz, *Nonlinear equations invariant under the Poincaré, similitude, and conformal groups in two-dimensional space-time*, J. Math. Phys. **31**, 1095 (1990), doi:10.1063/1.528789.

[12] P. J. Olver, *Application of Lie groups to differential equations*, Springer, New York, USA, ISBN 9781468402742 (1987), doi:10.1007/978-1-4684-0274-2.

[13] N. Kamran and P. J. Olver, *Equivalence of differential operators*, SIAM J. Math. Anal. **20**, 1172 (1989), doi:10.1137/0520077.

[14] I. A. Yehorchenko, *Differential invariants of nonlinear representation of the Poincaré algebra. Invariant equations*, in *Proceedings of the second international conference: Symmetry in nonlinear mathematical physics*, Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv, Ukraine (1997), https://www.slac.stanford.edu/econf/C9707077/papers/art23.pdf.

SciPost