The Harmonic Oscillator, Enhancements and Applications

Charles Dunkl

University of Virginia

The simple harmonic oscillator is one of the first models taught in undergraduate courses. As simple as it is there are many real-world applications (Foucault pendulum, childrens swings etc.). The quantum version is an instructive model with Hermite functions as wavefunctions, and by using a classical asymptotic formula one can see how the quantum model tends to the classical one as the energy increases. Next, add a potential of inverse-square type and invariant under a reflection group to the N-fold product of the quantum oscillator. The simplest case is the group $\mathbb{Z}_2$ of sign changes of the real line; both the classical and quantum models can be solved, the latter with Laguerre functions as wavefunctions and again there are interesting asymptotic relations. Several studies of the N-dimensional oscillator with the abelian group $\mathbb{Z}_2^N$ of sign-changes (thus the coordinate hyperplanes repel the particle) have been carried out. By constructing different bases of wavefunctions interesting algebras arise in the problem of expressing one basis in terms of another. As well there are results about super- integrability and relativistic modifications. In contrast to the abelian symmetry groups consider a model where the reflection group is nonabelian - in particular the dihedral situation. Here the plane is divided into "pie slices" with equal angles. We present more details about this model, solving for the wavefunctions and describing associated algebras of operators.