Multipoint conformal blocks and Gaudin integrable models

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Based on [2009.11882], in collaboration with Ilija Burić, Jeremy Mann, Lorenzo Quintavalle and Volker Schomerus

Introduction

Introduction: conformal field theories

- Conformal field theories (CFT): quantum field theories (in d dimensions) invariant under conformal transformations
- Conformal transformations: preserve angles, e.g. translations, rotations, dilations

- Important quantities in CFT: N-point correlation functions
- Decompose into elementary building blocks
  $\rightarrow$ conformal blocks

- Bootstrap program: use conformal blocks decomposition to constrain the properties of CFT
- Important to study conformal blocks

Introduction: multipoint conformal blocks

- 4-point conformal blocks: well understood, explicit expressions
- Common eigenvectors of a complete set of commuting differential operators [Dolan Osborn '03]
- Integrable system, equivalent to Calogero-Sutherland model [Isachenkov Schomerus '16]

- Multipoint conformal blocks: not much known for $N > 4$
- Goal: characterise multipoint blocks as common eigenvectors of a complete set of commuting differential operators
- Idea: construct these operators using a limit of the Gaudin model

Contents

Conformal field theories: synopsis

Conformal transformations

- Space-time $\mathbb{R}^{p,q}$ of dimension $d = p + q$ and signature (p, q)
- Transformations preserving the metric (and thus the angles):
  - d translations $x^\mu \mapsto x^\mu + a^\mu$
  - $d(d-1)/2$ rotations/boosts $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$
- Transformations preserving the angles:
  - dilation $x^\mu \mapsto \lambda x^\mu$
  - d special conformal transformations

Conformal transformations

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- Infinitesimal transformations $\mathcal{T}_a$ with $a = 1, \cdots, (d+1)(d+2)/2$
- Commutation relations of the conformal Lie algebra

$$\mathfrak{g} \simeq \mathfrak{so}(p+1, q+1)$$

- Conformal field theory: QFT invariant under $\mathfrak{g}$

Fields

- For simplicity, consider Euclidian $\mathbb{R}^3$, hence $\mathfrak{g} \simeq \mathfrak{so}(4,1)$
- CFT described by quantum fields (operators) $\mathcal{O}(\vec{x})$ on $\mathbb{R}^3$
- Action of $\mathfrak{g}$ on fields?

- Translations $\mathcal{O}(\vec{x}) \mapsto \mathcal{O}(\vec{x}+\vec{a})$, with generators

$$P_\mu = \partial_\mu$$

- Dilation: $\mathcal{O}(\vec{x}) \mapsto \lambda^\Delta \mathcal{O}(\lambda \vec{x})$ with generator

$$D = x^\mu \partial_\mu + \Delta\,\mathrm{Id}$$

- $\Delta \in \mathbb{R}$: conformal weight, measures internal behaviour under dilation

Fields

- Rotations: $\mathcal{O}(\vec{x})$ can have internal rotational degrees of freedom
  $\rightarrow$ spin $\ell \in \frac{\mathbb{N}}{2}$ ($\ell = 0$ for scalar field, $\ell = 1$ for vector field, ...)
- For scalar fields: $\mathcal{O}(\vec{x}) \mapsto \mathcal{O}(R\vec{x})$ with generators

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

- For spinning fields ($\ell \neq 0$), $L_{\mu\nu}$ include internal transformations

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- Δ and ℓ characterise the behaviour of $\mathcal{O}$ under $\mathfrak{g}$
- Generators $T_a = P_\mu, D, L_{\mu\nu}, \cdots$ form a representation $V_{\Delta,\ell}$ of $\mathfrak{g}$
- Spectrum: different values of (Δ, ℓ) appearing in the CFT
  $\rightarrow$ one of the characteristic of the theory

Casimir operators

- How to measure the conformal weight Δ and spin ℓ?
  $\rightarrow$ Casimir operators
- Simpler example: rotations $\mathfrak{so}(3)$ with infinitesimal generators $\vec{L}$
- Possesses a Casimir operator $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ $\qquad \left(\left[\vec{L}^2, L_\mu \right] = 0 \right)$
- On a representation of spin ℓ:

$$\vec{L}^2 = \ell(\ell+1)\,\mathrm{Id}$$

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- Back to the conformal algebra $\mathfrak{g} \simeq \mathfrak{so}(4,1)$
- Two Casimir operators $\mathcal{D}_2$ and $\mathcal{D}_4$, quadratic and quartic

$$\mathcal{D}_2 = \kappa_2^{ab} \mathcal{T}_a \mathcal{T}_b \qquad \text{and} \qquad \mathcal{D}_4 = \kappa_4^{abcd} \mathcal{T}_a \mathcal{T}_b \mathcal{T}_c \mathcal{T}_d$$

- Take values $\mathcal{C}_2^{(\Delta,\ell)}$ and $\mathcal{C}_4^{(\Delta,\ell)}$ on $V_{\Delta,\ell}$

$$\mathcal{C}_2^{(\Delta,\ell)} = \Delta(\Delta-3) + \ell(\ell+1)$$

Correlation functions and conformal blocks

Correlation functions and Ward identities

- Correlation function of N scalar fields $\mathcal{O}_i$ (with weights Δ_i):

$$G_N(\vec{x}_1, \cdots, \vec{x}_N) = \langle \mathcal{O}_1(\vec{x}_1) \cdots \mathcal{O}_N(\vec{x}_N) \rangle$$

- Conformal Ward identities: G_N invariant under conformal transformations acting simultaneously on all $\mathcal{O}_i$

$$T_a^{\text{diag}}\, G_N = \left(\sum_{i=1}^{N} T_a^{(i)} \right) G_N = 0$$

- $T_a^{(i)}$ conformal generators acting on the field $\mathcal{O}_i(\vec{x}_i)$:

$$T_a^{(i)} \;=\; \partial_{\mu,i} \;,\; x_i^\mu \partial_{\mu,i} + \Delta_i\, \text{Id} \;,\; \cdots$$

2-point and 3-point correlation functions

- For scalar fields, conformal Ward identities completely fix the form of 2-point and 3-point functions
- For 2-point functions: $r_{ij} = |\vec{x}_i - \vec{x}_j|$

$$G_2(\vec{x}_1, \vec{x}_2) = \frac{\delta_{\Delta_1 \Delta_2}}{r_{12}^{\Delta_1 + \Delta_2}}$$

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- For 3-point functions:

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{\lambda_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}}{r_{12}^{\Delta_1 + \Delta_2 - \Delta_3} r_{13}^{\Delta_1 + \Delta_3 - \Delta_2} r_{23}^{\Delta_2 + \Delta_3 - \Delta_1}}$$

- Structure constants $\lambda_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_k}$: characterise the CFT together with the spectrum (Δ_i, ℓ_i)

4-point correlation functions

- For 4-point correlation functions, Ward identities are not enough to completely fix the form:

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \Omega(\vec{x}_i)\, F(u, v)$$

- Kinematical prefactor:

$$\Omega(\vec{x}_i) = \frac{1}{r_{12}^{\Delta_1+\Delta_2} r_{34}^{\Delta_3+\Delta_4}} \left(\frac{r_{24}}{r_{14}}\right)^{\Delta_1-\Delta_2} \left(\frac{r_{14}}{r_{13}}\right)^{\Delta_3-\Delta_4}$$

- Cross-ratios u, v: conformal invariants constructed from $\vec{x}_i$

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2} \qquad \text{and} \qquad v = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2}$$

- All the dynamical information in G_4 is contained in $F(u, v)$

Operator product expansion

- Operator product expansion (OPE):

$$\mathcal{O}_i(\vec{x})\mathcal{O}_j(\vec{y}) = \sum_{\Delta,\ell} \lambda_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_{\Delta,\ell}} C(\vec{x}-\vec{y}, \partial_{y^\mu})\mathcal{O}_{\Delta,\ell}(\vec{y})$$

- In principle, can be used to reduce any correlation function $G_N = \langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle$ to 2-point correlation functions

- For instance, applying the OPE to the product $\mathcal{O}_1\mathcal{O}_2$ in G_3 gives back

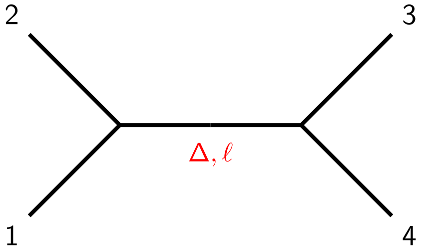
$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{13}^{\Delta_1+\Delta_3-\Delta_2} r_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

$\rightarrow$ OPE coefficients $\lambda_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}$ = structure constants in 3-pt functions

- CFT completely characterised by (Δ_i, ℓ_i) and $\lambda_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}$

Conformal block expansion of 4-point functions

- Apply OPE to the products $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$ in $G_4 = \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle$:



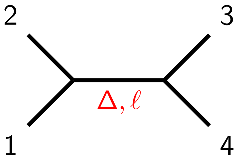
- Conformal block decomposition:

$$F(u,v) = \sum_{\Delta,\ell} \lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{\Delta,\ell}} \lambda_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{\Delta,\ell}} \, g_{\Delta,\ell}(u,v)$$

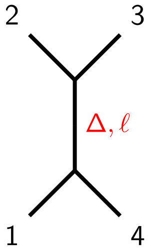
Conformal block expansion of 4-point functions

- Crossing identity: G_4 independent on the choice of OPE channel

$\sum_{\Delta,\ell}$



$=$ $\sum_{\Delta,\ell}$



- Example for all $\mathcal{O}_i$ equal to scalar field ϕ:

$$v^{\Delta_\phi} \sum_{\Delta,\ell} \lambda^2_{\phi\phi\mathcal{O}_{\Delta,\ell}} \, g_{\Delta,\ell}(u,v) = u^{\Delta_\phi} \sum_{\Delta,\ell} \lambda^2_{\phi\phi\mathcal{O}_{\Delta,\ell}} \, g_{\Delta,\ell}(v,u)$$

- Bootstrap: use crossing to constrain the possible values of $\lambda_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}$

Conformal blocks as eigenvectors of Casimir operators

- How to compute conformal blocks $g_{\Delta,\ell}(u,v)$?
  $\rightarrow$ approach of [Dolan Osborn '03]

- Before factoring out the prefactor Ω:

$$G_4 = \sum_{\Delta,\ell} \lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_{\Delta,\ell}} \lambda_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}_{\Delta,\ell}} \, h_{\Delta,\ell}, \qquad h_{\Delta,\ell}(\vec{x}_i) = \Omega(\vec{x}_i) g_{\Delta,\ell}(u,v)$$

- Operators $T_a^{(12)} = T_a^{(1)} + T_a^{(2)}$ also satisfy commutation relation of $\mathfrak{g}$
- Corresponding Casimir operators

$$\mathcal{D}_2^{(12)} = \kappa_2^{ab} \, T_a^{(12)} T_b^{(12)} \quad \text{and} \quad \mathcal{D}_4^{(12)} = \kappa_4^{abcd} \, T_a^{(12)} T_b^{(12)} T_c^{(12)} T_d^{(12)}$$

- $h_{\Delta,\ell}$ common eigenvector of $\mathcal{D}_2^{(12)}$ and $\mathcal{D}_4^{(12)}$: $\quad \left(\left[\mathcal{D}_2^{(12)}, \mathcal{D}_4^{(12)} \right] = 0 \right)$

$$\mathcal{D}_p^{(12)} \, h_{\Delta,\ell} = \mathcal{C}_p^{(\Delta,\ell)} \, h_{\Delta,\ell}, \qquad \text{for } p = 2, 4$$

Conformal blocks as eigenvectors of Casimir operators

- $\mathcal{D}_p^{(12)}$ acting on $h_{\Delta,\ell}$: differential operators in coordinates $\vec{x}_i$
- Since $\mathcal{D}_p^{(12)}$ commutes with $T_a^{\text{diag}} = \sum_{i=1}^4 T_a^{(i)}$:

$$T_a^{\text{diag}}\,\Psi(\vec{x}_i) = 0 \qquad \Rightarrow \qquad T_a^{\text{diag}}\big(\mathcal{D}_p^{(12)}\Psi\big)(\vec{x}_i) = 0,$$

$$\Psi(\vec{x}_i) = \Omega(\vec{x}_i)\widehat{\Psi}(u,v) \qquad \Rightarrow \qquad \mathcal{D}_p^{(12)}\Psi(\vec{x}_i) = \Omega(\vec{x}_i)\big(\widehat{\mathcal{D}}_p^{(12)}\widehat{\Psi}(u,v)\big)$$

with $\widehat{\mathcal{D}}_p^{(12)}$ differential operators in the cross-ratios u, v

- $g_{\Delta,\ell}$ common eigenvector of $\widehat{\mathcal{D}}_2^{(12)}$ and $\widehat{\mathcal{D}}_4^{(12)}$: $\quad \big(\big[\widehat{\mathcal{D}}_2^{(12)}, \widehat{\mathcal{D}}_4^{(12)}\big] = 0\big)$

$$\widehat{\mathcal{D}}_p^{(12)}\,g_{\Delta,\ell} = \mathcal{C}_p^{(\Delta,\ell)}\,g_{\Delta,\ell}, \qquad \text{for } p = 2, 4$$

Casimir equations for conformal blocks

$$\widehat{\mathcal{D}}_p^{(12)}\, g_{\Delta,\ell} = \mathcal{C}_p^{(\Delta,\ell)}\, g_{\Delta,\ell}, \qquad \text{for } p = 2,4$$

- Quadratic Casimir: $a = \frac{1}{2}(\Delta_2 - \Delta_1)$ and $b = \frac{1}{2}(\Delta_3 - \Delta_4)$

$$2\widehat{\mathcal{D}}_2^{(12)} = (1-u-v)\partial_v(v\partial_v + a + b) + u\partial_u(2u\partial_u - 3) \\ -(1+u-v)(u\partial_u + v\partial_v + a)(u\partial_u + v\partial_v + b)$$

- Casimir equations can be used to study $g_{\Delta,\ell}$

- Eigenfunctions of 2 commuting differential operators in 2 variables
  $\rightarrow$ complete (maximal) set of commuting operators
  $\rightarrow$ quantum integrable system

- Equivalent to BC_2 Calogero-Sutherland: [Isachenkov Schomerus '16]
  - $\widehat{\mathcal{D}}_2^{(12)} \sim$ CS Hamiltonian (Casimir equation $\sim$ Schrödinger equation)
  - $\widehat{\mathcal{D}}_4^{(12)} \sim$ additional quartic conserved charge

Multipoint conformal blocks

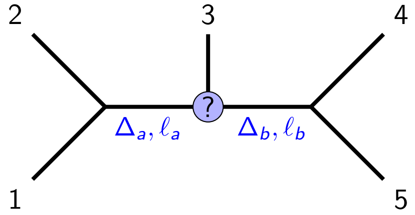
- Multipoint correlation functions: for $N > 4$, not much is known about the general conformal blocks with arbitrary exchanged fields
- Main goal:
  - generalise the approach of [Dolan Osborn '03] to multipoint blocks
  - characterise these blocks as eigenvectors of a complete set of commuting operators / integrable system
  - use this to study the properties of these blocks
- First progress in [Burić SL Mann Quintavalle Schomerus '20]
- Main difficulty: Casimir operators alone are not sufficient anymore
- For this talk: simplest example of $N = 5$ in $3d$

Conformal block expansion of 5-point functions

- 5-point function depends on 5 cross-ratios $u_1, \cdots, u_5$:

$$G_5(\vec{x}_1, \cdots, \vec{x}_5) = \langle \mathcal{O}_1(\vec{x}_1) \cdots \mathcal{O}_5(\vec{x}_5) \rangle = \Omega(\vec{x}_i) F(u_1, \cdots, u_5)$$

- Apply OPE to the products $\mathcal{O}_1 \mathcal{O}_2$ and $\mathcal{O}_4 \mathcal{O}_5$:



- Δ_k, ℓ_k measured by the Casimir operators $\widehat{\mathcal{D}}_p^{(12)}$ and $\widehat{\mathcal{D}}_p^{(123)}$ $(p = 2, 4)$
- 4 operators for 5 variables u_i

  $\rightarrow$ one operator missing: measures tensor structures at the vertex

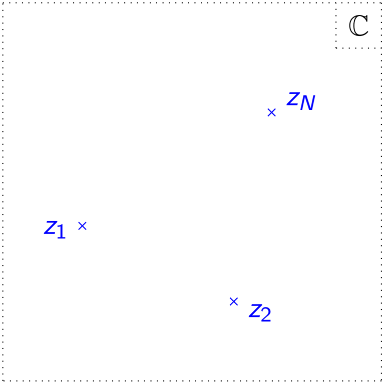
Missing vertex operator in 5-point conformal blocks

- Goal: find a fifth operator $\widehat{\mathcal{V}}$ commuting with the Casimir operators $\widehat{\mathcal{D}}_p^{(12)}$ and $\widehat{\mathcal{D}}_p^{(123)}$
- $\widehat{\mathcal{D}}_p^{(12)}$ and $\widehat{\mathcal{D}}_p^{(123)}$ come from the operators $\mathcal{D}_p^{(12)}$ and $\mathcal{D}_p^{(123)}$ in terms of $\vec{x}_i$ by factorising out the kinematical prefactor Ω
- Equivalently find an operator $\mathcal{V}$ in terms of $\vec{x}_i$ that commutes with $\mathcal{D}_p^{(12)}$, $\mathcal{D}_p^{(123)}$ and T_a^{diag}

  $\rightarrow$ integrable system with a global $\mathfrak{g}$-symmetry
- We will construct $\mathcal{V}$ using a well-chosen limit of Gaudin models

Gaudin models and applications to conformal blocks

- Gaudin model historically introduced as a spin system [Gaudin 76']



- Sites $z_1, z_2, \cdots, z_N$ in $\mathbb{C}$

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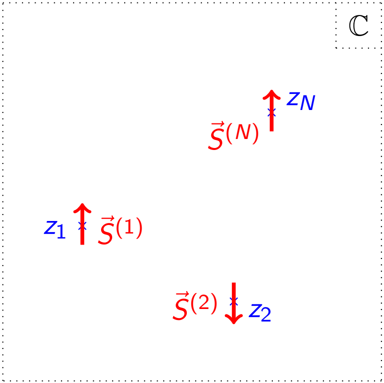

- Sites $z_1, z_2, \cdots, z_N$ in $\mathbb{C}$
- Spins $\vec{S}^{(i)}$

$$\left[S_a^{(i)}, S_b^{(j)}\right] = \delta_{ij}\epsilon_{abc}S_c^{(i)}$$

$\rightarrow$ Lie algebra $\mathfrak{su}(2)$

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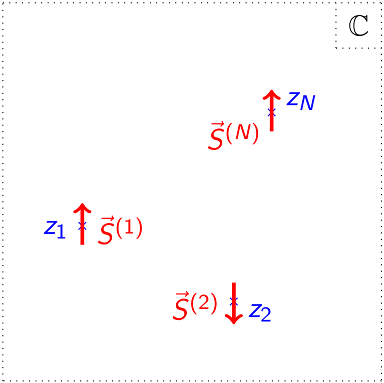

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$\rightarrow$ Lie algebra $\mathfrak{su}(2)$

- Hamiltonian of site i:

$$\mathcal{H}_i = \sum_{j \neq i} \frac{\vec{S}^{(i)} \cdot \vec{S}^{(j)}}{z_i - z_j}$$

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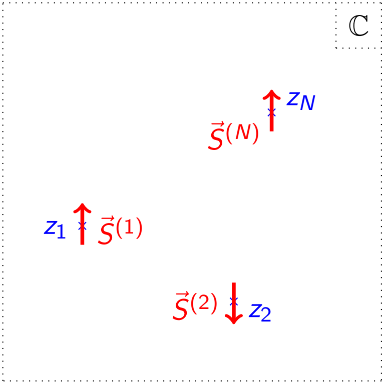

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$$[\mathcal{H}_i, \mathcal{H}_j] = 0$$

Gaudin models on simple Lie algebras $\mathfrak{g}$

Finite-dimensional simple Lie algebra $\mathfrak{g}$:

- Basis $\{\mathcal{T}_a\}$, structure constants $f_{ab}{}^c$: $[\mathcal{T}_a, \mathcal{T}_b] = f_{ab}{}^c\, \mathcal{T}_c$
- Invariant bilinear form κ_{ab}, inverse κ^{ab}:

$$f_{de}{}^a\, \kappa^{eb} + f_{de}{}^b\, \kappa^{ae} = 0$$

Gaudin model on $\mathfrak{g}$: [Gaudin '83] (with sites $z_1, \cdots, z_N$ in $\mathbb{C}$)

- Observables $\mathcal{A} = U(\mathfrak{g})^{\otimes N}$: N independent copies of $\mathfrak{g}$, generators $T_a^{(i)}$

$$\big[T_a^{(i)}, T_b^{(j)} \big] = \delta_{ij}\, f_{ab}{}^c\, T_c^{(i)}$$

- Commuting quadratic Hamiltonians:

$$\mathcal{H}_i = \sum_{j \neq i} \frac{\kappa^{ab}\, T_a^{(i)}\, T_b^{(j)}}{z_i - z_j}, \qquad [\mathcal{H}_i, \mathcal{H}_j] = 0$$

Gaudin models on simple Lie algebras $\mathfrak{g}$

- Lax matrix: (z auxiliary complex parameter: spectral parameter)

$$\mathcal{L}_a(z) = \sum_{r=1}^{N} \frac{T_a^{(i)}}{z - z_i}$$

Gaudin models on simple Lie algebras $\mathfrak{g}$

- Lax matrix: (z auxiliary complex parameter: spectral parameter)

$$\mathcal{L}_a(z) = \sum_{r=1}^{N} \frac{T_a^{(i)}}{z - z_i}$$

- Quadratic Hamiltonians $\mathcal{H}_i = \sum_{j \neq i} \frac{\kappa^{ab} T_a^{(i)} T_b^{(j)}}{z_i - z_j}$ extracted from

$$\mathcal{H}(z) = \frac{1}{2}\kappa^{ab}\mathcal{L}_a(z)\mathcal{L}_b(z) = \sum_{r=1}^{N} \left(\frac{1}{2} \frac{\mathcal{D}_2^{(i)}}{(z - z_i)^2} + \frac{\mathcal{H}_i}{z - z_i} \right)$$

with $\mathcal{D}_2^{(i)} = \kappa^{ab} T_a^{(i)} T_b^{(i)}$ Casimir operator of site i

- Commutation:

$$\big[\mathcal{H}(z), \mathcal{H}(w)\big] = 0, \qquad \forall\, z, w \in \mathbb{C}$$

Higher order Hamiltonians

- Higher order invariant symmetric tensors: $(\kappa_2^{ab} = \kappa^{ab})$

$$f_{bc}{}^{a_1}\,\kappa_p^{c\,a_2\cdots a_p} + f_{bc}{}^{a_2}\,\kappa_p^{a_1\,c\,a_3\cdots a_p} + \cdots + f_{bc}{}^{a_p}\,\kappa_p^{a_1\cdots a_{p-1}c} = 0$$

- Remark: directly related to Casimir operators of $\mathfrak{g}$

$$\mathcal{D}_p = \kappa_p^{a_1\cdots a_p}\mathcal{T}_{a_1}\cdots\mathcal{T}_{a_p}$$

- Higher order Hamiltonians: [Feigin Frenkel Reshetikhin '94, Talalaev '04, Chervov Talalaev '06, Molev '13, Molev Ragoucy Rozhkovskaya '16]

$$\mathcal{H}^{(p)}(z) = \kappa_p^{a_1\cdots a_p}\mathcal{L}_{a_1}(z)\cdots\mathcal{L}_{a_p}(z) \;+\; \ldots$$

with $\ldots$ quantum corrections involving derivatives of $\mathcal{L}(z)$

- Commutation:

$$\left[\mathcal{H}^{(p)}(z), \mathcal{H}^{(q)}(w)\right] = 0, \qquad \forall\, z, w \in \mathbb{C}, \;\; \forall\, p, q$$

Diagonal symmetry and representations

- Diagonal generator:

$$T_a^{\text{diag}} = \sum_{i=1}^{N} T_a^{(i)}$$

- Commute with Gaudin Hamiltonians:

$$\left[T_a^{\text{diag}}, \mathcal{H}^{(p)}(z) \right] = 0, \qquad \forall\, z \in \mathbb{C}, \;\; \forall\, p$$

- Gaudin model: integrable system with $\mathfrak{g}$-diagonal symmetry

- So far, $T_a^{(i)}$ abstract generators in $\mathcal{A} = U(\mathfrak{g})^{\otimes N}$
- One can make $T_a^{(i)}$ acts on a representation V_i of $\mathfrak{g}$. Hilbert space:

$$H = V_1 \otimes \cdots \otimes V_N$$

$\rightarrow \mathcal{H}^{(p)}(z)$ operators on H

Gaudin model and 5-point conformal blocks

Applications to 5-point conformal blocks:

- Gaudin model with 5 sites on the conformal Lie algebra $\mathfrak{g} = \mathfrak{so}(4,1)$
- Representations $V_{\Delta_i,0}$ corresponding to the scalar fields $\mathcal{O}_i$

  $\rightarrow$ $T_a^{(i)}$ differential operators $\partial_{x_i^\mu}$, $x_i^\mu \partial_{x_i^\mu} + \Delta_i \, \mathrm{Id}$, $\cdots$

- Commuting differential operators: coefficients of $\mathcal{H}^{(2)}(z)$ and $\mathcal{H}^{(4)}(z)$
- To relate to conformal blocks, we want these operators to contain the Casimir operators $\mathcal{D}_p^{(12)}$ and $\mathcal{D}_p^{(123)}$

  $\rightarrow$ particular limit of the parameters z_i

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- To relate to conformal blocks, we want these operators to contain the Casimir operators $\mathcal{D}_p^{(12)}$ and $\mathcal{D}_p^{(123)}$
  $\rightarrow$ particular limit of the parameters z_i

- More precisely, consider $\varpi \to 0$, with

$$z_1 = \varpi^3, \qquad z_2 = \varpi^2, \qquad z_3 = \varpi, \qquad z_4 = 1, \qquad z_5 = \frac{1}{\varpi}$$

Gaudin model and 5-point conformal blocks

$$z_1 = \varpi^3, \qquad z_2 = \varpi^2, \qquad z_3 = \varpi, \qquad z_4 = 1, \qquad z_5 = \frac{1}{\varpi}$$

- Quadratic Casimir operators from Hamiltonians $\mathcal{H}_i = \sum_{j \neq i} \frac{\kappa^{ab} T_a^{(i)} T_b^{(j)}}{z_i - z_j}$:

$$\left(\sum_{i=1}^{2} \Delta_i(\Delta_i - 3) \right) \mathrm{Id} + 2\varpi^2 \mathcal{H}_2 \xrightarrow{\varpi \to 0} \mathcal{D}_2^{(12)}$$

$$\left(\sum_{i=1}^{3} \Delta_i(\Delta_i - 3) \right) \mathrm{Id} + 2\varpi^2 \mathcal{H}_2 + 2\varpi \mathcal{H}_3 \xrightarrow{\varpi \to 0} \mathcal{D}_2^{(123)}$$

- Similarly, one obtains $\mathcal{D}_4^{(12)}$ and $\mathcal{D}_4^{(123)}$

- We also get a fifth independent commuting operator:

$$\mathcal{V} = \kappa_4^{abcd} S_a S_b S_c S_d, \qquad S_a = T_a^{(1)} + T_a^{(2)} - T_a^{(3)}$$

Summary and generalisations

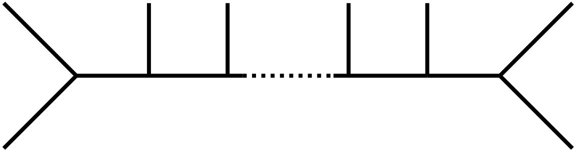
- Limit of $\mathfrak{so}(4,1)$-Gaudin model: 5-point conformal blocks in 3d
  $\rightarrow$ differential operators $\mathcal{D}_2^{(12)}$, $\mathcal{D}_2^{(123)}$, $\mathcal{D}_4^{(12)}$, $\mathcal{D}_4^{(123)}$, $\mathcal{V}$ in x_i^μ
  $\rightarrow$ five commuting differential operators in cross-ratios u_i
  $\rightarrow$ conformal blocks eigenvectors of these operators

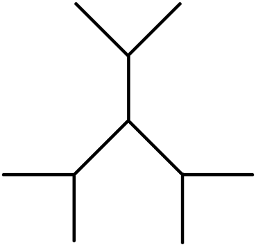
- Generalises for 5-point conformal blocks in higher dimension d
  $\rightarrow$ still 5 five cross-ratios and 5 commuting operators
  (Gaudin model based on the algebra $\mathfrak{so}(d+1,1)$)
  $\rightarrow$ d appears polynomially in the coefficients of the operators

- Generalisation to higher number of points

Higher number of points

- Generalisation to higher number of points and other topologies:





Perspectives and open questions

Perspectives and open questions

- Use of integrability techniques (separation of variables, ...) to study these integrable systems?
  - Spectrum of the operators?
  - Expression of the multipoint conformal blocks?
  - Series expansion?
  - ...

- Limit related to bending flows [Kapovich Millson '96, Falqui Musso '03]
- Vertex operator related to elliptic $\mathbb{Z}/4\mathbb{Z}$ Calogero-Moser system of [Etingof Felder Ma Veselov '10]

- Applications to multipoint conformal bootsrap?
- Recently: light-cone multipoint bootstrap in conformal gauge theories [Vieira Gonçalves Bercini '20]

Thank you for your attention!