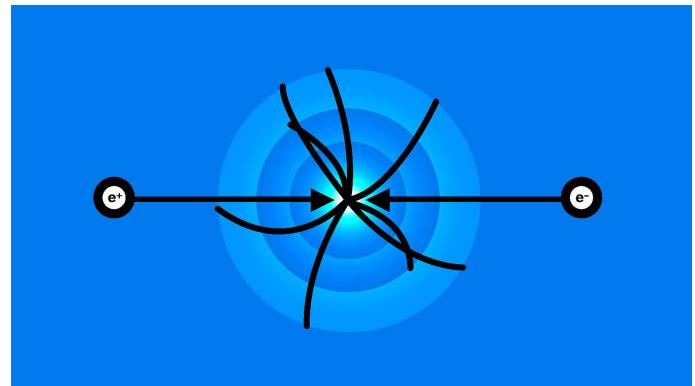


Electroweak measurements: Comparing theory and experiments

A. Freitas

University of Pittsburgh

- Electroweak precision at Z pole & WW
- EW precision tests at future colliders
- SM input parameters

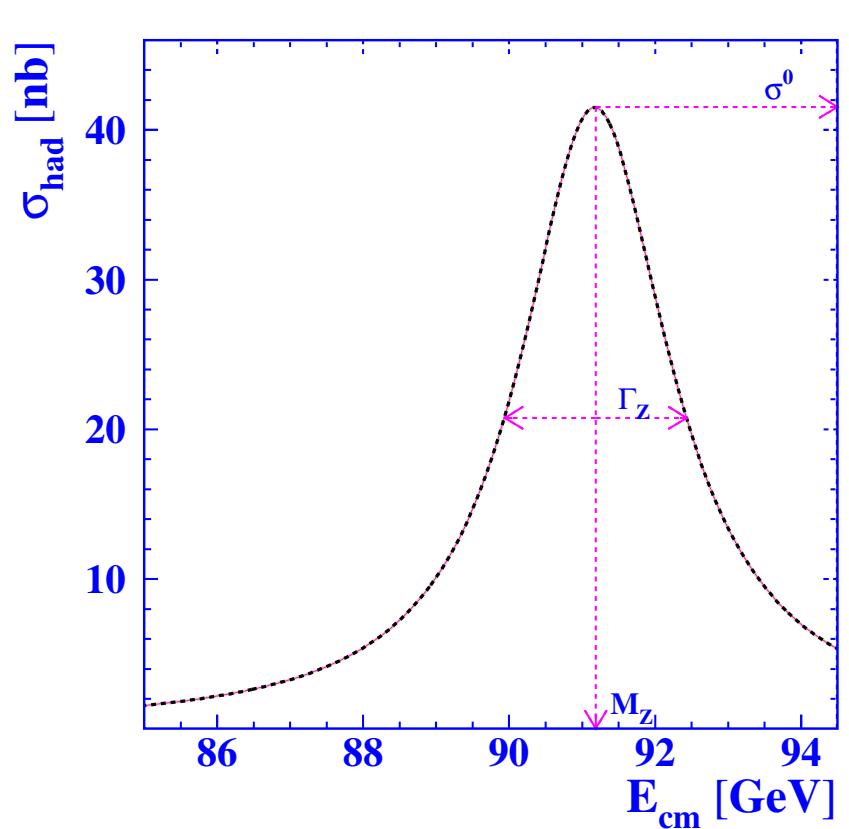
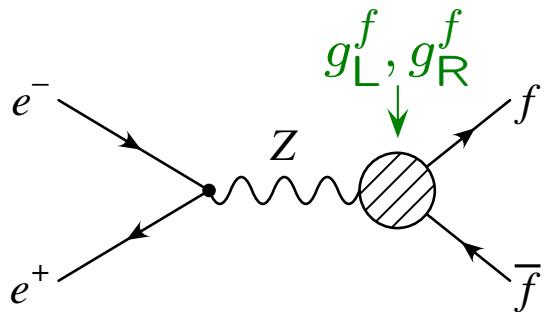


Z cross section and branching fractions

$e^+e^- \rightarrow f\bar{f}$ for $E_{\text{CM}} \sim M_Z$:

- Mass M_Z
- Width $\Gamma_Z = \sum_f \Gamma_{ff}$
- Braching ratio $R_f = \Gamma_{ff}/\Gamma_Z$
- $\sigma^0 \approx \frac{12\pi \Gamma_{ee} \Gamma_{ff}}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} = \frac{12\pi}{M_Z^2} R_e R_f$

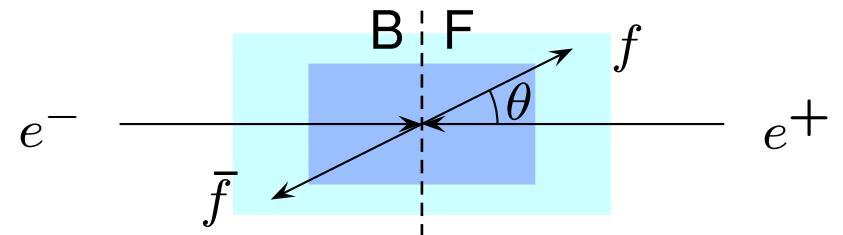
$$\Gamma_{ff} = C \left[(g_L^f)^2 + (g_R^f)^2 \right]$$



Forward-backward asymmetry:

$$A_{FB} \equiv \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4} \mathcal{A}_e \mathcal{A}_f$$

$$\mathcal{A}_f = \frac{2(1 - 4 \sin^2 \theta_{\text{eff}}^f)}{1 + (1 - 4 \sin^2 \theta_{\text{eff}}^f)^2}$$



$$\sin^2 \theta_{\text{eff}}^f = \frac{g_R^f}{2|Q_f|(g_R^f - g_L^f)}$$

Left-right asymmetry:

With polarized e^- beam: $A_{LR} \equiv \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \mathcal{A}_e$

Polarization asymmetry:

Average τ pol. in $e^+ e^- \rightarrow \tau^+ \tau^-$: $\langle \mathcal{P}_\tau \rangle = -\mathcal{A}_\tau$

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \rightarrow f\bar{f}] = \mathcal{R}_{\text{ini}}(s, s') \otimes \sigma_{\text{hard}}(s')$$

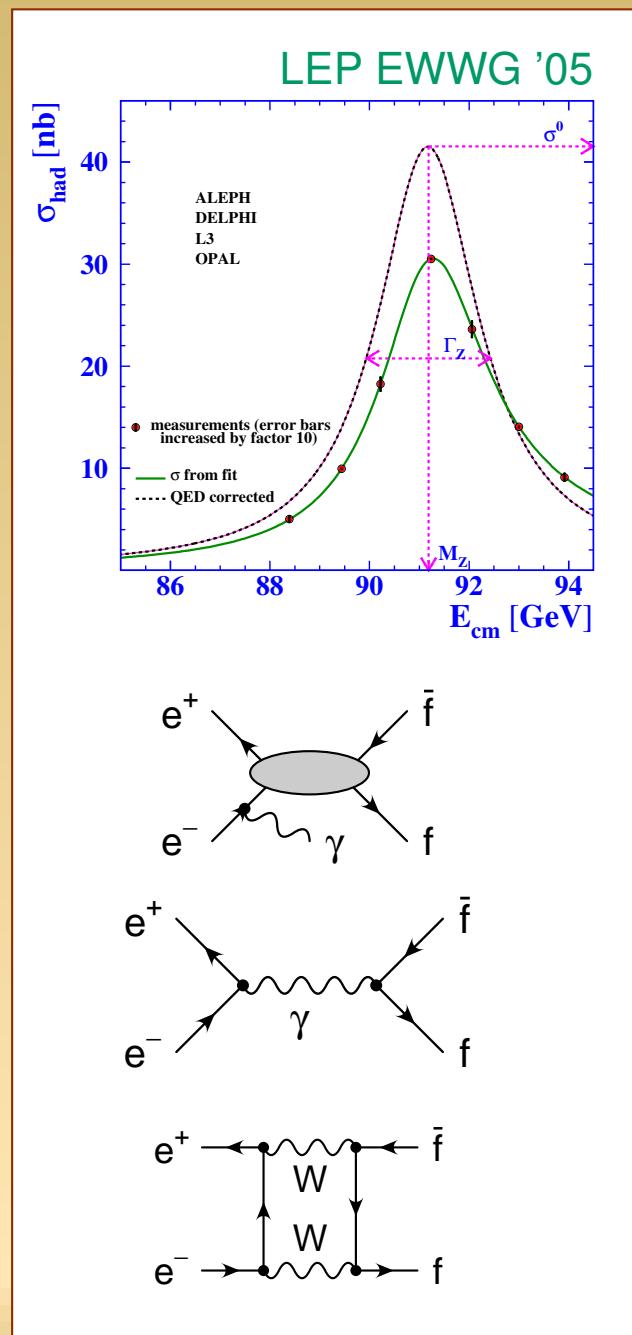
- Subtraction of γ -exchange, $\gamma-Z$ interference, box contributions:

$$\sigma_{\text{hard}} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\text{box}}$$

- Z -pole contribution:

$$\sigma_Z = \frac{R}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} + \sigma_{\text{non-res}}$$

$\sigma_\gamma, \sigma_{\gamma Z}, \sigma_{\text{box}}, \sigma_{\text{non-res}}$ known at NLO
 → need consistent pole expansion framework
 → leading NNLO may be needed for future e^+e^-



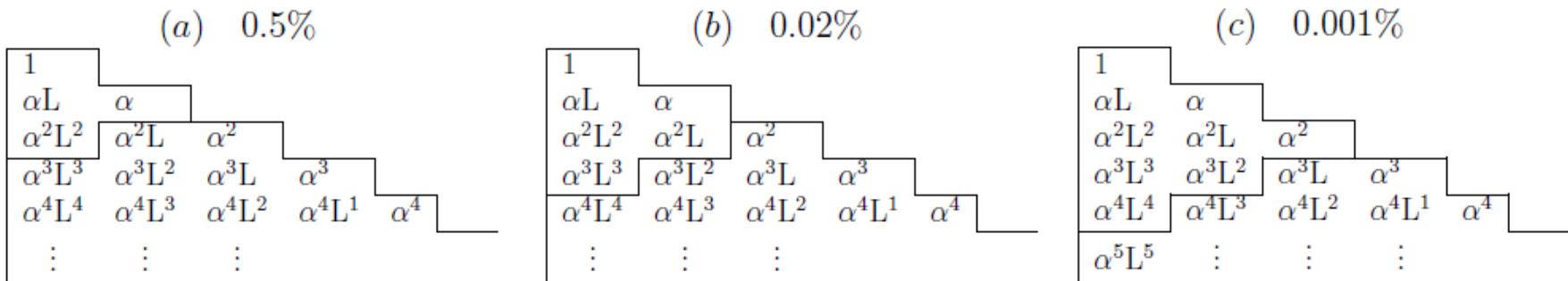
Monte-Carlo methods for QED effects

4/15

- Implementation in MC program to evaluate exp. efficiency and particle ID
- Current state of art: e.g. KORALZ, KKMC
 $\rightarrow \mathcal{O}(\alpha^2 L)$ accuracy [$L = \ln(s/m_e^2)$]Jadach, Ward, ...
- One to two orders improvement needed:

Observable	Where from	Present (LEP)	FCC stat.	FCC syst	Now FCC
M_Z [MeV]	Z linesh. [28]	$91187.5 \pm 2.1\{0.3\}$	0.005	0.1	3
Γ_Z [MeV]	Z linesh. [28]	$2495.2 \pm 2.1\{0.2\}$	0.008	0.1	2
$R_l^Z = \Gamma_h/\Gamma_l$	$\sigma(M_Z)$ [33]	$20.767 \pm 0.025\{0.012\}$	$6 \cdot 10^{-5}$	$1 \cdot 10^{-3}$	12
σ_{had}^0 [nb]	σ_{had}^0 [28]	$41.541 \pm 0.037\{0.25\}$	$0.1 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	6
N_ν	$\sigma(M_Z)$ [28]	$2.984 \pm 0.008\{0.006\}$	$5 \cdot 10^{-6}$	$1 \cdot 10^{-3}$	6
$\sin^2 \theta_W^{eff} \times 10^5$	$A_{FB}^{\text{lept.}}$ [33]	$23099 \pm 53\{28\}$	0.3	0.5	55
$A_{FB,\mu}^{M_Z \pm 3.5 \text{ GeV}}$	$\frac{d\sigma}{d\cos\theta}$ [28]	$\pm 0.020\{0.001\}$	$1.0 \cdot 10^{-5}$	$0.3 \cdot 10^{-5}$	100

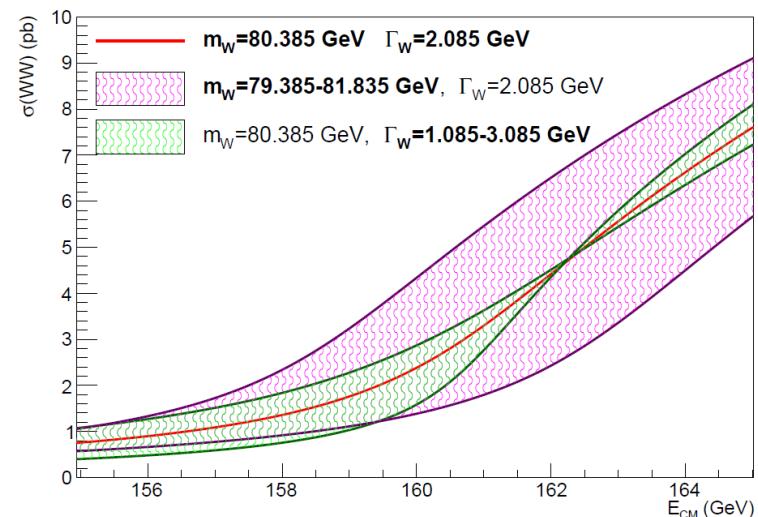
Jadach,
Skrzypek '19



→ Need matching of h.o. matrix elements with QED parton shower
(exclusive in all fs particles)

- High-precision measurement of M_W from $e^+e^- \rightarrow W^+W^-$ at threshold
- a) Corrections near threshold enhanced by $1/\beta$ and $\ln \beta$

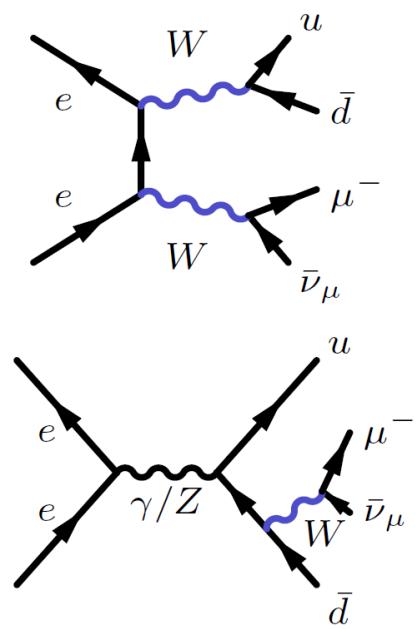
$$\beta \sim \sqrt{1 - 4 \frac{M_W^2 - i M_W \Gamma_W}{s}} \sim \sqrt{\Gamma_W/M_W}$$
- b) Non-resonant contributions are important



- Full $\mathcal{O}(\alpha)$ calculation of $e^+e^- \rightarrow 4f$
Denner, Dittmaier, Roth, Wieders '05

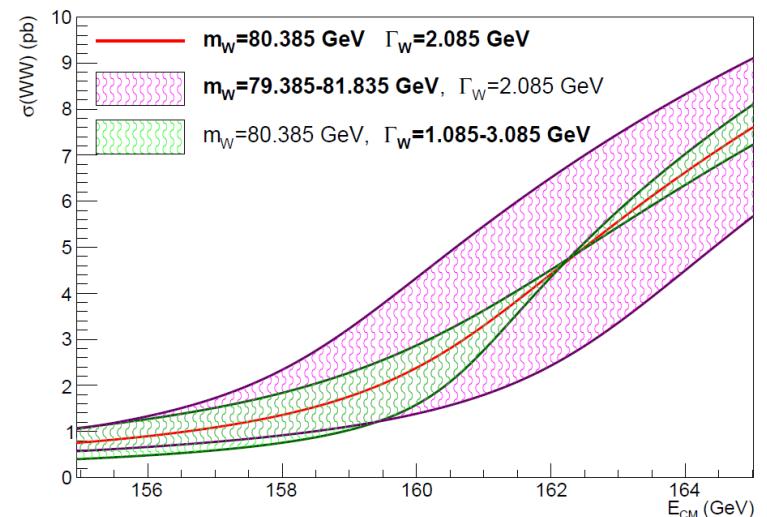
- EFT expansion in $\alpha \sim \Gamma_W/M_W \sim \beta^2$
Beneke, Falgari, Schwinn, Signer, Zanderighi '07

- NLO corrections with NNLO Coulomb correction ($\propto 1/\beta^n$): $\delta_{\text{th}} M_W \sim 3 \text{ MeV}$
Actis, Beneke, Falgari, Schwinn '08
- Adding NNLO corrections to $ee \rightarrow WW$ and $W \rightarrow f\bar{f}$ and NNLO ISR: $\delta_{\text{th}} M_W \lesssim 0.6 \text{ MeV}$



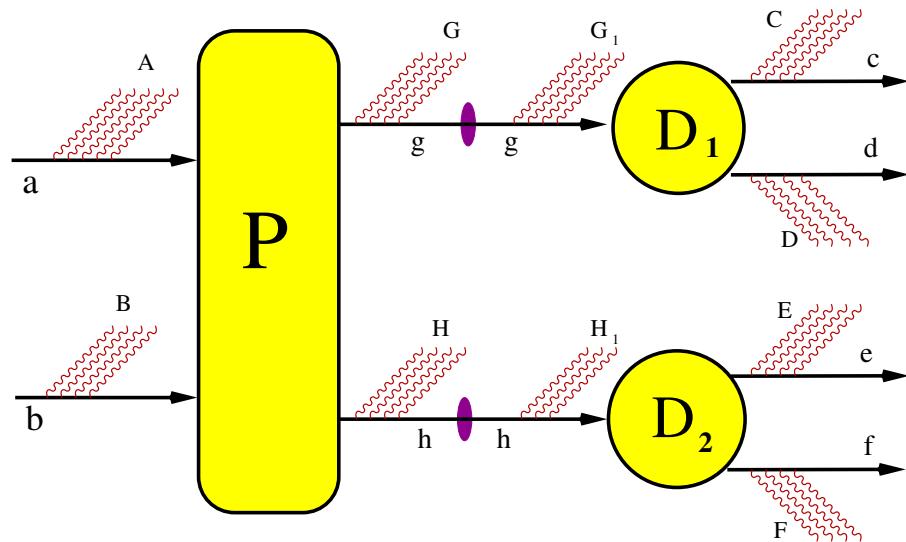
- High-precision measurement of M_W from $e^+e^- \rightarrow W^+W^-$ at threshold
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$$\beta \sim \sqrt{1 - 4 \frac{M_W^2 - i M_W \Gamma_W}{s}} \sim \sqrt{\Gamma_W/M_W}$$
- b) Non-resonant contributions are important



- Resummation of soft photon radiation

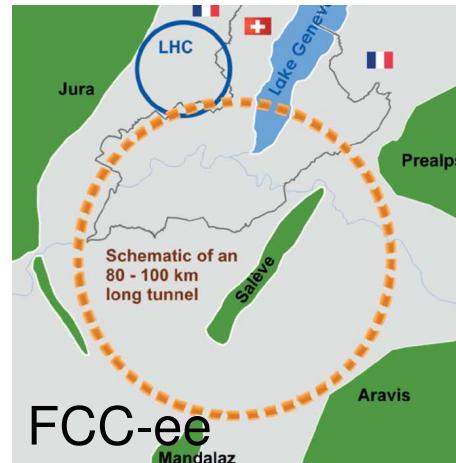
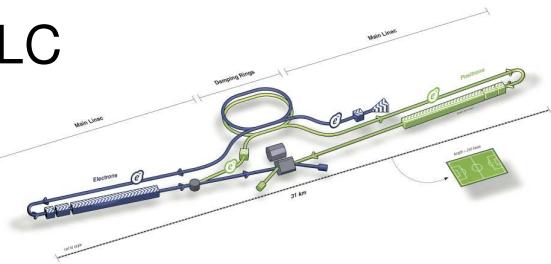
Jadach, Płaczek, Skrzypek '19



Electroweak precision tests at future colliders

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ILC



\sqrt{s}	M_Z	$2M_W$	
ILC/GigaZ	100 fb^{-1}	500 fb^{-1} (6 pts.)	beam pol. ($P_{e^-}=0.8$, $P_{e^+}=0.3$)
FCC-ee	230 ab^{-1}	10 ab^{-1} (2 pts.)	2 detectors
CEPC	45 ab^{-1}	2.6 ab^{-1} (3 pts.)	2 detectors

Anticipated precision for EWPOs:

	Current exp.	ILC/GigaZ	CEPC	FCC-ee
M_W [MeV]	15	1–2 ^{a,e}	1 ^e	1 ^e
M_Z [MeV]	2.1	–	0.5 ^e	0.1 ^e
Γ_Z [MeV]	2.3	1 ^a	0.5 ^e	0.1 ^e
$R_\ell = \Gamma_Z^{\text{had}} / \Gamma_Z^\ell$ [10^{-3}]	25	6 ^b	2 ^b	1 ^b
$R_b = \Gamma_Z^b / \Gamma_Z^{\text{had}}$ [10^{-5}]	66	15 ^c	4.3 ^c	6 ^c
$\sin^2 \theta_{\text{eff}}^\ell$ [10^{-5}]	16	1 ^d	<1 ^e	0.5 ^e

Systematics:

^a energy scale

^b acceptance

^c flavor tagging

^d polarization

^e beam energy calibration / beam-beam interactions

Comparison of EWPOs with theory

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- To probe new physics, compare EWPOs with SM theory predictions
- Need to take theory error into account:

	Current exp.	Current th. [†]	CEPC	FCC-ee
M_W [MeV]	15	4 *	1	1
Γ_Z [MeV]	2.3	0.4	0.5	0.1
$R_\ell = \Gamma_Z^{\text{had}} / \Gamma_Z^\ell [10^{-3}]$	25	5	2	1
$R_b = \Gamma_Z^b / \Gamma_Z^{\text{had}} [10^{-5}]$	66	10	4.3	6
$\sin^2 \theta_{\text{eff}}^\ell [10^{-5}]$	16	4.5	<1	0.5

* computed from G_μ

† full NNLO and leading NNNLO

- Theory error estimate is not well defined, ideally $\Delta_{\text{th}} \ll \Delta_{\text{exp}}$
- Common methods:
 - Count prefactors (α, N_c, N_f, \dots)
 - Extrapolation of perturbative series
 - Renormalization scale dependence
 - Renormalization scheme dependence

- **Electroweak precision tests** at future e^+e^- colliders require 1–2 orders improvement in SM theory calculations and tools
 - **Z-pole**: 3-loop & leading 4-loop EW + multi-loop/leg merging for QED MC
 - **off Z-pole / backgrounds**: (≥ 2)-loop EW
 - **WW** 2-loop EW for $2 \rightarrow 2$ processes (+ 4-loop QCD)
(≥ 1)-loop for backgr. and non-resonant terms

SMEFT: Gauge-invariant operators with SU(2) Higgs doublet

$$\mathcal{L} = \sum_i \frac{c_i}{\Lambda^2} \mathcal{O}_i + \mathcal{O}(\Lambda^{-3}) \quad (\Lambda \gg M_Z)$$

$$\mathcal{O}_{\phi 1} = (D_\mu \Phi)^\dagger \Phi \Phi^\dagger (D^\mu \Phi)$$

$$\alpha \Delta \textcolor{blue}{T} = -\frac{v^2}{2} \frac{c_{\phi 1}}{\Lambda^2}$$

$$\mathcal{O}_{BW} = \Phi^\dagger B_{\mu\nu} W^{\mu\nu} \Phi$$

$$\alpha \Delta S = -e^2 v^2 \frac{c_{BW}}{\Lambda^2}$$

$$\mathcal{O}_{LL}^{(3)e} = (\bar{L}_L^e \sigma^a \gamma_\mu L_L^e)(\bar{L}_L^e \sigma^a \gamma^\mu L_L^e)$$

$$\Delta G_{\textcolor{blue}{F}} = -\sqrt{2} \frac{c_{LL}^{(3)e}}{\Lambda^2}$$

$$\mathcal{O}_R^f = i(\Phi^\dagger \overset{\leftrightarrow}{D}_\mu \Phi)(\bar{f}_R \gamma^\mu f_R)$$

$$f = e, \mu, \tau, b, lq$$

$$\mathcal{O}_L^F = i(\Phi^\dagger \overset{\leftrightarrow}{D}_\mu \Phi)(\bar{F}_L \gamma^\mu F_L)$$

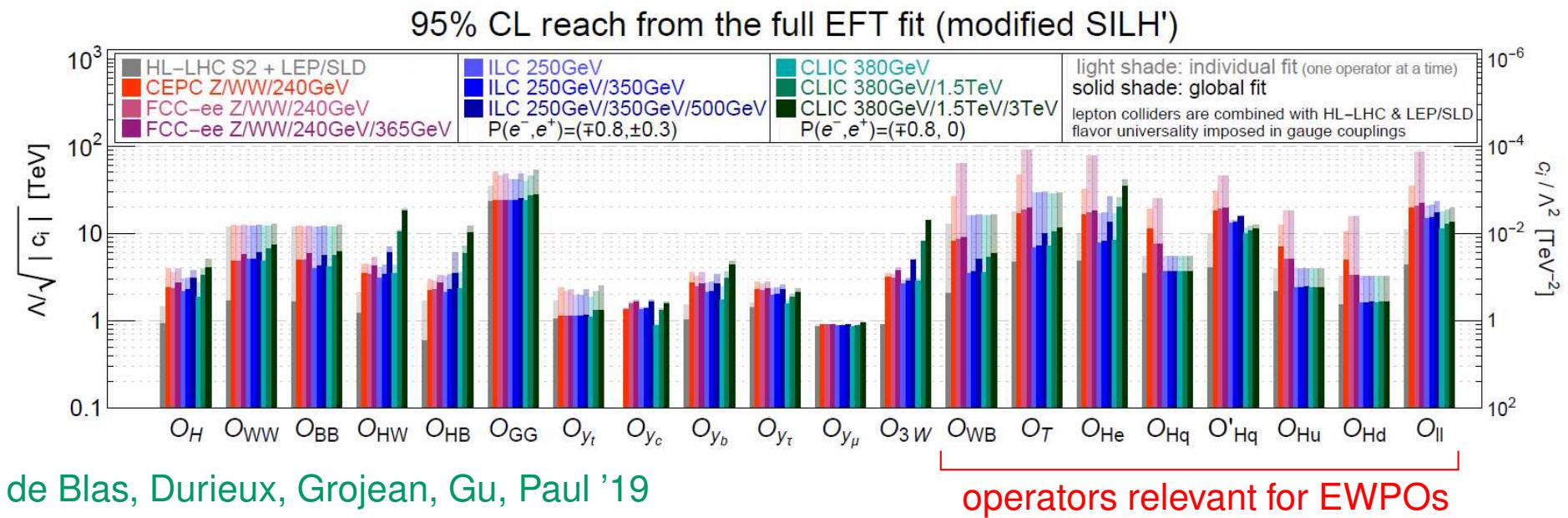
$$F = \binom{\nu_e}{e}, \binom{\nu_\mu}{\mu}, \binom{\nu_\tau}{\tau}, \binom{u, c}{d, s}, \binom{t}{b}$$

$$\mathcal{O}_L^{(3)F} = i(\Phi^\dagger \overset{\leftrightarrow}{D}_\mu^a \Phi)(\bar{F}_L \sigma_a \gamma^\mu F_L)$$

More operators than EWPOs

- Need to make flavor assumptions and/or
use other obs. (e.g. W production and decay)

Projected reach assuming Minimal Flavor Violation:



de Blas, Durieux, Grojean, Gu, Paul '19

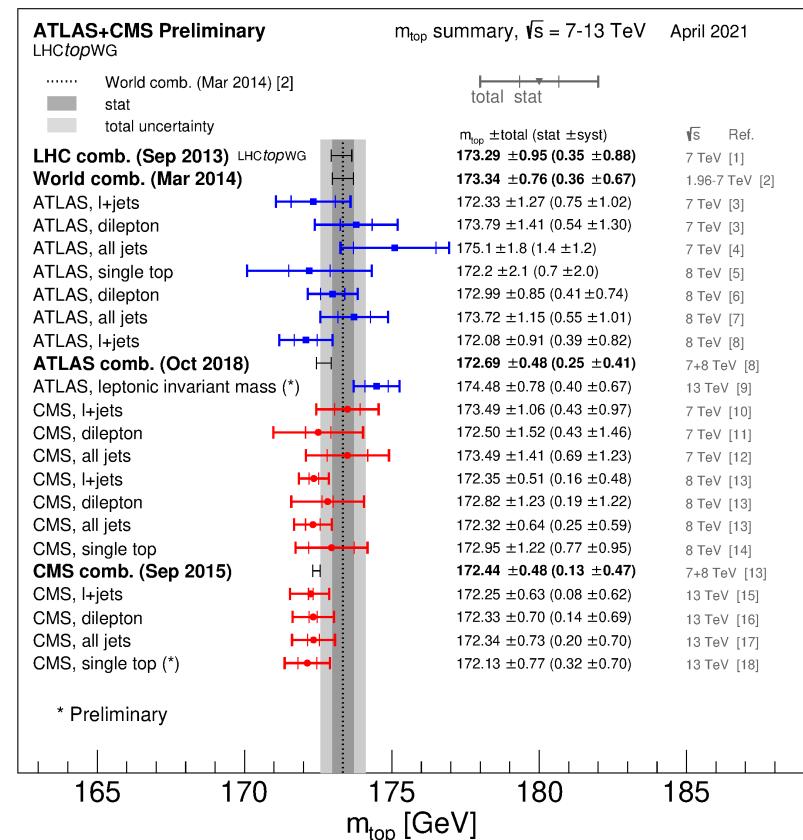
Reviews: 1906.05379, 2012.11642

- M_Z, Γ_Z : From $\sigma(\sqrt{s})$ lineshape; $\delta M_Z, \delta \Gamma_Z \sim 0.1$ MeV at FCC-ee
→ Main theory uncertainties: QED ISR
- m_t : Most precise measurement
at LHC: $\delta m_t \sim 0.3$ GeV PDG '20

Theoretical ambiguity in mass def.:

Hoang, Plätzer, Samitz '18

$$\begin{aligned}
 m_t^{\text{CB}}(Q_0) - m_t^{\text{pole}} \\
 &= -\frac{2}{3}\alpha_s(Q_0)Q_0 + \mathcal{O}(\alpha_s^2 Q_0) \\
 &\approx 0.5 \pm 0.2_{\text{pert.}} \pm 0.2_{\text{np. GeV}}
 \end{aligned}$$



Reviews: 1906.05379, 2012.11642

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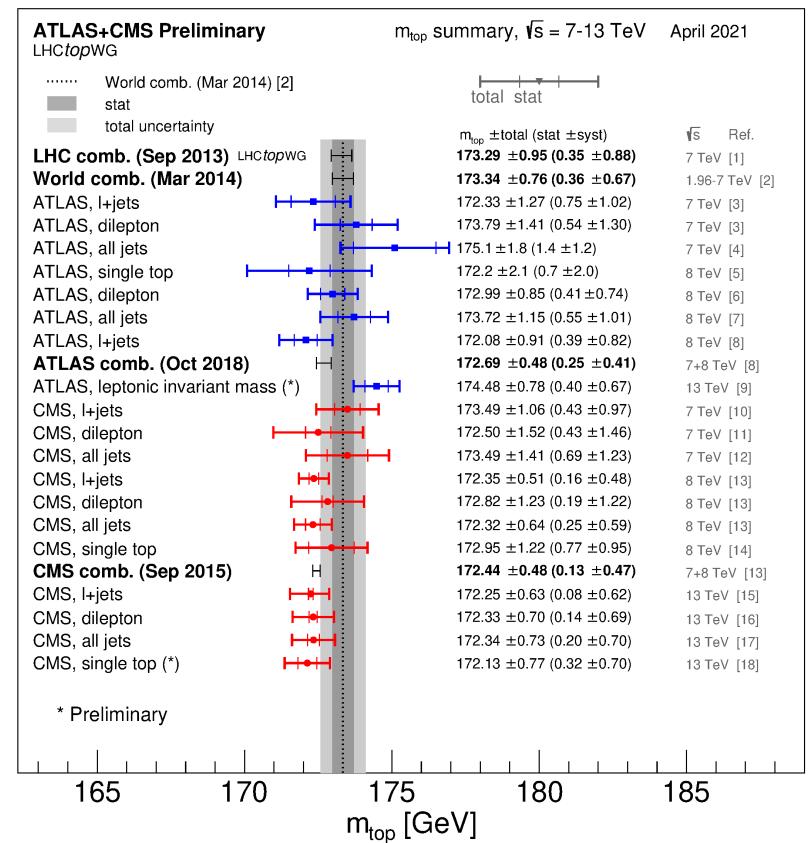
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 &\approx 0.5 \pm 0.2_{\text{pert.}} \pm 0.2_{\text{np. GeV}}
 \end{aligned}$$

Impact on SM prediction for EWPOs:

$$\begin{aligned}
 \delta m_t = 0.5 \text{ GeV} \Rightarrow \delta M_W \approx 3 \text{ MeV} \\
 \delta \sin^2 \theta_{\text{eff}}^\ell \approx 1.5 \times 10^{-5}
 \end{aligned}$$



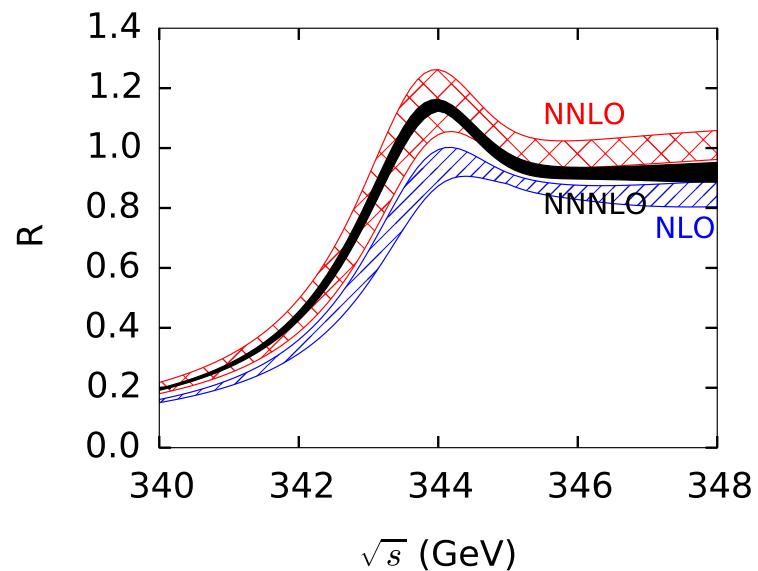
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- m_t : Most precise measurement
at LHC: $\delta m_t \sim 0.3$ GeV

From $e^+e^- \rightarrow t\bar{t}$ at $\sqrt{s} \sim 350$ GeV:

Stat. uncertainty < 20 MeV

Theo. uncertainty < 50 MeV
(with future improvements)



Beneke et al. '15

- α_s :
 - Most precise determination using Lattice QCD:
 - $\alpha_s = 0.1184 \pm 0.0006$ HPQCD '10
 - $\alpha_s = 0.1185 \pm 0.0008$ ALPHA '17
 - $\alpha_s = 0.1179 \pm 0.0015$ Takaura et al. '18
 - $\alpha_s = 0.1172 \pm 0.0011$ Zafeiropoulos et al. '19
 - Difficulty in evaluating systematics
- e^+e^- event shapes and DIS: $\alpha_s \sim 0.114$
Alekhin, Blümlein, Moch '12; Abbate et al. '11; Gehrmann et al. '13
 - Subject to sizeable non-perturbative power corrections
 - Systematic uncertainties in power corrections?
- Hadronic τ decays: $\alpha_s = 0.119 \pm 0.002$ PDG '18
 - Non-perturbative uncertainties in OPE and from duality violation
Pich '14; Boito et al. '15,18

- α_s :

- Electroweak precision ($R_\ell = \Gamma_Z^{\text{had}} / \Gamma_Z^\ell$):

$$\alpha_s = 0.120 \pm 0.003$$

PDG '18

→ No (negligible) non-perturbative QCD effects

FCC-ee: $\delta R_\ell \sim 0.001$

⇒ $\delta \alpha_s < 0.0002$ (subj. to theory error)

Caviat: R_ℓ could be affected by new physics

- $R = \frac{\sigma[ee \rightarrow \text{had.}]}{\sigma[ee \rightarrow \mu\mu]}$ at lower \sqrt{s}

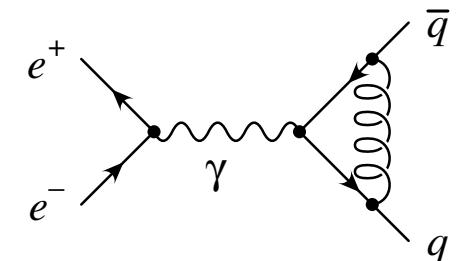
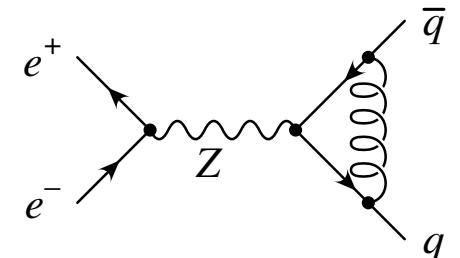
e.g. CLEO ($\sqrt{s} \sim 9$ GeV): $\alpha_s = 0.110 \pm 0.015$

Kühn, Steinhauser, Teubner '07

→ dominated by s -channel photon, less room for new physics

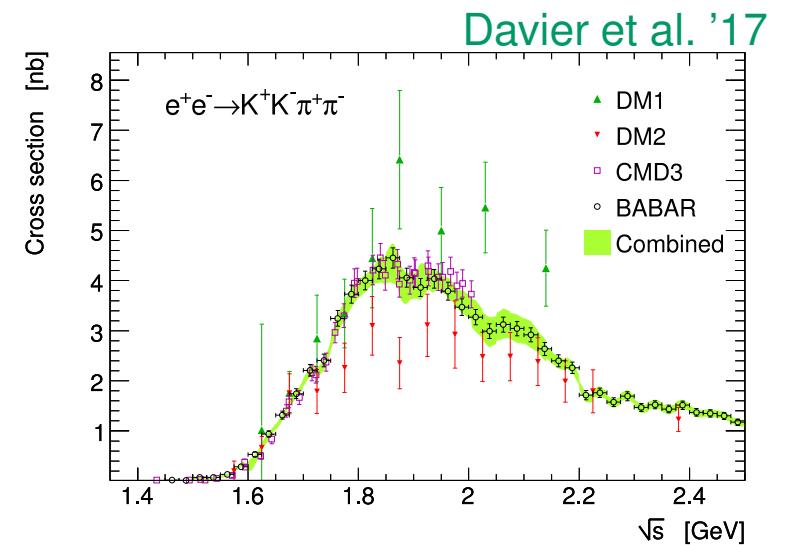
→ QCD still perturbative

naive scaling to 50 ab^{-1} (BELLE-II): $\delta \alpha_s \sim 0.0001$



- $\Delta\alpha \equiv 1 - \frac{\alpha(0)}{\alpha(M_Z)} \approx 0.059 = 0.0315_{\text{lept}} + 0.0276_{\text{had}}$

- a) From $e^+e^- \rightarrow \text{had.}$
using dispersion relation
- New data from BaBar, VEPP, BES, KLOE



Shift of finestructure constant

14/15

- $\Delta\alpha \equiv 1 - \frac{\alpha(0)}{\alpha(M_Z)} \approx 0.059 = 0.0315_{\text{lept}} + 0.0276_{\text{had}}$

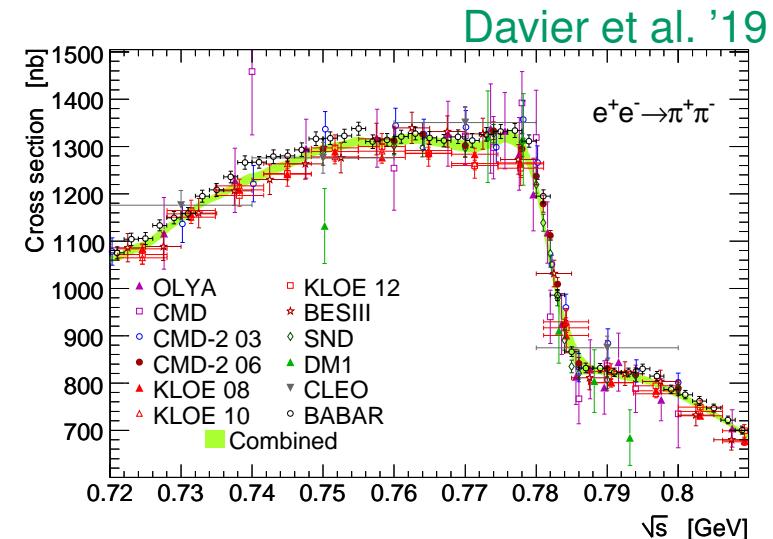
a) From $e^+e^- \rightarrow \text{had}$.
using dispersion relation

- New data from BaBar, VEPP, BES, KLOE
- Discrep. between BaBar & KLOE has small impact

→ Consistent results $\Delta\alpha_{\text{had}} \approx 0.0276 \pm 0.0001$

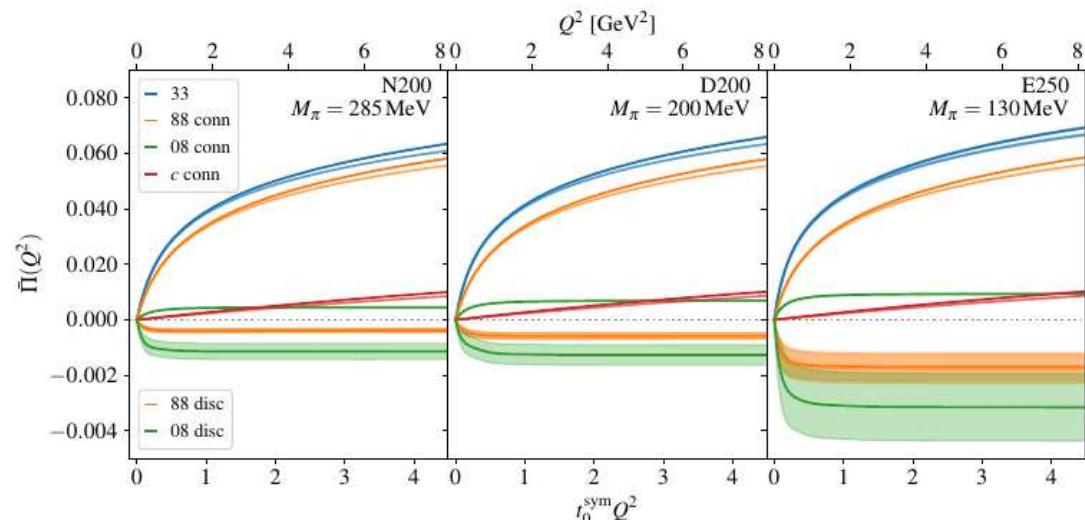
Davier et al. '19; Jegerlehner '19; Keshavarzi, Nomura, Teubner '19

→ Improvement to $\delta(\alpha_{\text{had}}) \sim 5 \times 10^{-5}$ likely



- b) From Lattice QCD
(work in progress)

Burger et al. '15
Cè et al. '19

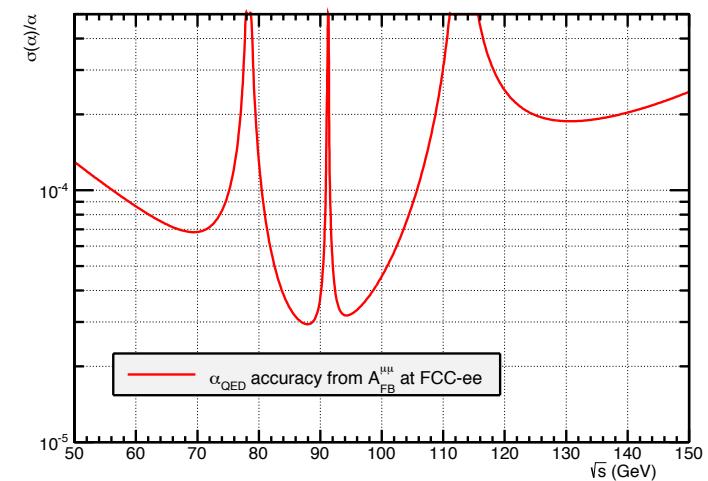


- c) Direct determination at FCC-ee from $e^+e^- \rightarrow \mu^+\mu^-$ off the Z peak
(i.e. $A_{\text{FB}}^{\mu\mu}$ at $\sqrt{s} \sim 88$ GeV and $\sqrt{s} \sim 95$ GeV)

$\rightarrow \delta(\Delta\alpha_{\text{had}}) \sim 3 \times 10^{-5}$ with $\mathcal{L}_{\text{tot}} = 85 \text{ ab}^{-1}$

Janot '15

Requires high-precision theory
prediction for $e^+e^- \rightarrow \mu^+\mu^-$
including 2/3-loop corrections for
 γ -exchange and box contributions



- Electroweak precision tests require theory input for **measurements of pseudo-observables** (BRs, widths, masses, cross-sections, ...) and their **SM/BSM interpretation**
- **Future e^+e^- colliders** improve precision by 1–2 orders of magnitude
- Uncertainties from **perturbative** and **non-perturbative** theory and **input parameters** require much work, but can also be mitigated through choice of measurements and analysis
- Theory progress needed both for **fixed-order loop corrections** as well as **MC tools**
- **Direct determination** of α_s , m_t , $\Delta\alpha$ at e^+e^- colliders is important
- Other lower-energy experiments can provide additional input:
BELLE II, BES, ...

Backup slides

Z lineshape

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \rightarrow f\bar{f}] = \mathcal{R}_{\text{ini}}(s, s') \otimes \sigma_{\text{hard}}(s')$$

- Subtraction of γ -exchange, $\gamma-Z$ interference, box contributions:

$$\sigma_{\text{hard}} = \sigma_Z + \sigma_\gamma + \sigma_{\gamma Z} + \sigma_{\text{box}}$$

- Z-pole contribution:

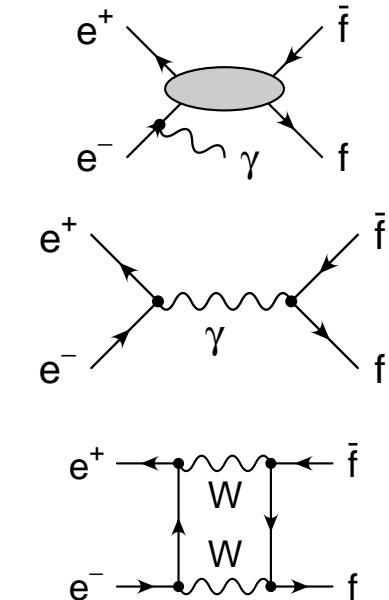
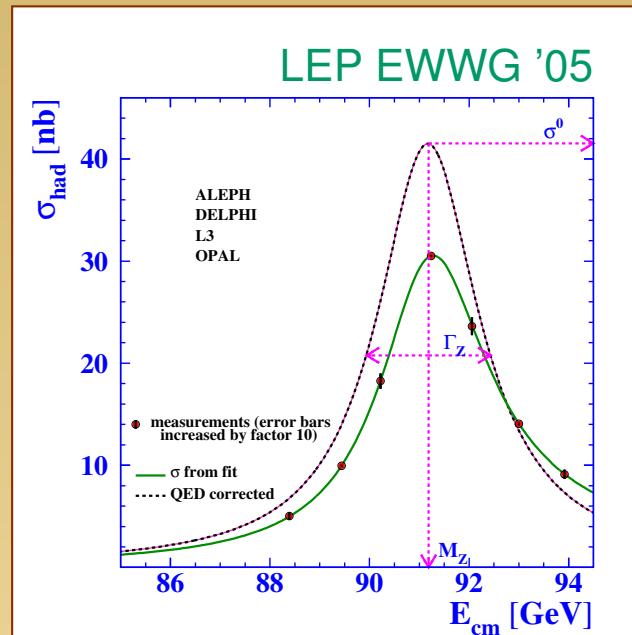
$$\sigma_Z = \frac{R}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} + \sigma_{\text{non-res}}$$

- In experimental analyses:

$$\sigma \sim \frac{1}{(s - M_Z^2)^2 + s^2 \Gamma_Z^2 / M_Z^2}$$

$$\overline{M}_Z = M_Z / \sqrt{1 + \Gamma_Z^2 / M_Z^2} \approx M_Z - 34 \text{ MeV}$$

$$\overline{\Gamma}_Z = \Gamma_Z / \sqrt{1 + \Gamma_Z^2 / M_Z^2} \approx \Gamma_Z - 0.9 \text{ MeV}$$



“Hard” matrix element

Consistent (gauge-invariant) theory setup:

Expansion of $\mathcal{A}[e^+ e^- \rightarrow \mu^+ \mu^-]$ about $s_0 = M_Z^2 - iM_Z\Gamma_Z$:

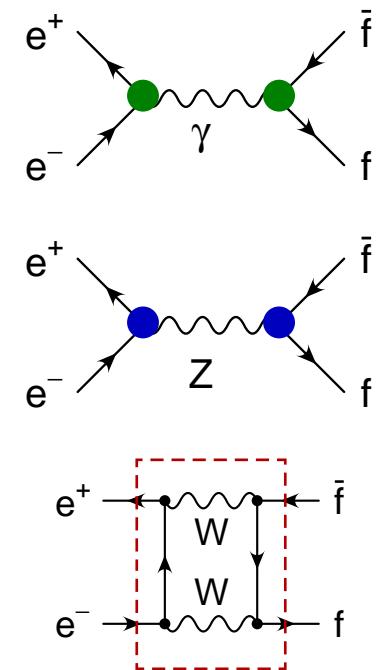
$$\mathcal{A}[e^+ e^- \rightarrow f\bar{f}] = \frac{R}{s - s_0} + S + (s - s_0)T + \dots$$

$$R = g_Z^e(s_0)g_Z^f(s_0)$$

$$S = \left[\frac{1}{M_Z^2} g_\gamma^e g_\gamma^f + g_Z^e g_Z^{f'} + g_Z^{e'} g_Z^f + S_{\text{box}} \right]_{s=s_0}$$

$g_V^f(s)$: effective $V f\bar{f}$ couplings

At NNLO: Need R at $\mathcal{O}(\alpha^2)$, S at $\mathcal{O}(\alpha)$, etc.



Z-pole asymmetries

Blondel scheme:

(if e^- and e^+ polarization available)

Blondel '88

Four independent measurements for $P_{e^+}/P_{e^-} = ++, +-, -+, --$

$$A_{LR} = \sqrt{\frac{(\sigma_{++} + \sigma_{-+} - \sigma_{+-} - \sigma_{--})(-\sigma_{++} + \sigma_{-+} - \sigma_{+-} + \sigma_{--})}{(\sigma_{++} + \sigma_{-+} + \sigma_{+-} + \sigma_{--})(-\sigma_{++} + \sigma_{-+} + \sigma_{+-} - \sigma_{--})}}$$

Note: No need to know $|P_{e^\pm}|$!

Main systematic uncertainties:

- Difference of $|P|$ for $P > 0$ and $P < 0$
- Difference of \mathcal{L} for $P > 0$ and $P < 0$

$$\delta A_{LR} \approx 10^{-4} \quad \Rightarrow \quad \delta \sin^2 \theta_{\text{eff}}^\ell \approx 1.3 \times 10^{-5}$$

Mönig, Hawkings '99

Theory calculations: Status

- Many seminal works on 1-loop and leading 2-loop corrections

Veltman, Passarino, Sirlin, Marciano, Bardin, Hollik, Riemann, Degrassi, Kniehl, ...

- Full 2-loop results for M_W , Z -pole observables

Freitas, Hollik, Walter, Weiglein '00

Hollik, Meier, Uccirati '05,07

Awramik, Czakon '02

Awramik, Czakon, Freitas, Kniehl '08

Onishchenko, Veretin '02

Freitas '14

Awramik, Czakon, Freitas, Weiglein '04

Dubovsky, Freitas, Gluza, Riemann, Usovitsch '16,18

Awramik, Czakon, Freitas '06

- Approximate 3- and 4-loop results (enhance by Y_t and/or N_f)

Chetyrkin, Kühn, Steinhauser '95

Chetyrkin et al. '06

Faisst, Kühn, Seidensticker, Veretin '03

Boughezal, Czakon '06

Boughezal, Tausk, v. d. Bij '05

Chen, Freitas '20

Schröder, Steinhauser '05

Theory and parametric uncertainties

	CEPC	perturb. error with 3-loop [†]	Param. error CEPC*	main source
M_W [MeV]	1	1	2.1	$m_t, \Delta\alpha$
Γ_Z [MeV]	0.5	0.15	0.15	m_t, α_s
R_b [10^{-5}]	4.3	5	< 1	
$\sin^2 \theta_{\text{eff}}^\ell$ [10^{-5}]	<1	1.5	2	$m_t, \Delta\alpha$

[†] **Theory scenario:** $\mathcal{O}(\alpha\alpha_s^2)$, $\mathcal{O}(N_f\alpha^2\alpha_s)$, $\mathcal{O}(N_f^2\alpha^2\alpha_s)$, leading 4-loop
 $(N_f^n = \text{at least } n \text{ closed fermion loops})$

Parametric inputs:

***CEPC:** $\delta m_t = 600$ MeV, $\delta \alpha_s = 0.0002$, $\delta M_Z = 0.5$ MeV,
 $\delta(\Delta\alpha) = 5 \times 10^{-5}$

Theory and parametric uncertainties

	CEPC	perturb. error with 3-loop [†]	Param. error CEPC*	main source
M_W [MeV]	1	1	0.6	$\Delta\alpha$
Γ_Z [MeV]	0.5	0.15	0.1	α_s
R_b [10^{-5}]	4.3	5	< 1	
$\sin^2 \theta_w^\ell$ [10 $^{-5}$]	<1	1.5	1	$\Delta\alpha$

[†] **Theory scenario:** $\mathcal{O}(\alpha\alpha_s^2)$, $\mathcal{O}(N_f\alpha^2\alpha_s)$, $\mathcal{O}(N_f^2\alpha^2\alpha_s)$, leading 4-loop
 $(N_f^n = \text{at least } n \text{ closed fermion loops})$

Parametric inputs:

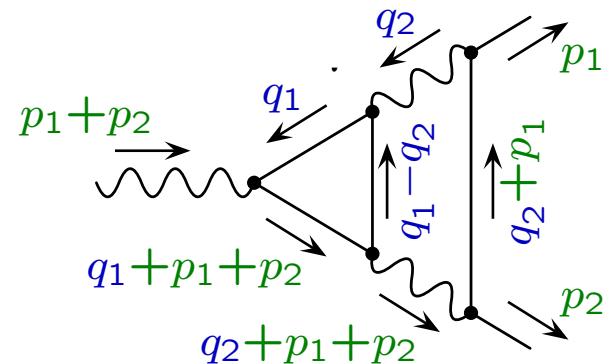
***FCC-ee:** $\delta m_t = 50$ MeV, $\delta\alpha_s = 0.0002$, $\delta M_Z = 0.5$ MeV,
 $\delta(\Delta\alpha) = 3 \times 10^{-5}$

Calculational techniques

Experimental precision requires inclusion of **radiative corrections** in theory
(1-loop, 2-loop, and partial 3-loop)

Integrals over loop momenta:

$$\int d^4 q_1 d^4 q_2 f(q_1, q_2, p_1, p_2, \dots, m_1, m_2, \dots)$$



Computer algebra tools:

- Generation of diagrams, $\mathcal{O}(100) - \mathcal{O}(10000)$
- Lorentz and Dirac algebra
- Integral simplification (and expansion)

Evaluation of loop integrals:

- In general not possible analytically
- Numerical methods are more general, but computing intensive
- Special numerical techniques can balance precision and evaluation time

Analytic calculations

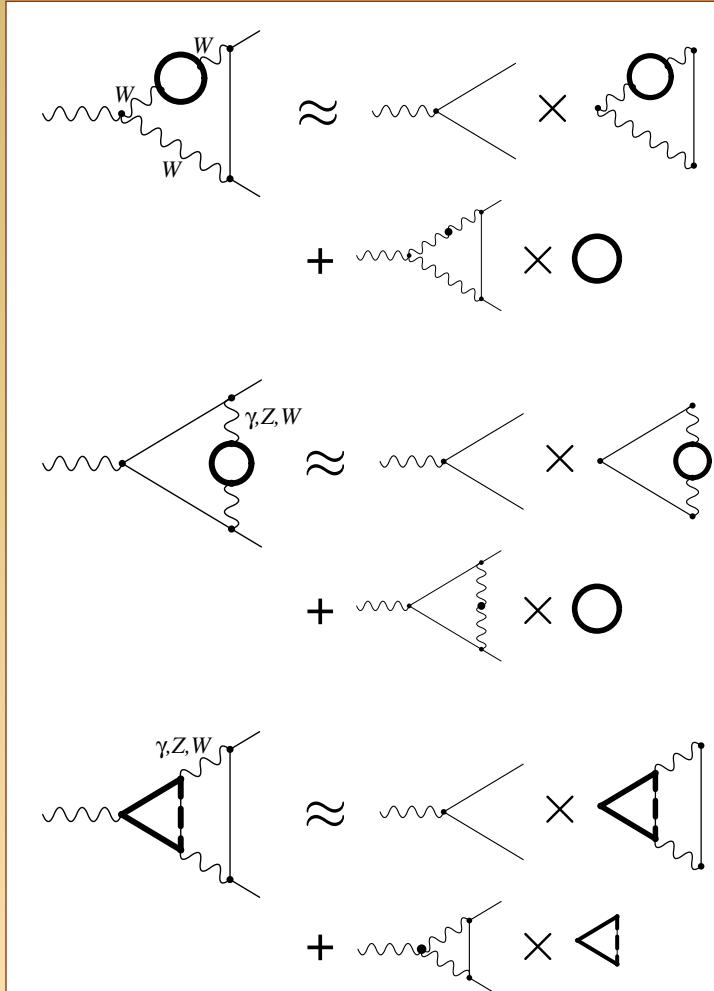
- Mostly used for diagrams with few mass scales
- Reduce to **master integrals** with integration-by-parts and other identities
[Chetyrkin, Tkachov '81](#); [Gehrmann, Remiddi '00](#); [Laporta '00](#); ...

Public programs:	Reduze	von Manteuffel, Studerus '12
	FIRE	Smirnov '13,14
	LiteRed	Lee '13
	KIRA	Maierhoefer, Usovitsch, Uwer '17

- Large need for computing time and memory
- Evaluate master integrals with differential equations or Mellin-Barnes rep.
[Kotikov '91](#); [Remiddi '97](#); [Smirnov '00,01](#); [Henn '13](#); ...
 - Result in terms of Goncharov polylogs / multiple polylogs
 - Some problems need iterated elliptic integrals / elliptic multiple polylogs
[Broedel, Duhr, Dulat, Trancredi '17,18](#)
[Ablinger et al. '17](#)
 - Even more classes of functions needed in future?

Asymptotic expansions

- Exploit large mass ratios,
e. g. $M_Z^2/m_t^2 \approx 1/4$
 - Evaluate coeff. integrals analytically
 - Fast numerical evaluation
- Used in some 2/3-scale problems
- Public programs:
exp Harlander, Seidensticker, Steinhauser '97
asy Pak, Smirnov '10
- Possible limitations:
- Difficult coefficient integrals
 - bad convergence



Numerical integration

Two general approaches:

- Automated treatment of UV/IR divergencies
- No restriction on number of loops or legs

■ Sector decomposition:

Public programs:	SecDec	Carter, Heinrich '10; Borowka et al. '12,15,17
	FIESA	Smirnov, Tentyukov '08; Smirnov '13,15

■ Mellin-Barnes representations:

Public programs:	MB/MBresolve	Czakon '06; Smirnov, Smirnov '09
	AMBRE/MBnumerics	Gluza, Kajda, Riemann '07 Dubovyk, Gluza, Riemann '15 Usovitsch, Dubovyk, Riemann '18

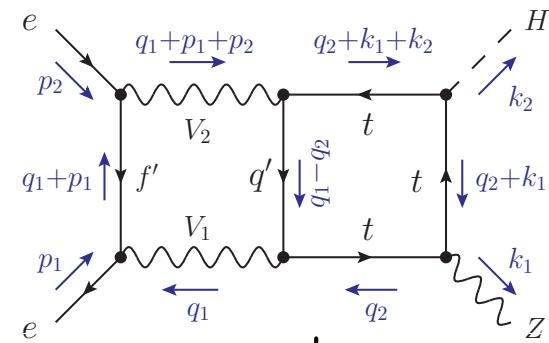
- Diagrams with internal thresholds can cause numerical instabilities
- Specialized techniques (for some type of diagrams) often improve computing time, robustness, precision (but not automated)

Specialized numerical techniques

Example: HZ double boxes

Song, Freitas '21

- Introduce Feynman parameters and disp. rel.
- Expressions for second loop from, e.g., LoopTools
Hahn, Perez-Victoria '98
- 3-dim. numerical integral with adaptive Gaussian integration
- $\mathcal{O}(0.1\%)$ precision in $\mathcal{O}(\text{min.})$ on laptop



$$\int dx dy$$

Feynman diagram showing the loop variables for the HZ double box loop. The top-right diagonal gluon has a momentum $k'_2 = xk_1 + (1-y)k_2$ and the bottom-right diagonal gluon has a momentum $k'_1 = (1-x)k_1 + yk_2$. The other momenta remain the same as in the previous diagram.

$$\int dx dy d\sigma$$

Feynman diagram showing the final result after numerical integration. A red vertical line labeled "mass σ " is inserted into the bottom-right diagonal gluon line. The top-right diagonal gluon has a momentum k'_2 and the bottom-right diagonal gluon has a momentum $q_1 + k'$. The other momenta remain the same as in the previous diagram.