Stochastic Collapse

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Introduction

Modern view of cosmology

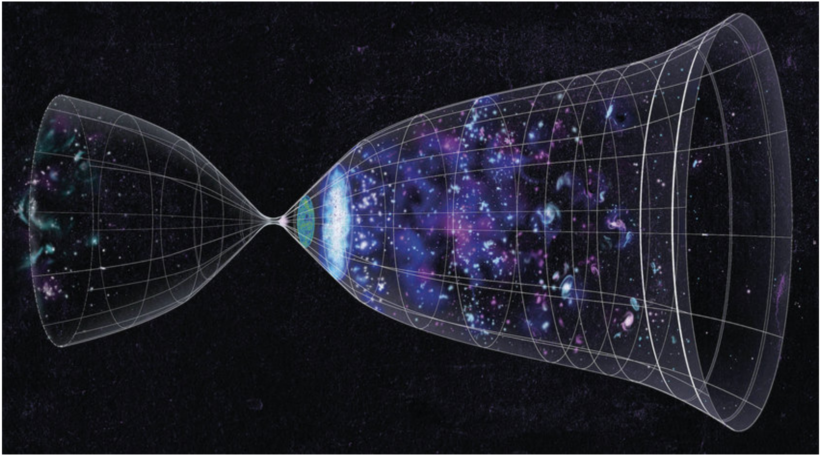
- Origin of large-scale structures from quantum vacuum fluctuations
- Small-scale initial perturbations stretched by accelerated expansion (Inflation)
- Classical inflation: slowly-rolling, self-interacting scalar field, almost scale-invariant spectrum. Very successful paradigm

But inflation is not a complete theory

- Ignores initial singularity
- Trans-Planckian modes
- Fine-tuning of the potential, etc...

Bouncing models can resolve some inflation problems.
Need for: contracting phase+bounce mechanism+expanding phase

How a bouncing universe could look like



Credits: https://www.aei.mpg.de/gravitation-and-cosmology

Scalar Field Collapse

Collapse scenario depends on potential

- Non-stiff collapse: $P < \rho$ with $V > 0$; (including scale-invariant collapse)
- Pre-Big Bang collapse: $P = \rho$ with $V = 0$; (blue tilted)
- Ekpyrotic collapse: $P \gg \rho$ with $V < 0$; (ultra-stiff fast-roll collapse)

Classical stability well-known (Heard & Wands, 2002).

Objective of this work

Study classical collapse scenarios with quantum fluctuations

FLRW Collapse

Homogeneous and isotropic background

$$L = \sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right] \quad \text{and} \quad ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$$

Scalar field with energy density and pressure

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\,, \quad P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\,, \tag{1}$$

Constant equation of state w

$$P = w\rho\,. \tag{2}$$

For simplicity: $V(\varphi) = V_0 e^{-\kappa\lambda\varphi} \implies$ scaling solution with

$$a \propto |t|^p \quad \text{where} \quad p = \frac{2}{\lambda^2} \quad \text{and} \quad \lambda^2 = 3(1+w)\,. \tag{3}$$

Dynamical system

Reducing dynamics to a one-dimensional problem

Klein-Gordon equation + Friedmann constraint

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 \qquad ; \quad H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V\right) . \tag{4}$$

Changing to dimensionless variables

$$x = \frac{\kappa\dot{\varphi}}{\sqrt{6}H} , \quad y = \frac{\kappa\sqrt{\pm V}}{\sqrt{3}H} , \tag{5}$$

The Friedmann constraint becomes

$$x^2 \pm y^2 = 1 , \tag{6}$$

Dynamical system (prime: $N = \ln(a)$)

$$x' = -3x(1 - x^2) \pm \lambda\sqrt{3/2}y^2 , \tag{7}$$

$$y' = xy(3x - \lambda\sqrt{3/2}) . \tag{8}$$

1D Phase-space I.Heard and D.Wands [Arxiv:0206085v1]

Stability analysis

Equation of state

$$w = \frac{x^2 \mp y^2}{x^2 \pm y^2} \ . \quad (9)$$

Critical points

$$(A_\pm)\ x_{A_\pm} = \pm 1 \ , \quad y_A = 0 \ ; \quad (10)$$

$$(B)\ x_B = \frac{\lambda}{\sqrt{6}} \ , \quad y_B = \sqrt{1 - \frac{\lambda^2}{6}} \ ; \quad (11)$$

the solution (B) exists for $\pm(6 - \lambda^2) > 0$.

- $\lambda^2 < 6$: flat positive potential
- $\lambda^2 > 6$: steep negative potential

Linear perturbations around x_B

$$x' = \frac{(\lambda^2 - 6)}{3}(x - x_B) \ . \quad (12)$$

What if we add noise to x?

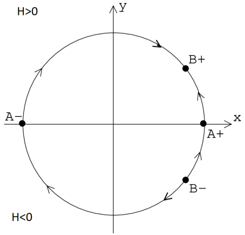


Figure: Phase-space for flat positive potentials, $\lambda^2 < 6$. Friedmann constraint $x^2 + y^2 = 1$. Arrows indicate evolution in cosmic time, t.

1D Phase-space

In Summary

- Expanding universe ($N \to +\infty$):
  - Scaling solution stable for positive, flat potential $\lambda^2 < 6$ (including inflation, $\lambda^2 \ll 1$).
  - Scaling solution unstable for negative, steep potential $\lambda^2 > 6$.
- Contracting universe ($N \to -\infty$):
  - Scaling solution stable for negative steep potential $\lambda^2 > 6$ (including ekpyrosis, $\lambda^2 \gg 6$).
  - Scaling solution unstable for positive flat potential $\lambda^2 < 6$ (including matter collapse, $\lambda^2 = 3$).

Field perturbations

First-order perturbations

- Scalar field perturbations $\varphi = \varphi(t) + \delta\varphi(t, \vec{x})$ in a linearly-perturbed FLRW metric

  $$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dx^i dt + a^2(t)\left[(1-2\psi)\delta_{ij} + 2\partial_{ij}E + h_{ij}\right] dx^i dx^j \ ,$$

  with A, B, ψ and E scalar potentials and h_{ij} tensor perturbations.
- Wave equation for first-order scalar field perturbations (spatially-flat gauge ($\psi = 0$))

  $$\frac{d^2 v}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2}\right) v = 0 \ , \tag{13}$$

  with $dt = a d\eta$, $v = a\delta\varphi$ and $z = a\dot{\varphi}/H$.

Field perturbations

Solutions at small (sub-Hubble) and large (super-Hubble) scales

$$\delta\varphi \simeq \frac{e^{-ikt/a}}{a\sqrt{2k}} \qquad \text{for } k^2/a^2 \gg H^2 \quad (\text{sub-Hubble scales})\,, \tag{14}$$

$$\delta\varphi \simeq \frac{C\dot{\varphi}}{H} + \frac{D\dot{\varphi}}{H}\int \frac{H^2}{a^3\dot{\varphi}^2}dt \qquad \text{for } k^2/a^2 \ll H^2 \quad (\text{super-Hubble scales})\,. \tag{15}$$

with quantum vacuum normalisation for under-damped oscillations on sub-Hubble scales (14).

Field perturbations

During accelerated expansion or collapse

- $|aH|$ increases $\rightarrow$ modes starting on sub-Hubble scales ($k^2 > a^2H^2$) stretched up to super-Hubble scales ($k^2 < a^2H^2$).

Result

- Quantum vacuum fluctuations $k^2/a^2 \gg H^2$ at early times[a] $\rightarrow$ well-defined predictions for the power spectrum of perturbations on super-Hubble scales.

[a] which means $\delta\varphi \simeq \frac{e^{-ikt/a}}{a\sqrt{2k}}$ for $k^2/a^2 \gg H^2$.

Field perturbations
D.Wands [Arxiv:0809.4556]

Perturbations evolution

Evolution of the perturbed scalar field ($\mathrm{v} = \mathrm{a}\delta\varphi$)

$$\frac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}\eta^2} + \left(\mathrm{k}^2 - \frac{\nu^2 - 1/4}{\eta^2}\right)\mathrm{v} = 0\ . \tag{16}$$

In power-law cosmology

$$\mathrm{a} \propto |\mathrm{t}|^{\mathrm{p}} \quad \text{where} \quad \nu = \frac{3}{2} + \frac{1}{\mathrm{p}-1}\ . \tag{17}$$

The growing mode solution of quantum fluctuations for a given k is

$$\delta\varphi_{\mathrm{k}} = \frac{\mathrm{i}}{\mathrm{a}}\sqrt{\frac{1}{4\pi\mathrm{k}}}\frac{\Gamma(|\nu|)2^{|\nu|}}{|\mathrm{k}\eta|^{|\nu|-1/2}}\ . \tag{18}$$

Power-law collapse

Predictions

- Power spectrum on super-Hubble scales as $\eta \to 0$

$$\mathcal{P}_{\delta\varphi} = \left[\frac{\Gamma(|\nu|)2^{|\nu|}}{(\nu - 1/2)2^{3/2}\Gamma(3/2)}\right]^2 \left(\frac{H}{2\pi}\right)^2 |k\eta|^{3-2|\nu|} \,. \tag{19}$$

- Power-law collapse $\implies$ power-law spectrum

$$\Delta n_{\delta\varphi} = \frac{d \ln \mathcal{P}_{\delta\varphi}}{d \ln k} = 3 - 2|\nu| \,. \tag{20}$$

- $\Delta n_{\delta\varphi} = 0$ for
  - Slow-roll inflation ($w = -1$ and $\nu = 3/2$);
  - Pressureless collapse ($w = 0$ and $\nu = -3/2$);

Stochastic Formalism

Quantifying how quantum noises modify the long-wavelength (or coarse-grained) field

Coarse-grained field and momentum (J.Grain and V.Vennin, JCAP 05(2017)045)

$$\dot{\overline{\varphi}} = \mathrm{a}^{-3}\overline{\pi}_{\varphi} + \xi_{\varphi} \ , \quad \dot{\overline{\pi}} = -\mathrm{a}^{3}\mathrm{V}_{,\overline{\varphi}} + \xi_{\pi} \ . \tag{21}$$

Time-dependent cut-off scale (coarse-graining scale)

$$\mathrm{k}_{\sigma} = \sigma \mathrm{a}\mathrm{H} \ . \tag{22}$$

Noises (small-wavelength part) described by two-points correlation matrix $\Xi_{\mathrm{f},\mathrm{g}}$

$$\Xi_{\mathrm{f},\mathrm{g}} = \langle 0|\xi_{\mathrm{f}}\xi_{\mathrm{g}}|0\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}\mathrm{k}_{\sigma}^{3}(\mathrm{N})}{\mathrm{d}\mathrm{N}}\mathrm{f}_{\mathrm{k}}(\mathrm{N})\mathrm{g}_{\mathrm{k}}^{\star}(\mathrm{N}) \ . \tag{23}$$

Noise growth in a collapsing universe?

Quantum noise

Perturbing EOS (note the relation $\delta \mathrm{w} = 4\mathrm{x_B}\delta\mathrm{x}$)

$$\delta \mathrm{x} = \frac{\kappa}{\sqrt{6}\mathrm{H}}\left(\delta\dot{\varphi} - \mathrm{A}\dot{\varphi} - \frac{\dot{\varphi}}{\mathrm{H}}\delta\mathrm{H}\right) . \tag{24}$$

Correlation matrix of the noise at critical point (B)

$$\Xi_{\mathrm{x,x}} = \mathrm{g}(\nu,\lambda)\frac{(|\nu| - \nu)^2}{\sigma^{2|\nu|-3}}\kappa^2\mathrm{H}_\star^2 \exp\left[-\frac{3-2\nu}{\nu - 1/2}(\mathrm{N}_\star - \mathrm{N})\right] . \tag{25}$$

No noise for $\nu > 0$: adiabatic perturbations (includes power-law inflation ($\nu = 3/2$) and ekpyrosis ($\nu = 1/2$)).
True at leading and next-to-leading order!

Kinetic-dominated solution (critical point A), $\lambda^2 = 6$ or $\nu = 0$

Always $\delta \mathrm{x} = 0$ at first order!

Variance of Langevin equations

Formal solutions

Langevin equation at $\mathrm{x} = \mathrm{x_B}$

$$\bar{\mathrm{x}}' = \mathrm{m}(\bar{\mathrm{x}} - \mathrm{x_B}) + \hat{\xi}_{\mathrm{x}} \quad \text{with} \quad \mathrm{m} = \frac{\lambda^2 - 6}{2} \,. \tag{26}$$

Variance split into classical/quantum parts

$$\begin{aligned}\sigma^2_{\mathrm{x}}(\mathrm{N}) = \left\langle (\bar{\mathrm{x}}(\mathrm{N}) - \mathrm{x_B})^2 \right\rangle &= \sigma^2_{\mathrm{x,cl}}(\mathrm{N}) + \sigma^2_{\mathrm{x,qu}}(\mathrm{N}) \\ &= \sigma^2_{\mathrm{x}}(\mathrm{N}_\star) e^{2\mathrm{m}(\mathrm{N}-\mathrm{N}_\star)} + \int_{\mathrm{N}_\star}^{\mathrm{N}} \mathrm{dS}\, e^{2\mathrm{m}(\mathrm{N}-\mathrm{S})} \Xi_{\mathrm{x,x}}(\mathrm{S})\end{aligned} \tag{27}$$

- For $\nu \neq -3/2$

$$\sigma^2_{\mathrm{x,qu}}(\mathrm{N}) = \mathrm{h}(\nu, \lambda, \sigma)\kappa^2 \mathrm{H}^2(\mathrm{N}) \left\{ 1 - \exp\left[\frac{3 + 2\nu}{\nu - 1/2} (\mathrm{N}_\star - \mathrm{N}) \right] \right\} \tag{28}$$

- For $\nu = -3/2$ (scale-invariant/pressureless collapse)

$$\sigma^2_{\mathrm{x,qu}}(\mathrm{N}) = \frac{3}{2^7 \pi} \frac{\mathrm{H}^2(\mathrm{N})}{\mathrm{M}^2_{\mathrm{pl}}} (\mathrm{N}_\star - \mathrm{N}) \;, \tag{29}$$

Quantum Diffusion and Power Spectrum

- Quantum part of variance decays when

$$\frac{3+2\nu}{\nu-1/2} > 0\,. \tag{30}$$

This is the case if either $\nu > 1/2$ or $\nu < -3/2$ (ignore first case: adiabatic!)

- Shift in spectral index:

$$\mathrm{n_s} - 1 = \frac{12\mathrm{w}}{1+3\mathrm{w}} = \frac{4(2\nu+3)}{3}\,. \tag{31}$$

For small positive deviation ϵ, red spectrum when $\nu = -3/2 - \epsilon$.

Spectral index

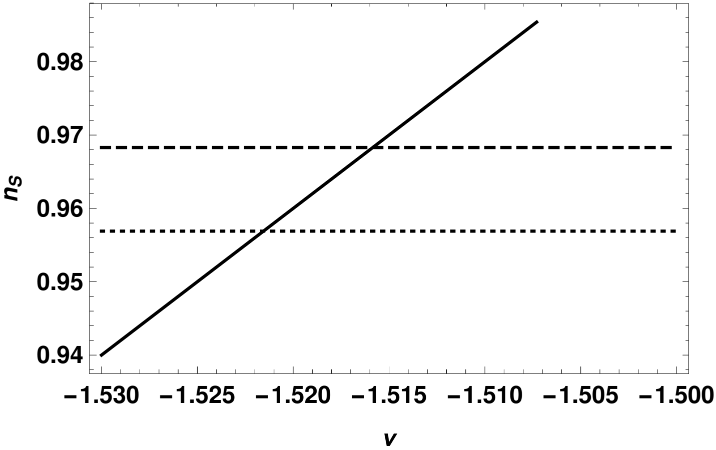


Figure: Spectral index vs Bessel index. Dotted lines enclose 68% CL measurements by Planck Collaboration.

Maximum lifetime of the collapse phase at the fixed point

Backreaction condition

If $\sigma^2_{\mathrm{x,qu}} = 1 \implies$ does quantum noise change the dynamics?

- Pressureless collapse ($\nu = -3/2$)

$$|\mathrm{H(N)}| \approx \sqrt{\frac{134}{\mathrm{N}_\star - \mathrm{N}}}\mathrm{M_{pl}} \,. \tag{32}$$

  Drives away from fixed point below Planck scale if $(\mathrm{N}_\star - \mathrm{N}) > 134$.

- For slightly red spectrum ($\nu = -3/2 - \epsilon$, $\mathrm{n_s} < 1$): classical perturbations grow faster
- Example from general solution: radiation-dominated collapse ($\nu = -1/2$)

$$|\mathrm{H(N)}| \approx \frac{13}{\sigma}\mathrm{M_{pl}} \,. \tag{33}$$

  Cannot escape fixed point since $\sigma < 1$.

Hubble rate Evolution

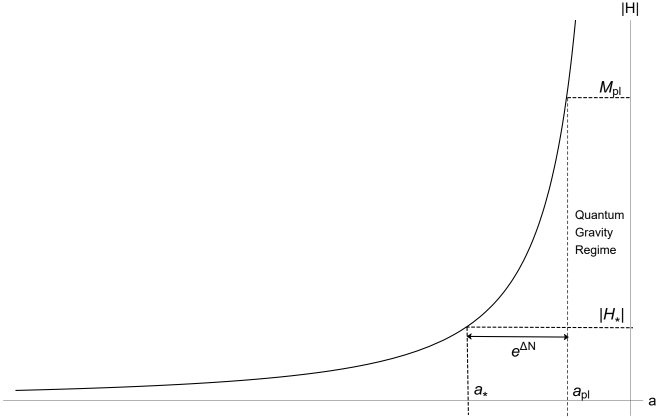


Figure: Quantum diffusion dominates if starts from low energy scales.

Summary

In summary

Inflation / Ekpyrotic collapse ($\nu > 0$) — Pressureless collapse ($\nu < 0$)

δx $= 0$ (adiabatic perturbation) — δx $\neq 0$ (non-adiabatic perturbation)

- Inflation / Ekpyrotic collapse both classical and quantum stable.
- Pressureless collapse: quantum diffusion may change dynamics before Planck scale for large number of e-folds.
- Classical perturbations dominate in almost scale-invariant collapse

What's next?

- Connect these results to expanding phase (extend stochastic formalism to non-monotonic time variable)
- Bounce from stochastic geometry?
- Gauge corrections in collapse?