

Stochastic Effects of Electromagnetic Fields during Inflation

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Outline

1 Stochastic Differential Equations

- Examples

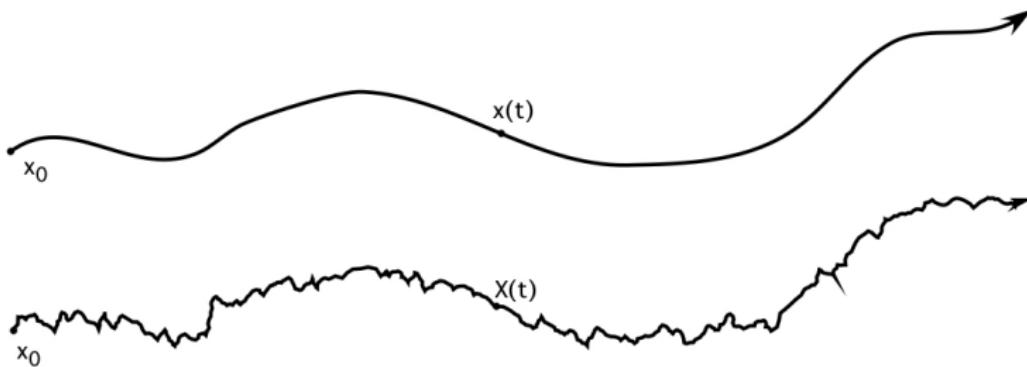


2 Stochastic Processes in Early Universe

- Inflation
- Anisotropic Inflation
- Primordial Magnetic Fields

3 Summary and Future Works

Stochastic Differential Equations Wiener Process



stochastic process $X(\cdot)$: solution of SDE

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t)) \xi(t), & (t > 0) \\ X(0) = X_0, \end{cases}$$

White noise:

$$\langle \xi(t) \xi(t') \rangle = \delta(t - t') .$$

Stochastic Differential Equations Wiener Process

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t)) \xi(t), & (t > 0) \\ X(0) = X_0, \end{cases}$$

White noise:

$$\langle \xi(t) \xi(t') \rangle = \delta(t - t').$$

$$X(t) = X_0 + \int_0^t b(X, s) \, ds + \int_0^t B(X, s) \, dW$$

Wiener Process (Brownian motion):

$$\frac{dW}{dt} = \xi(t)$$

$$\langle W(t) \rangle = 0, \quad \langle W(t)W(s) \rangle = \min\{t, s\}, \quad \langle W^2(t) \rangle = t,$$

Examples of linear SDE

Examples

- Wiener process
(Brownian motion)
- Stock prices
(geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process
(mean-reverting process)

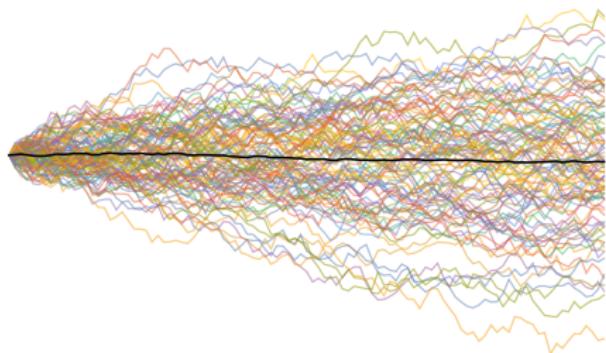


Figure: Wiener

$$dX_t = \sigma dW_t$$

$\sigma : \text{constant.}$

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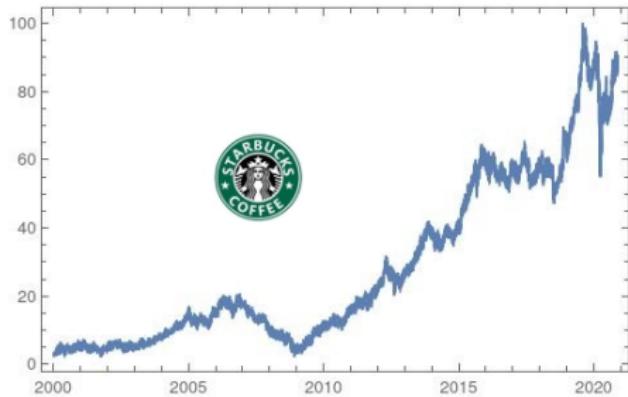


Figure: Starbucks Corporation (SBUX) Stock Price

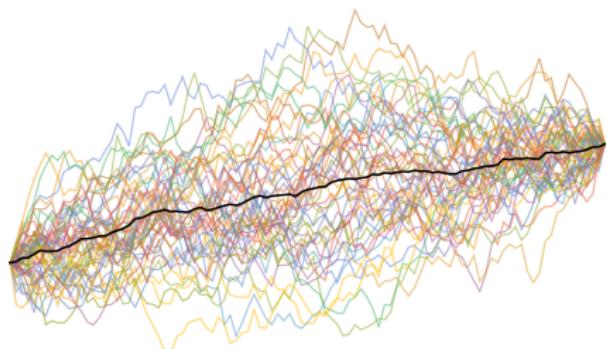
$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t, \\ S(0) = s_0, \end{cases}$$
$$S(t) = s_0 e^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t}$$

$\mu > 0$: Drift , σ : volatility .

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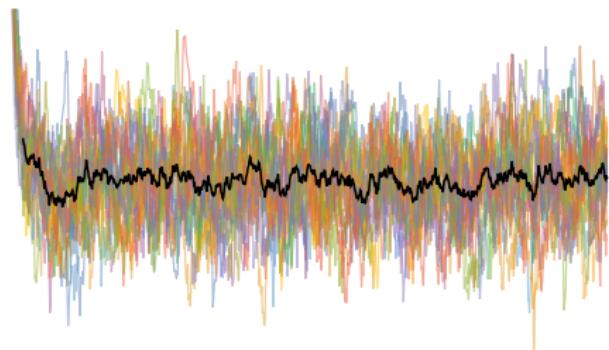
$$\begin{cases} dB_t = -\frac{B_t}{1-t} dt + dW_t, \\ B(0) = 0, \end{cases}$$

$$B(t) = (1-t) \int_0^t \frac{1}{1-s} dW_s$$

Examples of linear SDE

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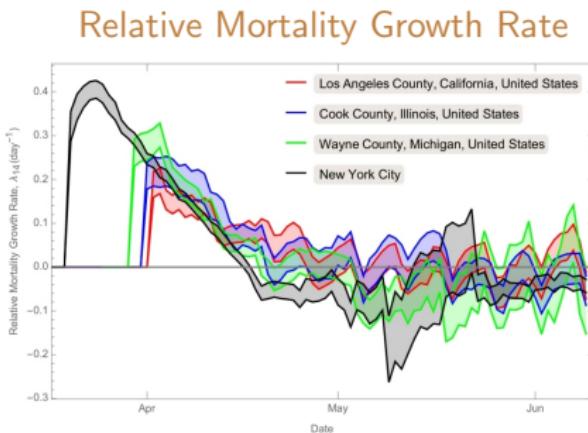
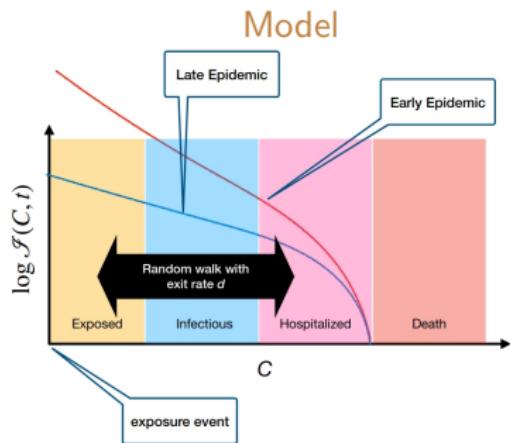
$$\begin{cases} dX_t = -b X_t dt + \sigma dW_t, \\ X(0) = X_0, \end{cases}$$

$$X(t) = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW_s$$

Friction $b > 0$, diffusion: σ

COVID-19

N. Afshordi, B. Holder, M. Bahrami and D. Lichtblau, arXiv:2007.00159



$\mathcal{I}(C, t)$: density of infected per stage at time t

C : stage of infection

τ : characteristic time scale of the random walk

d^{-1} : Time from exposure to quarantine/recovery

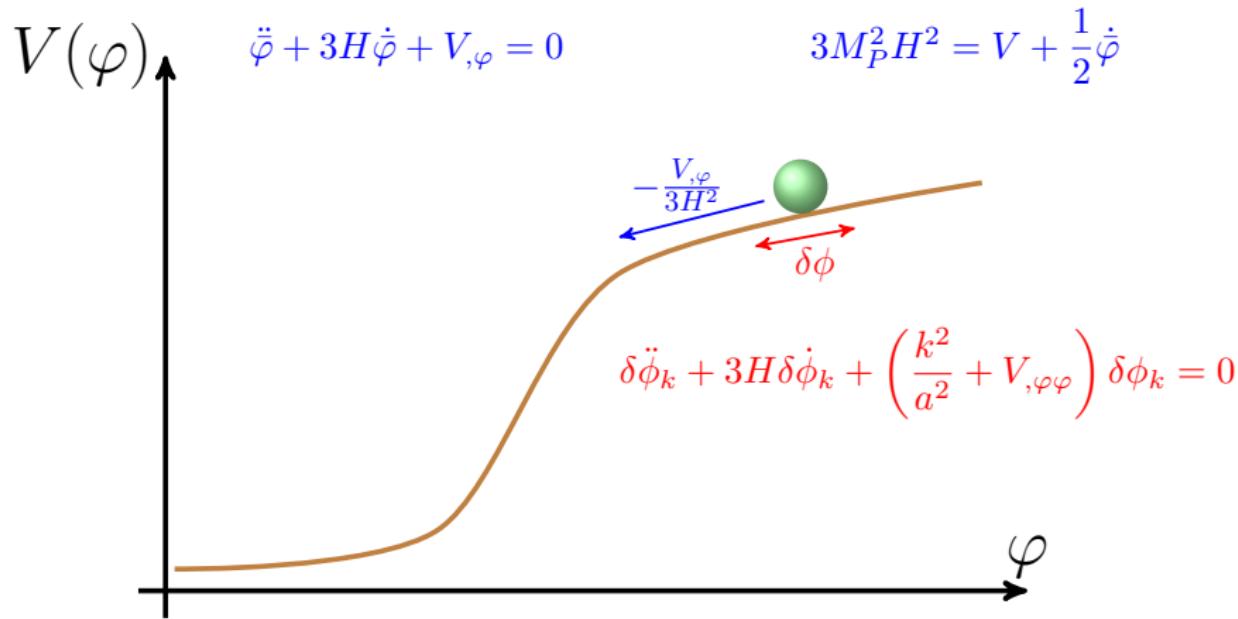
$$\frac{\partial \mathcal{I}}{\partial t} = \frac{1}{2\tau} \frac{\partial^2 \mathcal{I}}{\partial C^2} - d \mathcal{I}$$

Inflation: formal approach

Inflaton (quantum) field: $\hat{\Phi} = \bar{\varphi} + \delta\phi$

Classical background (fixed) : $\bar{\varphi} \Rightarrow$ acceleration \Rightarrow Homogeneous

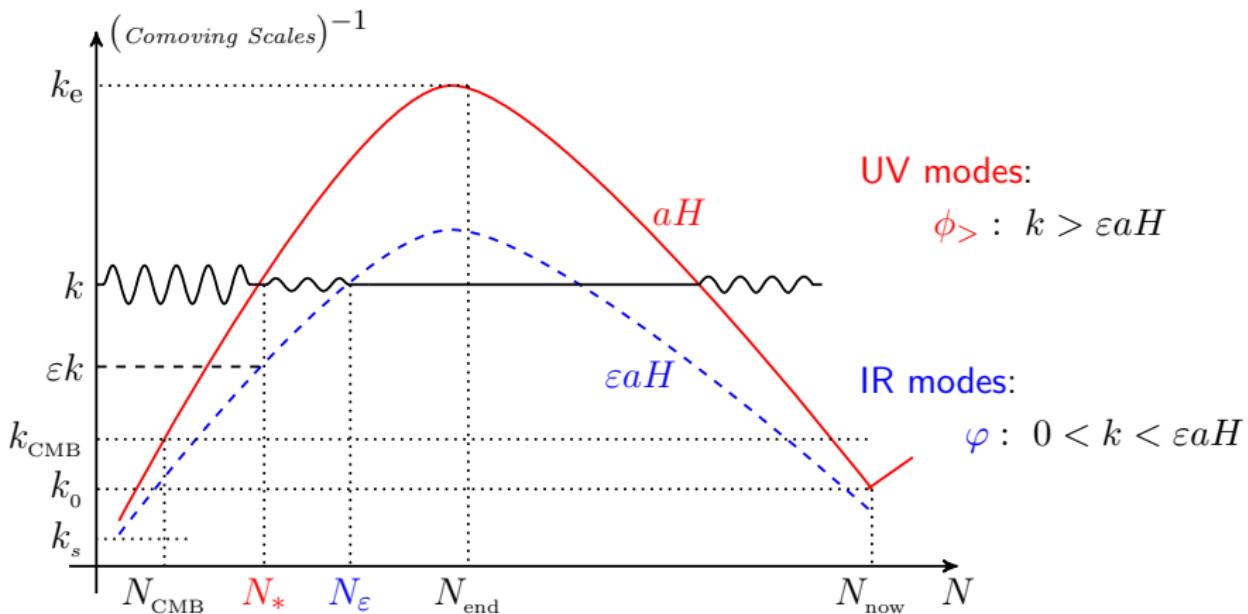
Small quantum perturbations : $\delta\phi \Rightarrow$ Structure



Stochastic Inflation

$$\hat{\Phi} = \varphi + \phi_>$$

Coarse-graining: $W_H(k, t) = \Theta(k - \varepsilon aH)$



Stochastic Inflation Langevin equation

Split a quantum fields $(\hat{\Phi}, \hat{\Pi})$ into **long** and **short** modes

$$\hat{\Phi} = \varphi + \int \frac{d^3k}{(2\pi)^3} \Theta(k - \varepsilon a(t)H) \hat{\phi}_k(t) e^{ik \cdot x}$$
$$\dot{\hat{\Phi}} = \dot{\hat{\Pi}} = \pi + \int \frac{d^3k}{(2\pi)^3} \Theta(k - \varepsilon a(t)H) \dot{\hat{\phi}}_k(t) e^{ik \cdot x}$$

time-dependent window function

Langevin's equation

$$\begin{cases} \frac{d\varphi}{dN} = \pi + \hat{\xi}_\varphi \\ \frac{d\pi}{dN} = -(3 - \epsilon_H)\pi - \frac{V_{,\varphi}}{H^2} + \hat{\xi}_\pi \end{cases}$$

e -folding number: $dN = H dt$

Stochastic inflation: Stochastic Noises

For a light scalar field, **Qunatum** stochastic noises:

$$\hat{\xi}_\varphi(t, \mathbf{x}) = -\frac{dk_c}{dt} \int \frac{d^3k}{(2\pi)^3} \delta(k - k_c) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\phi}_{\mathbf{k}}(t),$$

$$\hat{\xi}_\pi(t, \mathbf{x}) = -\frac{dk_c}{dt} \int \frac{d^3k}{(2\pi)^3} \delta(k - k_c) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\dot{\phi}}_{\mathbf{k}}(t), \quad k_c = \varepsilon a(t) H$$

commute as $\varepsilon \rightarrow 0$

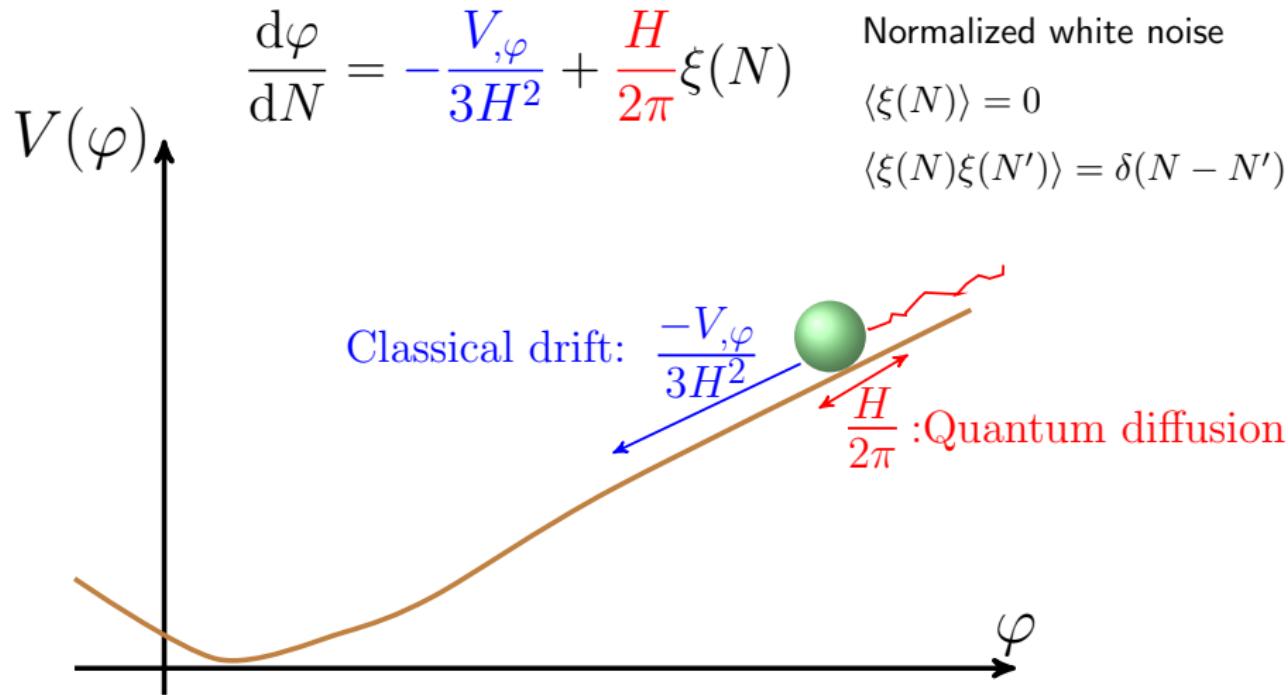
$$\langle [\hat{\xi}_\varphi, \hat{\xi}_\pi] \rangle = 0$$

and become **classic** noises:

$$\langle \hat{\xi}_\varphi(N_1) \hat{\xi}_\varphi(N_2) \rangle = \left(\frac{H}{2\pi}\right)^2 \delta(N_1 - N_2)$$

$$\langle \hat{\xi}_\pi(N_1) \hat{\xi}_\pi(N_2) \rangle \sim \mathcal{O}(\varepsilon^4)$$

Stochastic inflation: Slow-roll inflation



Electromagnetic fields during inflation

Motivations $\begin{cases} \text{Statistical anisotropy} & \text{CMB: } |g_*| \leq 10^{-2} \\ \text{Primordial magnetic field} & \text{Blazars: } B_{\text{Mpc}} \gtrsim 10^{-16} G \end{cases}$

Model $f^2 F^2$:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{f^2(\phi)}{4} F^{\mu\nu} F_{\mu\nu} \right]$$

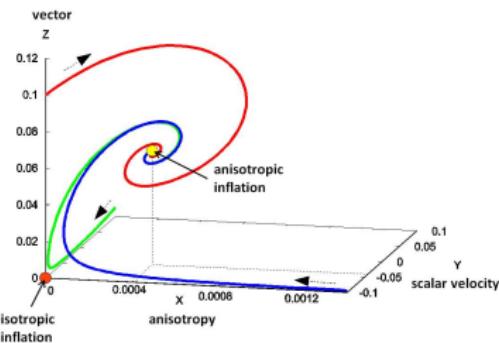
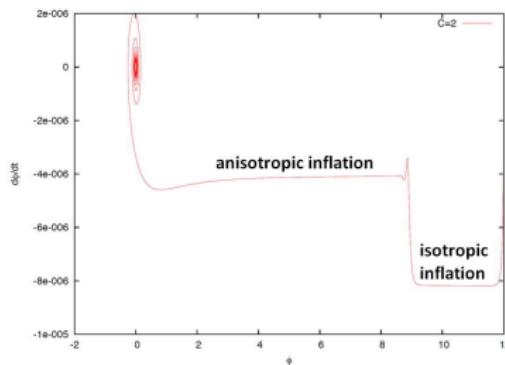
Inflationary Universe with Anisotropic Hair

M. Watanabe, S. Kanno and J. Soda, Phys. Rev. Lett. 102, 191302 (2009)

Electromagnetic energy decays as $\frac{1}{f^2 a^4}$

$$f_c \propto a^{-2c}, \quad c - 1 \sim 10^{-7} \Rightarrow |g_*| \sim 0.01$$

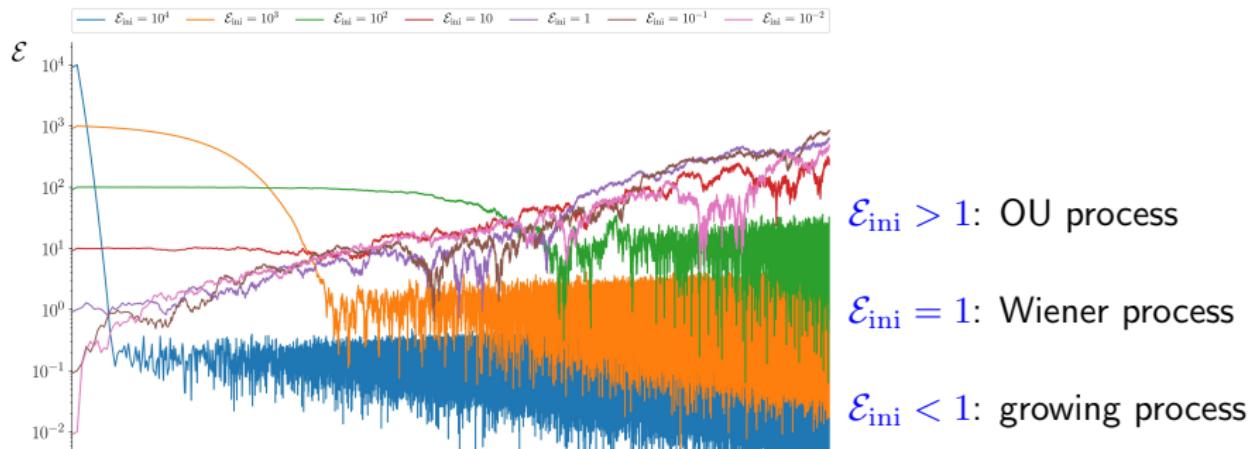
quadrupolar statistical anisotropy



$$E_{\text{att}} \simeq 10^{50} \left(\frac{r_t}{0.01} \right) V/m$$

Stochastic Effects in Anisotropic Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, Phys. Rev. D 101, 023524 (2020)



$\mathcal{E}_{\text{ini}} > 1$: OU process

$\mathcal{E}_{\text{ini}} = 1$: Wiener process

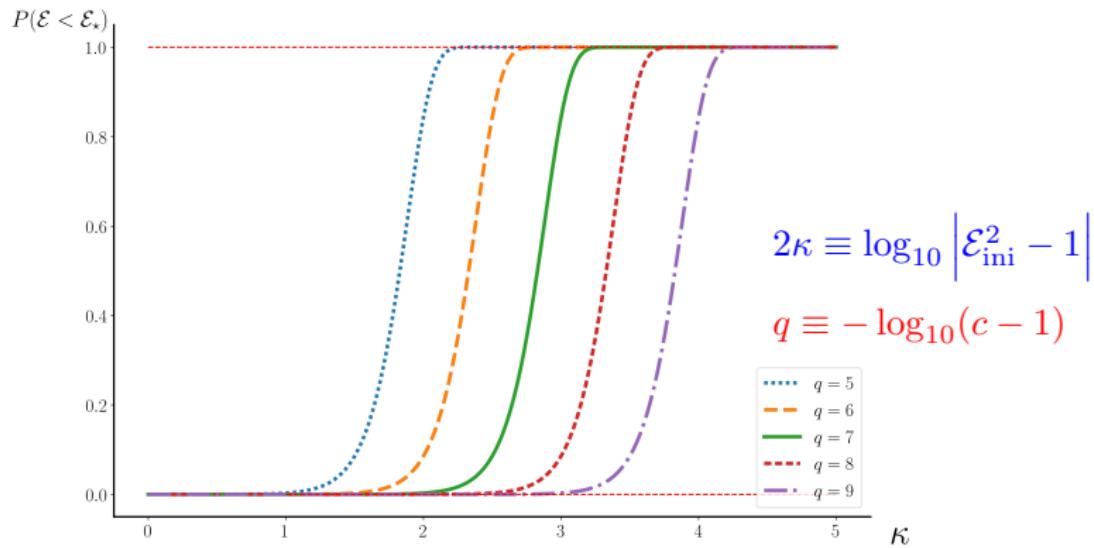
$\mathcal{E}_{\text{ini}} < 1$: growing process

$$\mathcal{E}' = 2(c-1)(1 - \mathcal{E}_{\text{ini}}^2) \mathcal{E} + 2\sqrt{\frac{\mathcal{P}_{\mathcal{R}}}{c-1}} \xi_N, \quad \mathcal{E} \equiv \frac{E}{E_{\text{att}}}$$

Stochastic Effects in Anisotropic Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, Phys. Rev. D 101, 023524 (2020)

Does Anisotropic Inflation Produce a Small Statistical Anisotropy? with the probability of $P(g_\star < 0.01) = P(\mathcal{E} < \mathcal{E}_\star)$ Yes



Observational facts and Primordial Magnetic Field

A. Bonafede *et al.*, Astron.Astrophys. 513 (2010)

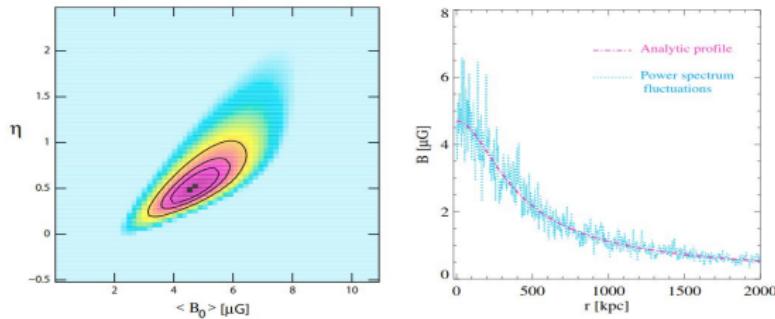


Figure: Bonafede et al. 2010

$$10^{-9}G \gtrsim B_{\text{obs}} \gtrsim 10^{-16}G \times \begin{cases} 1 & \lambda_B \gtrsim 1\text{Mpc} \\ \sqrt{\frac{1\text{Mpc}}{\lambda_B}} & \lambda_B \lesssim 1\text{Mpc}. \end{cases}$$

Correlation length λ_B

Primordial Magnetic Fields

V. Demozzi, V. Mukhanov and H. Rubinstein, JCAP 0908(2009) 025

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{f^2(\eta)}{4} F^{\mu\nu} F_{\mu\nu} \right]$$

$$f(\eta) = f_{\text{end}} \left(\frac{\eta}{\eta_{\text{end}}} \right)^n, \quad \eta \in (-\infty, 0), \quad \begin{cases} n < 0 & \text{strong coupling regime} \\ n > 0 & \text{weak coupling regime.} \end{cases}$$

Slow-roll limit: $3M_P^2 H^2 \simeq V(\phi) + \frac{1}{2}(E^2 + B^2)$,

Back-reaction parameter: $R \equiv \frac{E^2 + B^2}{6M_P^2 H^2} \ll 1$,

$$B_{\text{now}} = 2.5 \times 10^{-57} \left(\frac{r_t}{0.01} \right)^{-\frac{1}{2}} B_{\text{end}}$$

Magnetic fields from inflation?

V. Demozzi, V. Mukhanov and H. Rubinstein, JCAP 0908(2009) 025

$$\lambda_B = 1 \text{Mpc} \quad 10^{-9}G \gtrsim B_{\text{obs}} \gtrsim 10^{-16}G$$

Weak coupling regime

$$B_{\text{end}} \simeq \frac{H^2}{2\pi} \left(\frac{\lambda_{\text{ph}}}{H^{-1}} \right)^{n-3}$$

$$n=2: B_{\text{now}} \simeq 10^{-35}G$$

$$n=2.2: B_{\text{now}} \simeq 10^{-30}G$$

$$n=3: B_{\text{now}} \simeq 10^{-11}G \star$$

\star : Electric back-reaction problem

No back-reaction limit: $n \leq 2.2$

Strong coupling regime

$$B_{\text{end}} \simeq \frac{H^2}{2\pi} \left(\frac{\lambda_{\text{ph}}}{H^{-1}} \right)^{-n-2}$$

$$n=-3: B_{\text{now}} \simeq 10^{12}G! \star$$

$$n=-2.2: B_{\text{now}} \simeq 10^{-7}G$$

$$n=-2: B_{\text{now}} \simeq 10^{-11}G$$

\star : Magnetic back-reaction problem

No back-reaction limit: $n \geq -2.2$

Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066

Unified electric and magnetic field using an auxiliary vector field X_i satisfying

$$\ddot{X}_i - \frac{\nabla^2}{a^2} X_i + 5H\dot{X}_i - \left[(\nu - \frac{5}{2})(\nu + \frac{5}{2}) - 2(\nu^2 - \frac{5}{4})\epsilon_H \right] H^2 X_i \simeq 0$$

mode function $X(k, \eta) = i \frac{\sqrt{\pi}}{2} k H^2 \eta^{5/2} H_\nu^{(1)}(-k\eta)$.

$$E = X|_{\nu \rightarrow n + \frac{1}{2}}, \quad B = X|_{\nu \rightarrow n - \frac{1}{2}},$$

Long-Short decomposition:

$$\mathbf{X}(t, \mathbf{x}) = \mathbf{X}^{\text{IR}}(t, \mathbf{x}) + \int \frac{d^3 k}{(2\pi)^3} \Theta(k - \varepsilon a H) \mathbf{X}_k(t) e^{i \mathbf{k} \cdot \mathbf{x}}$$

Langevin equation:

$$\mathcal{X}' = b_\nu \mathcal{X} + D_\nu(\varepsilon) \xi, \quad \mathcal{X} = \frac{\mathbf{X}^{\text{IR}}}{\sqrt{2\epsilon_H M_P H}}$$

Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066

Electromagnetic fields have **no classical background** values,

$$\mathcal{X}(N) = D_\nu(\varepsilon) e^{b_\nu N} \int_0^N e^{-b_\nu s} d\mathbf{W}(s),$$

$$D_\nu(\varepsilon) = \sqrt{6\mathcal{P}_\zeta} \times \begin{cases} \frac{2^{|\nu|}}{3} \frac{\Gamma(|\nu|)}{\sqrt{2\pi}} \left(1 + \frac{\left|\frac{5}{2} - |\nu|\right|}{Q_\nu}\right) \varepsilon^{-b_\nu} & |\nu| \neq 5/2 \\ 1 & \nu = \pm 5/2. \end{cases}$$

$$b_\nu \equiv |\nu| - \frac{5}{2} + \mathcal{O}(\epsilon_H)$$

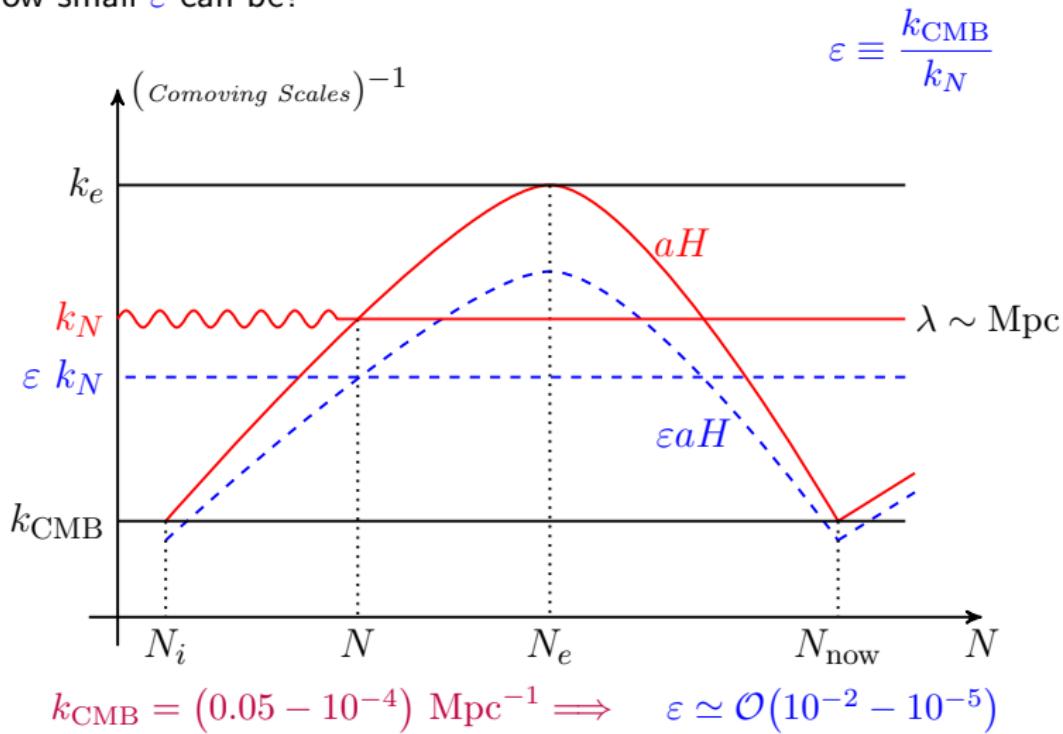
Ornstein-Uhlenbeck (OU) process: $b_\nu < 0$
frictional drift force $-|b_\nu|\mathcal{X} \sim$ the random force $D_\nu\xi$

Equilibrium state: $\langle \mathcal{X}^2 \rangle_{\text{eq}} = \frac{3D_\nu^2}{2|b_\nu|}$ at around $N_{\text{eq}} \simeq \mathcal{O}(\ln 10/|b_\nu|)$

Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066

How small ε can be?

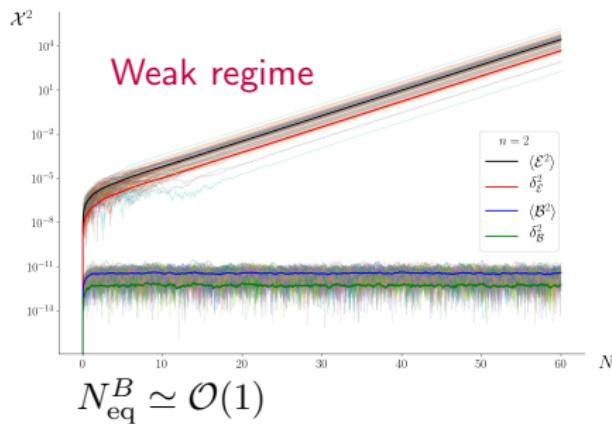


Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066

$$\mathcal{B}' = -\mathcal{B} + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}} \varepsilon \xi$$

$$\mathcal{E}' = -\epsilon_H \mathcal{E} + \sqrt{6\mathcal{P}_\zeta} \xi$$

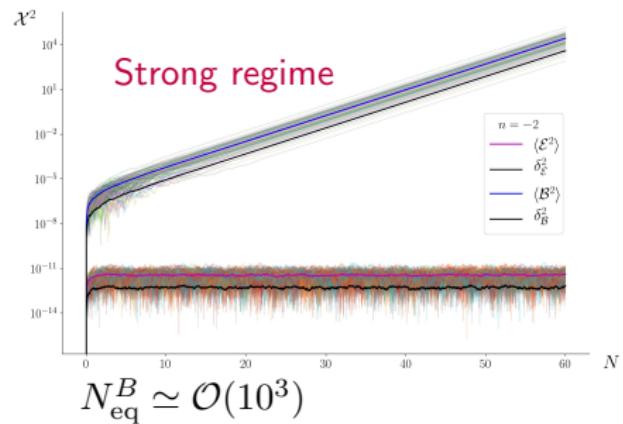


$$R(N=60) \simeq 10^{-10}$$

$$B_{\text{now}} \simeq 10^{-13} \text{ G}$$

$$\mathcal{E}' = -\mathcal{E} + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}} \varepsilon \xi$$

$$\mathcal{B}' = -\epsilon_H \mathcal{B} + \sqrt{6\mathcal{P}_\zeta} \xi$$

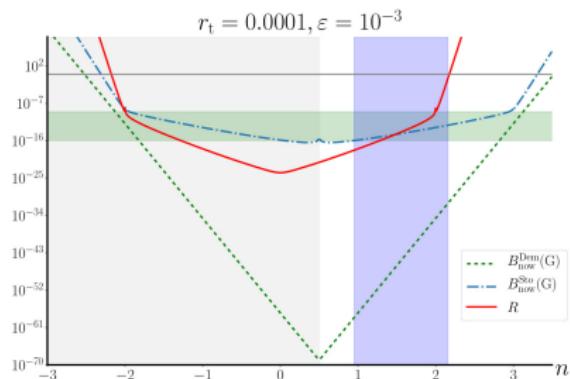
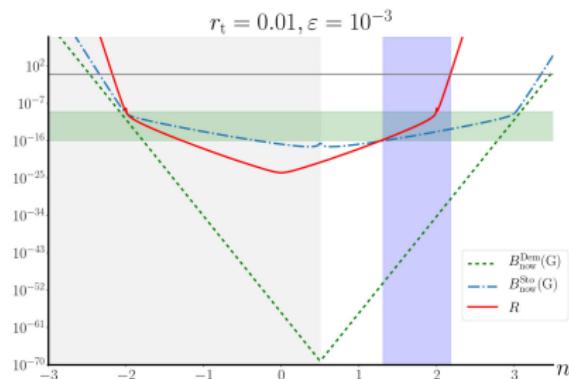


$$R(N=60) \simeq 10^{-10}$$

$$B_{\text{now}} \simeq 10^{-9} \text{ G}$$

Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066



Gray region: Strong regime

Green band: $10^{-9}G \gtrsim B_{\text{obs}} \gtrsim 10^{-16}G$

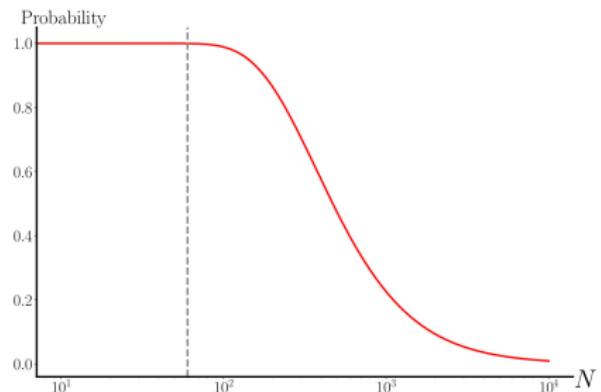
Blue band: Healthy parameter space

Revisiting Magnetogenesis during Inflation

A. T. A. Nassiri-Rad and H. Firouzjahi, arXiv:2007.11066

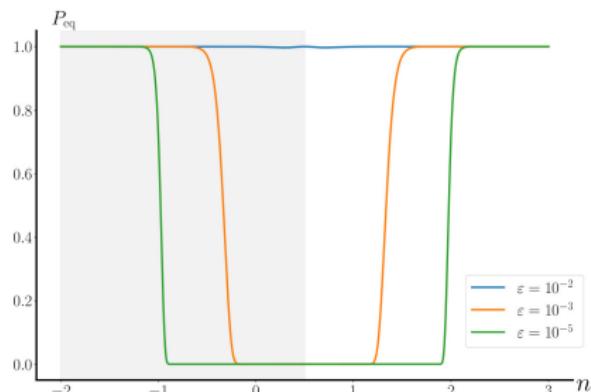
Wiener Process

$$P(10^{-16} \lesssim B_{\text{now}} \lesssim 10^{-9}; N)$$



OU Process

$$P_{\text{eq}}(10^{-16} \lesssim B_{\text{now}} \lesssim 10^{-9})$$



$n = 3$ and $n = -2$

$-2 < n < 3$

Summary and Conclusion

- The **stochastic formalism** consists of an **effective theory** for **IR modes** of the quantum fields, which are **coarse grained** at a fixed physical scale.
- In the presence of **stochastic noises** from small scales, the fate of **electric field**, in the anisotropic inflation $f^2 F_{\mu\nu} F^{\mu\nu}$ in which $f \sim a^{-2c}$, is very sensitive to its **initial value** of electric field.
- The amplitude of the **electromagnetic noises**, in $f^2 F_{\mu\nu} F^{\mu\nu}$ models in which $f \propto \eta^n$, depends on the cutoff parameter ε and n which indicates the scale dependency of the electromagnetic fields spectra.
- The main reason for the amplification of the magnetic field in this case is due to the **Ornstein-Uhlenbeck** process which settles the fields into an **equilibrium state** and prevent them from decaying.

Future Works

- Stochastic effects in primordial magneticgenesis models for $F\tilde{F}$ (in progress)

$$-\frac{f^2(\eta)}{4} \left(F^{\mu\nu} F_{\mu\nu} + \gamma F^{\mu\nu} \tilde{F}_{\mu\nu} \right)$$

- Stochastic effects Axion inflation. (in progress)

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\alpha}{f} \phi F^{\mu\nu} \tilde{F}_{\mu\nu}$$

THANK YOU FOR YOUR Attention!