Adam Falkowski

GDR lectures on EFT

Lectures given for the GDR Intensity Frontier

21-25 September 2020

Original Timetable

A series of 1 hour lectures, at a fairly elementary level

- Lecture 1 (today):
  Illustrated Philosophy of EFT

- Lecture 2 (Wed 11:00):
  Effective Toy Story, or an exercise in matching and running

- Lecture 3 (Wed 14:00):
  Effective theory above the electroweak scale: SMEFT et al.

- Lecture 4 (Fri 14:00):
  Chain of effective theories below the electroweak scale

More Probable Timetable

A series of 1 hour lectures, at a fairly elementary level

- Lecture 1 (today and Wed 11:00):
  Illustrated Philosophy of EFT
- Lecture 2 (Wed 14:00):
  Effective Toy Story, or an exercise in matching and running
- Lecture 3 (Fri 14:00):
  Effective theory above the electroweak scale: SMEFT et al.
- Lecture 4 (Fri 14:00):
  Chain of effective theories below the electroweak scale

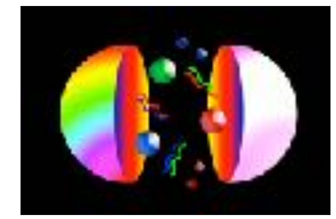
Standard Model

100 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c, b**

5 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c**

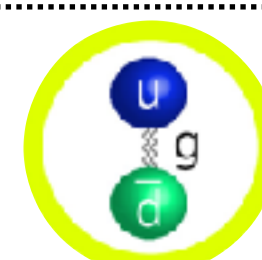


2 GeV

γ, ν_i, e, μ + **hadrons**

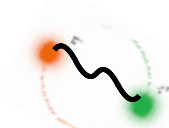
1 GeV

γ, ν_i, e, μ + **pions and kaons**



100 MeV

γ, ν_i, e



1 MeV

γ, ν_i

Recommended reading

General education

- Kaplan [nucl-th/0510023]
- Rothstein [hep-ph/0308266]
- Manohar [1804.05863]

Also my lecture notes from 2017 uploaded on the indico page

Recommended reading

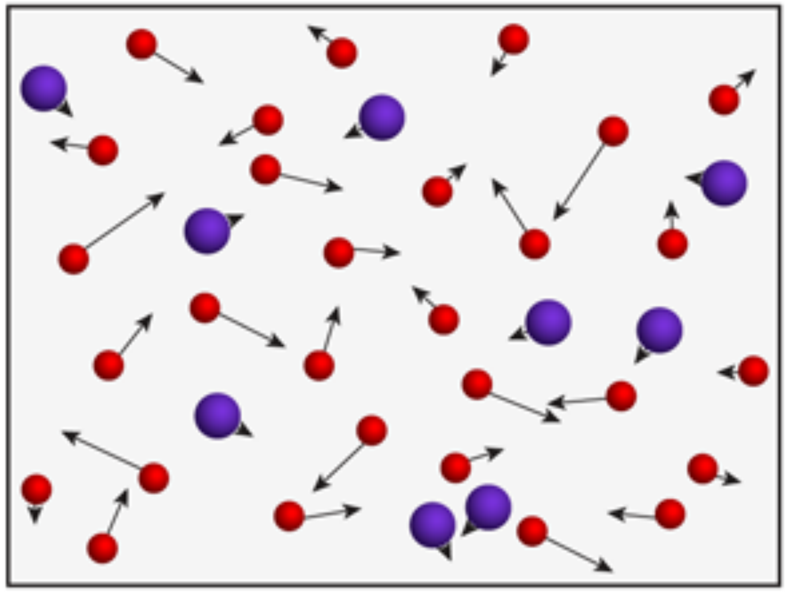
Specific EFTs

- EFT for superconductors: Polchinski [hep-th/9210046]
- EFT for heavy mesons: Grinstein [hep-ph/9411275]
- EFT for binary inspirals: Goldberger [hep-ph/07101129]
- EFT for low-energy QCD: Pich [1804.05664]
- EFT for nuclei: Van Kolck [1902.03141]
- ...

Lecture 1

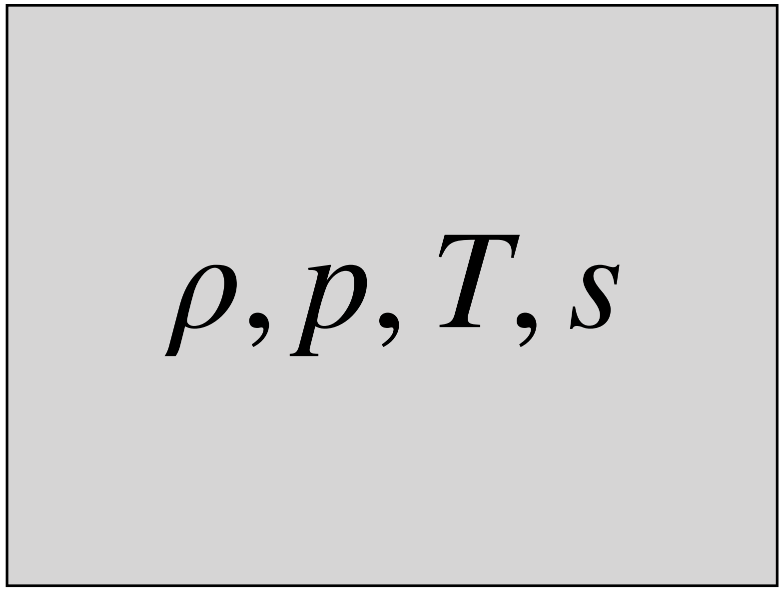
Illustrated Philosophy of EFT

Scale in physical problems



$\mapsto\ = 10^{-10}$ m

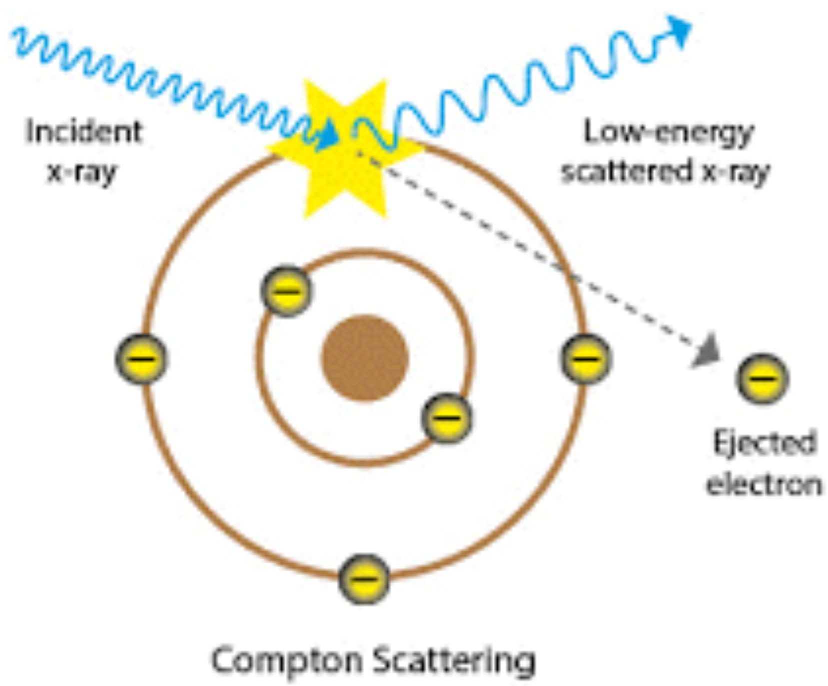
At small scales,
the degrees of freedom of gas
are positions and velocities
of its component atoms



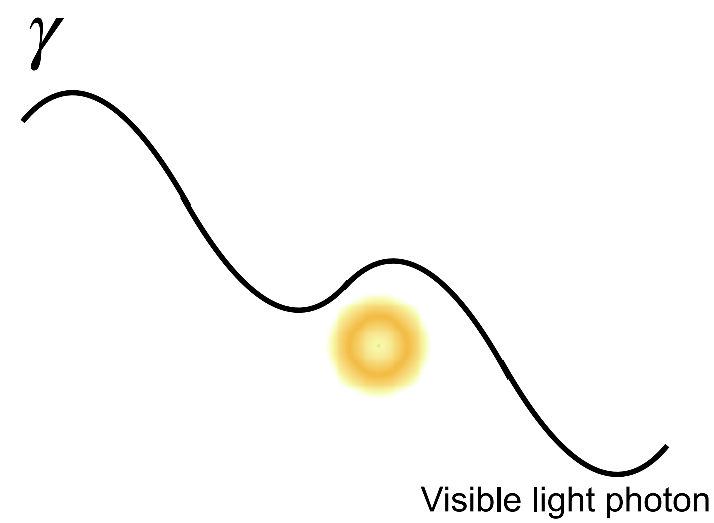
$\mapsto\ = 10^{-2}$ m

At large scales,
the useful degrees of freedom
are its macroscopic properties
like density, pressure,
temperature, or entropy

Scale in microscopic problems



$$\vdash\!\!\dashv = \frac{1}{m_e \alpha}$$



$$\vdash\!\!\dashv = \frac{10}{m_e \alpha}$$

X-ray photons see the atomic structure and scatter on the orbiting electrons

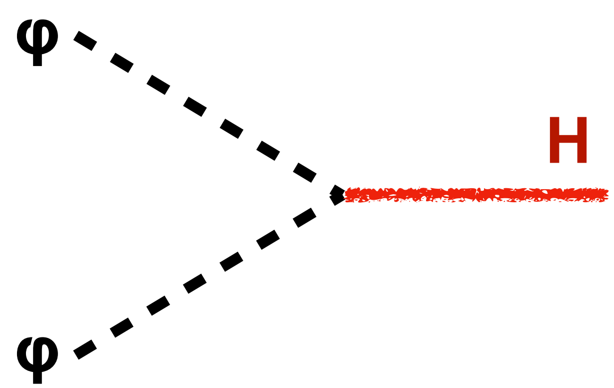
Lower-energy photons see atoms as neutral objects which are basically transparent

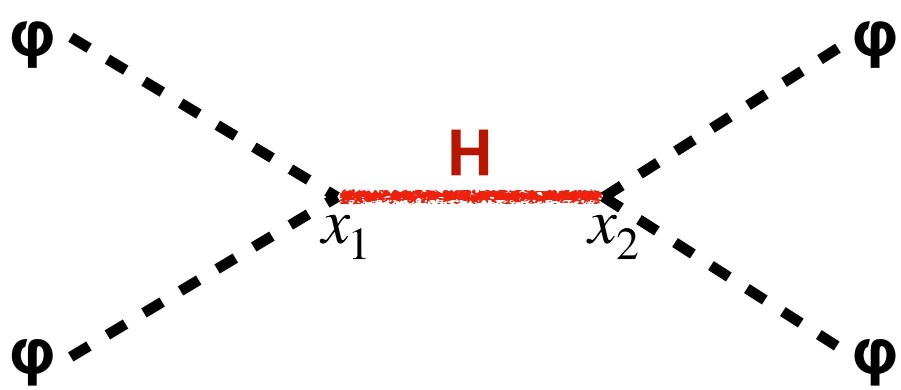
(that's how the universe becomes transparent to photons right after recombination)

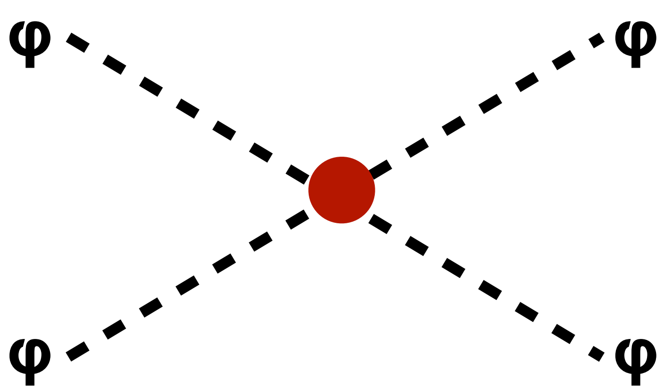
Scale in particle theory

Consider a theory of a light particle φ interacting with a heavy particle H

$$P(x_1, x_2) \sim \exp(-m_H |x_1 - x_2|)$$







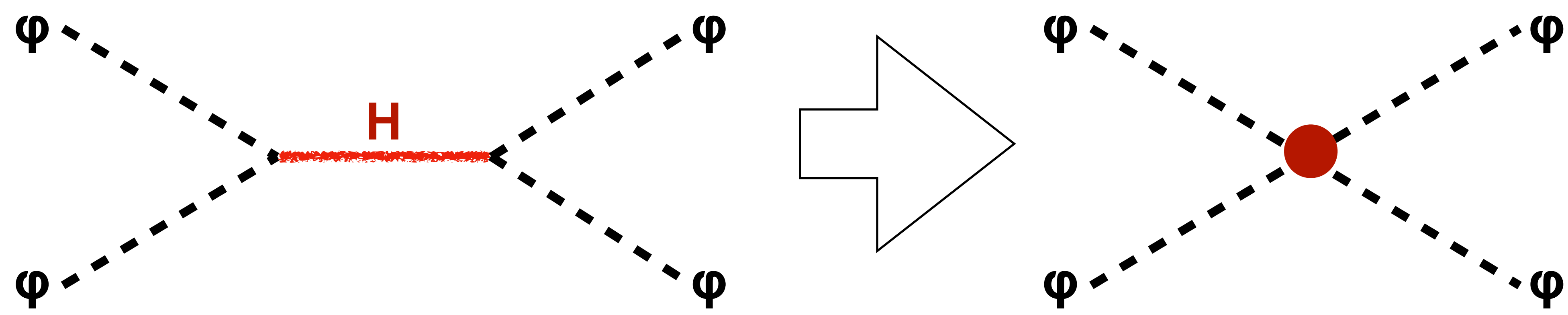
At small scales, $|x_1-x_2| \ll 1/m_H$, propagation of the heavy particle H leaves an imprint in the correlation function of the light particle φ

At large scales, $|x_1-x_2| \gg 1/m_H$, propagation of the heavy particle H can be approximated by a contact self-interaction of the light particle φ

$$m_H \sim \Delta E \ll \frac{1}{|x_1 - x_2|} \sim \frac{1}{\Delta t} \quad \Rightarrow \quad \Delta E \Delta t \ll 1$$

$$m_H \sim \Delta E \gg \frac{1}{|x_1 - x_2|} \sim \frac{1}{\Delta t} \quad \Rightarrow \quad \Delta E \Delta t \gg 1$$

Scale in particle theory



- Propagation of heavy particle H with mass m_H is suppressed at distance scale above its inverse mass

- Processes probing distance scales >> $1/m_H$, equivalently for energies << m_H, cannot resolve the propagation of H

- Then, intuitively, exchange of heavy particle H between light particles φ should be indistinguishable from a contact interaction of φ

- In other words, the effective theory describing φ interactions should be well approximated by a local Lagrangian, that is, by a polynomial in φ and its derivatives

This is the generic way how the effective theory description arise in particle physics, which will be repeated in all the examples that follow

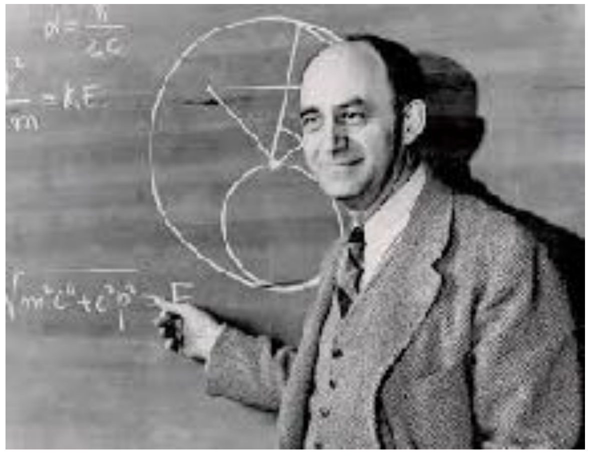


Illustration #1

Fermi EFT

Standard Model

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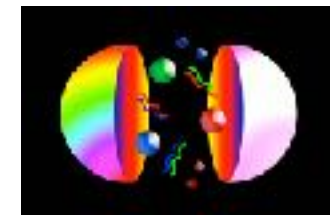

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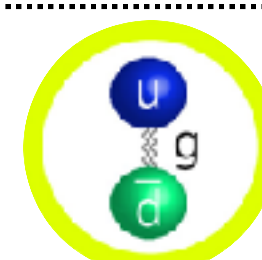
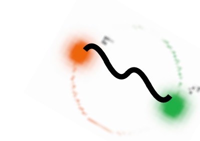


100 MeV

γ, ν_i, e



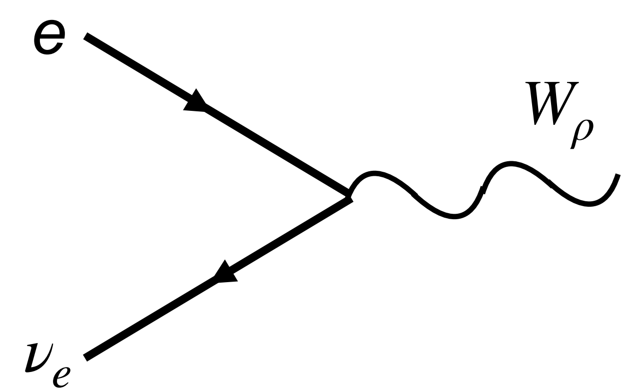
1 MeV

γ, ν_i

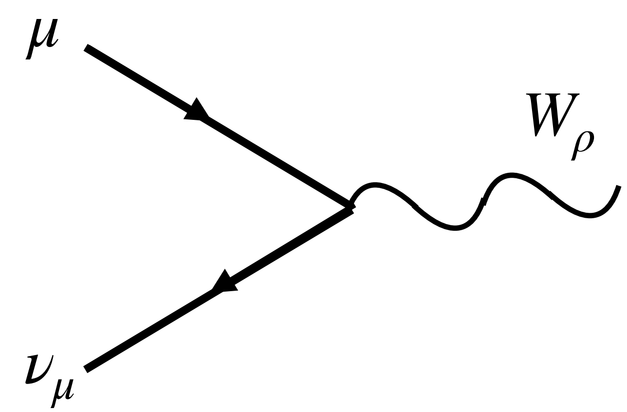
Fermi EFT

In the SM, weak interactions are mediated by W and Z bosons:

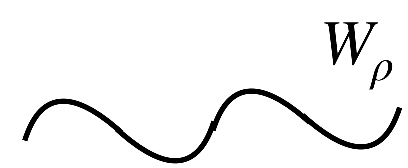
$$\mathscr{L}_{\mathrm{SM}} \supset \frac{g_L}{\sqrt{2}} \left[\bar{\nu}_e \gamma_\rho e_L + \bar{\nu}_\mu \gamma_\rho \mu_L \right] W_\rho^+ + \mathrm{h.c.}$$



$$i \frac{g_L}{\sqrt{2}} \gamma_\rho P_L$$

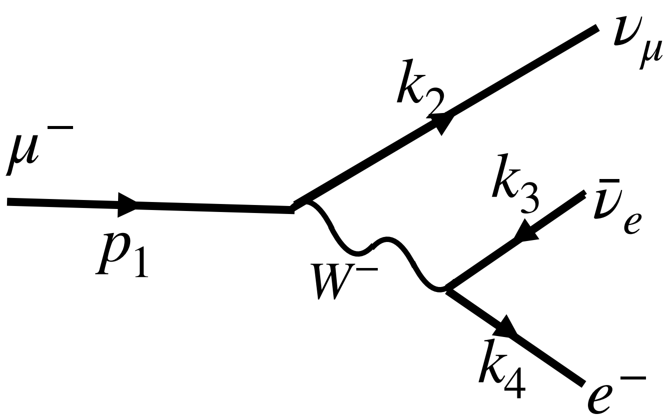


$$i \frac{g_L}{\sqrt{2}} \gamma_\rho P_L$$



$$-i \frac{\eta_{\mu\nu}}{p^2 - m_W^2}$$

In this theory, calculate muon decay



Tree-level amplitude:

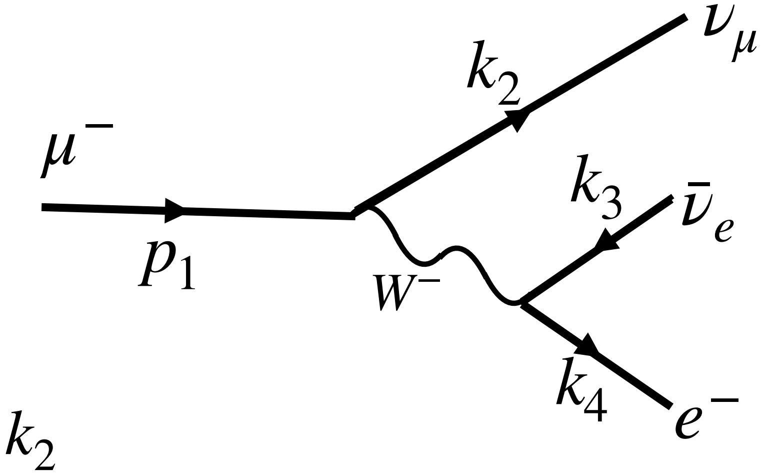
$$\mathscr{M} = \frac{g_L^2}{2} \bar{u}(k_2) \gamma_\rho P_L u(p_1) \frac{1}{q^2 - m_W^2} \bar{u}(k_4) \gamma_\rho P_L \mathrm{v}(k_3)$$

$$q = p_1 - k_2$$

u(p) and v(p) are spinor wave functions for particles and anti-particles

Fermi EFT

Tree-level amplitude:



$$\mathscr{M} = \frac{g_L^2}{2}\bar{u}(k_2)\gamma_\rho P_L u(p_1)\frac{1}{q^2 - m_W^2}\bar{u}(k_4)\gamma_\rho P_L \mathrm{v}(k_3)$$

$q = p_1 - k_2$

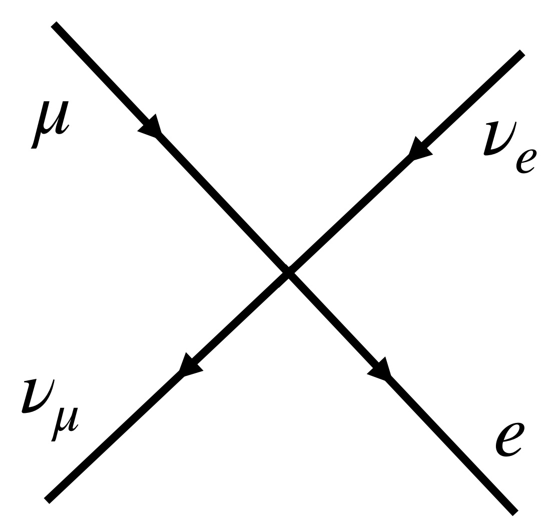
But kinematics of muon decay puts the constraint $q^2 \lesssim m_\mu^2 \ll m_W^2$

For all practical purpose one can thus approximate

$$\mathscr{M} = -\frac{g_L^2}{2m_W^2}\bar{u}(k_2)\gamma_\rho P_L u(p_1)\bar{u}(k_4)\gamma_\rho P_L \mathrm{v}(k_3) + \mathcal{O}(q^2/m_W^4)$$

This approximate amplitude can be equally well obtained from the <u>effective</u> Lagrangian

$$\mathscr{L}_{\text{eff}} \supset -\frac{g_L^2}{2m_W^2}(\bar{\nu}_\mu\gamma_\rho\mu_L)(\bar{e}_L\gamma_\rho\nu_e) + \text{h.c.}$$



$$-i\frac{g_L^2}{2m_W^2}[\gamma_\rho P_L][\gamma_\rho P_L]$$

The Lagrangian coefficient here is called the <u>Wilson coefficient</u> in this context

Interlude

EFT and path integrals

EFT and path integrals

Integrating out heavy particles is particularly transparent using the path integral formulation of QFT, because then it's literally integrating over the heavy fields…

The generating functional in the UV theory of light fields φ and heavy fields H

$$Z_{\mathrm{UV}}[J_\phi, J_H] = \int [D\phi][DH] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{UV}}(\phi, H) + J_\phi \phi + J_H H \right) \right]$$

The generating functional in the EFT of light fields φ

$$Z_{\mathrm{EFT}}[J_\phi] = \int [D\phi] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{EFT}}(\phi) + J_\phi \phi \right) \right]$$

Matching consists in imposing the condition

$$Z_{\mathrm{EFT}}[J_\phi] = Z_{\mathrm{UV}}[J_\phi, 0]$$

At leading order (tree-level), the field configurations contributing to the path integral are the ones that extremize the action:

$$Z_{\mathrm{UV}}[J_\phi, 0] = \int [D\phi] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{UV}}(\phi, H_{\mathrm{cl}}(\phi)) + J_\phi \phi \right) \right] \qquad 0 = \frac{\delta S}{\delta H}\Big|_{H=H_{\mathrm{cl}}(\phi)}$$

that is, $H_{cl}(φ)$ solves the classical equations of motion in the UV Lagrangian

Hence

$$\boxed{\mathscr{L}_{\mathrm{EFT}}(\phi) = \mathscr{L}_{\mathrm{UV}}(\phi, H_{\mathrm{cl}}(\phi))}$$

Fermi EFT

Previously we obtained the effective Lagrangian by matching amplitudes.
Let us now apply the path integral trick, instead.
Namely, we integrate out the heavy field directly at the Lagrangian level,
by solving its equations of motion and plug the solution back into the Lagrangian

Starting point: $$\mathscr{L}_{\mathrm{UV}} \supset -W_\rho^+(\Box - m_W^2)W_\rho^- + \frac{g_L}{\sqrt{2}}\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right]W_\rho^+ + \mathrm{h.c.}$$

e.o.m: $$-(\Box - m_W^2)W_\rho^- + \frac{g_L}{\sqrt{2}}\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right] = 0$$

solution: $$W_\rho^- = \frac{g_L}{\sqrt{2}}(\Box - m_W^2)^{-1}\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right]$$

(Non-local) Effective Lagrangian:

$$\mathscr{L}_{\mathrm{eff}} = \frac{g_L^2}{2}\left[\bar{e}_L\gamma_\rho\nu_e + \bar{\mu}_L\gamma_\rho\nu_\mu\right](\Box - m_W^2)^{-1}\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right]$$

Leading (local) Effective Lagrangian: $$\frac{1}{\Box - m_W^2} = -\frac{1}{m_W^2} - \frac{\Box}{m_W^4} - \frac{\Box^2}{m_W^6} - \dots$$

$$\mathscr{L}_{\mathrm{eff}} = -\frac{g_L^2}{2m_W^2}\left[\bar{e}_L\gamma_\rho\nu_e + \bar{\mu}_L\gamma_\rho\nu_\mu\right]\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right] + \mathcal{O}\left(\frac{1}{m_W^4}\right)$$

Fermi EFT

**Important comment: effective Lagrangian is <u>systematically improvable</u>.
Previously, we truncated the effective Lagrangian at order $(1/m_W)^2$.
However, nothing stops us from going to higher orders in $1/m_W$.**

(Non-local) Effective Lagrangian:

$$\mathscr{L}_{\rm eff} = \frac{g_L^2}{2}\left[\bar{e}_L\gamma_\rho\nu_e + \bar{\mu}_L\gamma_\rho\nu_\mu\right](\Box - m_W^2)^{-1}\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right]$$

Leading and subleading effective Lagrangian:

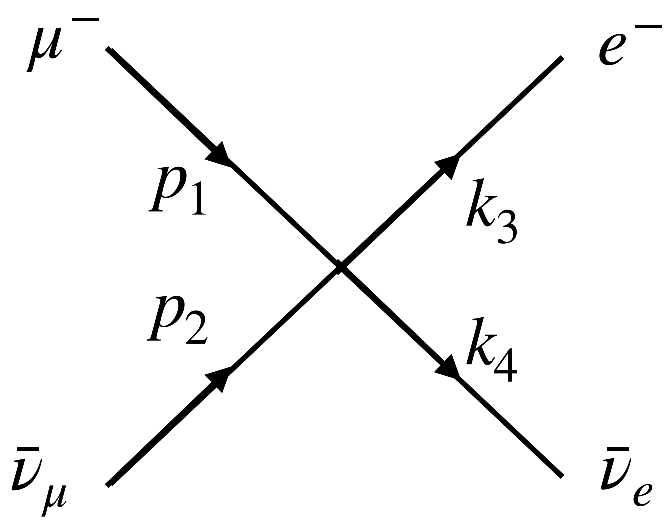
$$\mathscr{L}_{\rm eff} = -\frac{g_L^2}{2m_W^2}\left[\bar{e}_L\gamma_\rho\nu_e + \bar{\mu}_L\gamma_\rho\nu_\mu\right]\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right]$$
$$-\frac{g_L^2}{2m_W^4}\left[\bar{e}_L\gamma_\rho\nu_e + \bar{\mu}_L\gamma_\rho\nu_\mu\right]\Box\left[\bar{\nu}_e\gamma_\rho e_L + \bar{\nu}_\mu\gamma_\rho\mu_L\right] + \dots$$

**Effective Lagrangian is organized as a systematic expansion in powers of $1/m_W$.
The user decides at which order in $1/m_W$ the effective Lagrangian is truncated, depending on the accuracy of the calculations they want to achieve**

Fermi EFT

Every EFT carries
the seeds of its own destruction
(Mark Twain)

Consider a scattering rather than a decay process



$$\mathscr{L}_{\rm eff} \supset -\frac{g_L^2}{2m_W^2}(\bar{\nu}_\mu \gamma_\rho \mu_L)(\bar{e}_L \gamma_\rho \nu_e) + \mathrm{h.c.}$$

$$\Downarrow$$

$$\mathscr{M} = -\frac{g_L^2}{2m_W^2}\bar{v}(p_2)\gamma_\rho P_L u(p_1)\bar{u}(k_3)\gamma_\rho P_L v(k_4)$$

By dimensional analysis, for s >> μ², the amplitude behaves as

$$\mathscr{M} \sim \frac{g_L^2}{m_W^2}s$$

$$s = (p_1 + p_2)^2$$

This clashes with unitarity for $\sqrt{s} \sim \Lambda_{\rm max}$ **where**

$$\boxed{\Lambda_{\rm max} = \frac{4\pi m_W}{g_L} \sim 1.5\ \mathrm{TeV}}$$

Λ_{max} is the maximum cutoff scale from the consistency viewpoint of the EFT
In reality, the true cutoff $\Lambda = m_W$ is lower, which can be traced to the fact that the UV completion (the Standard Model) is weakly coupled

Fermi EFT

Summary and lessons learned:

- EFT can be a great and simple tool to study low-energy consequences of more complete theories as long as $E \ll \Lambda$, where E is the characteristic energy scale of the process of interest, and Λ is the mass scale of the UV theory (in our example $\Lambda = m_W$).

- EFT is organized in an expansion in powers of $1/\Lambda$, and is systematically improvable, that is to say, we can choose at which order we truncate the expansion depending on the required precision

- EFT predicts correlations between rates of different processes (in our example, processes related by crossing symmetry, but it is less trivial in other examples)

- Every EFT has necessarily a limited validity range. It stops making sense as a perturbative theory above energies of order $\Lambda_{max} = 4\pi\Lambda/g$, where g is the coupling strength in the UV theory. Note that Λ_{max} is always larger than the true cutoff Λ.

- As one approaches $E = \Lambda$ from the EFT side, higher-dimensional operators become more and more relevant, and expansion in $1/\Lambda$ becomes impractical. For $E \sim \Lambda$, resonances in the UV theory can be resolved and the EFT description becomes useless

Interlude

Dimensional analysis

Dimensional analysis

- Effective Lagrangians by construction must contain infinite number of terms. Therefore any useful EFT comes with a set of power counting rules which allow one to organize the Lagrangian in a consistent expansion and single out the most relevant terms

- Relativistic effective theories are obtained by integrating out heavy fields H with mass of order Λ, and the inverse of the latter provides a natural expansion parameter to organize the effective Lagrangian.

- The effective Lagrangian is then organized according to canonical dimensions of its interactions terms, where the powers of the mass scale multiplying each term are identified with Λ. The observables computed fare then expanded in E/Λ where E is the typical energy scale of the experiment

- Warning: different power counting rules may apply to non-relativistic theories, or relativistic systems with one heavy component (such as e.g. B-mesons), or to theories with non-linearly realized symmetry. These cases will be discussed later.

Dimensional analysis

To isolate UV and IR limits, consider rescaling of spacetime coordinates

$$x \to \xi x'$$

$\xi \to 0$ is zooming in on small distances (UV limit)

$\xi \to \infty$ is zooming in on large distances (IR limit)

$$S = \int d^4x \left((\partial_\mu \phi)^2 - m^2\phi^2 - \lambda\phi^4 - \frac{c}{\Lambda^{n+d-4}}\phi^n \partial^d \right)$$

$$\to \int d^4x' \left(\xi^2 (\partial'_\mu \phi)^2 - m^2 \xi^4 \phi^2 - \lambda \xi^4 \phi^4 - \frac{c\xi^{4-d}}{\Lambda^{n+d-4}}\phi^n \partial'^{\,d} \right)$$

Since path integral is dominated by kinetic terms to easily compare the original and rescaled actions it is convenient normalize the kinetic terms canonically

$$\phi(x) \to \xi^{-1}\phi'(\xi x)$$

$$S \to \int d^4x' \left((\partial'_\mu \phi')^2 - m^2 \xi^2 \phi'^{\,2} - \lambda \phi'^{\,4} - \frac{c_{n,d}\xi^{4-d-n}}{\Lambda^{n+d-4}}\phi'^{\,n} \partial'^{\,d} \right)$$

Dimensional analysis

$$\phi(x) \to \xi^{-1}\phi'(\xi x)$$

$$x \to \xi x'$$

$$S = \int d^4x \left((\partial_\mu \phi)^2 - m^2\phi^2 - \lambda\phi^4 - \frac{c_{n,d}}{\Lambda^{n+d-4}}\phi^n\partial^d \right)$$

$$\to \int d^4x' \left((\partial'_\mu \phi')^2 - m'^2\phi'^2 - \lambda'\phi'^4 - \frac{c'_{n,d}}{\Lambda^{n+d-4}}\phi'^n\partial'^d \right)$$

Mass term is relevant operator: it gets more important in IR

$$\begin{aligned} m'^2 &= m^2\xi^2 \\ \lambda' &= \lambda \\ c'_{n,d} &= c_{n,d}\xi^{4-d-n} \end{aligned}$$

Quartic coupling is marginal operator: it is (approximately) the same in UV and in IR

Higher dimensional interactions (for d+n>4) are irrelevant operators: they get less important in IR

Power counting in relativistic EFT, determining the importance of various interactions, can be organized based on canonical dimension of interactions

Dimensional analysis

Relativistic field theory

$$S = \int d^4x \left[\partial_\mu \phi^\dagger \partial_\mu \phi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + i\bar{\psi}\gamma_\mu \partial_\mu \psi + \dots \right]$$

$$[\partial] = \text{mass}^1$$

$$[\phi] = \text{mass}^1$$

$$[A_\mu] = \text{mass}^1$$

$$[\psi] = \text{mass}^{3/2}$$

Illustration #2

Euler-Heisenberg EFT

Standard Model

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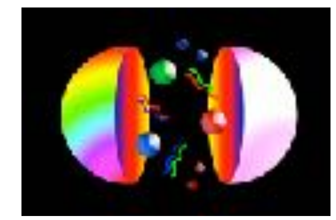
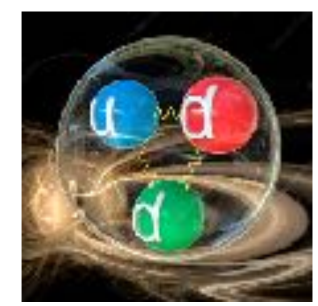


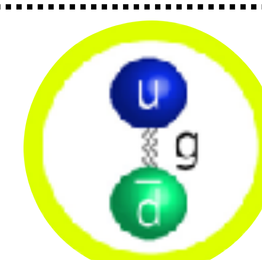
2 GeV

γ, ν_i, e, μ + hadrons



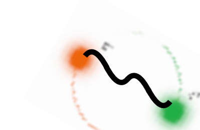
1 GeV

γ, ν_i, e, μ + pions and kaons



100 MeV

γ, ν_i, e



1 MeV

γ, ν_i

Euler-Heisenberg EFT

Consider effective theory for photons propagating in vacuum with $E_\gamma \ll 2m_e \approx 1$ MeV

- **At these energies all charged particles are integrated out, hence the effective Lagrangian must be a function of only the photon field A_μ**

- **Photons are massless, so the only explicit mass scale in this construction is the EFT cutoff scale Λ**

- **Gauge and Lorentz invariance requires the effective Lagrangian to be a function of the field strength $F_{\mu\nu}$ and its derivatives**

$$\mathscr{L}_{\text{eff}} = \mathscr{L}\left(F_{\mu\nu}, \tilde{F}_{\mu\nu}, \partial_\mu, \Lambda\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

We will build the effective Lagrangian as an expansion in 1/Λ

$$\mathscr{L}_{\text{eff}} = \Lambda^2 \mathscr{L}_{D=2} + \mathscr{L}_{D=4} + \frac{1}{\Lambda^2}\mathscr{L}_{D=6} + \frac{1}{\Lambda^4}\mathscr{L}_{D=8} + \ldots$$

Here D denotes the canonical dimension of each term
(no odd dimensions because $[F_{\mu\nu}]=2$, and derivatives must always come in pairs)

Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = \Lambda^2 \mathscr{L}_{D=2} + \mathscr{L}_{D=4} + \frac{1}{\Lambda^2}\mathscr{L}_{D=6} + \frac{1}{\Lambda^4}\mathscr{L}_{D=8} + \dots$$

D=2: $F_{\mu\mu} = \tilde{F}_{\mu\mu} = 0$ **No possible invariants thus** $\mathscr{L}_{D=2} = 0$

D=4: **One invariant** $F_{\mu\nu}F_{\mu\nu}$

$\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}$

$F_{\mu\nu}\tilde{F}_{\mu\nu}$ **is a total derivative**

$\partial_\mu \partial_\nu F_{\mu\nu} = 0$

$$\mathscr{L}_{D=4} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

the numerical coefficient is pure convention, except for the sign, which is required to avoid ghost instability

D=6: **Again, no non-trivial invariants! Hence** $\mathscr{L}_{D=6} = 0$

$$F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} = 0 = F_{\mu\nu}F_{\nu\rho}\tilde{F}_{\rho\mu} = \dots$$

$$F_{\mu\nu}\partial_\alpha F_{\mu\alpha}\partial_\beta F_{\nu\beta} = 0$$

$\mathscr{L}_{D=6} = cF_{\mu\nu}\Box F_{\mu\nu}$ **can be eliminated by the change of variables** $A_\mu \to A_\mu + \frac{2c}{\Lambda^2}\Box A_\mu$

Non-trivial interactions between photons can arise only at order 1/ Λ⁴ in the EFT!

Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{\Lambda^4} \mathscr{L}_{D=8} + \ldots$$

D=8: **The most general non-redundant Lagrangian at D=8 is**

$$\boxed{\mathscr{L}_{D=8} = c_1 (F_{\mu\nu} F_{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}_{\mu\nu})^2 + c_3 (F_{\mu\nu} F_{\mu\nu})(F_{\alpha\beta} \tilde{F}_{\alpha\beta})}$$

Other possible structures can be shown to be <u>redundant</u>, that is they can be eliminated or expressed by the three above. E.g.

$$F_{\mu\alpha} F_{\alpha\nu} F_{\mu\beta} F_{\beta\nu} = \frac{1}{4} (F_{\mu\nu} F_{\mu\nu})^2 + \frac{1}{2} (F_{\mu\nu} \tilde{F}_{\mu\nu})^2$$

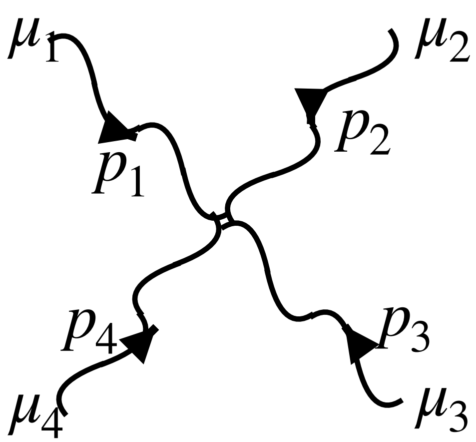
The high-school version of the same Lagrangian:

$$\mathscr{L}_{D=8} = 4c_1 \left(\vec{E}^2 - \vec{B}^2 \right)^2 + 16 c_2 \left(\vec{E}\, \vec{B} \right)^2 + 8 c_3 \left(\vec{E}^2 - \vec{B}^2 \right) \left(\vec{E}\, \vec{B} \right)$$

Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{\Lambda^4}\left\{c_1(F_{\mu\nu}F_{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + c_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta})\right\} + \ldots$$

This Lagrangian defines a completely healthy and consistent quantum field theory with quartic (and possibly higher-point) self-interactions between photons.



$$\frac{32ic_1}{\Lambda^4}\left(p_1^{\mu_2}p_2^{\mu_1} - p_1p_2\eta^{\mu_1\mu_2}\right)\left(p_3^{\mu_4}p_4^{\mu_3} - p_3p_4\eta^{\mu_3\mu_4}\right) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$
$$+\frac{32ic_2}{\Lambda^4}(\ldots) + \frac{32ic_3}{\Lambda^4}(\ldots)$$

Scattering amplitudes can be calculated in a systematic expansion in $1/\Lambda^4$. E.g.

$$\mathscr{M}(\gamma^+\gamma^+\gamma^+\gamma^+) = 8\frac{c_1 - c_2 + ic_3}{\Lambda^4}\left[s^2 + t^2 + u^2\right]$$

$$\mathscr{M}(\gamma^+\gamma^+\gamma^-\gamma^-) = 8\frac{c_1 + c_2}{\Lambda^4}s^2$$

$$\mathscr{M}(\gamma^-\gamma^-\gamma^-\gamma^-) = 8\frac{c_1 - c_2 - ic_3}{\Lambda^4}\left[s^2 + t^2 + u^2\right]$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 + p_3)^2$$
$$s = (p_1 + p_4)^2$$

Note that a non-zero c_3 violates parity!

The only difference between this effective theory and a renormalizable QFT is that counterterms of order $1/\Lambda^n$, also with n>4, are generated at loop level, thus these higher-order terms have to be added to the Lagrangian if we require precision beyond the $1/\Lambda^4$ order

Euler-Heisenberg EFT

$$\mathscr{L}_{\rm eff} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{\Lambda^4}\left\{c_1(F_{\mu\nu}F_{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + c_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta})\right\} + \dots$$

Scattered comments:

- This is the effective theory of light at low energies (UV, visible, IR, microwaves, radio) at the leading non-trivial order
- The quartic photon interaction terms in this EFT lead to non-linear field equations for the electromagnetic field. Thus, electrodynamics is really non-linear, and the superposition principle they taught you in school is not exactly true!
- One potentially observable effect of the D=8 terms is the so-called vacuum birefringence, that is rotation of light polarization propagating in vacuum in strong magnetic field. This effect was possibly observed in 2016 in a neutron star light.
- Another potentially observable effect is light-by-light scattering. This has been routinely observed in colliders, however at higher energies where this EFT is no longer valid.
- In the absence of new physics, the ordinary QED is the UV completion of this EFT, in which case the cutoff Λ can be identified with $2m_e$. However, in the presence of light axions or light milli-charged particles, this may no longer be the case.
- The Wilson coefficients c_1, c_2, c_3 can be calculated theoretically by matching this EFT to its UV completion, e.g. QED. However, I'm not aware of a systematic experimental measurement of these Wilson coefficients. A future such measurement will be a non-trivial result, as some unknown light particles could in principle contribute to it, along with the electron and other SM charged particles

Interlude

$\hbar$ counting

$\hbar$ counting

- We expect that a higher-dimensional operator with the canonical dimension D depends on the mass scale Λ in UV theory as $1/\Lambda^{(D-4)}$. But, in general we do not know how their Wilson coefficients depend on couplings of UV theory - that is a model dependent question

- However, there are general rules that in some cases allow us to estimate the coupling dependence

- To this end, a useful trick is to restore explicitly $\hbar$ in action.

In natural units $\hbar$=1 distance is inverse of energy

But the two are not equivalent and are instead related by the Planck constant if $\hbar\neq1$ is restored

$$\Delta x \Delta p \sim \hbar \quad \Rightarrow \quad [L] = \frac{\hbar}{[E]}$$

While the action is dimensionless in natural units, it caries the dimension [S] = $\hbar^1$ if the Planck constant $\hbar\neq1$ is restored

$$Z \sim \int D\phi \exp\left[i\frac{S}{\hbar}\right]$$

$\hbar$ counting

Consider now a generic action for a scalar field

$$\frac{S}{\hbar} \sim \frac{1}{\hbar}\int d^4x \left[(\partial_\mu \phi)^2 - m^2\phi^2 - C_{n,k}\partial^k\phi^n\right]$$

Eliminate $\hbar$ from the action via the field redefinition

$$\phi \to \hbar^{1/2}\phi$$

$$\frac{S}{\hbar} \sim \int d^4x \left[(\partial_\mu \phi)^2 - m^2\phi^2 - \hbar^{n/2-1}C_{n,k}\partial^k\phi^n\right]$$

Now each coupling secretly carries a power of $\hbar$,
which only depends on the number of fields in the vertex:

$$[C_{n,k}] = \hbar^{1-n/2}$$

e.g.

$$[C_{3,k}] = \hbar^{-1/2}$$

$$[C_{4,k}] = \hbar^{-1}$$

$$\cdots$$

Moreover, one can proved that each loop comes with another factor of $\hbar$ (in original variables, each propagator brings $\hbar$, each vertex brings $1/\hbar$, thus each diagram comes with the power of $\hbar$ equal to: N(propagotors)-N(vertices) = N(loops)-1

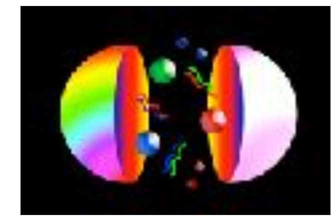
Standard Model

100 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c, b

5 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c

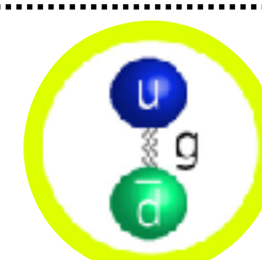


2 GeV

γ, ν_i, e, μ + hadrons

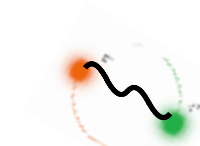
1 GeV

γ, ν_i, e, μ + pions and kaons



100 MeV

γ, ν_i, e



1 MeV

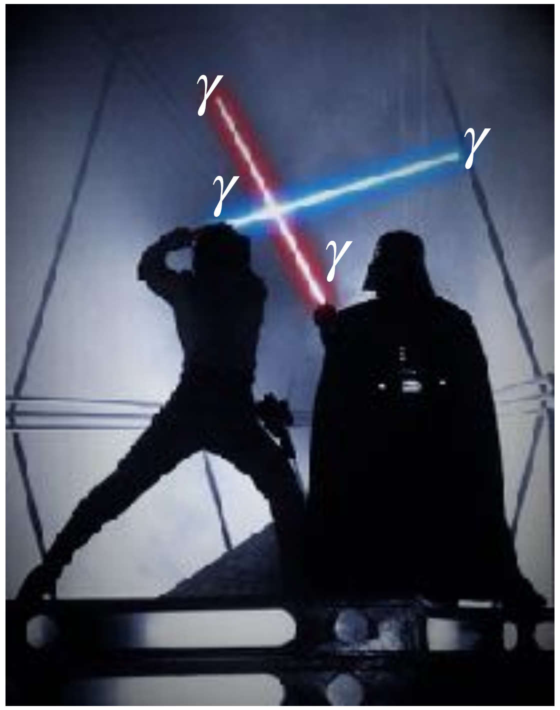
γ, ν_i

Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{\Lambda^4}\left\{c_1(F_{\mu\nu}F_{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + c_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta})\right\} + \ldots$$

This Lagrangian describes the effective theory of light at low energies (UV, visible, IR, microwaves, radio) at the leading order beyond the Maxwell approximation

This is the effective theory underlying the physics of light sabers



In its validity regime, it is also appropriate to describe vacuum birefringence, photon-photon scattering at low energies, and more

Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{\Lambda^4}\left\{c_1(F_{\mu\nu}F_{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + c_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta})\right\} + \ldots$$

$\hbar$ counting applied to the D=8 Euler-Heisenberg Lagrangian

$$[C_{n,k}] = \hbar^{1-n/2}$$

here, n=4 thus

$$[c_i] = \hbar^{-1}$$

Let **e** be a cubic coupling in the UV completion, e.g. the electromagnetic coupling in the QED

$$[e] = \hbar^{-1/2}$$

It follows

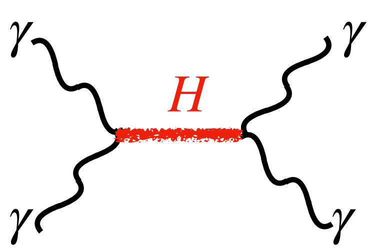
$$c_i \sim \frac{e^{2n+2}}{(16\pi^2)^n}$$

if this Wilson coefficient is generated at n-loops

In particular

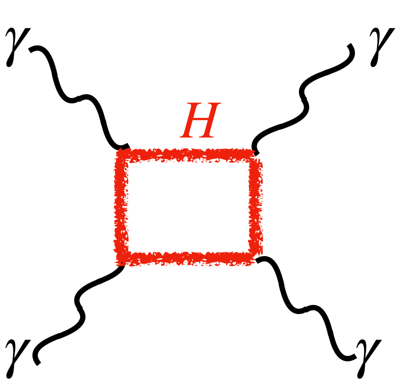
$$c_i \sim e^2$$

if this Wilson coefficient is generated at tree level



$$c_i \sim \frac{e^4}{16\pi^2}$$

if this Wilson coefficient is generated at 1-loop level

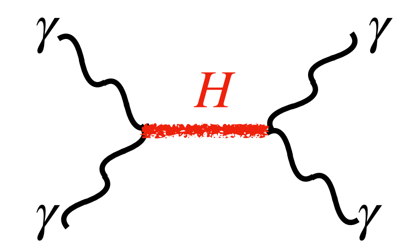


Euler-Heisenberg EFT

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{\Lambda^4}\{c_1(F_{\mu\nu}F_{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + c_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta})\} + \ldots$$

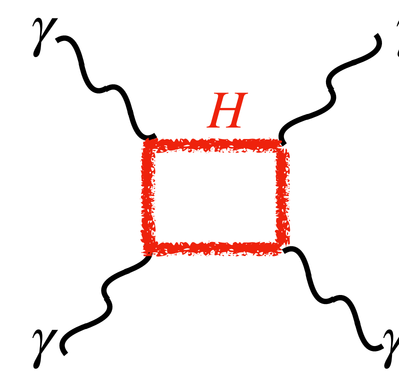
$$c_i \sim e^2$$

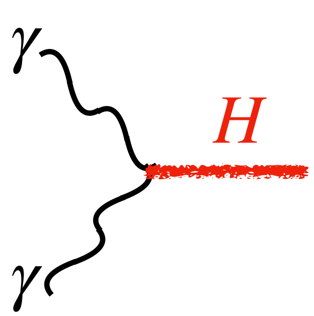
if this Wilson coefficient is generated at tree level



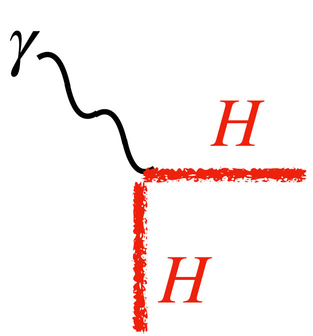
$$c_i \sim \frac{e^4}{16\pi^2}$$

if this Wilson coefficient is generated at 1-loop level





Doesn't exist in QED !



Does exist in QED !

Thus, <u>in QED</u>

$$\frac{c_i}{\Lambda^4} \sim \frac{e^4}{16\pi^2 m_e^4} = \frac{\alpha^2}{m_e^4}$$

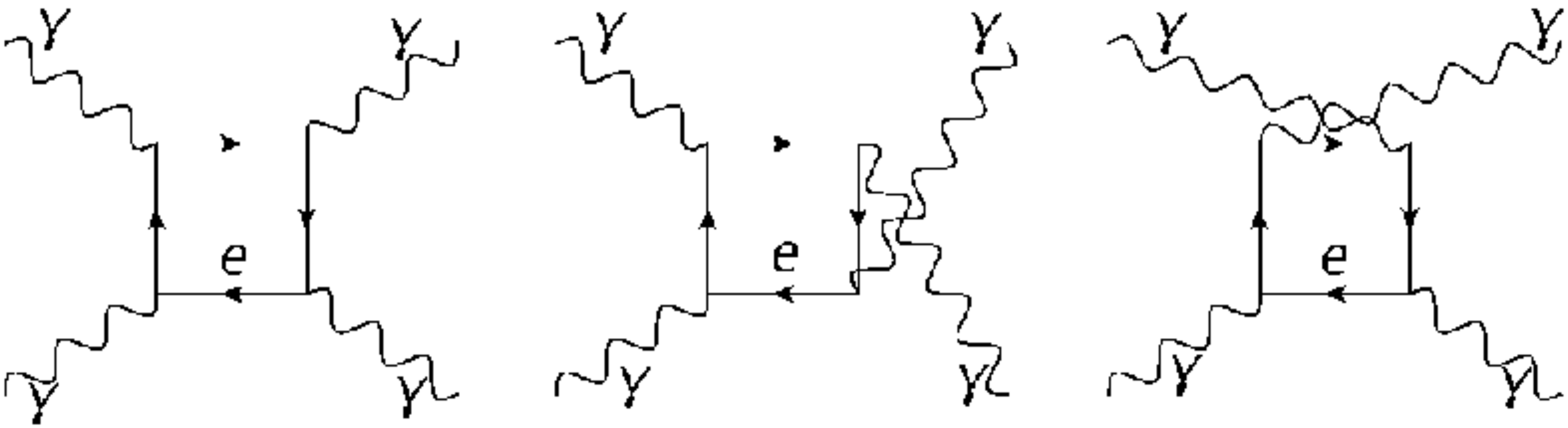
Moreover, c_3=0 in QED, due to parity conservation

Euler-Heisenberg EFT

QED UV completion

$$\mathscr{L}_{\mathrm{UV}} \supset i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m_e\bar{\psi}\psi + eA_{\mu}\bar{\psi}\gamma^{\mu}\psi$$

In this example, the UV completion of our effective theory is a renormalizable theory, which could in principle be valid to very high energy scales



+reversed fermion line

$$\mathscr{M}_{\mathrm{UV}}(\gamma^+\gamma^+\gamma^+\gamma^+) = -\frac{\alpha^2}{15m_e^4}\left[s^2+t^2+u^2\right] + \mathscr{O}(m_e^{-6})$$

$$\mathscr{M}_{\mathrm{UV}}(\gamma^+\gamma^+\gamma^-\gamma^-) = \frac{11\alpha^2}{45m_e^4}s^2 + \mathscr{O}(m_e^{-6})$$

$$\mathscr{M}_{\mathrm{UV}}(\gamma^-\gamma^-\gamma^-\gamma^-) = -\frac{\alpha^2}{15m_e^4}\left[s^2+t^2+u^2\right] + \mathscr{O}(m_e^{-6})$$

$$\mathscr{M}_{\mathrm{EFT}}(\gamma^+\gamma^+\gamma^+\gamma^+) = 8\frac{c_1 - c_2 + ic_3}{\Lambda^4}\left[s^2+t^2+u^2\right]$$

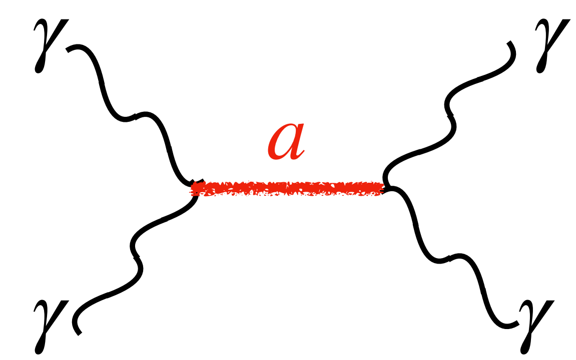
$$\mathscr{M}_{\mathrm{EFT}}(\gamma^+\gamma^+\gamma^-\gamma^-) = 8\frac{c_1 + c_2}{\Lambda^4}s^2$$

$$\mathscr{M}_{\mathrm{EFT}}(\gamma^-\gamma^-\gamma^-\gamma^-) = 8\frac{c_1 - c_2 - ic_3}{\Lambda^4}\left[s^2+t^2+u^2\right]$$

Thus, integrating out the electron at one-loop level yields:

$$\frac{c_1}{\Lambda^4} = \frac{\alpha^2}{90m_e^4}, \qquad \frac{c_2}{\Lambda^4} = \frac{7\alpha^2}{360m_a^4}, \qquad \frac{c_3}{\Lambda^4} = 0$$

Euler-Heisenberg EFT



ALP UV completion

$$\mathscr{L}_{\mathrm{UV}} \supset \frac{1}{2}(\partial_\mu a)^2 - \frac{m_a^2}{2}a^2 + \frac{a}{f_a}\left\{ gF_{\mu\nu}F_{\mu\nu} + \tilde{g}F_{\mu\nu}\tilde{F}_{\mu\nu} \right\}$$

Integrating out the axion at tree-level:

$$\frac{c_1}{\Lambda^4} = \frac{g^2}{2f_a^2 m_a^2}, \qquad \frac{c_2}{\Lambda^4} = \frac{\tilde{g}^2}{2f_a^2 m_a^2}, \qquad \frac{c_3}{\Lambda^4} = \frac{g\tilde{g}}{f_a^2 m_a^2}$$

Note that

$$\Lambda = \sqrt{f_a m_a}$$

In this example, the usual power counting, $\Lambda \sim m_a$, is disrupted, because the UV completion of an effective theory is itself an effective theory and contains other mass parameters than m_a

Interlude

Analyticity constraints

Analyticity constraints

From the low-energy point of view, the Wilson coefficients are arbitrary, within perturbativity limits

However, assuming the UV theory is causal, Poincaré invariant, and local one can surprisingly find additional constraints on the Wilson coefficients

Given $\mathscr{M}_{EFT}(X_1 X_2 \to X_1 X_2) = M(s, t, u)$

and $M_{\text{forward}}(s) = M(s, 0, -s)$

$$\frac{d^2 M_{\text{forward}}(s)}{s^2}\Big|_{s\to 0} > 0$$

Proof using dispersion relations

Euler-Heisenberg EFT

$$\mathcal{M}_{\rm EFT}(\gamma^+\gamma^+\gamma^+\gamma^+) = 8\frac{c_1 - c_2 + ic_3}{\Lambda^4}\left[s^2+t^2+u^2\right]$$
$$\mathcal{M}_{\rm EFT}(\gamma^+\gamma^+\gamma^-\gamma^-) = 8\frac{c_1 + c_2}{\Lambda^4}s^2$$
$$\mathcal{M}_{\rm EFT}(\gamma^-\gamma^-\gamma^-\gamma^-) = 8\frac{c_1 - c_2 - ic_3}{\Lambda^4}\left[s^2+t^2+u^2\right]$$

Applying this to the Euler-Heisenberg effective Lagrangian

$$\mathcal{M}(\gamma^+\gamma^+\to\gamma^+\gamma^+) = \mathcal{M}(\gamma^+\gamma^+\gamma^-\gamma^-) = 8\frac{c_1+c_2}{\Lambda^4}s^2 \quad \Rightarrow \quad c_1+c_2>0$$

One can actually get stronger bounds, by considering amplitude in the linear polarization basis

$$\mathcal{M}_{\rm forward}(\gamma^x\gamma^x\to\gamma^x\gamma^x) = \frac{16c_1}{\Lambda^4}s^2 \quad \Rightarrow \quad c_1>0$$

$$\mathcal{M}_{\rm forward}(\gamma^x\gamma^y\to\gamma^x\gamma^y) = \frac{16c_2}{\Lambda^4}s^2 \quad \Rightarrow \quad c_2>0$$

Of course, this bound is respected in our examples

$$\frac{c_1^{\rm ALP}}{\Lambda^4} = \frac{g^2}{2f_a^2m_a^2}>0, \qquad \frac{c_2^{\rm ALP}}{\Lambda^4} = \frac{\tilde{g}^2}{2f_a^2m_a^2}>0$$

$$\frac{c_1^{\rm QED}}{\Lambda^4} = \frac{\alpha^2}{90m_e^4}>0, \qquad \frac{c_2^{\rm QED}}{\Lambda^4} = \frac{7\alpha^2}{360m_a^4}>0$$

Euler-Heisenberg EFT

Summary and lessons learned

- Symmetries of a low-energy system often determine the structure of the effective theory at leading orders, up to a few unknown numerical parameters
- Furthermore, dimensional analysis and hbar counting often fixes the parameteric dependence of these Wilson coefficients on the UV parameters, up to O(1) coefficients
- Furthermore squared, in some case even the sign of the Wilson coefficients is fixed, given some plausible assumptions about the UV theory
- The EFT Lagrangian can be used for perturbative calculations of low-energy scattering amplitudes. But it is also a useful tool to work out subtle effects of classical field configurations

Illustration #3

Fermi EFT, again

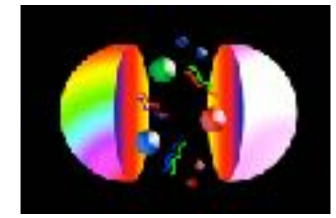
Standard Model

100 GeV

γ,g,$ν_i$,e,μ,τ + u, d, s, c, b

5 GeV

γ,g,$ν_i$,e,μ,τ + u, d, s, c

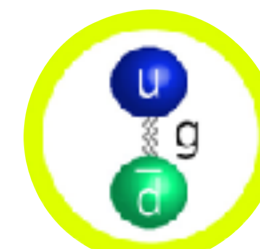


2 GeV

γ,$ν_i$,e,μ + hadrons

1 GeV

γ,$ν_i$,e,μ + pions and kaons



100 MeV

γ,$ν_i$,e

1 MeV

γ,$ν_i$

Fermi EFT, again

Consider low-energy interactions between light quarks and leptons

Starting point: $$\mathcal{L}_{\mathrm{UV}} \supset -W^+_\rho(\Box - m_W^2)W^-_\rho + \frac{g_L}{\sqrt{2}}\left\{\left[\bar\nu_e\gamma_\rho e_L + V_{ud}\bar u_L\gamma_\rho d_L\right]W^+_\rho + \mathrm{h.c.}\right\}$$

SM CKM element

e.o.m: $$-(\Box - m_W^2)W^-_\rho + \frac{g_L}{\sqrt{2}}\left[\bar\nu_e\gamma_\rho e_L + V_{ud}\bar u_L\gamma_\rho d_L\right] = 0$$

solution: $$W^-_\rho = \frac{g_L}{\sqrt{2}}(\Box - m_W^2)^{-1}\left[\bar\nu_e\gamma_\rho e_L + V_{ud}\bar u_L\gamma_\rho d_L\right] \approx -\frac{g_L}{\sqrt{2}m_W^2}\left[\bar\nu_e\gamma_\rho e_L + V_{ud}\bar u_L\gamma_\rho d_L\right]$$

Leading effective 4-fermion interactions:

$$\mathcal{L}_{\mathrm{eff}} \supset -\frac{g_L^2}{2m_W^2}\left[\bar e_L\gamma_\rho\nu_e + V_{ud}\bar d_L\gamma_\rho u_L\right]\left[\bar\nu_e\gamma_\rho e_L + V_{ud}\bar u_L\gamma_\rho d_L\right] \supset -\frac{2V_{ud}}{\mathrm{v}^2}(\bar e_L\gamma_\rho\nu_e)(\bar u_L\gamma_\rho d_L)$$

$$\mathrm{v} \equiv \frac{2m_W}{g_L} \approx 246\ \mathrm{GeV}$$

Fermi EFT, again

$$\mathscr{L}_{\rm eff} \supset -\frac{2V_{ud}}{\mathrm{v}^2}(\bar{e}_L\gamma_\rho\nu_e)(\bar{u}_L\gamma_\rho d_L) + \mathrm{h.c.}$$

This interaction leads to beta decays, in particular to the neutron decay

$$d \to u e^- \bar{\nu}_e \quad \Rightarrow \quad n \to p e^- \bar{\nu}_e$$

Amplitude for the latter process is

$$\begin{aligned}
M(n \to p e^- \bar{\nu}_e) &= -\frac{2V_{ud}}{\mathrm{v}^2}\langle p e^- \bar{\nu}_e \,|\, (\bar{e}_L\gamma_\rho\nu_e)(\bar{u}_L\gamma_\rho d_L) \,|\, n > \\
&= -\frac{2V_{ud}}{\mathrm{v}^2}\langle e^- \bar{\nu}_e \,|\, (\bar{e}_L\gamma_\rho\nu_e) \,|\, 0 > \langle p \,|\, (\bar{u}_L\gamma_\rho d_L) \,|\, n > \\
&= -\frac{2V_{ud}}{\mathrm{v}^2}\big(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu)\big)\langle p \,|\, (\bar{u}_L\gamma_\rho d_L) \,|\, n > \\
&= -\frac{V_{ud}}{\mathrm{v}^2}\big(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu)\big)\Big\{ \langle p \,|\, (\bar{u}\gamma_\rho d) \,|\, n > - \langle p \,|\, (\bar{u}\gamma_\rho\gamma_5 d) \,|\, n > \Big\}
\end{aligned}$$

$$P_L \equiv \frac{1-\gamma_5}{2}$$

where u(p), v(p) are the usual spinor wave functions for particle and antiparticles

Fermi EFT, again

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu)\right)\left\{ \langle p\,|\,(\bar{u}\gamma_\rho d)\,|\,n > - \langle p\,|\,(\bar{u}\gamma_\rho\gamma_5 d)\,|\,n > \right\}$$

Due to strong QCD interaction, the quark matrix element cannot be calculated perturbatively

However, with the input from dimensional analysis and QCD (approximate) symmetries they can be reduced to a few unknowns, which can be subsequently calculated on the lattice or using phenomenological models

Lorentz invariance + Parity of QCD implies

$$q \equiv p_n - p_p$$

$$\langle p\,|\,(\bar{u}\gamma_\rho d)\,|\,n > = \bar{u}(p_p)\left[g_V(q^2)\gamma_\rho + \frac{\tilde{g}_{TV}(q^2)}{2m_n}\sigma_{\rho\nu}q^\nu + \frac{\tilde{g}_S(q^2)}{2m_n}q_\rho\right]u(p_n)$$

$$\langle p\,|\,(\bar{u}\gamma_\rho\gamma_5 d)\,|\,n > = \bar{u}(p_p)\left[g_A(q^2)\gamma_\rho + \frac{\tilde{g}_{TA}(q^2)}{2m_n}\sigma_{\rho\nu}q^\nu + \frac{\tilde{g}_P(q^2)}{2m_n}q_\rho\right]\gamma_5 u(p_n)$$

Fermi EFT, again

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu)\right)\left\{ \langle p \,|\, (\bar{u}\gamma_\rho d)\,|\, n > - \langle p\,|\,(\bar{u}\gamma_\rho\gamma_5 d)\,|\,n > \right\}$$

For beta decay processes, and especially for neutron decay, recoil is much smaller than nucleon mass. Therefore at the leading order one can approximate

$$\langle p \,|\, (\bar{u}\gamma_\rho d)\,|\, n > = g_V \bar{u}(p_p)\gamma_\rho u(p_n) + \mathcal{O}(q)$$

$$\langle p \,|\, (\bar{u}\gamma_\rho\gamma_5 d)\,|\, n > = g_A \bar{u}(p_p)\gamma_\rho\gamma_5 u(p_n) + \mathcal{O}(q)$$

$$q \equiv p_n - p_p$$

where g_V=g_V(0) and g_A=g_A(0) are now numbers, called the vector and axial charges

Furthermore, in the isospin symmetric g_V=1, because the quark current is the isospin current One can prove that departures of g_V from one are second order in isospin breaking, thus tiny

All in all

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu)\right)\left\{ \bar{u}(p_p)\gamma_\rho u(p_n) - g_A \bar{u}(p_p)\gamma_\rho\gamma_5 u(p_n) + \mathcal{O}(q) \right\}$$

Fermi EFT, again

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2} \left(\bar{u}(p_e)\gamma_\rho P_L \mathrm{v}(p_\nu) \right) \left\{ \bar{u}(p_p)\gamma_\rho u(p_n) - g_A \bar{u}(p_p)\gamma_\rho \gamma_5 u(p_n) + \mathcal{O}(q) \right\}$$

$$\mathscr{L}_{\mathrm{UV}} \supset -\frac{2V_{ud}}{\mathrm{v}^2}(\bar{e}_L \gamma_\rho \nu_e)(\bar{u}_L \gamma_\rho d_L) + \mathrm{h.c.}$$

Matching

$$\mathscr{L}_{\mathrm{eff}} \supset -\frac{V_{ud}}{\mathrm{v}^2}(\bar{e}_L \gamma_\rho \nu_e) \left\{ (\bar{p}\gamma_\rho n) - g_A (\bar{p}\gamma_\rho \gamma_5 n) \right\} + \mathrm{h.c.} + \mathcal{O}\left(\frac{q}{m_n} \right)$$

as our n→p e ν amplitude can be obtained from this effective Lagrangian

The non-perturbative parameter g_A appearing in this matching has to be calculated on the lattice or measured in experiment

Lattice	**Experiment**
$g_A = 1.271 \pm 0.013$	$g_A = 1.27536 \pm 0.00041$

Nucleaon EFT

Summary and lesson learned

- Matching of the Wilson coefficients cannot always be calculated analytically if the UV theory is strongly coupled at the matching scale
- In those cases, it pays off to use the arguments based on symmetries and dimensional analysis, to reduce the number of unknown parameters in the EFT
- The remaining unknown parameters can be taken from the lattice, phenomenological models, or from experiment

Illustration #4

Schrödinger EFT

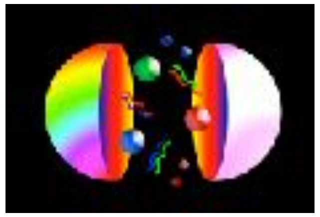
Standard Model

100 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c, b**

5 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c**

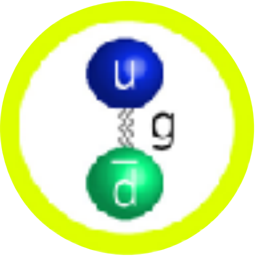


2 GeV

γ, ν_i, e, μ + **hadrons**

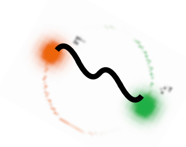
1 GeV

γ, ν_i, e, μ + **pions and kaons**



100 MeV

γ, ν_i, e



1 MeV

γ, ν_i

Schrödinger EFT

$$E_k \equiv \sqrt{\mathbf{k}^2 + m^2} \approx m, \qquad \leftrightarrow \qquad |\mathbf{k}| \ll m$$

- For energies close to particle's mass, that is at small momenta $|\mathbf{k}|$<<m, a quantum theory has qualitatively different properties than for E>>m. In particular, there is no particle production, and one can well ignore the existence of anti-particles.

- In the presence of long range forcers there are also qualitatively new phenomena appearing, such as the Sommerfeld enhancement or Coulomb bound state formation.

- While it is perfectly possible to use the usual relativistic QFT in this regime, there are advantages of using a simplified description where high-energy modes of the quantum field are integrated out

Schrödinger EFT

Consider a relativistic complex scalar field: $\mathscr{L} \supset |\partial_\mu \phi|^2 - m^2 |\phi|^2$

The quantum field contains creation/annihilation operators for both particles and anti-particles:

$$\phi = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(a_k e^{-ikx} + b_k^\dagger e^{ikx}\right), \qquad [a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{k}')$$

$$k = (E_k, \mathbf{k})$$

It is convenient to change variables such that particles and anti-particles are separated:

$$\psi = \frac{e^{imt}}{\sqrt{2}}\left[\hat{E}^{1/2}\phi + i\hat{E}^{-1/2}\partial_t \phi\right]$$

$$\hat{E} \equiv \sqrt{m^2 - \nabla^2}$$

$$\psi^c = \frac{e^{imt}}{\sqrt{2}}\left[\hat{E}^{1/2}\phi^\dagger + i\hat{E}^{-1/2}\partial_t \phi^\dagger\right]$$

In the new variables:

$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} a_k e^{-i(E_k - m)t} e^{i\mathbf{k}\mathbf{x}} \longleftarrow \quad \textbf{particle field}$$

$$\psi^c = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} b_k e^{-i(E_k - m)t} e^{i\mathbf{k}\mathbf{x}} \longleftarrow \quad \textbf{antiparticle field}$$

Schrödinger EFT

New fields satisfy non-local equations of motion:

$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} a_k e^{-i(E_k-m)t} e^{i\mathbf{kx}}$$

$$\psi^c = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} b_k e^{-i(E_k-m)t} e^{i\mathbf{kx}}$$

$$i\dot{\psi} = (\hat{E} - m)\psi, \qquad i\dot{\psi}^c = (\hat{E} - m)\psi^c$$

These can be obtained from the non-local Lagrangian

$$\mathscr{L} \supset i\bar{\psi}\dot{\psi} - \bar{\psi}(\hat{E} - m)\psi + i\bar{\psi}^c\dot{\psi}^c - \bar{\psi}^c(\hat{E} - m)\psi^c$$

Note that the particle and anti-particle fields do not mix at the quadratic level

Up to this point we only changed variables to a non-local and non-manifestly Lorentz-invariant description, without adding or removing any physics content

Non-relativistic theory is obtained by expanding this Lagrangian in powers of spatial derivatives

$$\hat{E} \equiv \sqrt{m^2 - \nabla^2} = m - \frac{\nabla^2}{2m} - \frac{\nabla^4}{2m^3} + \dots$$

$$\boxed{\mathscr{L} \supset i\bar{\psi}\dot{\psi} + \bar{\psi}\frac{\nabla^2}{2m}\psi + i\bar{\psi}^c\dot{\psi}^c + \bar{\psi}^c\frac{\nabla^2}{2m}\psi^c + \mathcal{O}(\nabla^4)}$$

In this approximation fields satisfy local (Schrödinger) equations:

$$i\dot{\psi} = -\frac{\nabla^2}{2m}\psi, \qquad i\dot{\psi}^c = -\frac{\nabla^2}{2m}\psi^c$$

Schrödinger EFT

Same story for a relativistic real scalar field: $$\mathscr{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$$

Relativistic field

$$\phi = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right), \qquad [a_k, a_{k'}^\dagger] = (2\pi)^3 2E_k \delta^3(\mathbf{k} - \mathbf{k}')$$

Non-relativistic field

$$\psi = \frac{e^{imt}}{\sqrt{2}}\left[\hat{E}^{1/2}\phi + i\hat{E}^{-1/2}\partial_t\phi\right] = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} a_k e^{-i(E_k - m)t} e^{i\mathbf{kx}}$$

Inverse transformation

$$\phi = \frac{1}{\sqrt{2}}\hat{E}^{-1/2}\left[e^{-imt}\psi + e^{imt}\bar{\psi}\right]$$

Non-relativistic Lagrangian

$$\boxed{\mathscr{L} \supset i\bar{\psi}\dot{\psi} + \bar{\psi}\frac{\nabla^2}{2m}\psi + \mathcal{O}(\nabla^4)}$$

We traded a real scalar field φ for a complex field ψ
The U(1) symmetry of the complex field is interpreted as the global particle number

Schrödinger EFT

Consider now cubic interactions

$$\mathscr{L} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - m\,\kappa\,\phi^3$$

Express this interaction in terms of non-relativistic variables:

$$\phi = \frac{1}{\sqrt{2}}\hat{E}^{-1/2}\left[e^{-imt}\psi + e^{imt}\bar{\psi}\right] = \frac{1}{\sqrt{2m}}\left[e^{-imt}\psi + e^{imt}\bar{\psi}\right] + \mathcal{O}(\nabla^2)$$

At the lowest, non-derivative order we find

$$-m\kappa\phi^3 \to -\frac{\kappa m}{(2m)^{3/2}}\left[e^{-3imt}\psi^3 + e^{-imt}\psi^2\bar{\psi} + \mathrm{h.c.}\right]$$

What the heck is this???

Actually, the non-relativistic theory is telling us something interesting, namely that to understand the non-relativistic dynamics of a scalar field we should first integrate out its high-frequency modes

Schrödinger EFT

Expand the non-relativistic field into frequency modes: $\psi(x,t) = \sum_n \psi_n(x,t)e^{imnt}$

Assumption: the fields ψ_n(x,t) are slowly varying $|\partial_t \psi_n| \ll m|\psi_n|$

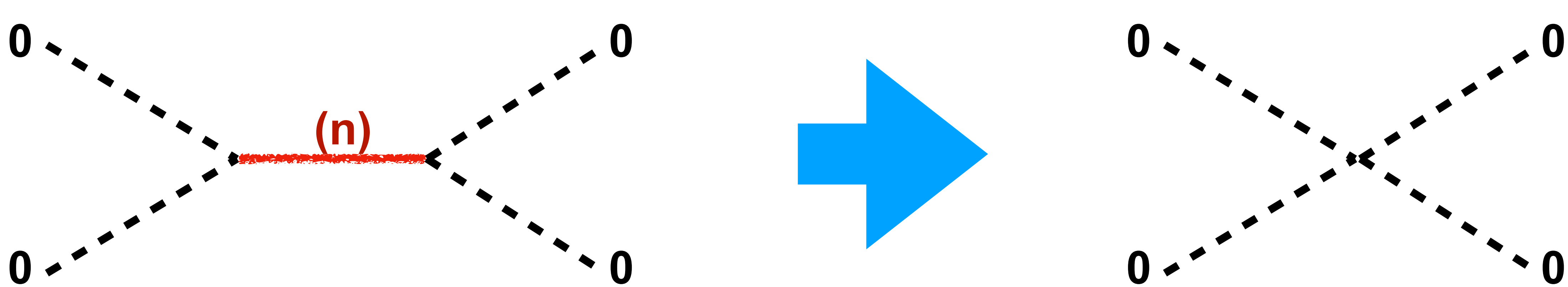
Take a small time interval $t \in [0, \frac{2\pi}{m}]$ **in which we can ignore time evolution of ψ_n(x,t)**

$$\mathscr{L} \supset i\bar{\psi}\dot{\psi} + \bar{\psi}\frac{\nabla^2}{2m}\psi - \frac{\kappa m}{(2m)^{3/2}}\left[e^{-3imt}\psi^3 + 3e^{-imt}\psi^2\bar{\psi} + \mathrm{h.c.}\right]$$

Integrating over that time interval:

$$L \equiv \int_0^{2\pi/m} dt\mathscr{L} \supset \frac{2\pi}{m}\left\{ \sum_n \bar{\psi}_n\left(-nm + \frac{\nabla^2}{2m}\right)\psi_n - \frac{\kappa m}{(2m)^{3/2}}\left[3\bar{\psi}_3\bar{\psi}_0^2 + 6\bar{\psi}_1|\psi_0|^2 + 3\bar{\psi}_{-1}\psi_0^2 + \mathrm{h.c.}\right]\right\}$$

Modes with n different from zero effectively have a large "mass", which suppresses their contribution to the path integral. Integrate them out!



Schrödinger EFT

$$L \supset \frac{2\pi}{m}\left\{\sum_n \bar{\psi}_n\left(-nm + \frac{\nabla^2}{2m}\right)\psi_n - \frac{\kappa m}{(2m)^{3/2}}\left[3\bar{\psi}_3\bar{\psi}_0^2 + 6\bar{\psi}_1|\psi_0|^2 + 3\bar{\psi}_{-1}\psi_0^2 + \mathrm{h.c.}\right]\right\}$$

Equations of motion for the high-frequency modes

$$\psi_1 = -6\frac{\kappa}{(2m)^{3/2}}|\psi_0|^2 + \mathcal{O}(\nabla^2)$$

$$\psi_{-1} = 3\frac{\kappa}{(2m)^{3/2}}\psi_0^2 + \mathcal{O}(\nabla^2)$$

$$\psi_3 = -\frac{\kappa}{(2m)^{3/2}}\bar{\psi}_0^2 + \mathcal{O}(\nabla^2)$$

Plugging this solution back into L:

$$L_{\mathrm{eff}} \supset \frac{2\pi}{m}\left\{\bar{\psi}_0\frac{\nabla^2}{2m}\psi_0 + \frac{15\kappa^2}{4m^2}|\psi_0|^4 + \mathcal{O}(\nabla^2)\right\}$$

Unfolding this into effective Lagrangian for the low-frequency mode:

$$\boxed{\mathscr{L}_{\mathrm{eff}} \supset \bar{\psi}_0\dot{\psi}_0 + \bar{\psi}_0\frac{\nabla^2}{2m}\psi_0 + \frac{15\kappa^2}{4m^2}|\psi_0|^4 + \mathcal{O}(\nabla^2)}$$

Schrödinger EFT

$$\mathscr{L}_{\mathrm{UV}} \supset \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - m\,\kappa\,\phi^3 \quad \Rightarrow \quad \mathscr{L}_{\mathrm{eff}} \supset \bar{\psi}_0 \dot{\psi}_0 + \bar{\psi}_0 \frac{\nabla^2}{2m}\psi_0 + \frac{15\kappa^2}{4m^2}|\psi_0|^4$$

- There is no cubic self-interaction in the non-relativistic effective Lagrangian! The relativistic scalar cubic coupling translates to a *quartic* coupling in the relativistic theory

- The same conclusion could be reached by calculating the ϕ ϕ→ ϕ ϕ scattering amplitude in the relativistic theory, and expanding it at low velocity

- This is a broad conclusion about physics of many non-relativistic systems (e.g. scalar condensates in cosmology) - a relativistic cubic potential corresponds to an *attractive* quartic potential in the non-relativistic regime

- The deeper reason is that there cannot be particle-number changing interactions in a non-relativistic EFT

- Self-interaction corresponds to an *irrelevant* operator in a non-relativistic EFT

Interlude

Non-relativistic power counting

Scaling in non-relativistic theories

$$S = \int dt d^3x \left[\phi^\dagger \partial_t \phi - \phi^\dagger \frac{\partial_x^2}{2m} \phi - c_4 (\phi^\dagger \phi)^2 + \dots \right]$$

$$x \to \xi x'$$
$$t \to \xi^2 t'$$
$$\phi(t,x) \to \xi^{-3/2} \phi'(\xi^2 t, \xi x)$$

In a non-relativistic theory
time and space are on different footing.
In order to keep kinetic terms invariant
they should be assigned different scaling dimensions

$$S \to \int dt' d^3x' \left[\phi'^\dagger \partial_{t'} \phi' - \phi'^\dagger \frac{\partial_{x'}^2}{2m} \phi' - c_4 \xi^{-1} (\phi'^\dagger \phi')^2 + \dots \right]$$

$$c_4' = c_4 \xi^{-1}$$

In usual non-relativistic theory
quartic self-interaction is
irrelevant!

There are other non-relativistic systems (e.g. z=3 fixed point) where scaling is yet different

Schrödinger EFT

Summary and lessons learned

- Non-relativistic effective Lagrangian can be build in a systematic expansion in ∇^2/m^2.
- Clever choice of variables may greatly facilitate derivation of the low-energy EFT Lagrangian
- In some cases, EFT interactions become transparent after integrating out high-frequency modes of the field describing the light particle participating in the EFT

Illustration #5

Chiral Perturbation Theory

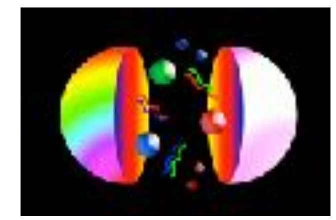
Standard Model

100 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c, b

5 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c

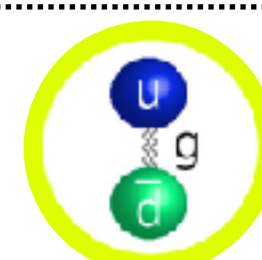


2 GeV

γ, ν_i, e, μ + hadrons

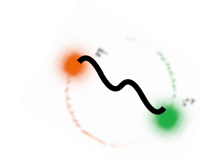
1 GeV

γ, ν_i, e, μ + pions and kaons



100 MeV

γ, ν_i, e



1 MeV

γ, ν_i

Chiral perturbation theory

- ChPT describes low energy interactions of pions.
- Underlying theory - QCD - is known, but coefficients of EFT operators cannot be calculated analytically.
- Approximate symmetries inherited from QCD provide a method to write down possible pion interactions in a systematic expansion

Chiral perturbation theory

$$\mathcal{L} = i\bar{u}\bar{\sigma}_\mu\partial_\mu u + i\bar{d}\bar{\sigma}_\mu\partial_\mu d + iu^c\sigma_\mu\partial_\mu\bar{u}^c + id^c\sigma_\mu\partial_\mu\bar{d}^c$$

- QCD has two nearly massless quarks: up and down. In massless limit, QCD Lagrangian has SU(2)LxSU(2)R symmetry corresponding to separate rotations of left-handed and right-handed components

$$\begin{pmatrix} u \\ d \end{pmatrix} \to L \begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} u_c \\ d_c \end{pmatrix} \to \begin{pmatrix} u_c \\ d_c \end{pmatrix} R^\dagger$$

SU(2) SU(2)

- This symmetry is explicitly and completely broken by quark masses

$$\mathcal{L}_{mass} = -m_u u u^c - m_d d d^c + \text{h.c.}$$

- There's even larger source of symmetry breaking due to QCD vacuum condensate, <u u^c> = < d d^c>

$$\langle q_i^c q_j\rangle = \delta_{ij} \to R^\dagger_{ik} L_{kj}$$

- This spontaneously breaks SU(2)LxSU(2)R down to diagonal SU(2) that rotates left-handed and right-handed quarks in the same way

- Therefore, there should 3 light Goldstone boson states (identified with pions), 1 for each spontaneously broken generator of symmetry

Chiral perturbation theory

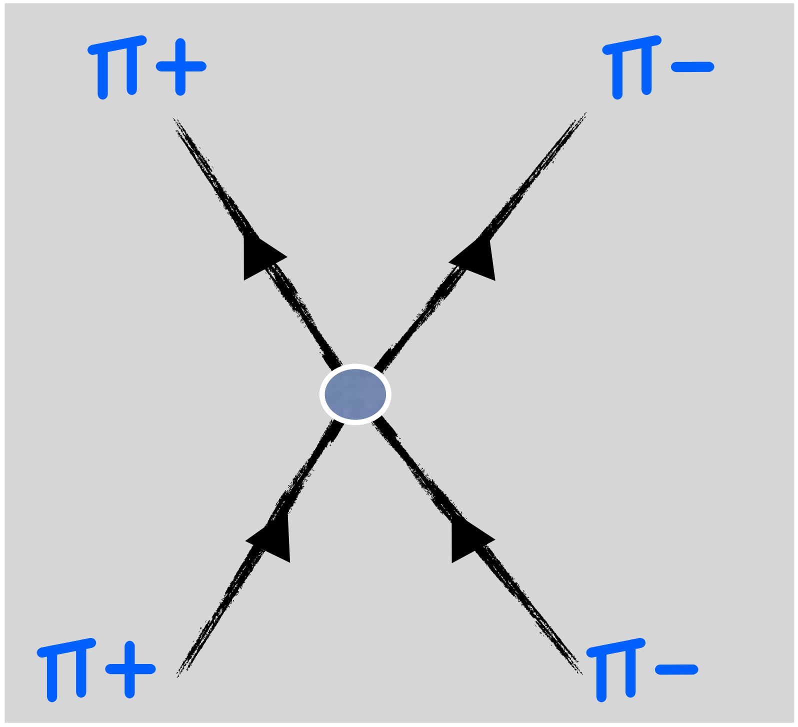
- Low energy theory of pions should inherit symmetries of QCD
- This means the theory should have non-linearly realized SU(2)LxSU(2)R symmetry such that diagonal (vector) part is linearly realized, and under axial part pions transform under shift symmetry
- Effective Lagrangian can then be written in derivative expansion
- Lowest order term that one can write has 2 derivatives. It describes kinetic terms of pions, but also infinite series of 2-derivative pion interaction terms
- These interactions can be tested in pion-pion scattering, which allows one to fit f≈93 MeV

$$U = \exp(i\pi^a\sigma^a/f)$$
$$= \exp\left[\frac{i}{f}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\right]$$

$$U \to LUR^\dagger, \qquad L, R \in SU(2)$$

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{f^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]$$

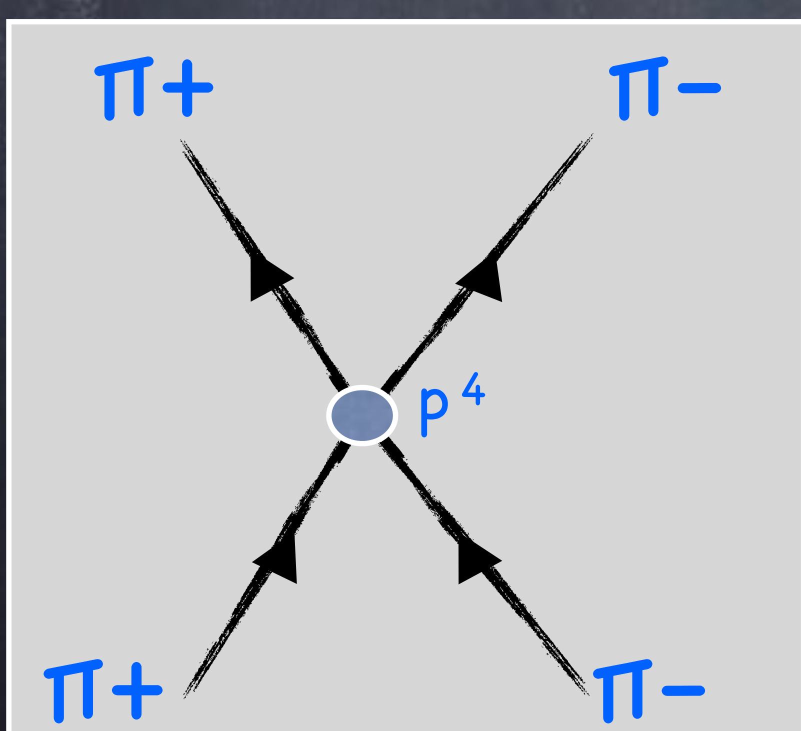
$$\mathcal{L}_{\text{eff}}^{(2)} = \partial_\mu\pi^+\partial_\mu\pi^- + \frac{1}{2}\partial_\mu\pi^0\partial_\mu\pi^0$$
$$+ \frac{1}{2f^2}(\partial_\mu\pi^+\pi^- + \partial_\mu\pi^-\pi^+ + \partial_\mu\pi^0\pi^0)^2$$
$$+ \dots$$

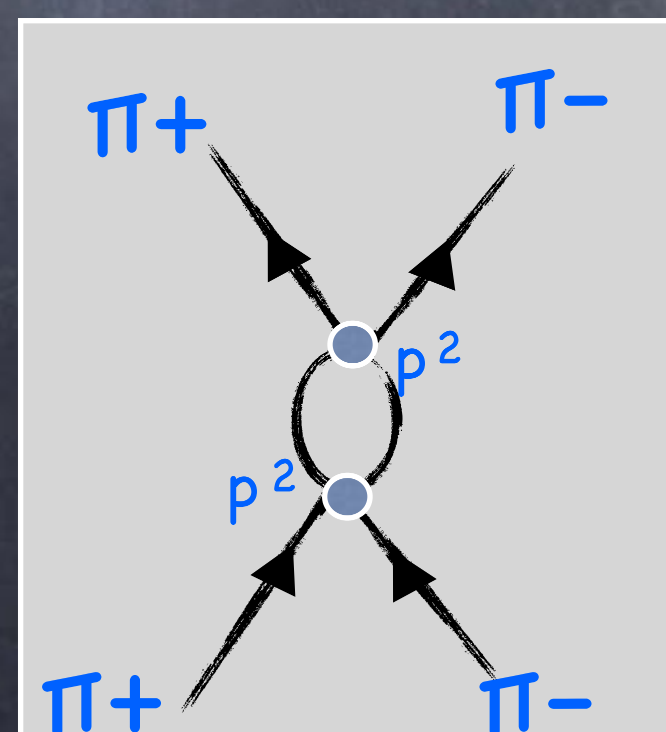


Chiral perturbation theory

$$\mathcal{L}_{\text{eff}}^{(4)} = L_1 \left(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\right)^2 + L_2 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U] + L_3 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\nu U]$$

- ChPT theory can be extended to 4-derivative level. This produces 4-derivative interactions terms of pions, in addition to 2-derivative ones
- By studying momentum dependence of pion scattering one can fit the parameters L1, L2, L3
- Note that in this case 1-loop diagrams with 2-derivative vertices have to included together with tree-level diagrams with 4-derivative vertices. In ChPT, derivative expansion is intimately tied to loop expansion.





Scherer, hep-ph/0210398

Coefficient	Empirical Value
L_1^r	0.4 ± 0.3
L_2^r	1.35 ± 0.3
L_3^r	-3.5 ± 1.1

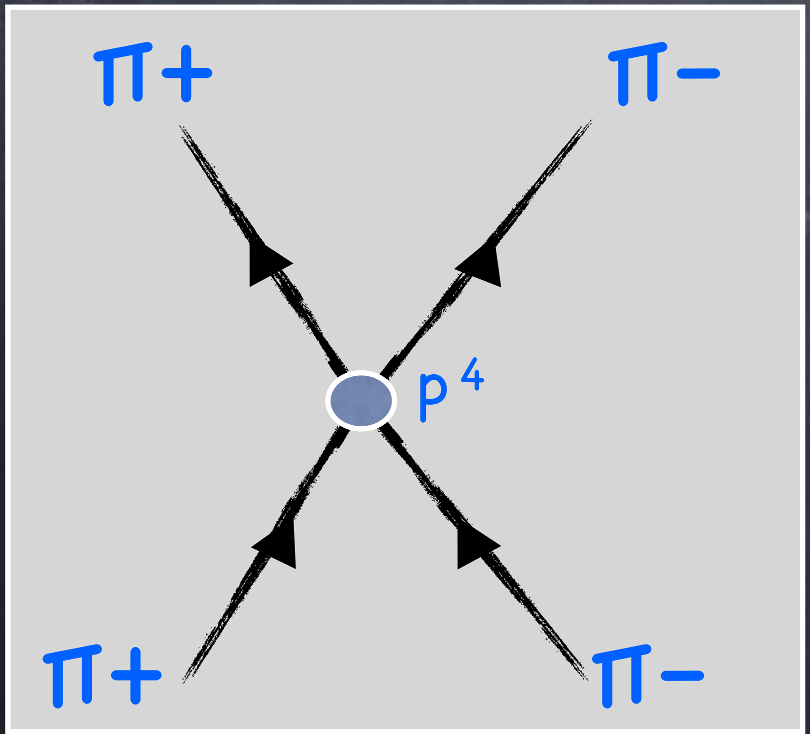
(In units of 10^-3, at scale mρ)

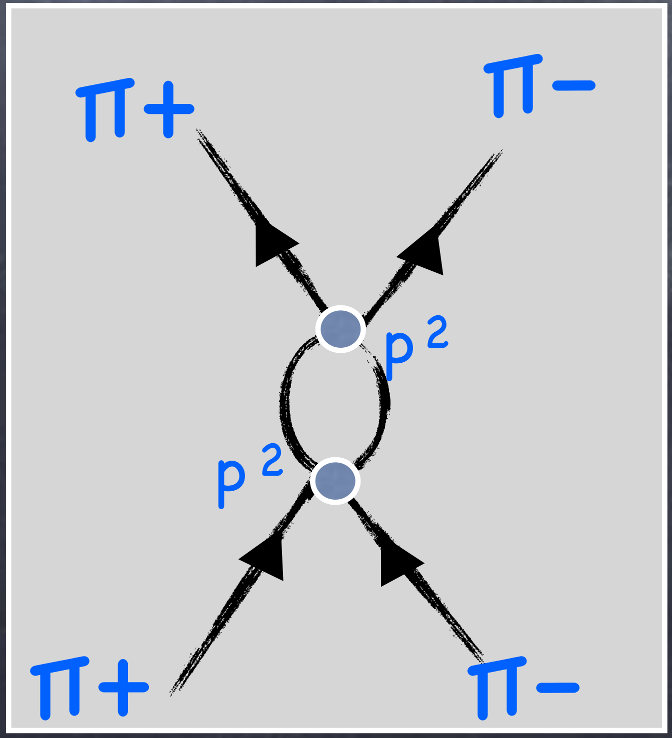
Chiral perturbation theory

$$\mathcal{L}_{\text{eff}}^{(4)} = L_1 \left(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\right)^2 + L_2 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U] + L_3 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\nu U]$$

? $\mathrm{Tr}[\partial^2 U^\dagger \partial^2 U]$

- Operators that can be eliminated or traded for other by equations of motion are not included in effective Lagrangian
- This is because they are redundant - all their effect on on-shell amplitudes can be described by other terms
- In this case, in the limit of massless pions, equation of motion is $\Box U = 0$, so the new term above does not contribute to on-shell amplitudes at all





Chiral perturbation theory

Lessons learned:

- It is often advantageous to work with EFT even when matching with UV theory cannot be calculated. Then one needs to write down all possible non-redundant interaction terms consistent with EFT symmetries in some systematic expansion, and determine their coefficients from experiment

- EFT is not renormalizable, therefore it formally has infinite number of parameter. However, at a fixed order in EFT expansion it is renormalizable. As soon as all coefficients are fixed at a given order from experiment, other observables can be predicted at that order

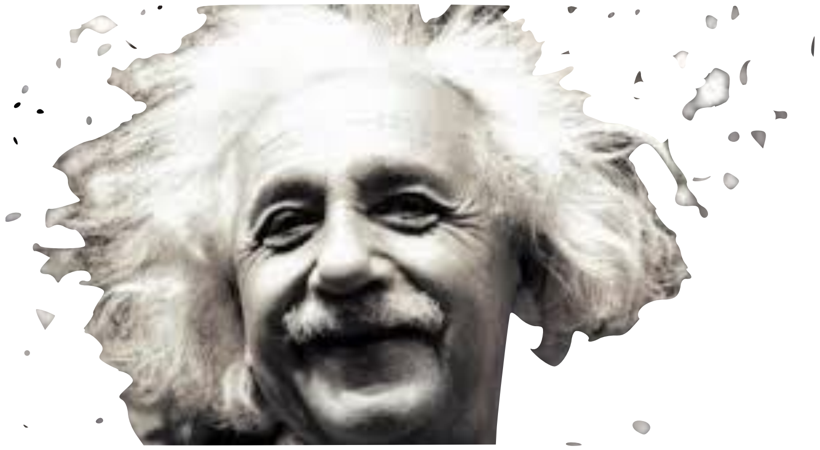


Illustration #6

Einstein EFT

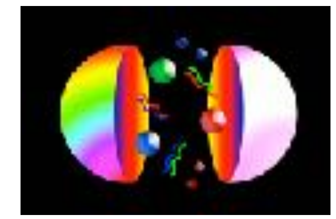
Standard Model

100 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c, b

5 GeV

$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c

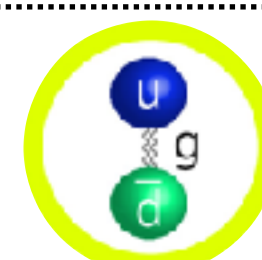


2 GeV

γ, ν_i, e, μ + hadrons

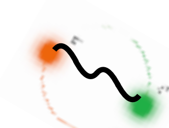
1 GeV

γ, ν_i, e, μ + pions and kaons



100 MeV

γ, ν_i, e



1 MeV

γ, ν_i

Einstein EFT

We want to write down an EFT for a massless spin-2 particle, aka the graviton

Such a particle can be described by a real and symmetric tensor field

$$h_{\mu\nu}(x)$$

For a massless spin-1 particle, QFT makes sense only in the presence of gauge invariance

Likewise, for a massless spin-2 particle, QFT makes sense only in the presence of <u>general coordinate invariance</u>
Otherwise, there is no way a 10-component symmetric tensor $h_{\mu\nu}$ can describe 2 components of the massless graviton

To implement the GC invariance, it is convenient to combine the graviton field with the Minkowski metric to write

$$h_{\mu\nu}(x) \to g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x)$$

and demand that $g_{\mu\nu}$ transform as a tensor under GC transformations

$$x \to y \quad \Rightarrow \quad g_{\mu\nu} \to \frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} g_{\alpha\beta}$$

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$S_{\mathrm{EFT}} = \int d^4x \mathscr{L}_{\mathrm{EFT}}$$

$$\mathscr{L}_{\mathrm{EFT}} = \Lambda^4 \mathscr{L}_{D=0}(g) + \Lambda^2 \mathscr{L}_{D=2}(g) + \Lambda^0 \mathscr{L}_{D=4}(g) + \frac{1}{\Lambda^2}\mathscr{L}_{D=6}(g) + \ldots$$

At the leading order the only possible invariant under GC transformations is

$$\mathscr{L}_{D=0}(g) = c_0\sqrt{-g}$$

This is the cosmological constant. Phenomenologically, this term is non-zero but tiny, though no one understand why….

It only plays a role at cosmological distance scale, so we ignore it in the following

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\rm EFT} = \Lambda^4 \cancel{\mathscr{L}_{D=0}}(g) + \Lambda^2 \mathscr{L}_{D=2}(g) + \Lambda^0 \mathscr{L}_{D=4}(g) + \frac{1}{\Lambda^2}\mathscr{L}_{D=6}(g) + \ldots$$

At the next-to-leading order the only possible invariant under GC transformations is

$$\mathscr{L}_{D=2}(g) = c_2\sqrt{-g}R$$

Let's rename variables, trading $c_2\,\Lambda^2$ for 1/2 $(M_{\rm Planck})^2$

$$\mathscr{L}_{D=2}(g) = \frac{1}{2}M^2_{\rm Planck}\sqrt{-g}R$$

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ R_{\mu\nu} &= R^{\alpha}_{\ \mu\alpha\nu} \\ R^{\alpha}_{\ \mu\nu\beta} &= \partial_\nu \Gamma^\alpha_{\mu\beta} - \partial_\beta \Gamma^\alpha_{\mu\nu} + \Gamma^\rho_{\mu\beta}\Gamma^\alpha_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\alpha_{\rho\beta} \\ \Gamma^\mu_{\nu\rho} &= \frac{1}{2}g^{\mu\alpha}\left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}\right) \end{aligned}$$

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\mathrm{EFT}} = \Lambda^4 \cancel{\mathscr{L}_{D=0}}(g) + \frac{1}{2} M^2_{\mathrm{Planck}} \sqrt{-g} R + \mathscr{L}_{D=4}(g) + \frac{1}{M^2_{\mathrm{Planck}}} \mathscr{L}_{D=6}(g) + \ldots$$

Expanding the next-to-leading order term in powers of the graviton field:

$$\mathscr{L}_{D=2}(1+h) = \mathscr{L}^{(2)}_{D=2} + \mathscr{L}^{(3)}_{D=2} + \mathscr{L}^{(4)}_{D=2} + \ldots$$

$$\mathscr{L}^{(2)}_{D=2} = \frac{M^2_{\mathrm{Planck}}}{4} \left[\frac{1}{2} (\partial_\rho h_{\mu\nu})^2 - \frac{1}{2} (\partial_\rho h)^2 - (\partial_\rho h_{\mu\rho})^2 + \partial_\mu h \partial_\rho h_{\mu\rho} \right]$$

This is the so-called Fierz-Pauli Lagrangian. Up to normalization, this is the unique ghost-free kinetic Lagrangian for a massless spin-2 particle

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\text{EFT}} = \cancel{\Lambda^4 \mathscr{L}_{D=0}(g)} + \frac{1}{2} M_{\text{Planck}}^2 \sqrt{-g} R + \mathscr{L}_{D=4}(g) + \frac{1}{M_{\text{Planck}}^2} \mathscr{L}_{D=6}(g) + \ldots$$

Expanding the next-to-leading order term in powers of the graviton field:

$$\mathscr{L}_{D=2}(1+h) = \mathscr{L}_{D=2}^{(2)} + \mathscr{L}_{D=2}^{(3)} + \mathscr{L}_{D=2}^{(4)} + \ldots$$

$$\mathscr{L}_{D=2}^{(2)} = \frac{M_{\text{Planck}}^2}{4} \left[\frac{1}{2} (\partial_\rho h_{\mu\nu})^2 - \frac{1}{2} (\partial_\rho h)^2 - (\partial_\rho h_{\mu\rho})^2 + \partial_\mu h \partial_\rho h_{\mu\rho} \right]$$

Kinetic terms

$$\mathscr{L}_{D=2}^{(3)} \sim M_{\text{Planck}}^2 h^3 \partial^2$$

Cubic interactions

$$\mathscr{L}_{D=2}^{(4)} \sim M_{\text{Planck}}^2 h^4 \partial^2$$

$$\ldots$$

Quartic interactions
and so on

We have built a consistent interacting effective theory of a massless spin-2 particle
This effective Lagrangian describes all known phenomenology of (pure) general relativity!

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\rm EFT} = \cancel{\Lambda^4 \mathscr{L}_{D=0}(g)} + \frac{1}{2} M^2_{\rm Planck} \sqrt{-g} R + \mathscr{L}_{D=4}(g) + \frac{1}{M^2_{\rm Planck}} \mathscr{L}_{D=6}(g) + \ldots$$

Expanding the next-to-leading order term in powers of the graviton field:

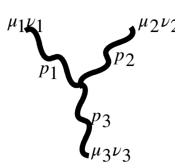
$$\mathscr{L}_{D=2}(1+h) = \mathscr{L}^{(2)}_{D=2} + \mathscr{L}^{(3)}_{D=2} + \mathscr{L}^{(4)}_{D=2} + \ldots$$

Canonical normalization: $h_{\mu\nu} \to \dfrac{2}{M_{\rm Planck}} \tilde{h}_{\mu\nu}$

$$\mathscr{L}^{(2)}_{D=2} = \frac{1}{2}(\partial_\rho \tilde{h}_{\mu\nu})^2 - \frac{1}{2}(\partial_\rho \tilde{h})^2 - (\partial_\rho \tilde{h}_{\mu\rho})^2 + \partial_\mu \tilde{h} \partial_\rho \tilde{h}_{\mu\rho}$$

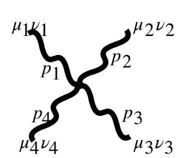
Canonically normalized kinetic terms

$$\mathscr{L}^{(3)}_{D=2} \sim \frac{1}{M_{\rm Planck}} \tilde{h}^3 \partial^2$$



Cubic interactions

$$\mathscr{L}^{(4)}_{D=2} \sim \frac{1}{M^2_{\rm Planck}} \tilde{h}^4 \partial^2$$



Quartic interactions

$$\ldots$$

and so on

This is a <u>consistent theory of quantum gravity</u>, organized as an EFT expanded in inverse powers of the Planck scale. It is arguably the best EFT ever, because its validity range is the largest of known EFTs, spanning from very low-energies all the way to the Planck scale

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\text{EFT}} = \Lambda^4 \cancel{\mathscr{L}_{D=0}}(g) + \frac{1}{2} M^2_{\text{Planck}} \sqrt{-g} R + \mathscr{L}_{D=4}(g) + \frac{1}{M^2_{\text{Planck}}} \mathscr{L}_{D=6}(g) + \ldots$$

Going to higher orders

$$\mathscr{L}_{D=4}(g) = \sqrt{-g} \left(c_1 R^2 + c_2 R^2_{\mu\nu} + c_3 R^2_{\mu\nu\alpha\beta} \right) \qquad ?$$

One can show that all these operators can be eliminated by using equations of motion, field redefinition, integration by parts !

Thus $$\mathscr{L}_{D=4}(g) = 0$$

First non-trivial EFT corrections to general relativity arise at dimension-6, that is at 6-derivative level !

This is arguably the best EFT ever, because corrections from higher-dimension operators are extremely suppressed

Einstein EFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathcal{L}_{\rm EFT} = \Lambda^4 \cancel{\mathcal{L}_{D=0}}(g) + \frac{1}{2} M_{\rm Planck}^2 \sqrt{-g} R + \frac{1}{M_{\rm Planck}^2} \mathcal{L}_{D=6}(g) + \ldots$$

Going to higher orders

$$\mathcal{L}_{D=6}(g) = c_1 C_{\mu\nu\alpha\beta} C_{\alpha\beta\rho\sigma} C_{\rho\sigma\mu\nu} + c_2 C_{\mu\nu\alpha\beta} C_{\alpha\beta\rho\sigma} \tilde{C}_{\rho\sigma\mu\nu}$$

We do not know what is the UV completion of this effective theory of gravity, so we do not know the numerical value coefficients c_1 and c_2. At this point, they parametrize our ignorance about nature. Maybe one day we will measure them experimentally, and that will give us a hint about the underlying, more fundamental theory of gravity

Einstein EFT

Summary and lessons learned

- Gravity is (to a large extent) like any other QFT, and can be treated by EFT methods. As usual, symmetry is the key to building the EFT.

- Einstein Gravity is not only a good classical theory. It is a consistent EFT at a quantum level, describing a self-interacting massless spin-2 particle. The theory is valid in the very broad energy regime up to the Planck scale

- Corrections from higher dimensional operators added to the Einstein-Hilbert Lagrangian are very small, because they are suppressed by many powers of the Planck scale. At this point, they seem unobservable

- One key difference to previous examples is that we don't know the Wilson coefficients of the higher-dimensional operators, because we don't know the UV completion of gravity