GDR-InF annual meeting 2020

Leading hadronic contribution to the muon magnetic moment from lattice QCD [arXiv:2002.12347]

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History of magnetic moments in a nutshell

▷ Leptons ℓ have magnetic moments $\vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell c}\vec{s}$ due to their spin.

1927 ▶ **Pauli** equation (g_ℓ was initially left as free parameter):

$$i\hbar\partial_t\phi(x) = \left[\frac{1}{2m_\ell}\left(-i\hbar\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2 - g_\ell\frac{e}{2m_\ell c}\vec{s}\cdot\vec{B} + e_\ell A_0\right]\phi(x)$$

1928 ▶ **Dirac** equation for relativistic spin-$\frac{1}{2}$ fermions:

$$i\hbar\partial_t\psi(x) = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi(x)$$

▷ Let $\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}$: **Dirac eq.** $\xrightarrow[v\ll c]{E\sim mc^2}$ **Pauli eq.** $\Leftrightarrow$ $g_\ell|_{\text{Dirac}} = 2$.

1948 ▶ A precision measurement of g_e reveals an anomalous magnetic moment:

$$a_e^{\text{exp (1948)}} = \frac{g_e - 2}{2} = 0.00119(5)$$

1948 ▶ QED + Renormalization $\to$ anomaly comes from loop corrections to QED vertex. Schwinger evaluates the dominant contribution to a_e:

$$a_e^{(1)\text{QED}} = \frac{\alpha}{2\pi} \approx 0.0011614 \quad \text{(99\% of the anomaly)}$$

Today ▶ The full SM is required to match $a_e^{(\text{exp})}$ and $a_\mu^{(\text{exp})}$... *But is the SM enough?*

Electromagnetic vertex in the Standard Model

Consider the matrix element of electromagnetic current between two lepton states:

$$i\Gamma^{\mu}_{\gamma\ell\ell}(p_1,p_2) = \langle \ell(p')|J_\mu(0)|\ell(p)\rangle = i\bar{u}(p_2)\Gamma^{\mu}_{\gamma\ell\ell}u(p_1)$$

Defining $P = p_1 + p_2$ and $q = p_2 - p_1$, assuming Poincaré invariance and current conservation $\partial_\mu j^\mu_{\text{em}} = 0$, $\Gamma^\mu_{\gamma\ell\ell}$ takes the general form:

$$i\Gamma^{\mu}_{\gamma\ell\ell}(p_1,p_2) = \bar{u}(p_2)\left[\gamma^\mu F_E(q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2m_\ell}F_M(q^2) + \left(\gamma^\mu - \frac{2m_\ell q^\mu}{q^2}\right)\gamma_5 F_A(q^2) + \sigma^{\mu\nu}\frac{q_\nu}{2m_\ell}\gamma_5 F_D(q^2)\right]u(p_1)$$

$F_E(q^2) \rightarrow$ *Electric charge form factor*. By charge renormalization $F_E(0) = 1$.

$F_M(q^2) \rightarrow$ *Magnetic form factor*. It leads to the anomalous magnetic moment:

$$g_\ell = 2F_E(0) + F_M(0) \Rightarrow \boxed{F_M(0) = a_\ell = \frac{g_\ell - 2}{2}}$$

$F_A(q^2) \rightarrow$ *Anapole moment*. P violating at vanishing at $q^2 = 0$.

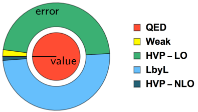
$F_D(q^2) \rightarrow$ *Electric dipole moment*. CP violating. $d_\ell = -F_D(0)/2m_\ell$

- F_M, F_A and F_D come from loops, but *UV finite* once theory is renormalized.
- a_ℓ is dimensionless.
- Form factors can be isolated by means of projection operators: $F_i = \mathrm{Tr}\{P_{\mu i}\Pi^\mu_{i\ell\ell}\}$.

a_e, a_μ and a_τ: why is a_μ special

- $\mathbf{a_e^{exp}} = 11596521.8073(28) \times 10^{-10}$ [D. Hanneke, S. Fogwell, G. Gabrielse (2008)]
  - $\tau_e = \infty$, $m_e = 0.511$ MeV
  - Dominated by QED effects up the 0.66 ppb precision level:
    sensitivity to hadronic and weak effects as well as to physics beyond SM is tiny.
  - a_e is known 829 more precisely than a_μ
  - Provides best measure of $\alpha = 137.035999046(27)$ [Parker, Yu, Zhong, Estey, Muller (2018)]
  - $a_e^{\text{exp}} - a_e^{\text{SM}} = -0.0087(28)^{\text{exp}}(23)^{\alpha}(2)^{\text{SM}} \times 10^{-10} \to -2.4\sigma$ discrepancy
- $\mathbf{a_\tau^{exp}} = ??$
  - $\tau_\tau = 3 \times 10^{-15}$s, $m_\tau = 1777$ MeV
    Very short lived $\Rightarrow$ no measurements yet
- $\mathbf{a_\mu} = 11659208.9(6.3) \times 10^{-10}$ [BNL '04]
  - latest measurement from experiment **Muon E821** at BNL (final report issued in 2006)
  - $\tau_\mu = 2 \times 10^{-6}$s, $m_\mu = 105$ MeV
  - $m_\mu^2/m_e^2 \approx 205^2$ times more sensitive to physics BSM.
  - $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 27.9(6.3)^{\text{exp}}(3.6)^{\alpha}(2)^{\text{SM}} \times 10^{-10} \to 3.6\sigma$ discrepancy

em	$(11658471.895(8) \pm 0.008) \times 10^{-10}$	[Kinoshita et al., (2012)]
weak	$(15.36 \pm 0.10) \times 10^{-10}$	[Gnendinger et al., (2013)]
HVP	$(693.26 \pm 2.46) \times 10^{-10}$	[Keshavarzi et al., (2018)]
HVP (α^3)	$(-9.84 \pm 0.06) \times 10^{-10}$	[Hagiwara et al., (2011)]
LbL	$(11658471.895(8) \pm 0.008) \times 10^{-10}$	[Prades et al., (2009)]



- To match the future experimental precision:

$$\Delta a_\mu^{\text{hvp}}/a_\mu^{\text{hvp}} \to 0.2\% \quad \text{and} \quad \Delta a_\mu^{\text{lbl}}/a_\mu^{\text{lbl}} \to 10\%$$

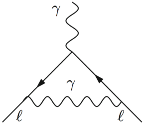
State of the art of SM predictions for a_μ

$$\begin{aligned} a_\mu^{\mathrm{SM}} &= a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{had}} + a_\mu^{\mathrm{weak}} \\ &= O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2\left(\frac{m_\mu}{M_W}\right)^2\right) \\ &= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right) \end{aligned}$$

QED contribution

- $a_\mu^{\mathrm{QED}} = 0.00116584718841(7)_m(17)_{\alpha^4}(6)_{\alpha^5}(28)_{\alpha(a_e)} \sim 10^{-3}$
- By far the largest contribution for all three a_ℓ (more than 99.99%)
- Computed to $O(\alpha^5)$ [Aoyama et al '18]
- 9 diagrams at $O(\alpha^2)$, 72 diagrams at $O(\alpha^3)$, 891 at $O(\alpha^4)$, 12672 at $O(\alpha^5)$

Fig. 4.1 The universal lowest order QED contribution to a_ℓ



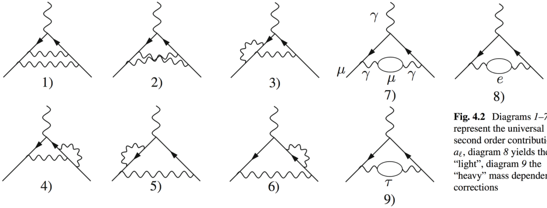


Fig. 4.2 Diagrams *1–7* represent the universal second order contribution to a_ℓ, diagram *8* yields the "light", diagram *9* the "heavy" mass dependent corrections

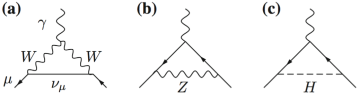
Pictures from *Jegerlehner*, "The anomalous magnetic moment of the muon", (2017)

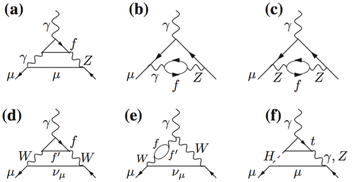
State of the art of SM predictions for a_μ

$$
\begin{aligned}
a_\mu^{\mathrm{SM}} &= a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{had}} + a_\mu^{\mathrm{weak}} \\
&= O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

Weak contribution:

- Collects all loop contributions involving at least one of $W^\pm, Z$ and H.
- $a_\mu^{\mathrm{weak}} = 0.000000001536(10) \sim 10^{-9}$
- Computed to 2 loops [Gnendiger et al '15 and refs therein]





$$O\left(\frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\right) = 19.481(1) \times 10^{-10}$$

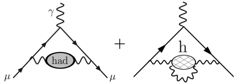
$$O\left(\frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right) = -4.12(60) \times 10^{-10}$$

State of the art of SM predictions for a_μ

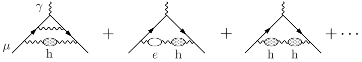
$$\begin{aligned} a_\mu^{\rm SM} &= a_\mu^{\rm QED} + a_\mu^{\rm had} + a_\mu^{\rm weak} \\ &= O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \\ &= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right) \end{aligned}$$

Hadronic contribution:

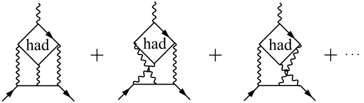
- Non-perturbative QCD because $q^2 = 0$ and $m_\mu \ll 1\,\mathrm{GeV}$
- $a_\mu^{\rm had} \stackrel{?}{=} a_\mu^{\rm exp} - a_\mu^{\rm QED} - a_\mu^{\rm weak} = 0.00000007219(63) \sim 10^{-7}$



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$



$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$



$\rightarrow a_\mu^{\rm HLbyL} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

Experimental measurement of a_μ

Two experiments aim to reduce precision of a_μ^{exp} to 0.14 ppb

- **Muon g-2** at Fermilab (operative since 2017, results waited for Nov '20)
- **Muon g-2/EDM** at J-PARC (planned for $\geq$ 2020)

A muon in a $\perp$ magnetic field experiences two frequencies (here $\vec{\omega} \parallel \vec{B}$):

1. $\omega_C = \frac{eB}{m_\mu c \gamma}$ (circular precession)
2. $\omega_S = g_\mu \frac{eB}{2m_\mu c} + \frac{1-\gamma}{\gamma}\frac{eB}{m_\mu c}$ (spin precession)

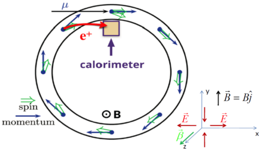
$\Rightarrow$ $\omega_a = \omega_S - \omega_C = a_\mu \frac{eB}{m_\mu c}$

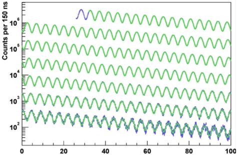
Also $\vec{E}$ contributes to $\vec{\omega}$:

$$\vec{\omega} = a_\mu \frac{e\vec{B}}{m_\mu} - a_\mu \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\frac{e\vec{B}}{m_\mu})\vec{\beta} + \underbrace{\left(-a_\mu + \frac{1}{\gamma^2+1}\right)}_{0 \text{ at } \gamma_{\text{magic}} \text{ (FNAL)}} \underbrace{\frac{\vec{\beta}\times\vec{E}}{c}}_{\vec{E}=0 \text{ (J-PARC)}} \frac{e}{m_\mu}$$

Muons decay preferentially in the spin direction: each detector will measure

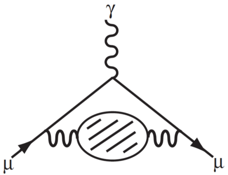
$$N(E,t) = N_0(E)e^{-t/\gamma\tau_\mu}\left[1 + A(E)\cos(\omega_a t + \phi)\right]$$





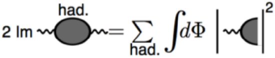
$$a_\mu^{\text{LO HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w\left(Q^2/m_\mu^2\right) \underbrace{\left(\Pi(Q^2) - \Pi(0)\right)}_{\hat{\Pi}(Q^2)}$$

$$w(r) = \pi \left(r + 2 - \sqrt{r(r+4)}\right)^2 / \sqrt{r(r+4)}$$

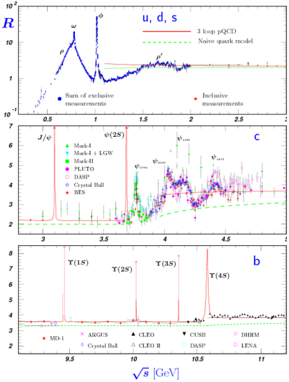


How to get $\hat{\Pi}(Q^2)$?

1. Dispersion relation to change $\hat{\Pi}(Q^2)$ with its imaginary part:
$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$
2. $\mathrm{Im}\Pi$ is related to $\sigma(\gamma \to \text{had})$ via optical theorem:



3. $\sigma(\gamma \to \text{had})$, or likewise the R-ratio
$$R^{\text{data}}(s) = \frac{\sigma(e^+e^- \to \text{had}, s)}{\sigma(e^+e^- \to e^+e^-)} = \frac{3s}{4\pi\alpha^2}\sigma(e^+e^- \to \gamma^* \to \text{had}, s)$$
can be extracted from experiments (BaBar, KLOE, NSK, BES-III,...) up to a certain E_{cut^2}
4. Over $E_{\text{cut}^2} \to R^{\text{pQCD}}(s)$.



Lattice QCD

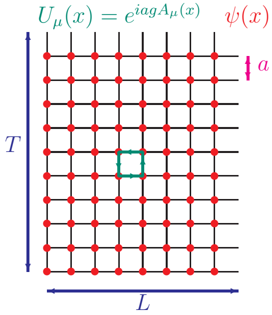
Recipe:

1. Lattice = euclidean tool $\Rightarrow t \to -ix_4$
2. Discretize spacetime on lattice Λ of size $L^3 \times T$ and spacing a
   $\Rightarrow$ IR and UV divergences are now regularized
3. Define discretized equivalents of continuum fields
   - $\phi(t, \vec{x}) \to \phi(x)$ with $x = a(n_1, n_2, n_3, n_4)$
   - $A_\mu(x) \to U_\mu(x) = P\{\exp \int_x^{x+a\hat{e}_\mu} dsA_\mu(s)\}$
4. Define lattice action so that $\mathcal{S}_{\text{lat}} \xrightarrow{a\to 0} \mathcal{S}_E$ Example: Wilson action

$$S_W = \frac{\beta}{2N} \sum_{x\in\Lambda,\mu\nu} \text{ReTr}\{U_{\mu\nu}(x)\} \xrightarrow{a\to 0} -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$$

$\text{Tr}\{U_{\mu\nu}(x)\} = \text{Tr}\{U_\mu(x)U_\nu(x + a\hat{e}_\mu)U_\mu^\dagger(x + a\hat{e}_\nu)U_\nu^\dagger(x)\}$ is the elementary plaquette.

- The equivalence holds iff $\beta = \frac{2N}{g^2}$
- Discretization procedure not unique
- Fermions are problematic (loss of chirality vs doubling problem)



Lattice QCD

5 The QFT partition function

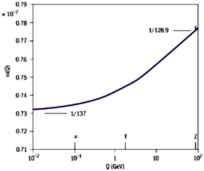
$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i[S_G + \int \bar{\psi} D[M]\psi]} = \int \mathcal{D}A_\mu \, \det(D[M]) e^{iS_G}$$

becomes on the lattice

$$\mathcal{Z} = \prod_{\rho,x} \int dU_\rho(x) \det(D_x[M]) e^{-\frac{\beta}{2N}\sum \mathrm{ReTr} U_{\mu\nu}}$$

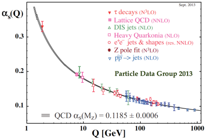
Looks like the partition function of a statistical system in the **canonical ensemble**

$\Rightarrow$ new tools from Statistical Mechanics, like stochastic methods (MC, ...) to perform numerical evaluations.



6 Asymptotic freedom implies $a \xrightarrow{g\to 0} 0$!

$$\begin{cases} \alpha_s(a) = \frac{g^2(a)}{4\pi} = -\frac{4\pi}{b_0 \log(a^2\Lambda^2)} \\ b_0 = 11 - \frac{2}{3} n_f \end{cases} \Rightarrow a(g) \sim \frac{1}{\Lambda} \exp\left(-\frac{8\pi^2}{b_0 g^2}\right)$$



7 Fix QCD parameters using $1 + n_f$ physical inputs.

8 Restore ∞-volume by extrapolation from simulations in different volumes.

Lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

1. From R-ratio we know that $w(Q)\Pi(Q)$ is peaked on $m_\mu/2 \sim 50\text{MeV} \Rightarrow$ NP regime
2. In Euclidean space the polarization tensor is

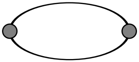
$$\begin{aligned}\Pi_{\mu\nu}(Q) &= \int d^4x e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle \quad \leftarrow \text{ measurable on the lattice} \\ &= \underbrace{(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\, \Pi(Q^2)}_{\text{O(4) inv. and current conservation}} \leftarrow \text{ what we need}\end{aligned}$$

with $J_\mu/e = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$

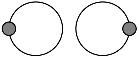
3. We define

$$\begin{aligned}C_L(t) &= \frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}} \langle J_i(x) J_i(0)\rangle \\ &= C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\mathrm{disc}}(t) = C_L^{I=0}(t) + C_L^{I=1}(t)\end{aligned}$$

where $C_L^{ud}(t), \ldots$ correspond to different Wick contractions:



quark-connected (qc)



quark-disconnected (qd)

Lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

4. Performing a Fourier transformation and subtracting $\Pi^L_{\mu\nu}(Q=0) \neq 0$, we get the connection between $\hat{\Pi}^f_L(Q^2)$ and $C_L(t)$:

$$\hat{\Pi}^f_L(Q^2) \equiv \Pi^f_L(Q^2) - \Pi^f_L(0) = \frac{1}{3}\sum_{i=1}^{3}\frac{\Pi^f_{ii,L}(0) - \Pi^f_{ii,L}(Q)}{Q^2} - \Pi^f_L(0) = 2a\sum_{t=0}^{T/2} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2}\right] \mathrm{Re}C^f_L(t)$$

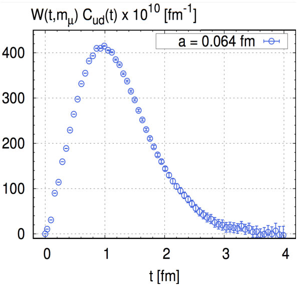
Note: $\Pi^L_{\mu\nu}(Q=0) \neq 0$ gives a FV contribution $\propto L^4 \exp(-EL/2)$

5. Therefore $a_{\ell,f}^{\text{LO-HVP}}$ from $C^f_L(t)$ is

$$\boxed{a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\max})} = \alpha^2 \int_0^{Q^2_{\max}} \frac{dQ^2}{Q^2}\, w\left(\frac{Q^2}{m^2_\mu}\right) \hat{\Pi}(Q^2)$$

$$\boxed{= \lim_{\substack{a \to 0 \\ L \to \infty \\ T \to \infty}} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m^2_\ell}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\max}/m^2_\ell)\, \mathrm{Re}C^f_L(t)}$$

$$W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$

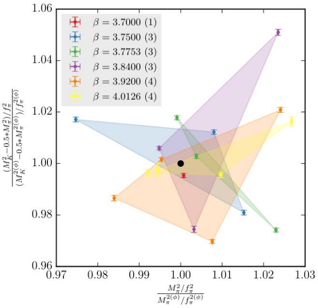


6. Finally, adding using pQCD for $Q > Q_{\max}$ (blue=measurable on the lattice)

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max}) + \gamma_\ell(Q_{\max})\,\hat{\Pi}^f(Q^2_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$$

Simulation details

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	64 $\times$ 48	904
3.7500	0.1191	96 $\times$ 56	2072
3.7753	0.1116	84 $\times$ 56	1907
3.8400	0.0952	96 $\times$ 64	3139
3.9200	0.0787	128 $\times$ 80	4296
4.0126	0.0640	144 $\times$ 96	6980



- 27 high-statistics simulations
- 4-stout staggered quarks
- $N_f = 2+1+1$ flavors
- m_{ud} and m_s chosen to bracket the physical point
- $m_c/m_s = 11.85$
- 6 gauge parameters β, i.e. a's: $0.134 \to 0.064\,\mathrm{fm}$
- $L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$
- Conserved EM current
- State-of-the-art techniques:
  - Low mode averaging (Giusti et al '04)
  - All mode averaging (Blum et al '13)
  - Solver truncation (Bali et al '09)
- Nearly 20,000 gauge configurations
- 10's of millions of measurements

Major challenges

1. **Scale determination :** a relative error in the lattice spacing propagates into about twice a relative error in the determination of a_μ.

   $\Rightarrow$ Severe requirements for scale setting variables:

   - Precisely determined on the lattice
   - Moderate quark dependence
   - Experimental value known to accuracy better than permil level

   We used M_Ω and ω_0.

2. **Noise reduction :** $C_L^{ud}(t)$ and $C_L^{disc}(t)$ become quite noisy for large t
   $\Rightarrow$ high statistical error

   Some solutions:

   - Lowest eigenmodes of the Dirac operators
   - Decrease noise by replacing $C_L^{ud}(t)$ by average of rigorous upper/lower bounds above $t_c = 4\,\mathrm{fm}$

   $$0 \leq C_L^{ud}(t) \leq C_L^{ud}(t_c)\, e^{-E_{2\pi}(t-t_c)}$$

5. **Infinite volume and continuum extrapolations :** a_μ is very sensitive to the lattice size L: the general rule $M_\pi L > 4$ is not satisfactory.

   Finite-volume analysis key points:

   - Evaluation of two-loop, finite volume, staggered chiral perturbation theory corrections to a_μ^{LO-HVP}
   - Lellouch-Lüscher-Gounaris-Sakurai model
   - Full lattice simulation at $L = 11\mathrm{fm}$

6. **QED and strong isospin breaking :** uquenched QCD in the isospin limit $m_u = m_d$ is not satisfactory for the desired level of precision
   $\Rightarrow$ we included all isospin breaking effects up to first order in isospin breaking parameters

   - $\delta m_l \equiv m_d - m_u$
   - The electric charge e_v^2, e_s^2, $e_v e_s$ where we separated sea and valence quark contributions.

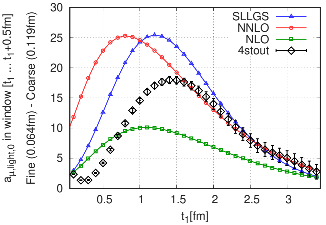
Taste corrections and finite volume corrections

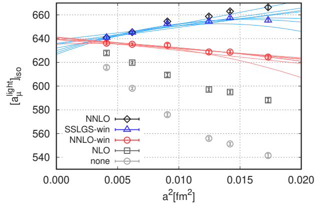
Long-distance discretization effects in $a_{\mu,ud}^{\text{LO-HVP}}$ due to taste violations (HPQCD '16)

Phenomenological models to solve the problem:

1. NNLO (2-loop) χPT (Aubin et al '19, BMWc '20)
2. Lellouch-Lüscher formalism w/ Gounaris-Sakurai model (LLGS) (Meyer '11, Francis '13, Giusti et al '18, BMWc '20)

- Reproduce observed discretization effects well
- Corrections vanish in continuum limit
- **Continuum extrapolations with NNLO and with SLLGS improvements are consistent and help reducing uncertainties**
- The two models can be used to evaluate FV correction that has to be added to the continuum extrapolation at reference volume. In addition, a lattice study with high statistics and $L = T = 11\text{fm}$ has been performed.





$$[a_\mu^{\text{light}}]_0(L,T,a) \quad\rightarrow\quad [a_\mu^{\text{light}}]_0(L,T,a) + \underbrace{\frac{10}{9}\Big[[a_\mu^{\text{ChPT}}]_0(L,T,a) - [a_\mu^{\text{SChPT}}]_0(L,T,a)\Big]}_{\text{correct taste artefacts}}$$

$$+\frac{10}{9}\Big[[a_\mu^{\text{ChPT}}]_0(L_{\text{ref}},T_{\text{ref}},a) - [a_\mu^{\text{ChPT}}]_0(L,T,a)\Big]$$

Including isospin breaking on the lattice

$$S_{\rm QCD+QED} = S_{\rm QCD}^{\rm iso} + \frac{1}{2}\delta m \int (\bar{d}d - \bar{u}u) + ie \int A_\mu j_\mu , \qquad j_\mu = \bar{q} Q \gamma_\mu q, \qquad \delta m = m_d - m_u$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle j_\mu j_\nu \rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and QCD scale

(1) operator insertion method (RM123 '12, '13, ...)

$$\langle \mathcal{O} \rangle_{\rm QCD+QED} = \langle \mathcal{O}_{\rm Wick} \rangle_{G_\mu}^{\rm iso} - \frac{\delta m}{2} \langle [\mathcal{O} \int (\bar{d}d - \bar{u}u)]_{\rm Wick} \rangle_{G_\mu}^{\rm iso} - \frac{e^2}{2} \langle [\mathcal{O} \int_{xy} j_\mu(x) D_{\mu\nu}(x-y) j_\nu(y)]_{\rm Wick} \rangle_{G_\mu}^{\rm iso}$$
$$+ e^2 \langle \langle \left[\mathcal{O} \partial_e \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \int_x j_\mu(x) A_\mu(x) - \frac{1}{2} \mathcal{O} \partial_e^2 \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \right]_{\rm Wick} \rangle_{A_\mu} \rangle_{G_\mu}^{\rm iso}$$

(2) direct method (Eichten et al '97, BMWc '14, ...)

Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

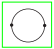
(3) combinations of (1) & (2) (BMWc '20)

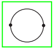
We include all $O(e^2)$ and $O(\delta m)$ effects

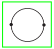
For valence e^2 effects use easier (2), and for δm and e^2 sea effects, (1)

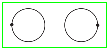
Summary of contributions to $a_\mu^{\text{LO-HVP}}$

Isospin symmetric

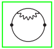
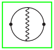

connected light
634.6(2.7)(3.7)

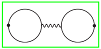
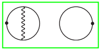

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)

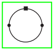

disconnected
-13.15(1.28)(1.29)

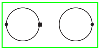
QED isospin-breaking: valence



connected -1.27(40)(33)

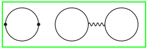
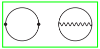


disconnected -0.55(15)(11)

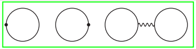
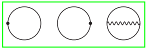
Strong isospin-breaking


connected
6.59(63)(53)


disconnected
-4.63(54)(69)

QED isospin-breaking: sea



connected 0.42(20)(19)

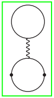


disconnected -0.047(33)(23)

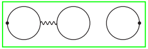
Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0095(86)(99)


disconnected 0.011(24)(14)

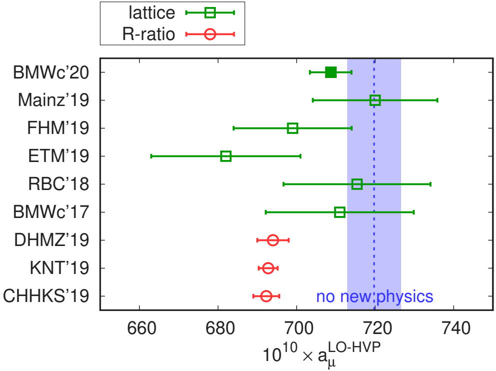
Finite-size effects

isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 708.7(2.8)_{\text{stat}}(4.5)_{\text{sys}}[5.3]_{\text{tot}}$$

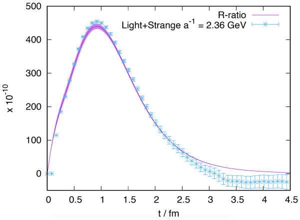
Comparison with phenomenology and with other lattice calculations



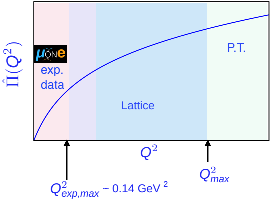
- Consistent with other lattice results
- Total uncertainty is $\sim \div 4$, comparable to R-ratio
- Consistent with BNL experiment ("no new physics" scenario)
- 2.2σ larger than DHMZ'19, and 2.7σ larger than KNT'19 ?

What next?

- FNAL E989 should put out first results very soon (Nov)
- This result be confirmed by other lattice groups
- Must be understood why we don't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18)
- Important to pursue $e^+e^- \rightarrow$ hadrons measurements (CMD-3, Belle, ...)
- $\mu e \rightarrow \mu e$ experiment MuOne very important for experimental crosscheck and complementarity with LQCD



(RBC/UKQCD '18)



(Marinkovic et al '19)