Stochastic « Inflation » in the phase space

Julien Grain & Vincent Vennin

Based on :
- Grain, Vennin, 2017, JCAP, 1705, 045 ; 2020, JCAP, 2002, 022 ; & arXiv : 2005.04222
- Grain, Vennin, many papers in prep. (~140 pages in notes and drafts)

Outline

- Going to the phase space

  motivation, Langevin equations in phase space

- Exemple of a test field

  Diffusion away from slow-roll, stochastic anisotropies

  Grain, Vennin, 2017, JCAP, 1705, 045 ; 2020, JCAP, 2002, 022 ; & arXiv : 2005.04222

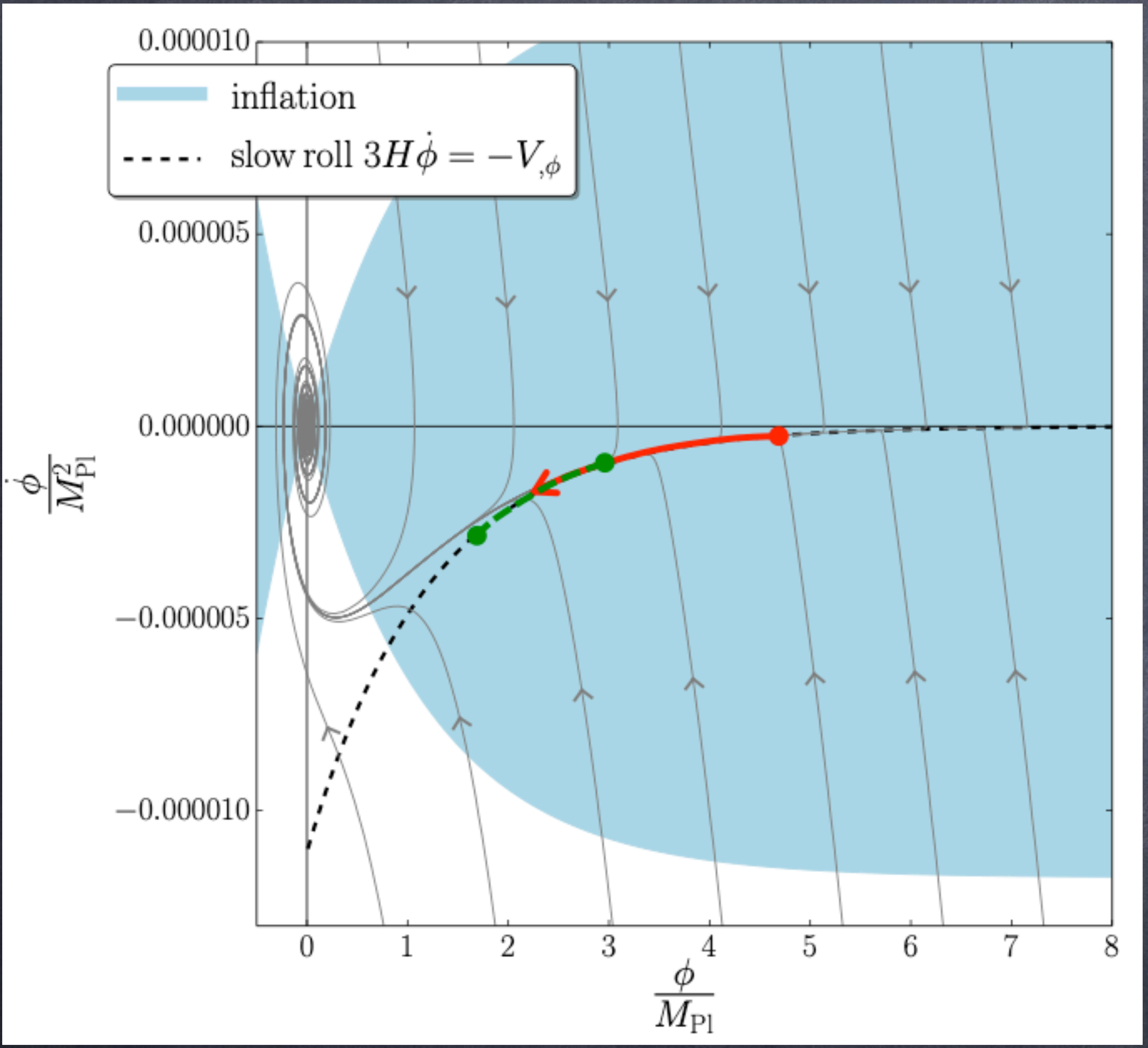
- Separate universe & gauge-correction

  Readdress Pattison et al. 2019 from the phase space perspective

  Work in progress: Grain, Vennin, many papers in prep. (~140 pages in drafts)

Slow-roll inflation

Slow-roll approximation: $3H\dot{\phi} + \frac{dV}{d\phi} = 0$



- SR is an attractor
- SR simplifies a 2D dynamics into a 1D dynamics
- SR attractor can be generalized to a test scalar field in quasi-dS background

Stochastic diffusion: $3H\dot{\bar{\phi}} + \frac{dV(\bar{\phi})}{d\bar{\phi}} = \tilde{\xi}_\sigma(t)$ ← White noise

Drift evolution and diffusion are 1-dimensional

Beyond slow-roll inflation

Ultra-slow-roll: $V(\phi) = V_0$

-> Field momentum is conserved

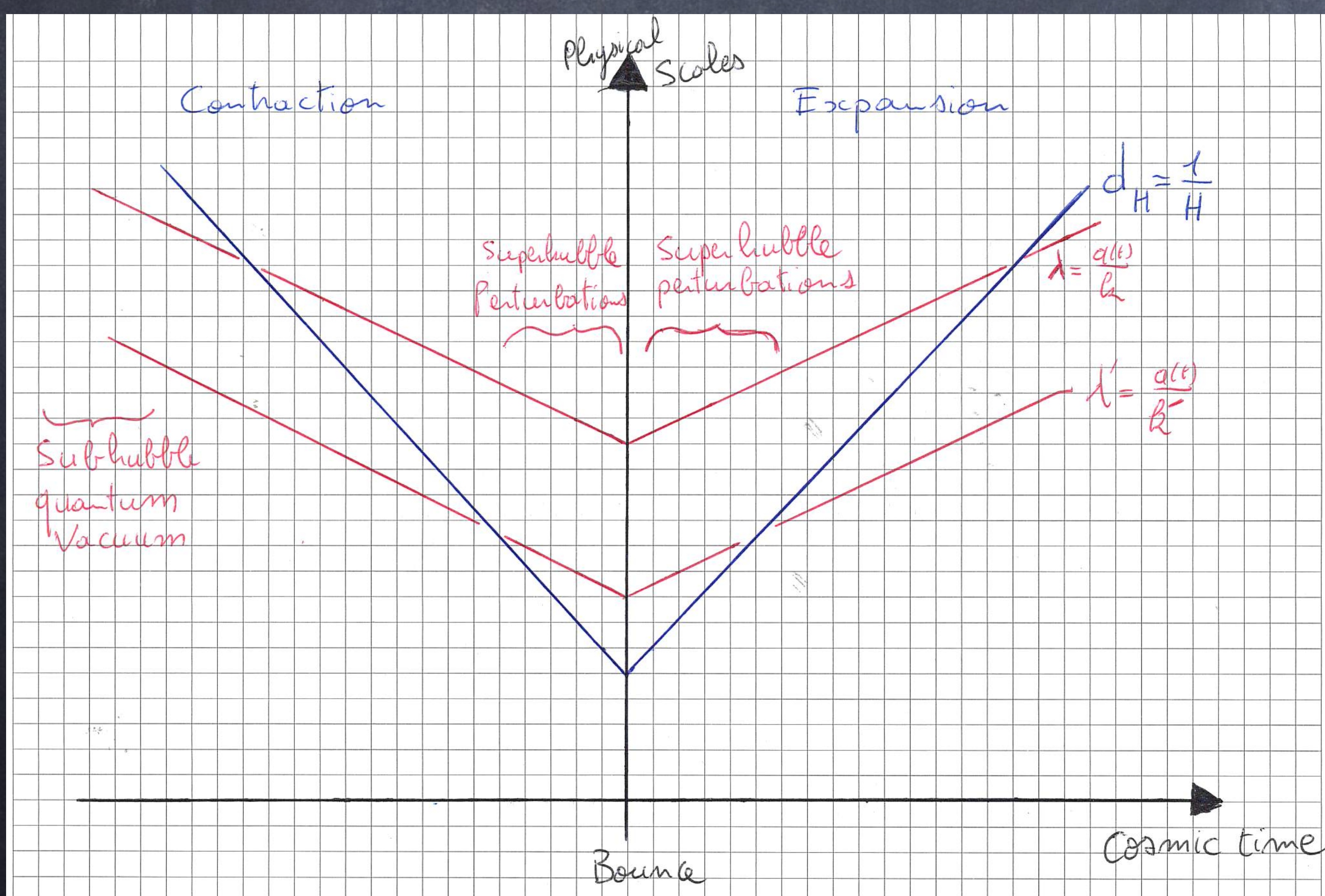
$$\phi(t) = \phi_0 + \pi_0 \int_{t_0}^{t} \frac{dt'}{a^3(t')}$$

$$\pi(t) = \pi_0$$

For instance
- Inoue et al., PLB, 2002, 524, 15
- Kinney, PRD, 2005, 72, 023515
- Pattison et al. JCAP, 2018, 1808, 048
- Biagetti et al., JCAP, 2018, 1807, 032
- Ezquiaga et al., JCAP, 2020, 2003, 029

Bouncing cosmologies:

-> inflow of Q-fluctuations to large-scale classical perturbations



For instance
- Khoury et al., PRD, 2001, 64, 123522
- Barrau et al., CQG, 2014, 31, 053001
- Brandenberger & Peter, Found. Phys, 2017, 47, 797
- Agullo & Singh, arXiv:1612.0123

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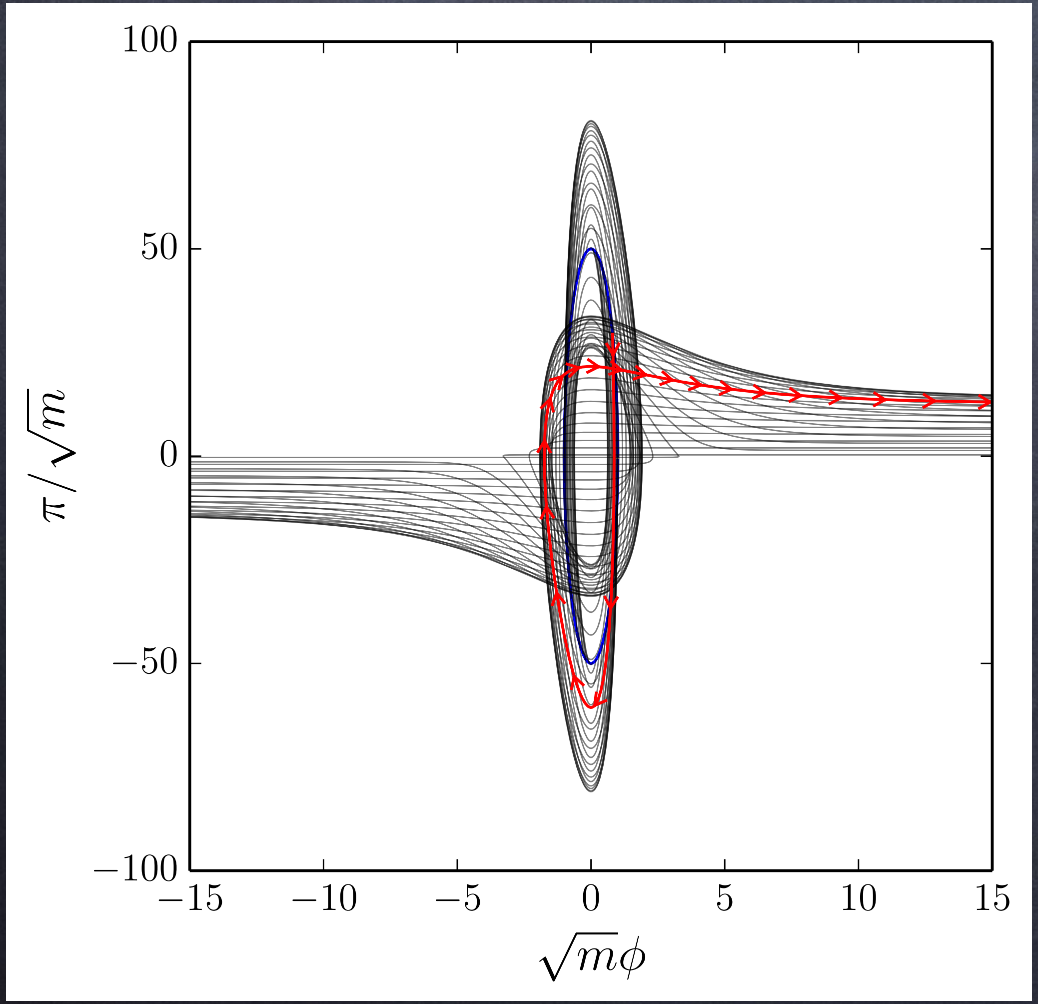
Bouncing cosmologies:

-> test free scalar-field in contracting universe (w=0)



- m>H: oscillatory regime in P.S.
- m<H: friction dominated regime à la USR

$$\phi(t) \simeq \phi_m + \pi_m \int_{t_m}^{t} \frac{dt'}{a^3(t')}$$

$$\pi(t) \simeq \pi_m$$

For « inflaton » field, see example in LQC:
- Linsefors & Barrau, PRD, 2013, 123509
- Bolliet et al., PRD, 2015, 91, 084035
- Schander et al., PRD, 2016, 93, 023531

Going to the phase space

Beyond slow-roll: 2-dimensional dynamics

- Full Klein-Gordon Equation (1 second order ODE)
- Hamiltonian formalism (2 coupled first order ODE's)

- Langevin equations are more easily formulated as 1st order system of ODE's

—> Fokker-Planck equation and PDF in the phase space, use of canonical transformation

- Noise is of quantum origin

—> quantization (and classicalization) is more « naturally » done using Hamiltonian formalism

- SR corresponds to a specific direction in the phase space

—> « Is there diffusion away from SR ? » becomes « is the noise strongly pointing away from SR in the phase space ? »

- Nakao, Nambu, Sasaki, 1988, Prog. Th. Phys, 80, 1041
- Kandrup, 1989, PRD, 39, 2245
- Habib, 1992, PRD, 46, 2408
- Grain, Vennin, 2017, JCAP, 1705, 045

Stochastic inflation in P.S.

1. Split the field in long- & small-wavelength $\Phi_i \equiv \begin{pmatrix} \phi \\ \pi \end{pmatrix}$

$$\Phi = \bar{\Phi} + \Phi_Q \text{ with } \Phi_Q = \int d^3k \Theta(k/k_\sigma - 1) \left[\hat{a}_{\vec{k}} \Phi_k(t) e^{i\vec{k}\vec{x}} + \text{c.c.} \right]$$

Solution of Hamilton EoM

2. Plug into Hamilton EoM –> Langevin equations in phase space

Linearize in ϕ_Q ; neglect gradients for $\bar{\phi}$

Stochastic large-scale dynamics : $\frac{d}{d\eta} \begin{pmatrix} \bar{\phi} \\ \bar{\pi} \end{pmatrix} = \begin{pmatrix} \bar{\pi}/a^2 \\ -a^4 V_{,\bar{\phi}}(\bar{\phi}) \end{pmatrix} + \xi$ with $\xi \equiv \begin{pmatrix} \xi_\phi \\ \xi_\pi \end{pmatrix} \propto \frac{dk_\sigma^3}{d\eta} \left[\hat{a}_{k_\sigma} \Phi_{k_\sigma}(\eta) e^{i\vec{k}\vec{x}} + \text{c.c.} \right]$

Small-wevelength dynamics : $\frac{d}{d\eta} \begin{pmatrix} \phi_k \\ \pi_k \end{pmatrix} = \begin{pmatrix} \bar{\pi}_k/a^2 \\ -\left[a^2 k^2 + a^4 V_{,\bar{\phi},\bar{\phi}} \right] \phi_k \end{pmatrix}$

3. Noise properties

Gaussian and classical stochastic 2D white noise

$$\langle 0 | \xi_i(\eta_1) \xi_j(\eta_2) | 0 \rangle \propto \frac{dk_\sigma^3}{d\eta} \Phi_{i,k_\sigma} \Phi^\star_{j,k_\sigma} \delta(\eta_1 - \eta_2)$$

$$\xi\xi^\dagger \propto \frac{1}{2} \left(|\phi_{k_\sigma}|^2 + |\pi_{k_\sigma}|^2 \right) \boldsymbol{I} + \frac{1}{2} \left(|\phi_{k_\sigma}|^2 - |\pi_{k_\sigma}|^2 \right) \boldsymbol{J}_z + \text{Re}\left[\phi_{k_\sigma} \pi^\star_{k_\sigma} \right] \boldsymbol{J}_x + \boldsymbol{J}_y$$

Quantum commutator

Power spectra

Stochastic inflation in P.S.

A concrete example : test field with $V(\phi) = \frac{1}{2}m^2\phi^2$

$$\frac{d}{d\eta}\begin{pmatrix}\bar{\phi}\\ \bar{\pi}\end{pmatrix} = \begin{pmatrix}0 & 1/a^2\\ -a^4m^2 & 0\end{pmatrix}\begin{pmatrix}\bar{\phi}\\ \bar{\pi}\end{pmatrix} + \xi$$

Drift:

--> rapidly aligned with SR

Diffusion: perturbations mass = field's mass

Large scales: gradients negligible

--> perturbations evolve like the background field: they align with SR attractor

Both drift evolution and noises (hence diffusion) are aligned with SR

--> Slow-roll is also stochastic attractor

For details: Grain, Vennin, 2017, JCAP, 1705, 045

Stochastic anisotropies

Consider a Bianchi I space-time: $ds^2 = -dt^2 + a^2(t)\exp[2\beta_i(t)]\delta_{ij}dx^i dx^j$

$$H^2 = \frac{\rho}{3M_{\rm Pl}^2} + \frac{1}{2}\sigma_{ij}\sigma^{ij}$$

$$\dot{H} = -\frac{\rho + P}{2M_{\rm Pl}^2} - \frac{1}{2}\sigma_{ij}\sigma^{ij}$$

$$(\sigma^i_j)^{\cdot} + 3H\sigma^i_j = \frac{\Pi^i_j}{M_{\rm Pl}^2}$$

Evolution of the shear sourced by the anisotropic stress

$$\dot{\rho} + 3H(\rho + P) = -\sigma_{ij}\Pi^{ij}$$

Solve for the shear: $\sigma^i_j(t) = \sigma^i_j(t_o)\left[\frac{a_o}{a(t)}\right]^3 + \frac{1}{M_{\rm Pl}^2}\int_{t_o}^t \left[\frac{a(t')}{a(t)}\right]^3 \Pi^i_j(t')dt'$

What about contributions from anisotropic stress ?

Shear as $1/a^6$ in Friedmann

--> Singularity in contraction

Stochastic massless scalar field in FLRW: $\Pi_{ij} = \partial_i\phi\partial_j - \frac{1}{3}\left(\gamma^{mn}\partial_m\phi\partial_n\phi\right)\gamma_{ij}$

$$\boldsymbol{\Phi}(t) := \begin{pmatrix} \phi \\ \pi_\phi \end{pmatrix} = \boldsymbol{G}(t, t_o)\boldsymbol{\Phi}_o + \int_{t_0}^t \boldsymbol{G}(t,s)\boldsymbol{\xi}(s)ds$$

Homogeneous bckg field $\Pi^i_j = 0$

Isotropic 2-pt correlation: $\left\langle \widehat{\Pi}^i_j(t) \right\rangle = 0$

Anisotropic fluctuations (4-pt): $\left\langle \widehat{\Pi}^i_j(t)\Pi^j_i(t') \right\rangle \neq 0$

Stochastic anisotropies

Unvoidable stochastic shear from quantum fluctuations:
-> Connected part of 4-point correlation

$$\rho_\sigma := \frac{1}{2}\left\langle \sigma_{ij}\sigma^{ij}\right\rangle = \frac{1}{2M_{\rm Pl}^2}\int^t dt_1 \int^t dt_2 \left[\frac{a(t_1)a(t_2)}{a^2(t)}\right]^3 \left\langle \Pi^i_j(t_1)\Pi^j_i(t_2)\right\rangle$$

Computed from the derivatives of the field 2-pt function

Contraction driven by a perfect fluid with arbitrary w:

$$\frac{\rho_\sigma}{\rho_{\rm tot}} = \epsilon^{4\frac{1+9w}{1+3w}} f(w)\left(\frac{H}{M_{\rm Pl}^2}\right)^4\left[\frac{1}{5+27w} - \left(\frac{a}{a_o}\right)^{1+9w} + \frac{3+9w}{(1-9w)^2}\left(\frac{a}{a_o}\right)^{2(1+9w)} + \cdots\right]$$

-> $H<M_{Pl}$ in bouncing cosmology

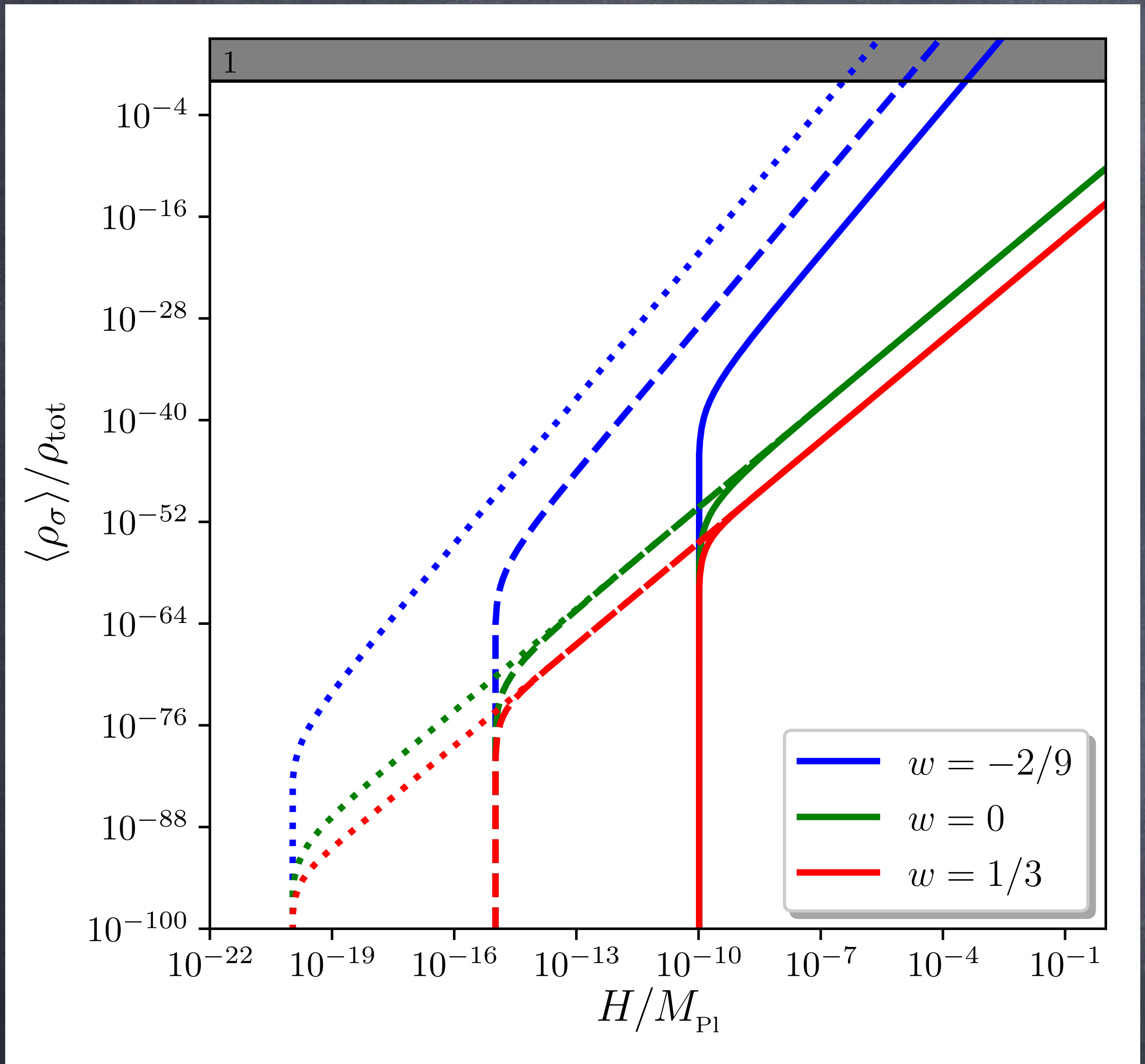
$w>-1/9$: shear never exceeds the Planck scale

$w<-1/9$: shear increase above M_{Pl} because fluctuations spectrum is too red

For details: Grain, Vennin, arXiv : 2005.04222

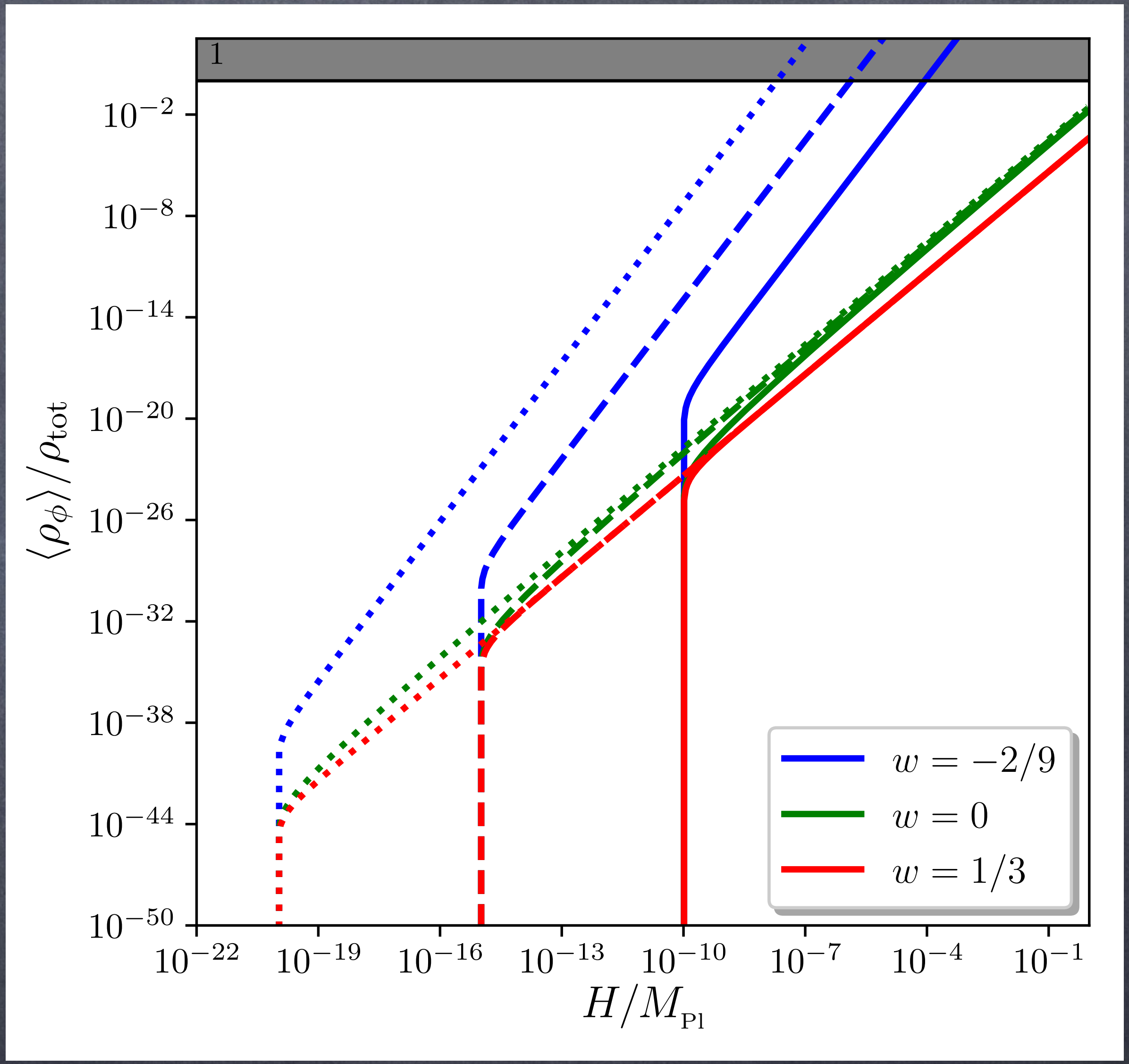
Stochastic anisotropies

Stochastic shear energy density



-> Stochastic instability if w<-1/9

Stochastic kinetic energy density



-> Stochastic instability if w<0

For details: Grain, Vennin, arXiv:2005.04222

Separate Universe in P.S.

Pattison et al. JCAP, 2019, 1907, 031

Conditions for using the stochastic formalism:

- Quantum-to-classical transition

  Does not assume slow-roll, always a 2-mode squeezed state

- Validity of the separate universe

  superhubble FLRW patches = cosmological perturbations at superhubble scales
  -> identify local time and expansion to time and expansion of FLRW patch

- Gauge-corrections to the noise

  Langevin equation in synchronous gauge but noise in other gauges
  -> Gauge corrections from e.g. spatially-flat to synchronous

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Separate Universe in P.S.

Two different perturbative expansions about FLRW

Cosmological perturbations:

Inhomogeneous and anisotropic

Lagrange multipliers: 2

$$N(\tau) + \int \frac{d^3k}{(2\pi)^{3/2}} \delta N(\vec{k},\tau) e^{i\vec{k}\vec{x}}$$

$$\int \frac{d^3k}{(2\pi)^{3/2}} \delta N_1(\vec{k},\tau) \left(\frac{ik^i}{k}\right) e^{i\vec{k}\vec{x}}$$

Scalar field: 2*1

$$\phi(\tau) + \int \frac{d^3k}{(2\pi)^{3/2}} \delta\phi(\vec{k},\tau) e^{i\vec{k}\vec{x}}$$

$$\pi_\phi(\tau) + \int \frac{d^3k}{(2\pi)^{3/2}} \delta\pi_\phi(\vec{k},\tau) e^{i\vec{k}\vec{x}}$$

Gravitational field: 2*2

$$v^{2/3}(\tau)\delta_{ij} + \int \frac{d^3k}{(2\pi)^{3/2}} \left[\delta\gamma_1(\vec{k},\tau)M^1_{ij} + \delta\gamma_2(\vec{k},\tau)M^2_{ij}\right] e^{i\vec{k}\vec{x}}$$

$$\frac{1}{2}v^{1/3}\theta(\tau)\delta^{ij} + \int \frac{d^3k}{(2\pi)^{3/2}} \left[\delta\pi_1(\vec{k},\tau)M^{ij}_1 + \delta\pi_2(\vec{k},\tau)M^{ij}_2\right] e^{i\vec{k}\vec{x}}$$

Separate Universe:

homogeneous and isotropic

Lagrange multipliers: 1

$$N(\tau) + \widetilde{\delta N}(\tau)$$

Scalar field: 2*1

$$\phi(\tau) + \widetilde{\delta\phi}(\tau)$$

$$\pi_\phi(\tau) + \widetilde{\delta\pi_\phi}(\tau)$$

Gravitational field: 2*1

$$v^{2/3}(\tau)\delta_{ij} + \widetilde{\delta\gamma_1}(\tau)M^1_{ij}$$

$$\frac{1}{2}v^{1/3}\theta(\tau)\delta^{ij} + \widetilde{\delta\pi_1}(\tau)M^{ij}_1$$

Separate Universe in P.S.

Two different perturbative expansions about FLRW

Cosmological perturbations:

$$C[N+\delta N, \delta N^i] = NS^{(0)} + \int d^3x \left(\delta N \mathcal{S}^{(1)} + \delta N^i \mathcal{D}_i^{(1)} + N\mathcal{S}^{(2)}\right)$$

Separate Universe:

$$C[N+\widetilde{\delta N}, 0] = NS^{(0)} + \widetilde{\delta N} S^{(1)} + NS^{(2)}$$

Comparison at super-Hubble scales, i.e. neglecting $(k/a)^2$

-> Cosmo. pert. constraints become homogeneous and isotropic

$$\mathcal{S}^{(1)} \simeq S^{(1)} + \mathcal{O}(k/a)^2[\delta\gamma_1 + \delta\gamma_2]$$

$$\mathcal{S}^{(2)} \simeq S^{(2)} + f(\delta\gamma_2, \delta\pi_2) + \mathcal{O}(k/a)^2[\delta\phi^2 + \delta\gamma_1^2 + \delta\gamma_2^2 + \delta\gamma_1\delta\gamma_2]$$

Anisotropic d.o.f. decouples from isotropic d.o.f. at large scales

-> Full dynamics is matched providing the gauges are matched

$$\delta N \simeq \widetilde{\delta N} + \mathcal{O}(k/a)^2[\delta z]$$ ← Matching of proper time

$$k\delta N_1 \simeq \mathcal{O}(k/a)^2[\delta z]$$ ← Matching of expansion rates

Validity of separate universe: question of gauge-matching

Separate Universe in P.S.

Is there an analog gauge-fixing in the separate universe?

Spatially-flat gauge: $\boxed{\delta\gamma_1 = 0 = \delta\gamma_2}$

$$\delta N \propto f(\tau)(\delta\pi_1 + \sqrt{2}\delta\pi_2)$$
$$k\delta N_1 = h(\tau)\delta\pi_2$$

$\boxed{\widetilde{\delta\gamma_1} = 0}$

$$\widetilde{\delta N} = \tilde{f}(\tau)\widetilde{\delta\gamma_1} + \tilde{g}(\tau)\widetilde{\delta\pi_1}$$
$$\widetilde{\delta N_1} = 0$$

Newtonian gauge: $\boxed{\delta\gamma_2 = 0 = \delta\pi_2}$

$$\delta N \propto f(\tau)\delta\gamma_1$$
$$k\delta N_1 = h(\tau)\delta\pi_2 = 0$$

No counterpart in separate universe

$$\widetilde{\delta N} = ?$$
$$\widetilde{\delta N_1} = 0$$

Synchronous gauge: $\boxed{\delta N = 0 = k\delta N_1}$ $\boxed{\widetilde{\delta N} = 0 = \widetilde{\delta N_1}}$

- Gauges with anisotropic pert. cannot be generated in the sep. univ.
- Synchronous gauge works -> Sep. Univ. valid for stochastic formalism

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Pattison et al. JCAP, 2019, 1907, 031

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Separate Universe in P.S.

- Stochastic formalism: Noise in the synchronous gauge
- Scalar-field perturbations: spatially-flat gauge

Gauge parameters from S.-F. gauge to the synchronous gauge

$$0 = \delta g_{00}^{\rm synch} = \delta g_{00} - 2N\dot{N}\xi^0 - 2N^2\dot{\xi}^0, \qquad \delta g_{00} = 2N\,\delta N$$

with

$$0 = \delta g_{0i}^{\rm synch} = \delta g_{0i} - N^2\partial_i\xi^0 + v^{2/3}\delta_{ij}\dot{\xi}^j \qquad \delta g_{0i} = v^{2/3}\delta_{ij}\,\delta N^j$$

Express lapse and shift in S.-F. as functions of field perturbations

$$\delta N^{\rm SF} = \frac{-N\pi_\phi}{v\theta}\delta\phi^{\rm SF}$$

$$k\delta N_1^{\rm SF} = N\left[\frac{3}{2M_{\rm Pl}^2}\frac{\pi_\phi}{v} - \frac{V,\phi}{\theta}\right]\delta\phi^{\rm SF} - \frac{N\pi_\phi}{v^2\theta}\delta\pi_\phi^{\rm SF}$$

Gauge transformation of the scalar-field perturbations

$$\delta\phi^{\rm synch} = \delta\phi^{\rm SF} + \left\{\delta\phi^{\rm SF}, \int d^3x\xi^0\mathcal{S}^{(1)}\right\} + \left\{\delta\phi^{\rm SF}, \int d^3x\xi^i\mathcal{D}_i^{(1)}\right\} = \delta\phi^{\rm SF} + \frac{\pi_\phi}{v}\xi^0$$

$$\delta\pi_\phi^{\rm synch} = \delta\pi_\phi^{\rm SF} + \left\{\delta\pi_\phi^{\rm SF}, \int d^3x\xi^0\mathcal{S}^{(1)}\right\} + \left\{\delta\pi_\phi^{\rm SF}, \int d^3x\xi^i\mathcal{D}_i^{(1)}\right\} = \delta\pi_\phi^{\rm SF} - vV_{,\phi}\xi^0 + \pi_\phi\partial_j\xi^j$$

Separate Universe in P.S.

Noise is given in the synchronous gauge: $\xi_\phi \propto \frac{dk_\sigma^3}{dt}\delta\phi^{\rm synch}(k_\sigma)$

$$\xi_{\pi_\phi} \propto \frac{dk_\sigma^3}{dt}\delta\pi_\phi^{\rm synch}(k_\sigma)$$

- Scalar-field perturbations in the synchronous gauge (N=1):

$$\delta\phi^{\rm synch}(k) = \delta\phi^{\rm SF}(k) - \dot\phi\int^t dt'\left[\frac{\dot\phi}{2M_{\rm Pl}^2 H}\right]\delta\phi^{\rm SF}(k)$$

$$\begin{aligned}\delta\pi_\phi^{\rm synch}(k) =&\,\delta\pi_\phi^{\rm SF}(k) + a^3 V_{,\phi}\int^t dt'\left[\frac{\dot\phi}{2M_{\rm Pl}^2 H}\right]\delta\phi^{\rm SF}(k)\\ &- a^3\dot\phi k^2\int^t\frac{dt'}{a^2}\left\{\int^{t'} dt''\left[\frac{\dot\phi}{2M_{\rm Pl}^2 H}\right]\delta\phi^{\rm SF}(k)\right\}\\ &+ a^3\dot\phi\int^t\frac{dt'}{2M_{\rm Pl}^2 a^2}\left[\left(3\dot\phi + \frac{V_{,\phi}}{H}\right)\delta\phi^{\rm SF}(k) + \frac{\dot\phi}{a^3 H}\delta\pi_\phi^{\rm synch}(k)\right]\end{aligned}$$

-> Gauge-corrections are proportional to $\sqrt{\epsilon_1}$ and ϵ_2

-> Comparison with Pattison et al. 2019 needs to be done

Next steps

- Application to U.S.R. and bounce
  - —> Gauge-corrections in the momentum direction
  - —> Phase-space alignments of the noise
  - —> Stochastic anisotropies for « contracton » field

- Test fields to explore stochastic contraction
  - —> Noise alignment in the absence of attractor ?
  - —> Non-Bunch-Davies vacuum states ?
  - —> Scale of « classicality » vs. horizon scale

- Role of anisotropic modes ; formal aspects
  - —> Gauge-fixing in separate universe vs. cosmo. pert.
  - —> Is flat FLRW the right separate universe ?
    - Bianchi I to capture the anisotropic modes
    - Close/open FLRW to capture bits on inhomogeneities