Stochastic Inflation: Primordial Black Hole Production and Ultra-Slow Roll

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Outline

- Inflation
- Characteristic function formalism
- Application to primordial black holes
- Stochastic ultra-slow-roll inflation
- Summary

Inflation

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Often work in the **slow-roll (SR) approximation**, which takes

$$\epsilon_{i+1} \qquad \frac{1}{\epsilon_i}\frac{\mathrm{d}\epsilon_i}{\mathrm{d}N} \qquad 1\,,$$

where $\epsilon_0 = H_{\mathrm{in}}/H$, and $\mathrm{d}N = H\mathrm{d}t$ is the number of e-folds.

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where $\epsilon_0 = H_{\rm in}/H$, and $\mathrm{d}N = H\mathrm{d}t$ is the number of e-folds.
In this case, the eom simplifies to

$$\dot{\phi}_{\rm SR} \;' \quad \frac{V^0(\phi)}{3H}\,.$$

Stochastic Formalism

Stochastic inflation (Starobinsky, 1986) treats the quantum fluctuations as white noise, ξ.

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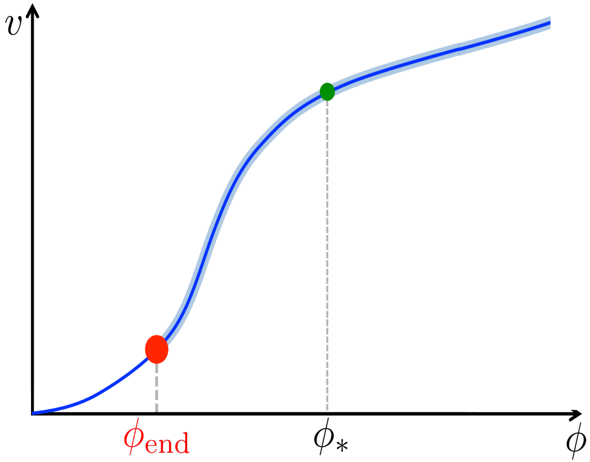
Then, in SR, ϕ is described by a Langevin equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = \quad \frac{V^{\prime}}{3H^2} + \frac{H}{2\pi}\xi\left(N\right) ,$$

where $\langle\xi\left(N\right)\rangle = 0$ and $\langle\xi\left(N\right)\xi\left(N^{\prime}\right)\rangle = \delta\left(N \quad N^{\prime}\right)$, $k < aH$ and $N = \int H\mathrm{d}t$.

Inflaton evolves under Langevin equation until ϕ reaches ϕ_{end} where inflation ends.



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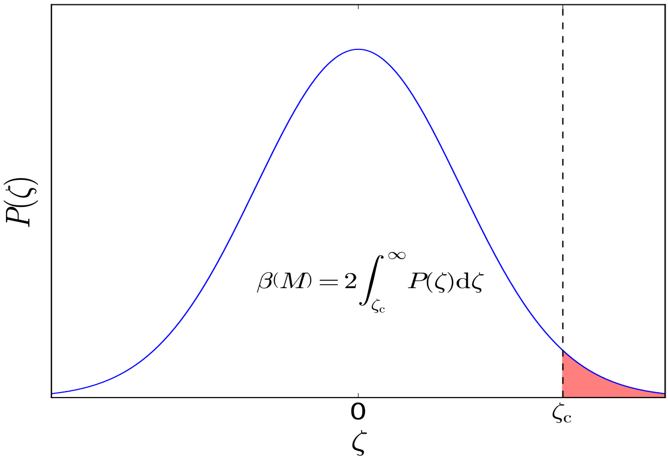
- Large density fluctuations during inflation can collapse to form PBHs.
- Such large fluctuations need a non-perturbative approach - the δN formalism.
- We use stochastic-δN to study how likely PBHs are to form (Pattison et al, 1707.00537).
- Number of PBHs formed is found from integrating the probability distribution of curvature (or density) perturbations

Gaussian Example

Typically assumed ζ has Gaussian distribution.



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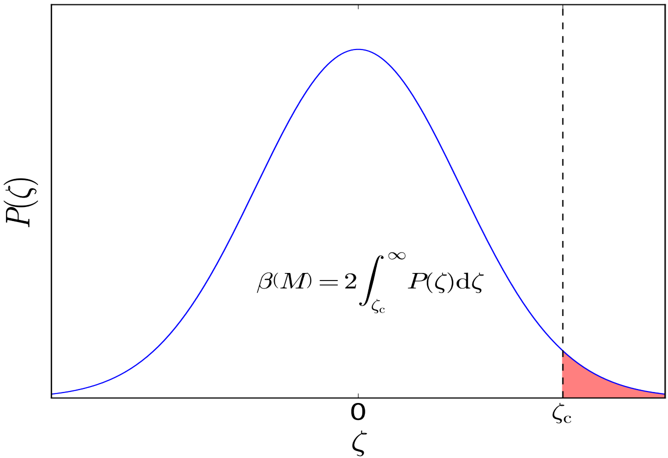

Let's not assume this...

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We set $f_n(\phi) = \langle \mathcal{N}^n(\phi) \rangle$ and construct the characteristic function $\chi_{\mathcal{N}}(t, \phi)$ as

$$
\begin{aligned}
\chi_{\mathcal{N}}(t, \phi) &= \left\langle e^{it\mathcal{N}(\phi)} \right\rangle \\
&= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} f_n(\phi) \,.
\end{aligned}
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$$\chi_N(t,\phi) = \left\langle e^{itN(\phi)} \right\rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} f_n(\phi)\,.$$

χ_N is related to the PDF $P(\delta N, \phi)$ by

$$P(\delta N, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it[\delta N + \langle N \rangle(\phi)]} \chi_N(t,\phi)\, \mathrm{d}t\,,$$

where $\delta N = N - \langle N \rangle = \zeta$ is the curvature perturbation.

We define the dimensionless potential

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We can derive (building on Vennin et al, 1506.04732) a differential equation for χ_N given by

$$\left[\frac{\partial^2}{\partial\phi^2} \quad \frac{v'}{v^2}\frac{\partial}{\partial\phi} + \frac{it}{vM_{\mathrm{Pl}}^2}\right]\chi_N(t,\phi) = 0\,.$$

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$$\left[\frac{\partial^2}{\partial\phi^2} \quad \frac{v^{0}}{v^2}\frac{\partial}{\partial\phi} + \frac{it}{v M_{\mathrm{Pl}}^2}\right]\chi_N\,(t,\phi) = 0\,.$$

This means we need to solve a hierarchy of uncoupled differential equations, to be solved at fixed t.

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As a toy model, let's take the potential

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We generally do *not* get a Gaussian solution.

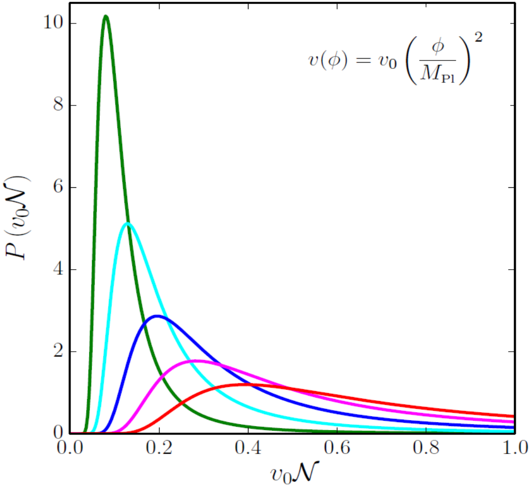

Figure 1: Plot of the PDF of N against N.

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Application to Primordial Black Holes (PBHs)

If $\zeta > \zeta_{\rm c}$, collapse to form PBHs

The number of PBHs produced is then calculated from the probability distribution $P(\delta\mathcal{N}, \phi)$ of these large perturbations using

$$\beta\left[M\left(\phi\right)\right] = 2\int_{\zeta_{\rm c}}^{\mathit{1}} P\left(\delta\mathcal{N}, \phi\right)\mathrm{d}\delta\mathcal{N}\ .$$

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$$\beta[M(\phi)] = 2 \int_{\zeta_c}^{\infty} P(\delta \mathcal{N}, \phi) \, \mathrm{d}\delta \mathcal{N} \, .$$

This gives the mass fraction of the universe contained in PBHs

Gaussian Example

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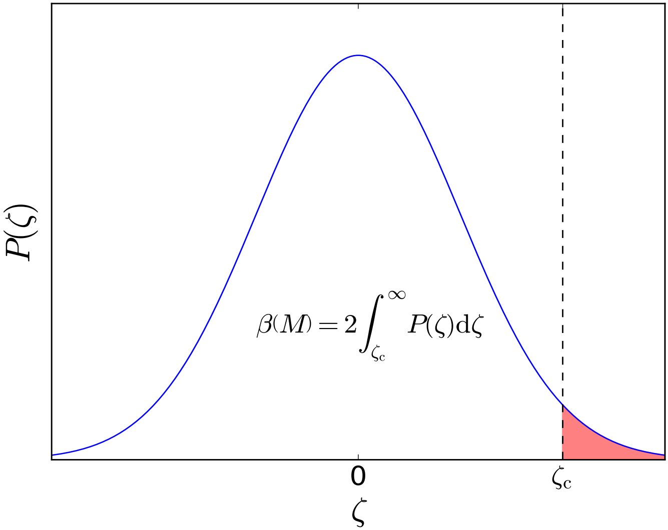
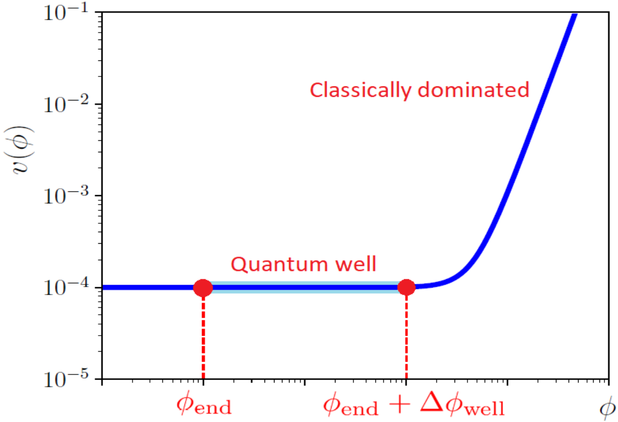


Stochastic Limit

Inflationary models that can produce $\zeta > \zeta_c$ are well approximated by a flat potential at the end of inflation, so v ' v_0 and

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} \;\text{'}\; \frac{H}{2\pi}\xi(N)\,.$$



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$$P\left(N, \phi\right) = \frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}N}\right),$$

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$$P(N, \phi) = \quad \frac{\pi}{2\mu^2} \vartheta_2^0 \left(\frac{\pi}{2} x, e \quad ^{\frac{\pi^2}{\mu^2} N} \right) ,$$

where

$$\mu^2 = \frac{\Delta\phi_{\mathsf{well}}^2}{v_0 M_{\mathrm{Pl}}^2} , \qquad x = \frac{\phi \quad \phi_{\mathsf{end}}}{\Delta\phi_{\mathsf{well}}} ,$$

and ϑ_2 is the second elliptic theta function.

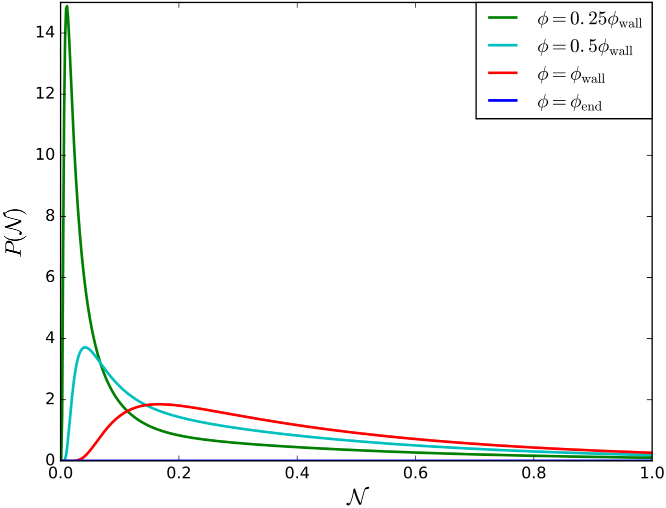

Figure 2: The PDF we obtain for a flat potential.

Mass fraction

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The expression we find depends on ϕ, μ and $\zeta_{\rm c}$.

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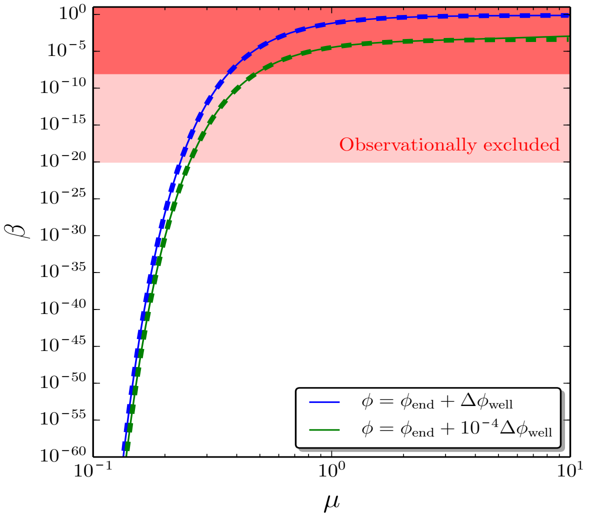
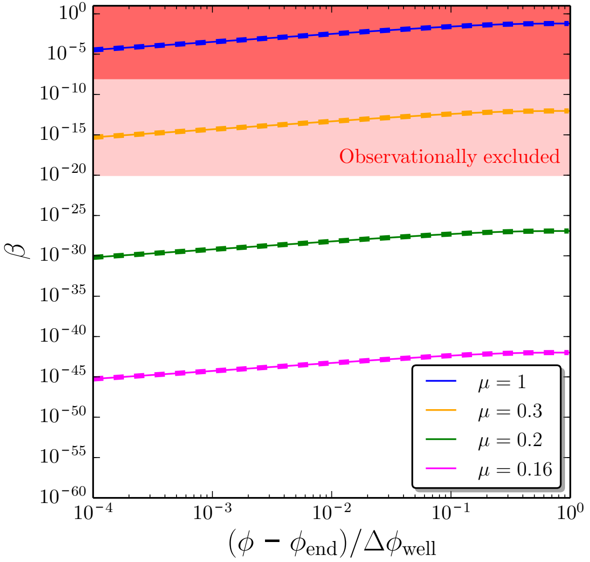


Figure 3: The mass fraction β is plotted as a function of μ, with $\zeta_c = 1$.



10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
10^{-35}
10^{-40}
10^{-45}
10^{-50}
10^{-55}
10^{-60}
β
Observationally excluded
$\mu = 1$
$\mu = 0.3$
$\mu = 0.2$
$\mu = 0.16$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
$(\phi - \phi_{\rm end})/\Delta\phi_{\rm well}$

We can compare our expression for β (taking $\zeta_{\rm c} = 1$) to observations and get constraints on our parameter μ.

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$$\mu < 0.21 \, .$$

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For light PBHs $M \quad 10^9 \quad 10^{16}\mathrm{g}$ a typical constraint $\beta < 10 \;\; {}^{24}$ gives

$$\mu < 0.21\,.$$

For heavier PBHs $M \quad 10^{16} \quad 10^{50}\mathrm{g}$, typically $\beta < 10 \;\; {}^{5}$, which gives

$$\mu < 0.47\,.$$

e-folds constraint

For an arbitrary mass PBH, $\mu < 1$ to prevent over-production.

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We can write the number of e-folds spent in the quantum well as

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Power spectrum is also $\propto \mu^2$, so μ determines everything.

Generic Recipe

The “recipe” for analysing a generic potential the following:

- identify the region of your potential that are flat and quantum dominated, and the parts where classical drift dominates;

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Generic Recipe

The "recipe" for analysing a generic potential the following:

- identify the region of your potential that are flat and quantum dominated, and the parts where classical drift dominates;
- in the classical regions, make use of the classical constraint $P_\zeta \Delta N < 10^{\ 2}$;
- in the "quantum wells", check if slow roll is violated. If not make use of our new stochastic constraint $\mu < 1$ ($\Delta N\ < 1$).

Running mass inflation (Stewart, 1996) has the potential

$$v\left(\phi\right) = v_0 \left\{ 1 \quad \frac{c}{2} \left[\quad \frac{1}{2} + \ln\left(\frac{\phi}{\phi_0}\right) \right] \frac{\phi^2}{M_{\mathrm{Pl}}^2} \right\} .$$

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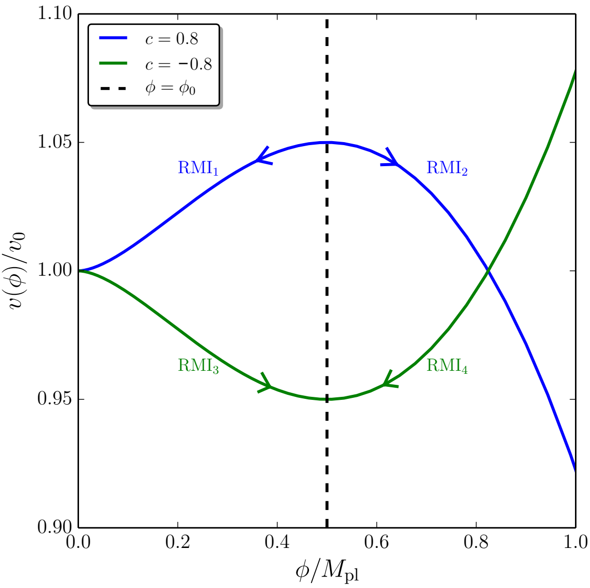
c is a dimensionless coupling constant, assumed to be $c \quad 1$, and ϕ_0 must be sub-Planckian, $\phi_0 \quad M_{\rm Pl}$.



1.10
1.05
1.00
0.95
0.90
0.0
0.2
0.4
0.6
0.8
1.0
$c = 0.8$
$c = -0.8$
$\phi = \phi_0$
RMI_1
RMI_2
RMI_3
RMI_4
$v(\phi)/v_0$
ϕ/M_{pl}

Begin by identifying the cases where the potential is flat and will be quantum dominated.

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In all three quantum wells, we find

$$\mu^2 \propto \frac{1}{|c|} \gg 1 \,.$$

Consequences

In all three cases, we see that in the quantum well we see over-production of PBHs.

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In both the classical and stochastic regimes, we find

$$P_\zeta \; / \; \mu^2 \,,$$

and so $\mu \quad 1$ gives a large power spectrum even in the classical regime.

Slow-roll violation

- Many models that produce PBHs also violate slow-roll!
- This means stochastic formalism needs to be extended to include these situations.
- We have checked that stochastic inflation is valid beyond slow roll (Pattison et al, 1905.06300), despite (incorrect) claims in the literature.

Ultra-slow-roll

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$$\dot{\phi} = 0\,,$$

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so we have no dynamics!
Take the case of $V^{0} = 0$ in

$$\ddot{\phi} + 3H\dot{\phi} + V^{0} = 0\,.$$

This is “ultra-slow-roll” (USR) inflation.

We then find

$$\dot{\phi}_{\rm USR} = \dot{\phi}_{\rm in} e^{\ \ 3Ht}\,,$$

which, unlike slow roll, depends on initial conditions.

Characteristic function in USR

Use the USR system for a flat potential rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}N} = \quad 3y + \frac{\sqrt{2}}{\mu}\xi(N)$$
$$\frac{\mathrm{d}y}{\mathrm{d}N} = \quad 3y\,,$$

where

$$x = \frac{\phi \quad \phi_{\mathrm{end}}}{\Delta\phi_{\mathrm{well}}}\,, y = \frac{\dot{\phi}}{\dot{\phi}_{\mathrm{crit}}}\,,$$

with $\dot{\phi}_{\mathrm{crit}} = \quad 3H\Delta\phi_{\mathrm{well}} \quad .$

Now, $N = N(x, y)$, and characteristic function equation becomes

$$\left[\frac{1}{\mu^2}\frac{\partial^2}{\partial x^2} \quad 3y\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + it\right]\chi_N(t; x, y) = 0\,,$$

with initial conditions

$$\chi_N(t; 0, y) = 1\,, \frac{\partial \chi_N}{\partial x}(t; 1, y) = 0\,.$$

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Lots of current work trying to solve this equation...

Classical limit

Neglecting diffusion:

$$\chi_N \Big|_{\mathrm{cl}}(t; x, y) = \left(1 \quad \frac{x}{y}\right)^{\quad \frac{it}{3}} \; .$$

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$$\chi_N \Big|_{\mathrm{cl}}(t; x, y) = \left(1 \quad \frac{x}{y}\right)^{\frac{it}{3}} .$$

Use this to find the number of e-folds:

$$\begin{aligned} hN\,i(x, y) &= \quad i \left.\frac{\partial \chi_N}{\partial t}\right|_{t=0} \\ &= \quad \frac{1}{3} \ln \left[1 \quad \frac{x}{y}\right] , \end{aligned}$$

which matches the known classical limit.
Can expand around this for corrections!

Late-time limit

This is the limit when $y \to 0$, and then DE for χ_N becomes

$$\left[\frac{1}{\mu^2}\frac{\partial^2}{\partial x^2} + it\right]\chi_N(t;x) = 0\,,$$

which is exactly the **same as stochastic SR limit!**
This means we know the solution and PDF in this limit:

$$P(N, x(\phi)) = \quad \frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{\ \frac{\pi^2}{\mu^2}N}\right).$$

Small-y limit

Without giving details and long equations, we can do a small-y expansion to calculate χ_N for small velocity.

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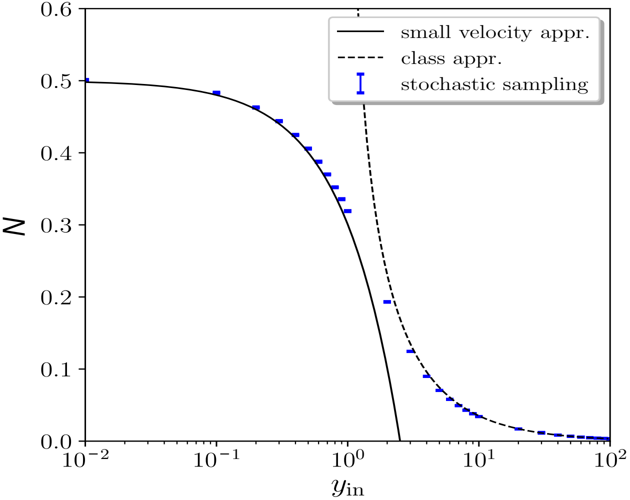

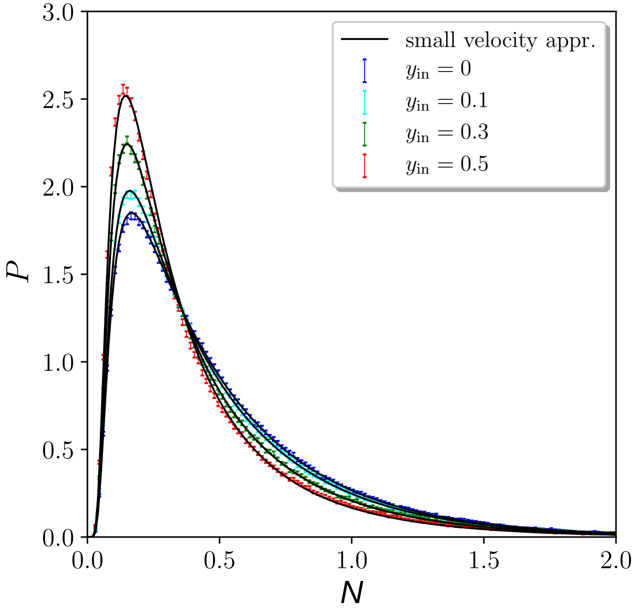
small velocity appr.
$y_{\rm in} = 0$
$y_{\rm in} = 0.1$
$y_{\rm in} = 0.3$
$y_{\rm in} = 0.5$
P
N
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0

Ongoing work

- We can recast stochastic USR equation to be pure diffusion but with moving barriers. Old system:

$$\frac{\mathrm{d}x}{\mathrm{d}N} = -3y + \frac{\sqrt{2}}{\mu}\xi(N)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}N} = -3y\,,$$

  If we take $z = x - y$ then our Langevin system becomes

$$\frac{\mathrm{d}z}{\mathrm{d}N} = \frac{\sqrt{2}}{\mu}\xi(N)\,, \quad \frac{\mathrm{d}y}{\mathrm{d}N} = -3y\,,$$

- Then use a new approach of a Volterra equation to calculate PDFs (Zhang and Hui astro-ph/0508384, Buonocore et al 1990[1])
- Provides easy and quick way to get full PDFs without weeks of simulations

[1] https://www.jstor.org/stable/3214598

Summary

- The stochastic-δN formalism is needed to analyse curvature perturbations and PBH formation.
- It is sensitive to large-scale quantum kicks, coming from new modes exiting the horizon
- The quantum effects are important for astrophysical objects such as PBHs
- Formalism can be used beyond slow roll, and we are working to use it in USR

Future Work

- Apply our USR formalism more complicated PBH models (eg Garcia-Bellido et al, 2017)
- Calculate PBH abundances and compare to constraints for USR models
- Extend the formalism to include multi-field inflation.

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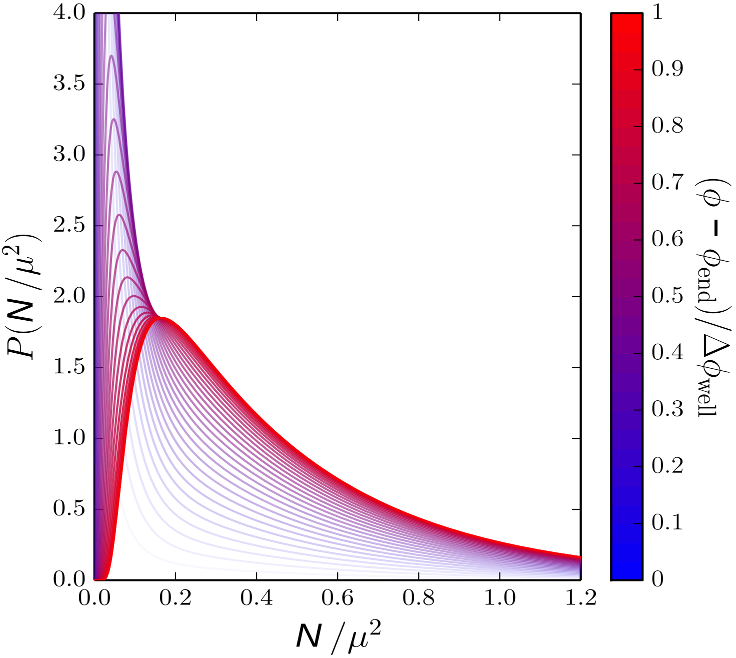
Taking $\beta < 10^{22}$, this gives $P_\zeta < 1.6 \quad 10^{\ 2}$.

Contrary to the classical condition $P_\zeta \Delta N < 10^{\ 2}$, we don't have the number of e-folds in the stochastic constrain, since μ determines everything.



$P(N/\mu^2)$
N/μ^2
$(\phi - \phi_{\rm end})/\Delta\phi_{\rm well}$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

A larger curvature power spectrum means more PBHs.
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A larger curvature power spectrum means more PBHs.
Classically P_ζ / $v^3/v^{\prime 2}$.

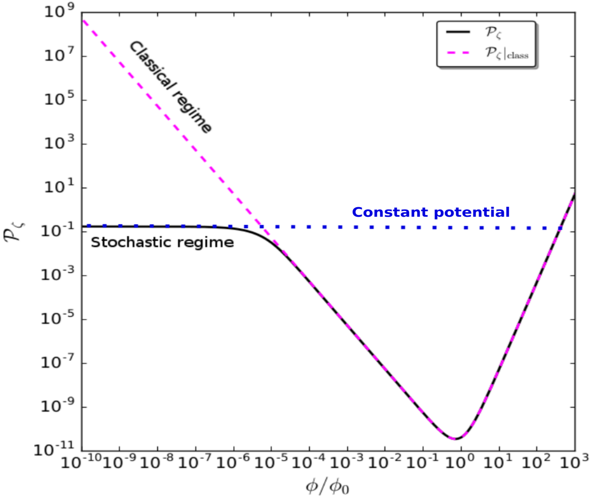

Figure 4: Power spectra for v / $1+\phi^2$ and $v = \text{constant}$.