New lattice QCD calculation of the hadronic vacuum polarization contribution to the muon magnetic moment

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arXiv:2002.12347
PRL 18 (2018) (Editors' Selection)

Lepton magnetic moments and BSM physics

Interaction with an external EM field: Dirac eqn

Dirac eqn w/ minimal coupling:

$$i\hbar\frac{\partial\psi}{\partial t}=\left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla}-e_\ell\vec{A}\right)+\beta c^2 m_\ell+e_\ell A_0\right]\psi$$

nonrelativistic limit $\downarrow$ (Pauli eqn)

$$i\hbar\frac{\partial\phi}{\partial t}=\Bigg[\frac{\left(\frac{\hbar}{i}\vec{\nabla}-\frac{e_\ell}{c}\vec{A}\right)^2}{2m_\ell}-\underbrace{\frac{e_\ell\hbar}{2m_\ell}\vec{\sigma}\cdot\vec{B}}_{\vec{\mu}_\ell\cdot\vec{B}}+e_\ell A_0\Bigg]\phi$$

with

$$\vec{\mu}_\ell=g_\ell\left(\frac{e_\ell}{2m_\ell}\right)\vec{S},\qquad \vec{S}=\hbar\frac{\vec{\sigma}}{2}$$

and

$$g_\ell\big|_{\text{Dirac}}=2$$

Interaction with an external EM field: SM & BSM

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

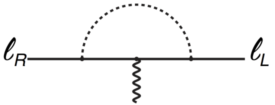
$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

$$\left. +\gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

$F_1(q^2)$	$\rightarrow$	Dirac form factor: $F_1(0) = 1$
$F_2(q^2)$	$\rightarrow$	Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell - \overbrace{2}^{g_\ell\vert_{\text{Dirac}}}}{2}$
$F_3(q^2)$	$\rightarrow$	$\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell/e_\ell$
$F_4(q^2)$	$\rightarrow$	$\not{P}$, anapole moment: $\vec{\sigma}\cdot(\vec{\nabla}\times\vec{B})$

- $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once theory's couplings are renormalized (in a renormalizable theory)
- a_ℓ dimensionless
  - $\Rightarrow$ corrections including only ℓ and γ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell/M)^{2p} \times \ln^q(m_\ell^2/M^2)$
  - $\rightarrow$ contributions from particles w/ $m \ll m_\ell$ are e.g. $\propto \ln^2(m_\ell^2/m^2)$

Why are a_ℓ special?



$$\longrightarrow \quad \frac{a_\ell}{2m_\ell} e F^{\mu\nu} [\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- Loop induced $\Rightarrow$ sensitive to new dofs
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, $b \to s\ell^+\ell^-$, $\mu \to e\gamma$, $B \to D^{(*)}\ell\nu_\ell$, EW precision observables, LHC direct searches, ...
- Chirality flipping $\Rightarrow$ a_ℓ related to mass generation (Czarnecki et al '01)
- In EW theory, only source of chirality flips is $y_\ell \, \bar{\ell}_L H \ell_R$

$$m_\ell = y_\ell \langle H \rangle, \qquad a_\ell^{\text{weak}} \propto \frac{\alpha}{4\pi} \left(\frac{m_\ell}{M_W} \right)^2$$

- BSM can be very different

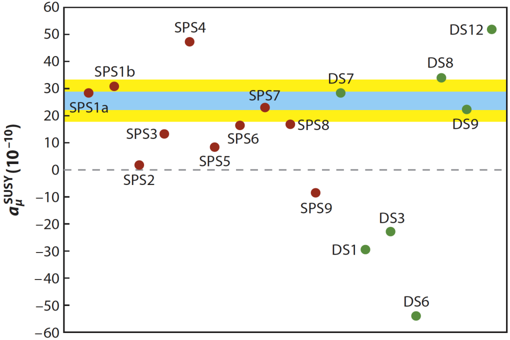
$$a_\ell^{\text{N}\Phi} = O(1) \left(\frac{\Delta^{\text{N}\Phi} m_\ell}{m_\ell} \right) \left(\frac{m_\ell}{M_{\text{N}\Phi}} \right)^2$$

e.g. SUSY $(\Delta^{\text{N}\Phi} m_\ell / m_\ell) \sim (\alpha/4\pi)\, \text{sign}\mu \, \tan\beta$

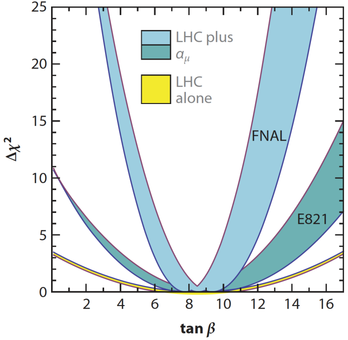
Why is a_μ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\mathrm{GeV}$ $\tau_e : \tau_\mu : \tau_\tau =$ "∞" $: 2.\cdot 10^{-6} : 3.\cdot 10^{-15}$ s

- a_μ is $(m_\mu/m_e)^2 \sim 4. \times 10^4$ times more sensitive to new Φ than a_e
- a_τ is even more sensitive to new Φ, but is too shortly lived



(Miller et al '12)



(SPS1a)

Big > 0 effect → $\mathrm{sign}\mu = 1$, $M_{N\phi} \sim 100 \div 500\,\mathrm{GeV}$, $\tan\beta \sim 3 \div 40$

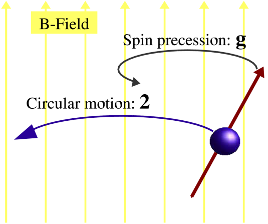
Big question

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, what is the new Φ and can it be seen elswhere?

Experimental measurement of a_μ

Measurement principle for a_μ



$$\vec{\mu}_\mu = 2(1+a_\mu)\frac{Qe}{2m_\mu}\vec{S}, \qquad \vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c}\vec{S}$$

$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} + \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] - \eta_\mu \frac{Qe}{2m_\mu}\left[\frac{\vec{E}}{c} + \vec{\beta}\times\vec{B}\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &

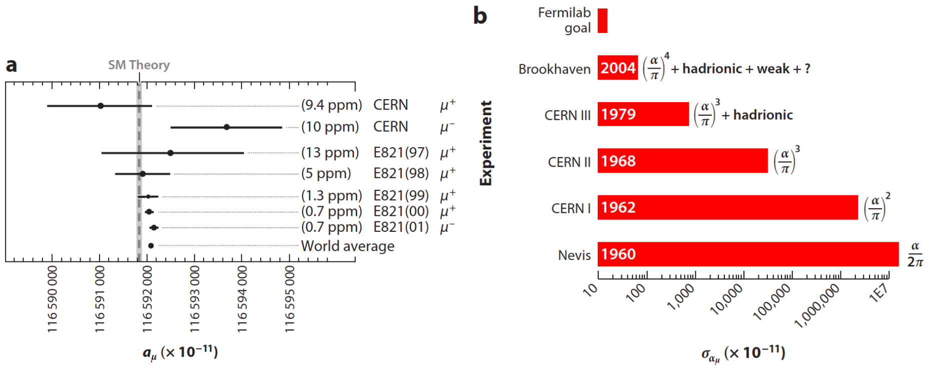
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9)\times 10^{-19} e\cdot\text{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E}=0$ (J-PARC)

$$\to \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$

a_μ experimental summary



(Miller et al '12)

Two new experiments plan to reduce error on a_μ to $\sim$ 0.14 ppm

- New $g-2$ (E989) @ Fermilab: has started taking data fall 2017
- $g-2$/EDM (E34) @ J-PARC: should start taking data $\geq$ 2021

Standard model calculation of a_μ

$$
\begin{aligned}
a_\mu^{\text{SM}} &= a_\mu^{\text{QED}} + a_\mu^{\text{had}} + a_\mu^{\text{weak}} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi \sin\theta_W}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

QED contributions to a_ℓ

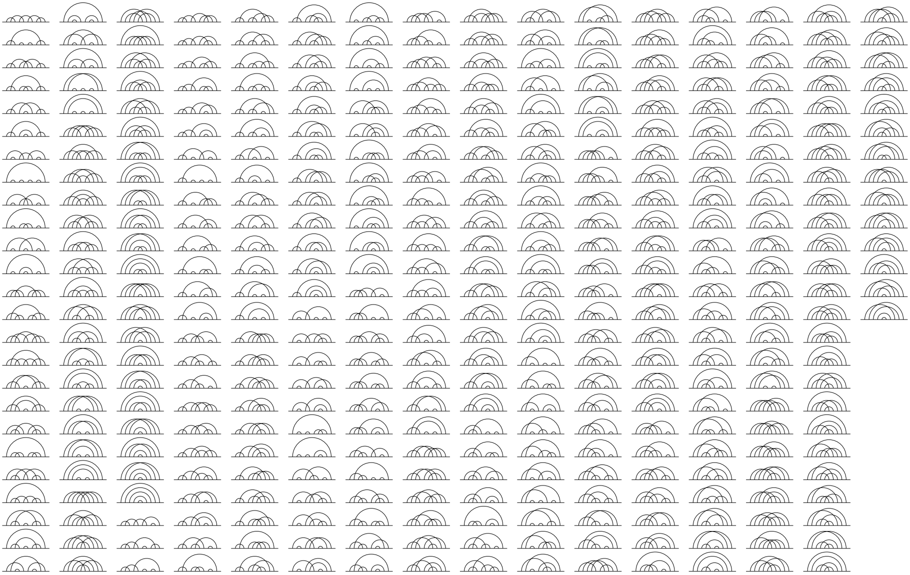
Loops with only photons and leptons

$$a_\ell^{\text{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}, A_1^{(4)}, A_1^{(6)}, A_2^{(4)}, A_2^{(6)}, A_3^{(6)}$ known analytically (Schwinger '48; Sommerfield '57, '58; Petermann '57; ...)
- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta '95; Aguilar et al '08; Aoyama, Kinoshita, Nio '96-'18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Only some diagrams are known analytically

5-loop QED diagrams



(Aoyama et al '15)

QED contribution to a_μ

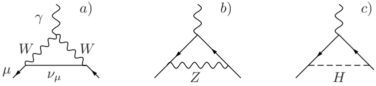
$$a_\mu^{\rm QED}(Rb) = 1\,165\,847\,189.51(9)_m(19)_{\alpha^4}(7)_{\alpha^5}(77)_{\alpha(Rb)} \times 10^{-12}\ [0.7\ \mathrm{ppb}]$$
$$a_\mu^{\rm QED}(a_e) = 1\,165\,847\,188.41(7)_m(17)_{\alpha^4}(6)_{\alpha^5}(28)_{\alpha(a_e)} \times 10^{-12}[0.3\ \mathrm{ppb}]$$

(Aoyama et al '12, '18)

$$a_\mu^{\rm exp} - a_\mu^{\rm QED} = 737.2(6.3) \times 10^{-10}$$
$$\stackrel{?}{=} a_\mu^{\rm weak} + a_\mu^{\rm had}$$

Weak contributions to a_μ: Z, W, H, etc. loops

1-loop

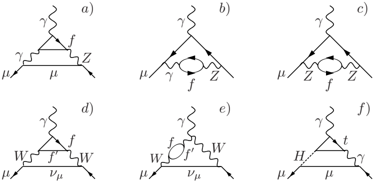


$$a_\mu^{\text{weak},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right)$$

$$= 19.480(1) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

2-loop



$$a_\mu^{\text{weak},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right)$$

$$= -4.12(60) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_\mu^{\text{weak}} = 15.36(10) \times 10^{-10}$$

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\rm exp} - a_\mu^{\rm QED} - a_\mu^{\rm weak} = 721.8(6.3) \times 10^{-10} \stackrel{?}{=} a_\mu^{\rm had}$$

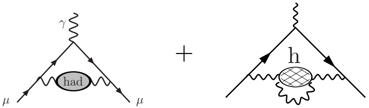
Clearly right order of magnitude:

$$a_\mu^{\rm had} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

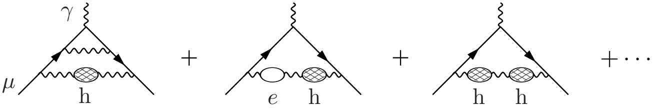
Write

$$a_\mu^{\rm had} = a_\mu^{\rm LO\text{-}HVP} + a_\mu^{\rm NLO\text{-}HVP} + a_\mu^{\rm HLbyL} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

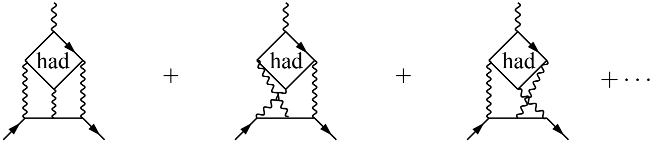
Hadronic contributions to a_μ: diagrams



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$



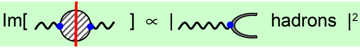
$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$



$\rightarrow a_\mu^{\text{HLbyL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

LO-HVP from $e^+e^- \to$ had

- Use (Bouchiat et al '61) optical theorem (unitarity)



$$\Rightarrow \mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \qquad R(s) \equiv \frac{\sigma(e^+e^- \to \mathrm{had})}{4\pi\alpha(s)^2/(3s)}$$

and a dispersion relation w/ data for $R(s)$ (CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds\, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) = \frac{Q^2}{12\pi^2} \int_0^\infty ds\, \frac{1}{s(s+Q^2)} R(s)$$

- Three recent determinations:

$$\begin{aligned} a_\mu^{\text{LO-HVP}} &= 692.78(2.42) \times 10^{-10} \quad [3.5‰] \quad \text{(KNT '19)} \\ &= 693.9(4.0) \times 10^{-10} \quad [5.8‰] \quad \text{(DHMZ '19)} \\ &= 692.3(3.3) \times 10^{-10} \quad [4.8‰] \quad \text{(CHHKS '19)} \end{aligned}$$

- Higher orders:

$$\begin{aligned} a_\mu^{\text{NLO-HVP}} &= -9.87(0.09) \times 10^{-10} \quad \text{(Kurz et al '14)} \\ a_\mu^{\text{NNLO-HVP}} &= 1.24(0.01) \times 10^{-10} \quad \text{(Kurz et al '14)} \end{aligned}$$

Standard model prediction and comparison to experiment

SM prediction vs experiment

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8841 ± 0.0034	[Aoyama et al '18]
HVP LO	692.8 ± 2.4	[KNT '19]
	693.9 ± 4.0	[DHMZ '19]
	692.3 ± 3.3	[CHHKS '19]
HVP NLO	-9.87 ± 0.09	[Kurz et al '14]
		[Kurz et al '14, Jegerlehner '16]
HVP NNLO	1.24 ± 0.01	[Kurz et al '14]
		[Jegerlehner '16]
HLbyL	10.5 ± 2.6	[Prades et al '09]
	$7.8 \pm 3.1 \pm 1.8$	[RBC '19]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '15]
SM Tot [0.30 ppm]	11659181.9 ± 3.6	[w/ KNT '19]
[0.41 ppm]	11659183.0 ± 4.8	[w/ DHMZ '19]
[0.36 ppm]	11659181.4 ± 4.2	[w/ CHHKS '19]
Exp [0.54 ppm]	11659209.1 ± 6.3	[Bennett et al '06]
Exp − SM	27.2 ± 7.2 [3.8σ]	[KNT '19]
	26.1 ± 7.9 [3.3σ]	[DHMZ '19]
	27.7 ± 7.6 [3.7σ]	[CHHKS '19]

a_μ present and future

- With HVP from R-ratio: $a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} \simeq 3.3 \div 3.8\,\sigma$
- Fermilab E989 began fall 2017 and aims for 0.14 ppm
- J-PARC E34 to begin $\geq$ 2021 and aims for similar precision
- If both central values stay the same:
  - E989 alone (exp. error $/4$) $\rightarrow\sim 5 \div 7\,\sigma$
  - E989 + new HLbyL (w/ error $\sim 10\%$) $\rightarrow\sim 6 \div 9\,\sigma$
  - E989 + new HLbyL + new HVP (w/ error $/2$) $\rightarrow\sim 9 \div 12\,\sigma$
- Must have a completely independent first principles determination of the 2 contributions w/ leading theory errors before concluding the presence of BSM physics
  $\Rightarrow$ Lattice QCD
- Discrepancy is large: $2\times$ electroweak contribution
- **No new physics scenario**, i.e. (experiment) $-$ (other contributions)
  - $a_\mu^{\text{LO-HVP}}|_{\text{no-N}\Phi} = 720.0(6.8) \times 10^{-10}$, i.e. 4% larger than DHMZ '17
  - $a_\mu^{\text{HLbyL}}|_{\text{no-N}\Phi} = 37.9(7.1) \times 10^{-10}$, i.e. $3.6\times$ (Prades et al '09)

A brief introduction to lattice QCD

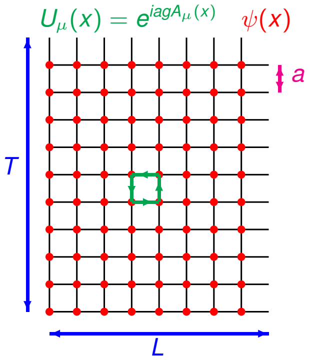
What is lattice QCD (LQCD)?

To describe ordinary matter, QCD requires $\geq$ 104 numbers at every point of spacetime
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G - \int \bar{\psi}D[M]\psi}\, O[U,\psi,\bar{\psi}]$$
$$= \int \mathcal{D}U\, e^{-S_G}\, \mathrm{det}(D[M])\, O[U]_{\mathrm{Wick}}$$

- $\mathcal{D}Ue^{-S_G}\,\mathrm{det}(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

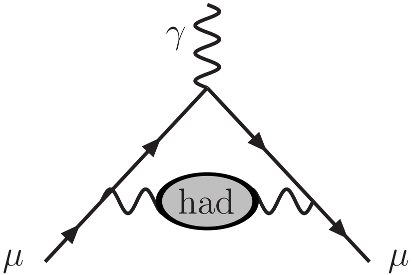
HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

Challenges of a full lattice calculation

To make contact with experiment need:

- **A valid approximation to the SM**
  - $\rightarrow$ at least u, d, s in the sea w/ $m_u = m_d \ll m_s$ (N_f=2+1) $\Rightarrow \sigma \sim 1\%$
  - $\rightarrow$ better also include c (N_f=2+1+1) & $m_u \le m_d$ (N_f=4×1) & EM (N_f=4×1 + QED) $\Rightarrow \sigma \sim 0.1\%$
- **u & d w/ masses well w/in $SU(2)$ chiral regime** : $\sigma_\chi \sim (M_\pi/4\pi F_\pi)^2$
  - $\rightarrow$ $M_\pi \sim 135\,\mathrm{MeV}$ or many $M_\pi \le 400\,\mathrm{MeV}$ w/ $M_\pi^{\mathrm{min}} < 200\,\mathrm{MeV}$ for $M_\pi \to 135\,\mathrm{MeV}$
- **a → 0** : $\sigma_a \sim (a\Lambda_{\mathrm{QCD}})^n$, $(am_q)^n$, $(a|\vec{p}|)^n$ w/ $a^{-1} \sim 2 \div 4\,\mathrm{fm}$
  - $\rightarrow$ at least 3 a's $\le 0.1\,\mathrm{fm}$ for $a \to 0$
- **L → ∞** : $\sigma_L \sim (M_\pi/4\pi F_\pi)^2 \times e^{-LM_\pi}$ for stable hadron pties, $\sim 1/L^n$ for resonances, QED, ...
  - $\rightarrow$ many L w/ $(LM_\pi)^{\mathrm{max}} \gtrsim 4$ for stable hadrons & better otherwise to allow for $L \to \infty$
- These requirements $\Rightarrow$ $O(10^9)$ **dofs** that have to be integrated over
- **Renormalization** : best done nonperturbatively
- **A signal** : $\sigma_{\mathrm{stat}} \sim 1/\sqrt{N_{\mathrm{meas}}}$, reduce w/ $N_{\mathrm{meas}} \to \infty$

Lattice QCD calculation of a_μ^{HVP}



Consider in Euclidean spacetime (Blum '02)

$$\Pi_{\mu\nu}(Q) \;=\;$$

$$= \int d^4x\, e^{iQ\cdot x}\langle J_\mu(x)J_\nu(0)\rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

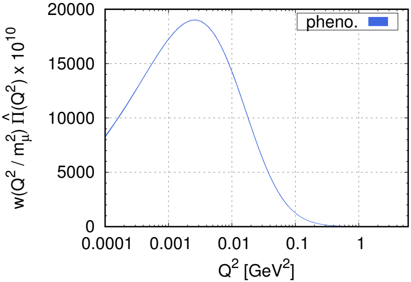
Then (Lautrup et al '69, Blum '02)

$$a_\ell^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, w(Q^2/m_\ell^2)\hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\mathrm{MeV}$ for μ

However, $Q_{\text{min}} \equiv \frac{2\pi}{T} \sim 135\,\mathrm{MeV}$ for lattice w/ $T = \frac{3}{2}L \sim 9\,\mathrm{fm}$



(HVP from Jegerlehner, "alphaQEDc17" (2017))

Low-Q^2 challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q=0) \neq 0$ in FV that contaminates $\Pi(Q^2) \sim \Pi_{\mu\nu}(Q)/Q^2$ for $Q^2 \to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\min} = 2\pi/T \sim 135\,\mathrm{MeV} > \frac{m_\mu}{2} \sim 50\,\mathrm{MeV}$ for $T \sim 9\,\mathrm{fm}$

$$\Downarrow$$

- Compute on $T \times L^3$ lattice in $N_f = 2+1+1$ QCD

$$C_L^{\text{iso}}(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

- Decompose ($C_L^{I=1} = \frac{9}{10} C_L^{ud}$)

$$C_L^{\text{iso}}(t) = C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\text{disc}}(t) = C_L^{I=1}(t) + C_L^{I=0}(t)$$

- Define (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C) (see also Charles et al '17)

$$\hat{\Pi}_L^f(Q^2) \equiv \Pi_L^f(Q^2) - \Pi_L^f(0) = \frac{1}{3}\sum_{i=1}^{3} \frac{\Pi_{ii,L}^f(0) - \Pi_{ii,L}^f(Q)}{Q^2} - \Pi_L^f(0) = 2a \sum_{t=0}^{T/2} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2}\right] \mathrm{Re}\, C_L^f(t)$$

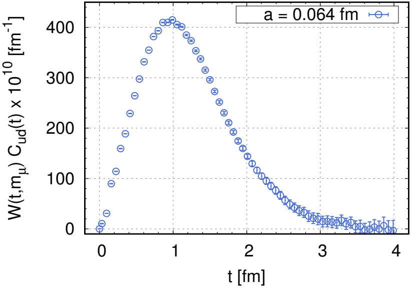
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_L^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\text{max}}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\text{max}}/m_\ell^2)\ \mathrm{Re}C_L^f(t)$$

where

$$W(\tau, x_{\text{max}}) = \int_0^{x_{\text{max}}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$



(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light π

E. Two types of contributions

quark-connected (qc) quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

F. $\langle J_\mu^{ud}(x) J_\nu^{ud}(0)\rangle_{qc}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ many algorithmic improvements + very high statistics + rigorous bounds

G. Must control $\langle J_\mu(x) J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_\mu$ $\rightarrow L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: $0.134 \rightarrow 0.064\,\mathrm{fm}$

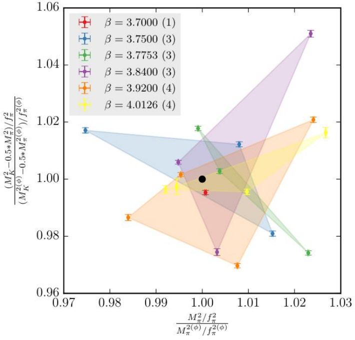
$\rightarrow$ improve approach to continuum limit w/ 2-loop SU(2) SχPT and/or phenomenological model (SLLGS)

Simulation details: ad D - I

27 high-statistics simulations w/ N_f=2+1+1 flavors of 4-stout staggered quarks:

- Bracketing physical m_{ud}, m_s, m_c
- 6 a's: 0.134 → 0.064 fm
- $L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$
- Conserved EM current

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	64 × 48	904
3.7500	0.1191	96 × 56	2072
3.7753	0.1116	84 × 56	1907
3.8400	0.0952	96 × 64	3139
3.9200	0.0787	128 × 80	4296
4.0126	0.0640	144 × 96	6980



- State-of-the-art techniques:
  - EigCG
  - Low mode averaging (Giusti et al '04)
  - All mode averaging (Blum et al '13)
  - Solver truncation (Bali et al '09)

⇒ Nearly 20,000 gauge configurations

⇒ 10's of millions of measurements

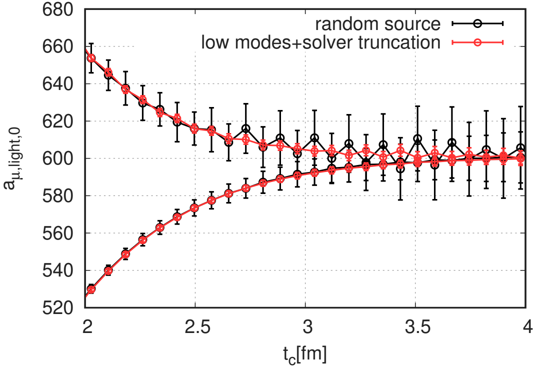
Noise reduction: ad F-G

N/S in $C_L^{ud}(t)$ grows like $e^{(M_\rho - M_\pi)t}$

- LMA: use exact (all-to-all) quark propagators in IR and stochastic in UV

- Decrease noise by replacing $C_L^{ud}(t)$ by average of rigorous upper/lower bounds above $t_c = 4\,\mathrm{fm}$

  $0 \leq C_L^{ud}(t) \leq C_L^{ud}(t_c)\, e^{-E_{2\pi}(t-t_c)}$



$\Rightarrow$ few pemil accuracy on each ensemble

More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$

$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\, \hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

using $O(\alpha_s^4)$ results from `rhad` package (Harlander et al '03)

J. Include c quark for higher precision and good matching onto perturbation theory $\rightarrow$ done

K. Even in our large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects can be significant

$\rightarrow$ 1-loop SU(2) χPT (Aubin et al '16) suggests 2% even in our large volumes

$\rightarrow$ perform dedicated FV study in even larger volumes

$\rightarrow$ check and supplement w/ 2-loop χPT and pheno. model (LLGS)

L. Our $N_f = 2 + 1 + 1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at permil-level precision

$\rightarrow$ perform lattice calculation of ALL $O(\alpha)$ and $O(\delta m = m_d - m_u)$ effects

Yet more challenges

M. Need permil determination of lattice spacing

$\Rightarrow$ 2‰ calculation of Ω^- baryon mass

$\Rightarrow$ Use intermediate Wilson-flow scale w_0 (Lüscher '10, BMWc '12)

N. Need need thorough and robust determination of statistical and systematic errors

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach (BMWc '08, '14)
  - Millions of different analyses of correlation functions
  - Each one is weighted by AIC weight

$$\text{AIC} \sim \exp\left[-\frac{1}{2}(\chi^2 - 2n_{\text{dof}})\right]$$

  - Simplify w/ importance sampling
  - Use median of distribution for central values
  - Use $16 \div 84\%$ confidence interval to get total error

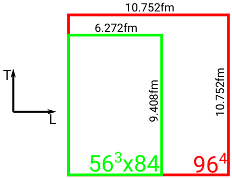
(2002.12347 has 75 pp. appendix detailing methods)

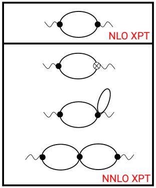
Finite-volume corrections: ad K

Early estimate of these e^{-LM_π} effects (Aubin et al '16): 2% on $a_\mu^{\text{LO-HVP}}$ in our $L = 6$ simulations

→ Perform dedicated lattice study

- 4 very-high statistics $N_f = 2 + 1$, super-smeared (4HEX) simulations
- Tuned so that staggered M_π^{HMS} brackets physical M_π
- L up to 11 fm ($a \simeq 0.112$ fm)!





→ Check w/ EFTs and models: dominated by long-distance $\pi\pi$ effects

- NNLO (2-loop) χPT (Aubin et al '19, BMWc '20)
- Lellouch-Lüscher formalism w/ Gounaris-Sakurai model (LLGS) (Meyer '11, Francis '13, Giusti et al '18, BMWc '20)
- QFT relation to Compton scattering (HP) (Hansen et al '19-'20)

$[\times 10^{-10}]$	lattice	NLO	NNLO	LLGS	HP
$a_\mu^{\text{LO-HVP}}(L_{\text{big}}) - a_\mu^{\text{LO-HVP}}(L_{\text{ref}})$	18.1(2.0)(1.4)	11.2	15.3	18.3	16.3

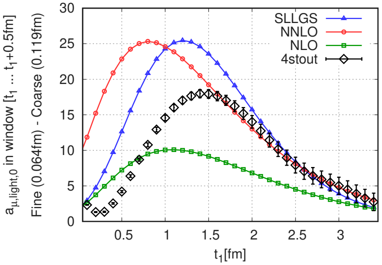
Model validation ⇒ $a_\mu^{\text{LO-HVP}}(\infty) - a_\mu^{\text{LO-HVP}}(L_{\text{big}}) = 1.4 \times 10^{-10}$ from NNLO χPT and HP

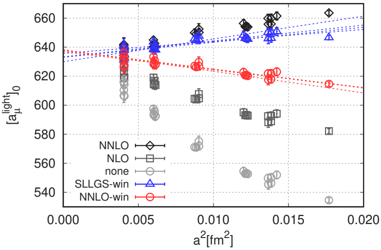
Continuum extrapolation: ad H

Long-distance discretization effects in $a_{\mu,ud}^{\text{LO-HVP}}$ due to taste violations in $\pi\pi$ states (HPQCD '16)

Correct w/ NNLO SχPT and SLLGS (BMWc '20)

- No free parameters
- Reproduces observed discretization effects well
- Corrections vanish in continuum limit
- 6 a's $\rightarrow$ full control over continuum limit





- Improves approach to continuum limit $\Rightarrow$ reduced uncertainties
- But does NOT modify this limit $\Rightarrow$ NO model dependence of result
- Systematics from cuts on a and different improvements

Including isospin breaking on the lattice: ad I

$$S_{\mathrm{QCD+QED}} = S^{\mathrm{iso}}_{\mathrm{QCD}} + \frac{1}{2}\delta m \int (\bar{d}d - \bar{u}u) + ie \int A_\mu j_\mu, \qquad j_\mu = \bar{q}Q\gamma_\mu q, \qquad \delta m = m_d - m_u$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle j_\mu j_\nu \rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and scale

(1) operator insertion method (RM123 '12, '13, ...)

$$\langle \mathcal{O} \rangle_{\mathrm{QCD+QED}} = \langle \mathcal{O}_{\mathrm{Wick}} \rangle^{\mathrm{iso}}_{G_\mu} - \frac{\delta m}{2} \langle [\mathcal{O} \int (\bar{d}d - \bar{u}u)]_{\mathrm{Wick}} \rangle^{\mathrm{iso}}_{G_\mu} - \frac{e^2}{2} \langle [\mathcal{O} \int_{xy} j_\mu(x) D_{\mu\nu}(x-y) j_\nu(y)]_{\mathrm{Wick}} \rangle^{\mathrm{iso}}_{G_\mu}$$
$$+ e^2 \langle\langle \left[\mathcal{O} \partial_e \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \int_x j_\mu(x) A_\mu(x) - \frac{1}{2} \mathcal{O} \partial_e^2 \frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu, 0]} |_{e=0} \right]_{\mathrm{Wick}} \rangle_{A_\mu} \rangle^{\mathrm{iso}}_{G_\mu}$$

(2) direct method (Eichten et al '97, BMWc '14, ...)

Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

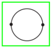
(3) combinations of (1) & (2) (BMWc '20)

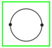
We include ALL $O(e^2)$ and $O(\delta m)$ effects

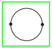
For valence effects use easier (2), and for sea effects, (1)

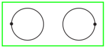
Summary of contributions to $a_\mu^{\text{LO-HVP}}$

Isospin symmetric

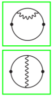

connected light
637.2(1.6)(3.0)

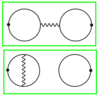

connected strange
53.421(58)(80)


connected charm
14.6(0)(1)

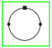

disconnected
-13.54(67)(39)

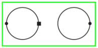
QED isospin-breaking: valence


connected -1.21(33)(30)

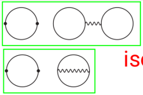

disconnected -0.58(13)(12)

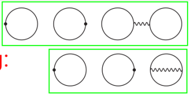
Strong isospin-breaking


connected
6.31(55)(48)


disconnected
-3.58(42)(41)

QED isospin-breaking: sea


connected 0.51(13)(22)

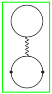

disconnected -0.032(31)(17)

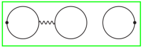
Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0065(60)(42)


disconnected 0.011(23)(12)

Finite-size effects

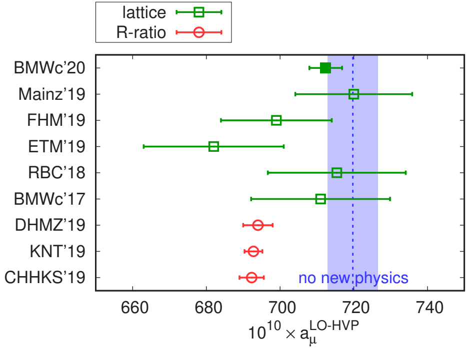
isospin-symmetric
18.7(2.0)(1.4)

isospin-breaking
0.0(0.1)

$$a_\mu^{\text{LO-HVP}} = 712.4(1.9)_{\text{stat}}(4.0)_{\text{syst}}\,[4.5]_{\text{tot}} \times 10^{-10}\ [0.6\%]$$

Comparison and outlook

Comparison



- Consistent with other lattice results
- Total uncertainty is $\sim \div 4$ . . .
- . . . and comparable to R-ratio
- Consistent w/ BNL experiment ("no new physics" scenario) !
- 3.1σ larger than DHMZ'19, 3.9σ than KNT'19 ?

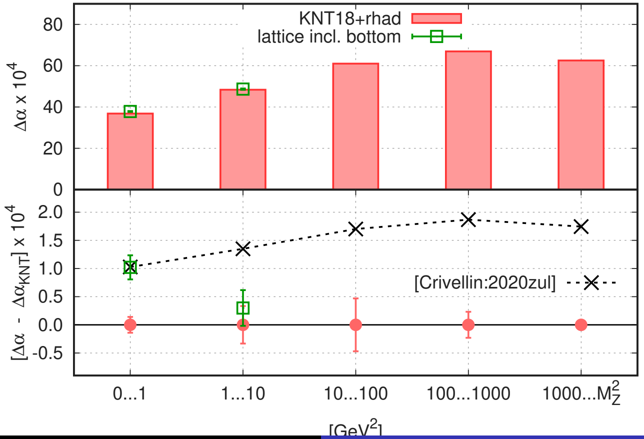
What next?

- FNAL E989 should put out first results very soon !!
- Will they still agree with our prediction ?
- Must be confirmed by other lattice groups
- Must understand why we don't agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18)
- Should be able to further divide total lattice error by 2 (i.e. 0.3%) in coming years
- Important to push FNAL E989 to the limit, to build J-PARC $g_\mu - 2$ and pursue a_e experiments

BACKUP

Do our results imply NP @ EW scale?

- Crivellin et al '20, most aggressive scenario: our results suggest a 4.2σ overshoot in $\Delta^{(5)}_{\rm had}\alpha(M_Z^2)$ compared to result of fit to EWPO
- Assume same 2.8% relative deviation from R-ratio as we find in $a_\mu^{\rm LO\text{-}HVP}$
- Hypothesis is not consistent w/ BMWc '17 nor new preliminary result



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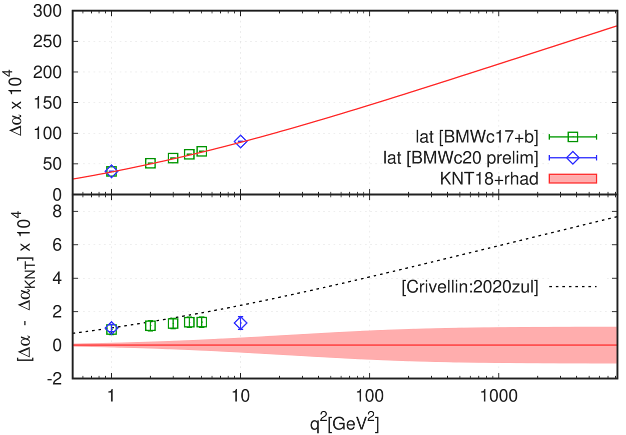
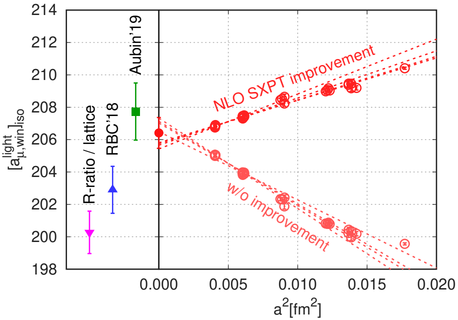
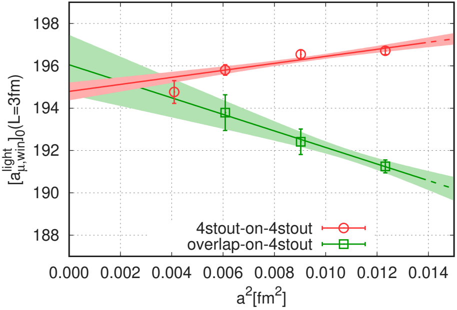


- Restrict $C_L^{ud}(t)$ to smooth window w/ t in range $0.4 \div 1.0\,\mathrm{fm}$
- Easier calculation $\rightarrow$ other LQCD groups have comparable error
- Good agreement w/ other LQCD groups but 3.8σ tension w/ R-ratio



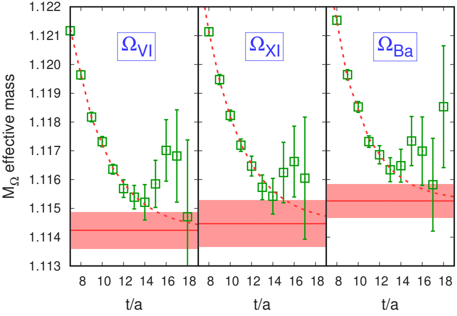


- Check universality of continuum limit
- Calculation using overlap valence quarks in same window
- Use renormalized local current
- Results of two discretizations consistent in $a \rightarrow 0$ limit

Setting the scale: ad M

Q^2 in $\hat{\Pi}(Q^2)$ must be converted to physical units $\to$ need a

- Use Ω^- baryon mass:
  $a \equiv (aM_{\Omega^-})^{\text{latt.}}/M_{\Omega^-}^{\text{PDG}}$
- Use multi-exponential fits and/or GEVP
- Reach permil level accuracy



Determine intermediate scale w_0 with two permil precision:

$$w_0 = 0.17180(18)(35)\,\text{fm}$$

- Use w_0 in analysis (small quark-mass dependence, very precise, tiny QED part)
- Consistent w/ $w_0 = 0.1715(9)\,\text{fm}$ (HPQCD'13)

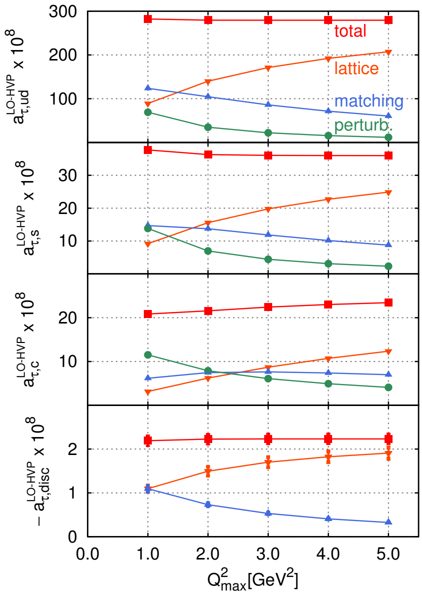
Matching to perturbation theory: ad I & J

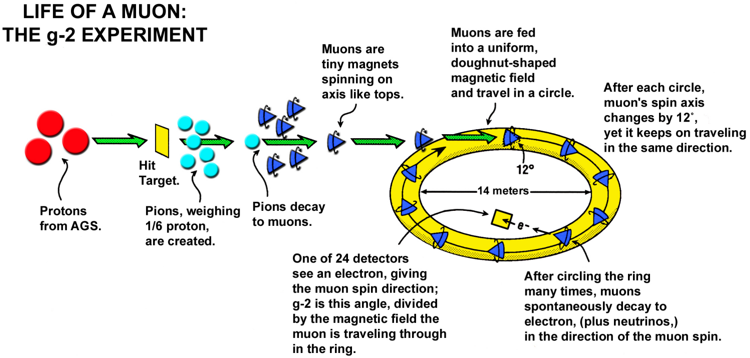
Consider separation ($\ell = e,\ \mu,\ \tau$)

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{\max}) + \gamma_\ell(Q_{\max})\ \hat{\Pi}^f(Q^2_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$$

- Compute $\Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$ using $R_{\text{pert}}(s)$ to $O(\alpha_s^4)$ from Harlander et al '03
- Not relevant for $\ell = e, \mu$ but important for τ
- Perfect matching of continuum lattice results for $Q^2_{\max} \geq 2\,\text{GeV}^2$

  $\rightarrow$ control $\hat{\Pi}(Q^2)$ up to $Q^2 \rightarrow \infty$
- Get matching systematic from considering $Q^2_{\max} = 2$ and $5\,\text{GeV}^2$



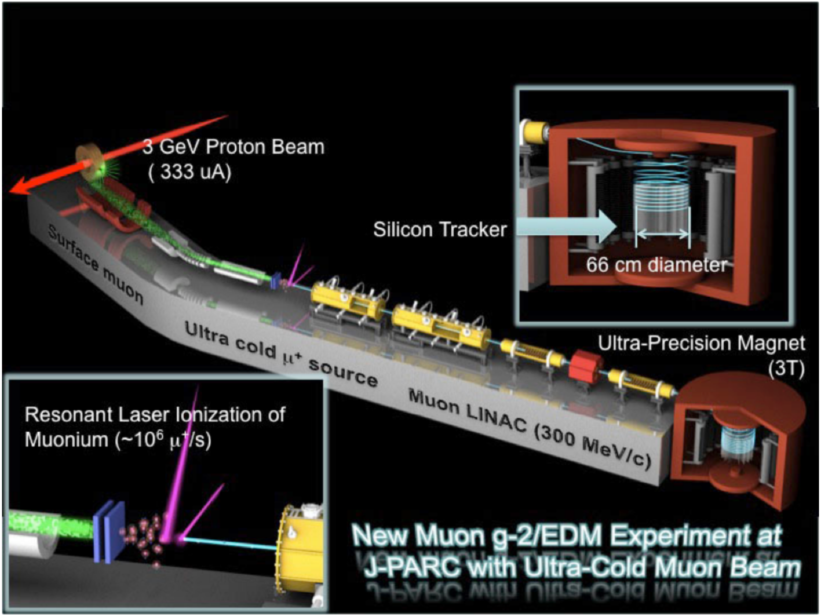


(ⓒBNL)

Advantages of Fermilab

- Up to $12\times$ more μ per p
- $4\times$ fill frequency
- $1/20\times$ hadronic-induced background at injection (π decay channel is 2 km vs 80 m)

$g-2$/EDM @ J-PARC: $\vec{E}=0$



Comparison: BNL vs FNAL vs J-PARC

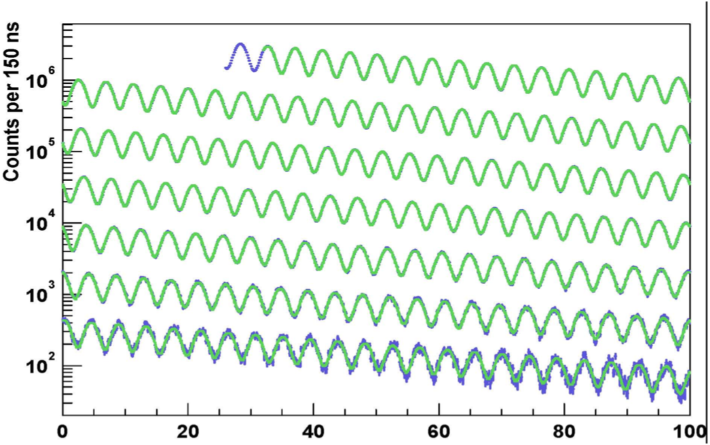
	BNL-E821	FNAL-E989	J-PARC-g-2/EDM
Muon momentum	3.09 GeV/c		0.3 GeV/c
γ	29.3		3
Polarization	100%		$> 90\%$
Storage field	$B = 1.45$ T		$B = 3.0$ T
Focusing field	Electric Quad.		very-weak magnetic
Cyclotron period	149 ns		7.4 ns
Anomalous spin precession period	4.37 μs		2.11 μs
# of detected e^+	5.0×10^9	1.8×10^{11}	1.5×10^{12}
# of detected e^-	3.6×10^9	–	–
Statistical precision	0.46 ppm	0.1 ppm	0.1 ppm

(CDR, g — 2/EDM expt, J-PARC '11)

BNL-E821 arrival-time spectrum ($E > 1.8\,\mathrm{GeV}$)

$$N(t,E) = N_0(E)\, e^{-t/\gamma\tau_\mu}\,[1 + A(E)\cos(\omega_a t + \phi(E))]$$

Fit to data out to about $640\,\mu$s $\sim 10\,\gamma\tau_\mu$



Arrival time modulo 100 μs