

# Tidal effects in the gravitational-wave phase of compact binaries

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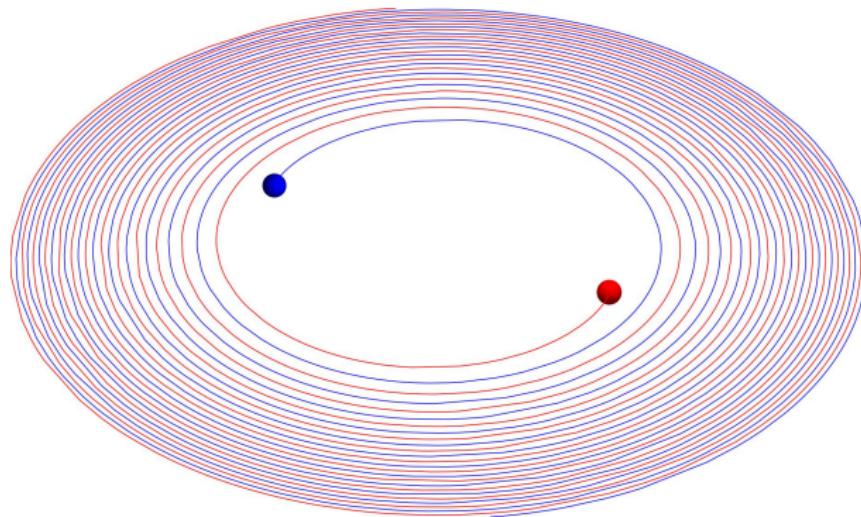
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# Introduction

**Topic :** Modelling the gravitational waves emitted by a binary neutron-star system including tidal effects up to the second relative post-Newtonian order.



HFB20a : [\[10.1103/PhysRevD.101.064047\]](https://doi.org/10.1103/PhysRevD.101.064047)

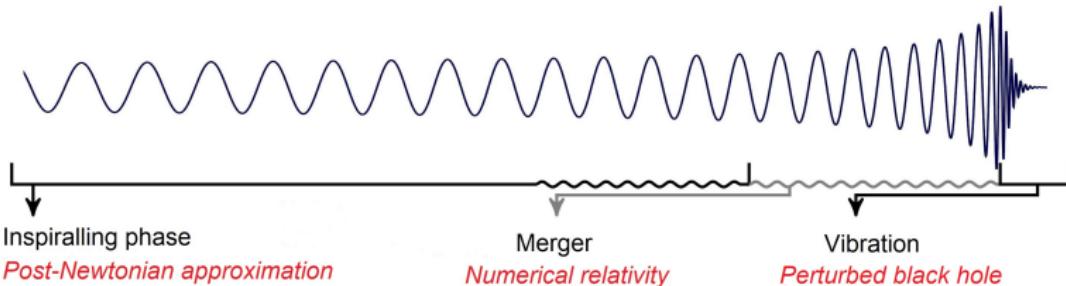
HFB20b : [\[10.1103/PhysRevD.102.044033\]](https://doi.org/10.1103/PhysRevD.102.044033)

HFB20c : [\[arXiv:2009.12332\]](https://arxiv.org/abs/2009.12332)

# Outline

- 1) Context and motivations.
- 2) Effective action at 2PN.
- 3) Equations of motion.
- 4) Gravitational field.
- 5) Balance equation.
- 6) Summary.

# The post-Newtonian formalism



→ Tidal effects negligible in the inspiralling phase but measurable in the pre-merger.

## PN formalism :

- Perturbative expansion of the equations of GR.

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu} \quad \text{with} \quad \tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

- Weak field, small velocities :  $(v/c) \ll 1$ .
- 2<sup>nd</sup> relative PN order  $\rightarrow O(1/c^4)$ .

# Motivations

- All GW detections from LIGO/Virgo came from compact binaries.

|                            | GW150914             | GW170817                  |
|----------------------------|----------------------|---------------------------|
| Chirp mass ( $M_{\odot}$ ) | $30.4^{+2.1}_{-1.9}$ | $1.188^{+0.004}_{-0.002}$ |
| Cycles                     | 8                    | $\sim 3000$               |

[LIGO, Virgo PRL 116, 061102 (2016);  
PRL 119, 161101 (2017)]

- NR inadequate for high number of cycles.
- Necessity to use analytical models for data analysis.
- Build more precise templates.
- Extend overlap between PN and NR.
- Compare with different analytical methods.
- Allows to constrain the equation of state of neutron stars through Love numbers  $\{k^{(\ell)}, j^{(\ell)}\}$ .

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# Effective action at 2PN

$$S = S_{\text{EH}} + S_{\text{pp}} + S_{\text{T}}$$

$$S_{\text{T}} = \sum_{A=1,2} \int d\tau_A \left[ \frac{\mu_A^{(2)}}{4} G_{\alpha\beta}^A G_A^{\alpha\beta} + \frac{\sigma_A^{(2)}}{6c^2} H_{\alpha\beta}^A H_A^{\alpha\beta} + \frac{\mu_A^{(3)}}{12} G_{\alpha\beta\gamma}^A G_A^{\alpha\beta\gamma} \right]$$

- $G_{\alpha\beta} \equiv -R_{\alpha\mu\beta\nu} u^\mu u^\nu$  : tidal quadrupole moment (mass type)
- $H_{\alpha\beta} \equiv 2R_{\alpha\mu\beta\nu}^* u^\mu u^\nu$  : tidal quadrupole moment (current type)
- $G_{\alpha\beta\gamma}$  : tidal octupole moment (mass type)

→ Tidal tensors regularized on the worldline of the bodies → removes proper field.

[HFB20a]

## Effective action at 2PN

$$S_T = \sum_{A=1,2} \int d\tau_A \left[ \frac{\mu_A^{(2)}}{4} G_{\alpha\beta}^A G_A^{\alpha\beta} + \frac{\sigma_A^{(2)}}{6c^2} H_{\alpha\beta}^A H_A^{\alpha\beta} + \frac{\mu_A^{(3)}}{12} G_{\alpha\beta\gamma}^A G_A^{\alpha\beta\gamma} \right]$$

- $\mu^{(\ell)}$  and  $\sigma^{(\ell)}$  linked to Love numbers :  $\mu_A^{(\ell)} = \frac{2k_A^{(\ell)}}{(2\ell - 1)!!} \frac{R_A^{2\ell+1}}{G}$ .
- $\mathcal{C} = \frac{Gm}{Rc^2} \sim 1$  for compact objects.
- $\mu^{(2)} \sim O\left(\frac{1}{c^{10}}\right) \sim \sigma^{(2)} \rightarrow$  Newtonian (leading) order (5PN)
- $\mu^{(3)} \sim O\left(\frac{1}{c^{14}}\right) \rightarrow$  2PN relative (7PN)

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# Computation of EOM and conserved quantities

Total action:  $S \equiv S_{\text{EH}} + S_{\text{pp}} + S_{\text{T}}$ .

- Build an action (called Fokker action)  $S_{\text{Fokker}}$  equivalent to  $S$ .
- Deduce the associated Lagrangian  $L_{\text{Fokker}}(\vec{y}_A, \vec{v}_A, \vec{a}_A, \dots, \vec{a}_A^{(n)})$ .
- Reduce  $L_{\text{Fokker}}$  to obtain an equivalent Lagrangian  $L(\vec{y}_A, \vec{v}_A, \vec{a}_A)$ .
- Compute the EOM.
- Compute the 10 conserved quantities  $\{P^i, J^i, G^i, E\}$ .

## 2PN CoM energy for circular orbits

$$\textcolor{green}{x} \equiv \left( \frac{GM\omega}{c^3} \right)^{2/3}$$

$$\begin{aligned}
 E = & -\frac{\mu c^2 \textcolor{green}{x}}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) \textcolor{green}{x} + \left( -\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24} \right) \textcolor{green}{x}^2 \right. \\
 & - 18 \tilde{\mu}_+^{(2)} \textcolor{green}{x}^5 + \left[ \left( -\frac{121}{2} + 33\nu \right) \tilde{\mu}_+^{(2)} - \frac{55}{2}\delta \tilde{\mu}_-^{(2)} - 176\tilde{\sigma}_+^{(2)} \right] \textcolor{green}{x}^6 \\
 & + \left[ \left( -\frac{20865}{56} + \frac{5434}{21}\nu - \frac{91}{4}\nu^2 \right) \tilde{\mu}_+^{(2)} + \delta \left( -\frac{11583}{56} + \frac{715}{12}\nu \right) \tilde{\mu}_-^{(2)} \right. \\
 & \left. \left. + \left( -\frac{2444}{3} + \frac{1768}{3}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{884}{3}\delta \tilde{\sigma}_-^{(2)} - 130\tilde{\mu}_+^{(3)} \right] \textcolor{green}{x}^7 \right\}
 \end{aligned}$$

[HFB20a]

# Hamiltonian for tidal effects

→ Hamiltonian computed by other groups with different methods (EFT, scattering amplitudes) in isotropic coordinates in the CoM.

- Transform  $L(\vec{y}_A, \vec{v}_A, \vec{a}_A)$  into  $L'(\vec{y}_A, \vec{v}_A)$ .
- Get a Hamiltonian in the CoM.
- Canonical transformation to get a Hamiltonian in isotropic coordinates
- Comparison: in full agreement with the litterature [Solon et al. 2020, Porto et al. 2020].
- Computation of the Delaunay Hamiltonian and the orbital precession for circular orbits.

$$\begin{aligned} K_{\text{tidal}} = & 45\tilde{\mu}_+^{(2)} \mathbf{x}^5 + \left[ \tilde{\mu}_+^{(2)} \left( \frac{1755}{4} - 120\nu \right) + \frac{315}{4}\tilde{\mu}_-^{(2)}\Delta + 624\tilde{\sigma}_+^{(2)} \right] \mathbf{x}^6 \\ & + \left[ \tilde{\mu}_+^{(2)} \left( \frac{64911}{16} - \frac{9381}{4}\nu + 140\nu^2 \right) + \tilde{\mu}_-^{(2)}\Delta \left( \frac{19191}{16} - \frac{945}{4}\nu \right) \right. \\ & \left. + \tilde{\sigma}_+^{(2)} \left( 6220 - 2544\nu \right) + 1156\tilde{\sigma}_-^{(2)}\Delta + 420\tilde{\mu}_+^{(3)} \right] \mathbf{x}^7 + O\left(\frac{1}{c^6}\right) \end{aligned}$$

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# Stress-energy tensor and potentials

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

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$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

We define  $\sigma \equiv \frac{T^{00} + T^{ii}}{c^2}$ ,  $\sigma_i \equiv \frac{T^{0i}}{c}$  and  $\sigma_{ij} \equiv T^{ij}$ ,

- $\square V = -4\pi G\sigma$
- $\square V_i = -4\pi G\sigma_i$
- $\square \hat{W}_{ij} = -4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V$

The metric is parametrized by the potentials :  $g_{\mu\nu} = g_{\mu\nu}[V, V_i, \hat{W}_{ij}]$

# Source multipolar moments and flux

**Multipolar moments :**

$$I_L = \text{FP} \int d^3x \int_{-1}^1 dz \left[ \delta_\ell \hat{x}_L \Sigma + \frac{a_\ell}{c^2} \delta_{\ell+1} \hat{x}_{iL} \dot{\Sigma}_i + \frac{b_\ell}{c^4} \delta_{\ell+2} \hat{x}_{ijL} \ddot{\Sigma}_{ij} \right] \left( x, u + \frac{zr}{c} \right)$$

$\Sigma$ ,  $\Sigma_i$  and  $\Sigma_{ij}$  contain the  $\sigma$ ,  $\sigma_i$ ,  $\sigma_{ij}$  and the potentials  $V$ ,  $V_i$  and  $\hat{W}_{ij}$ .

**Flux :**

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[ c_\ell \left( I_L^{(\ell+1)} \right)^2 + \frac{d_\ell}{c^2} \left( J_L^{(\ell+1)} \right)^2 + O \left( \frac{G}{c^3} \right) \right]$$

[Blanchet Living Review (2014)]

## 2.5PN CoM flux for circular orbits

$$\begin{aligned} \mathcal{F}_{\text{tidal}} = & \frac{192c^5\nu \textcolor{green}{x}^{10}}{5G} \left\{ (1+4\nu)\tilde{\mu}_+^{(2)} + \Delta\tilde{\mu}_-^{(2)} + \left[ \left( -\frac{22}{21} - \frac{1217}{168}\nu - \frac{155}{6}\nu^2 \right) \tilde{\mu}_+^{(2)} \right. \right. \\ & + \Delta \left( -\frac{22}{21} - \frac{23}{24}\nu \right) \tilde{\mu}_-^{(2)} + \left( -\frac{1}{9} + \frac{76}{3}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{1}{9}\Delta\tilde{\sigma}_-^{(2)} \Big] \textcolor{green}{x} \\ & + 4\pi \left[ (1+4\nu)\tilde{\mu}_+^{(2)} + \Delta\tilde{\mu}_-^{(2)} \right] \textcolor{green}{x}^{3/2} \\ & + \left[ \left( \frac{167}{54} - \frac{722429}{18144}\nu + \frac{15923}{336}\nu^2 + \frac{965}{12}\nu^3 \right) \tilde{\mu}_+^{(2)} + \Delta \left( \frac{167}{54} + \frac{66719}{2016}\nu \right. \right. \\ & - \frac{2779}{144}\nu^2 \Big) \tilde{\mu}_-^{(2)} + \left( -\frac{173}{756} + \frac{145}{3}\nu - 208\nu^2 \right) \tilde{\sigma}_+^{(2)} + \Delta \left( -\frac{173}{756} \right. \\ & \left. \left. + \frac{1022}{27}\nu \right) \tilde{\sigma}_-^{(2)} + \frac{80}{3}\nu\tilde{\mu}_+^{(3)} \right] \textcolor{green}{x}^2 + 4\pi \left[ \left( -\frac{22}{21} - \frac{5053}{1344}\nu - \frac{2029}{48}\nu^2 \right) \tilde{\mu}_+^{(2)} \right. \\ & + \Delta \left( -\frac{22}{21} - \frac{351}{64}\nu \right) \tilde{\mu}_-^{(2)} + \left( -\frac{1}{18} + \frac{226}{9}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{\Delta}{18}\tilde{\sigma}_-^{(2)} \Big] \textcolor{green}{x}^{5/2} \Big\} \end{aligned}$$

[HFB20b]

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# Balance equation

$$\mathcal{F} = -\frac{dE}{dt} \Rightarrow \varphi(\omega)$$

What is new in the phase?

| $\psi_{\text{tidal}}$ | Mass quadrupole | Current quadrupole | Mass octupole |
|-----------------------|-----------------|--------------------|---------------|
| 5PN (L)               | ✓               | ✗                  | ✗             |
| 6PN (NL)              | ✓               | ✓                  | ✗             |
| 7PN (NNL)             | new             | new                | ✓             |
| 6.5PN (tail)          | ✓               | ✗                  | ✗             |
| 7.5PN (tail)          | disagreement    | new                | ✗             |

## 2.5PN SPA phase for circular orbits

$$v = \left( \frac{\pi G m f}{c^3} \right)^{1/3} \text{ where } f \text{ is the orbital frequency.}$$

$$\begin{aligned} \psi_{\text{PP}} = & \frac{3}{128\nu v^5} \left\{ 1 + \left( \frac{3715}{756} + \frac{55}{9}\nu \right) v^2 - 16\pi v^3 + \left( \frac{15293365}{508032} + \frac{27145}{504}\nu + \frac{3085}{72}\nu^2 \right) v^4 \right. \\ & \left. + \left( \frac{38645}{252} - \frac{65}{3}\nu \right) \pi v^5 \ln \left( \frac{v}{v_0} \right) \right\} \end{aligned}$$

$$\begin{aligned} \psi_{\text{tidal}} = & - \frac{9v^5}{16\nu^2} \left\{ (1 + 22\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} + \left[ \left( \frac{195}{112} + \frac{1595}{28}\nu + \frac{325}{84}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left( \frac{195}{112} + \frac{4415}{336}\nu \right) \tilde{\mu}_-^{(2)} \right. \right. \\ & + \left( -\frac{5}{126} + \frac{1730}{21}\nu \right) \tilde{\sigma}_+^{(2)} - \frac{5}{126} \Delta \tilde{\sigma}_-^{(2)} \Big] v^2 - \pi \left[ (1 + 22\nu) \tilde{\mu}_+^{(2)} + \Delta \tilde{\mu}_-^{(2)} \right] v^3 \\ & + \left[ \left( \frac{136190135}{27433728} + \frac{975167945}{4572288}\nu - \frac{281935}{6048}\nu^2 + \frac{5}{3}\nu^3 \right) \tilde{\mu}_+^{(2)} + \Delta \left( \frac{136190135}{27433728} + \frac{211985}{2592}\nu \right. \right. \\ & \left. \left. + \frac{1585}{1296}\nu^2 \right) \tilde{\mu}_-^{(2)} + \left( -\frac{745}{4536} + \frac{1933490}{5103}\nu - \frac{3770}{81}\nu^2 \right) \tilde{\sigma}_+^{(2)} + \Delta \left( -\frac{745}{4536} + \frac{19355}{243}\nu \right) \tilde{\sigma}_-^{(2)} \right. \\ & \left. + \frac{1000}{27}\nu \tilde{\mu}_+^{(3)} \right] v^4 + \pi \left[ \left( -\frac{397}{112} - \frac{5343}{56}\nu + \frac{1315}{42}\nu^2 \right) \tilde{\mu}_+^{(2)} + \Delta \left( -\frac{397}{112} - \frac{6721}{336}\nu \right) \tilde{\mu}_-^{(2)} \right. \\ & \left. + \left( \frac{2}{21} - \frac{8312}{63}\nu \right) \tilde{\sigma}_+^{(2)} + \frac{2}{21} \Delta \tilde{\sigma}_-^{(2)} \right] v^5 \right\} \end{aligned}$$

[HFB20b]

## SPA phase for identical objects

**EOB tidal phase of the GW without spin (in the SPA) :**

$$\Psi^{\text{T,EOB}}(\textcolor{violet}{v}) = -\frac{117}{2}\tilde{\mu}^{(2)}\textcolor{violet}{v}^5 \left[ 1 + \frac{3115}{1248}\textcolor{violet}{v}^2 - \pi\textcolor{violet}{v}^3 + \left( \frac{23073805}{3302208} + \frac{20}{81}\bar{\alpha}_2^{(2)} \right. \right. \\ \left. \left. + \frac{20}{351}\beta_2^{22} \right) \textcolor{violet}{v}^4 - \frac{4283}{1092}\pi\textcolor{violet}{v}^5 \right]$$

[Damour, Nagar, Villain (2012)]

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[Damour, Nagar, Villain (2012)]

$$\psi_{\text{tidal}} = -\frac{117}{2} \tilde{\mu}^{(2)} v^5 \left[ 1 + \frac{3115}{1248} v^2 - \pi v^3 + \frac{379931975}{44579808} v^4 - \pi \frac{2137}{546} v^5 \right]$$

↪ Our work fixes  $\beta_2^{22} = \frac{642083}{1016064} \simeq 0.632$

↪ Slight disagreement on the tail term ( $\propto v^5$ ).

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# Summary

- Computation of the Lagrangian, EOM and conserved quantities.
  - ↪ Equations of motion
  - ↪ Energy in agreement with litterature
- Computation of the associated Hamiltonian.
  - ↪ Isotropic Hamiltonian in agreement with litterature
  - ↪ Delaunay Hamiltonian and orbital precession
- Computation of the flux and phase up to relative 2.5PN.
  - ↪ Flux
  - ↪ Phase in time domain
  - ↪ SPA phase

**Thank you !**