Noise propagation through TDI for the noise budget of LISA mission

Quang Nam Dam [1], Yves Lemière [2],
Antoine Petiteau [1], Joseph Martino [1]

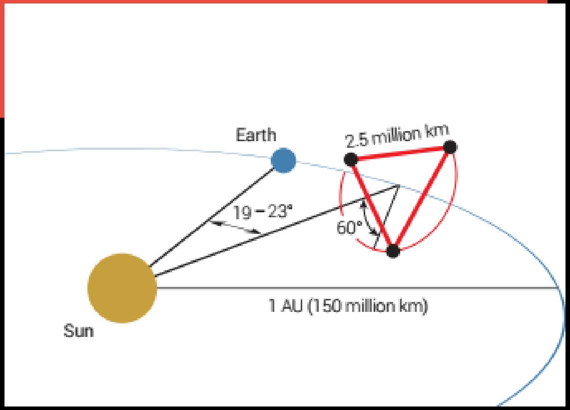
[1] Université de Paris, CNRS, Astroparticule et Cosmologie, F-75006 Paris, France
[2] Normandie Univ, ENSICAEN, UNICAEN, CNRS/IN2P3, LPC Caen, 14000 Caen, France

Troisième assemblée générale du
GdR Ondes Gravitationnelles,
15 October 2020

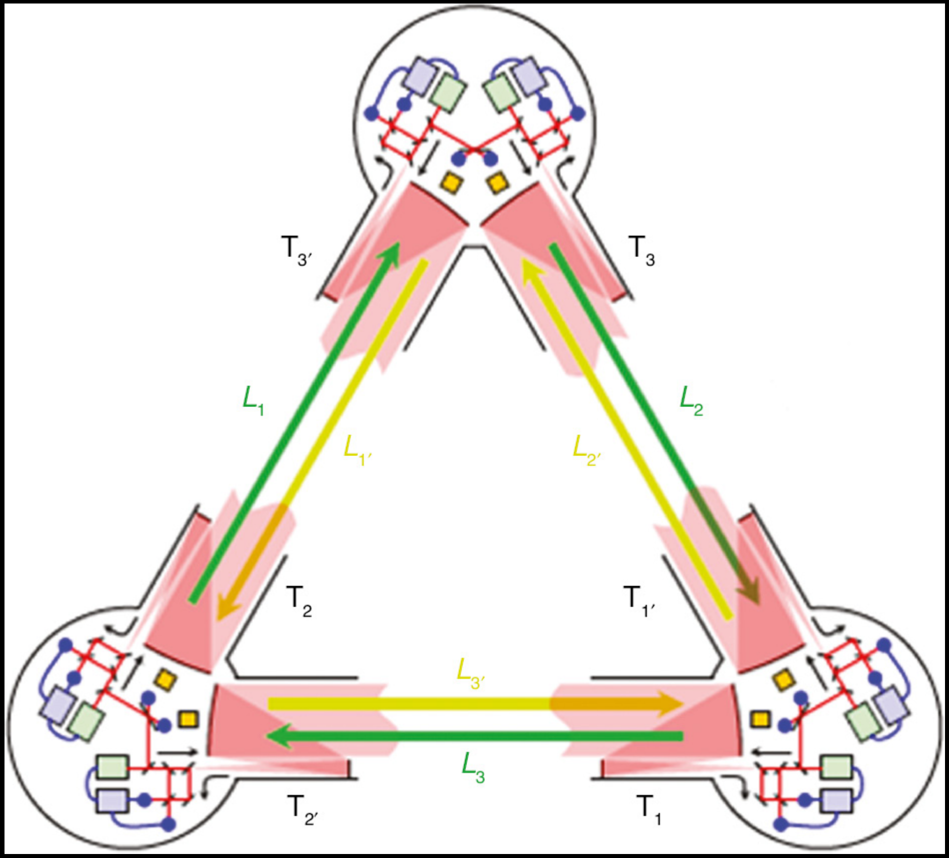
Outline

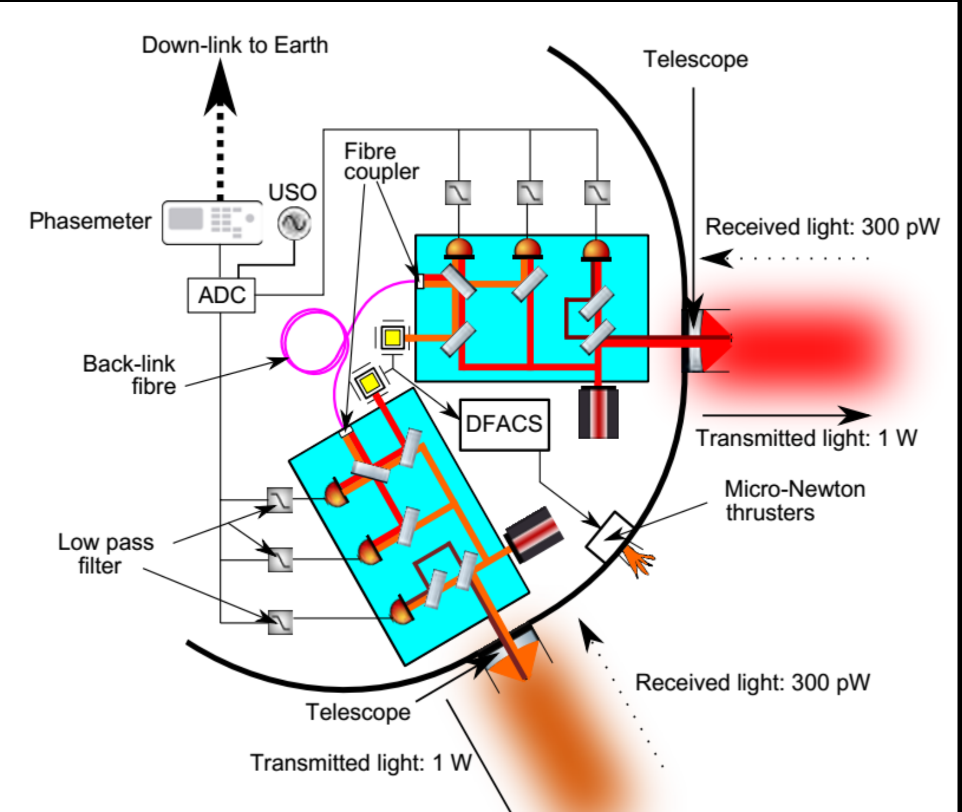
- Overview
- LISA measurement model
- TDI and noise propagation
- Conclusion

Overview of LISA



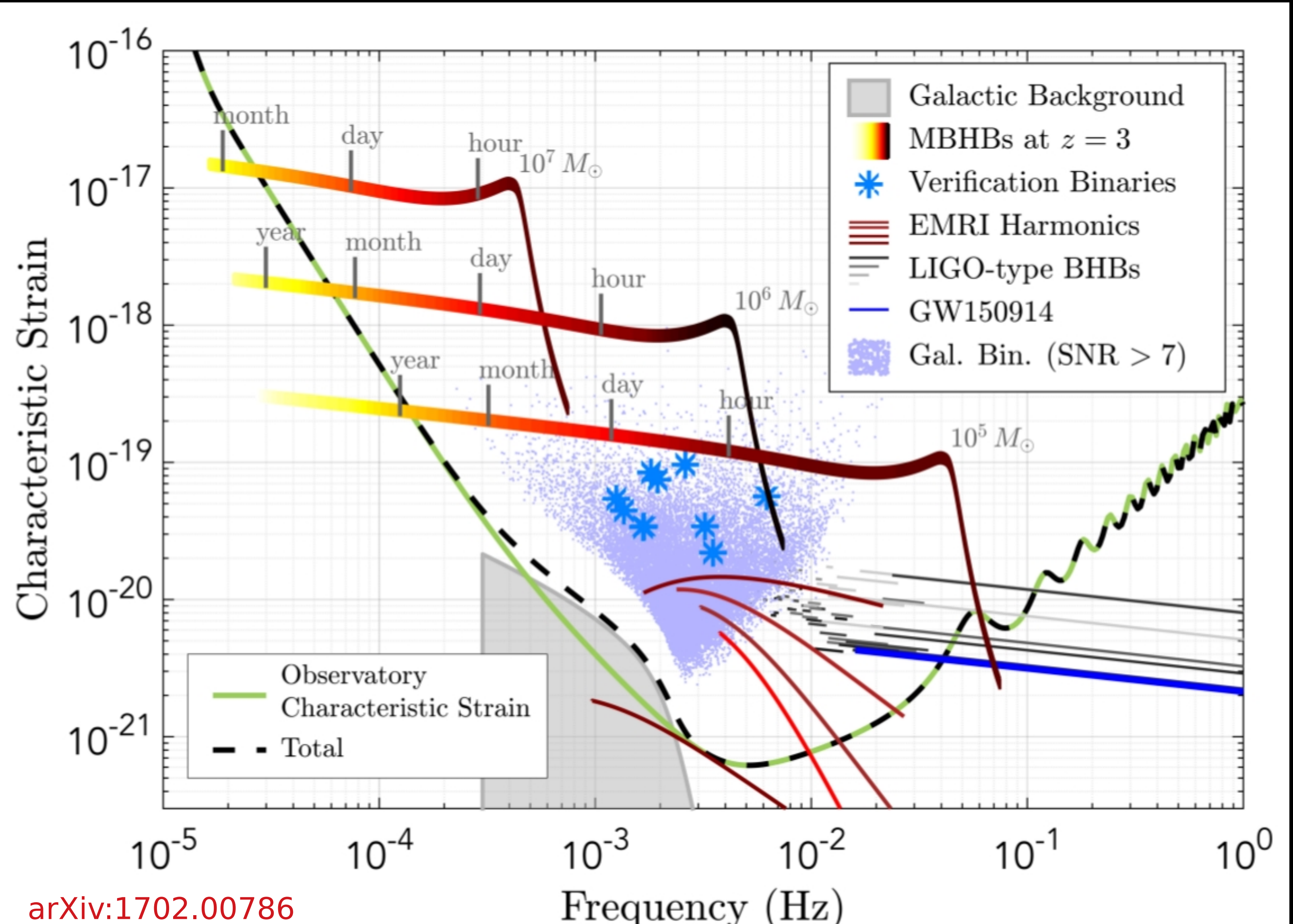
- Laser interferometer space antenna (LISA).
  - Detecting Gravitational Waves (GWs) from space
  - Use the laser links to measure the deformation of spacetime





From M. Otto, PhD thesis, 2016

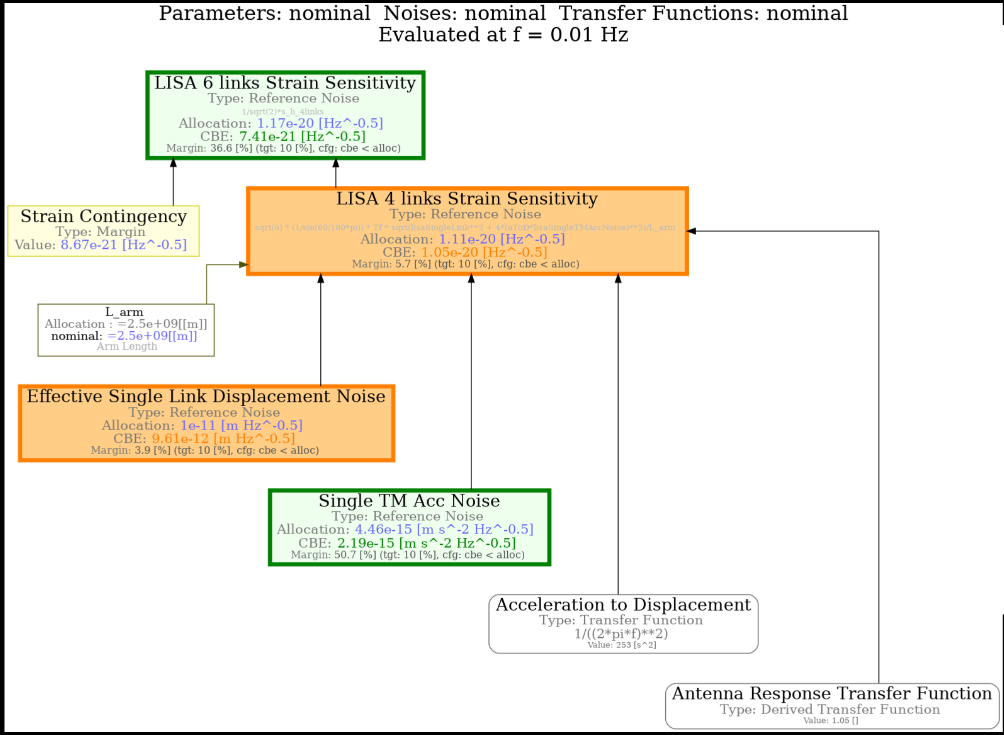
Overview of LISA

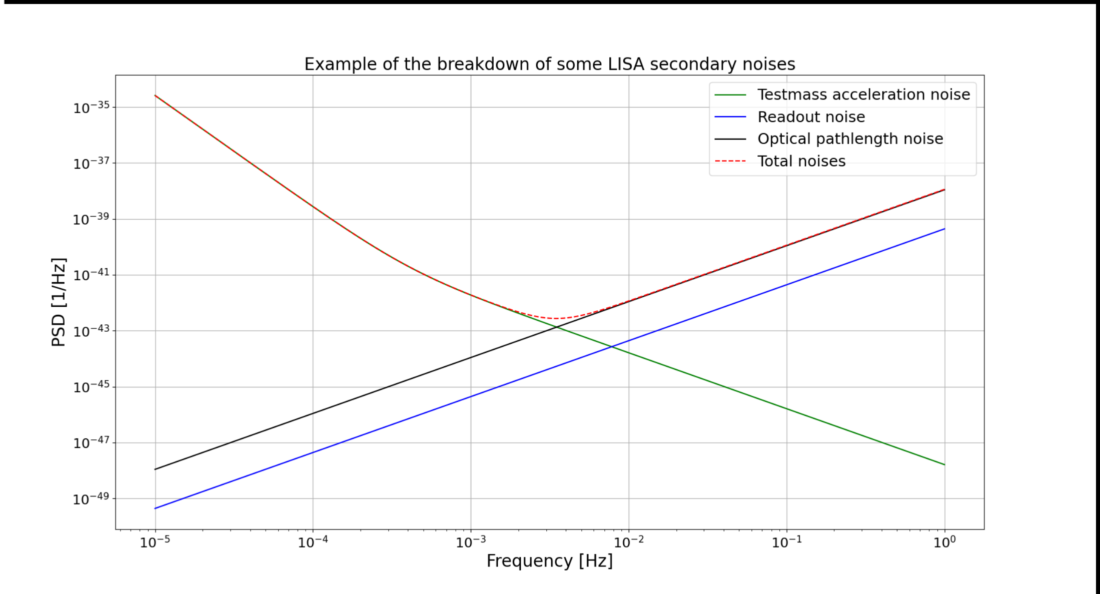


arXiv:1702.00786

LISA Performance model

- A frequency domain model of the instrument going from strain sensitivity to sub-system noises, which are built as the noise budget
- to parameterize all the noise contributors to the top-level strain sensitivity
- to study the performance trade-offs for the instrument design.





Goals, objectives and working team

- To study the performance of LISA instrument:
  - model the top-level noises that blur the GW signals
  - examine their propagation through Time-Delay Interferometry (TDI) algorithm, the aiming is to have the transfer function matrix of the TDI channels
- Focus on the non-suppressed noises, e.g. testmass acceleration, backlink, readout, optical pathlength noises.
- Investigate the impact of laser locking and noise correlation (WIP) to the transfer function
- This study has been working in collaboration with LDPG-WG6 Initial Noise Reduction Pipeline (INREP), LIG, Performance WG and Simulation WG.

Beam model

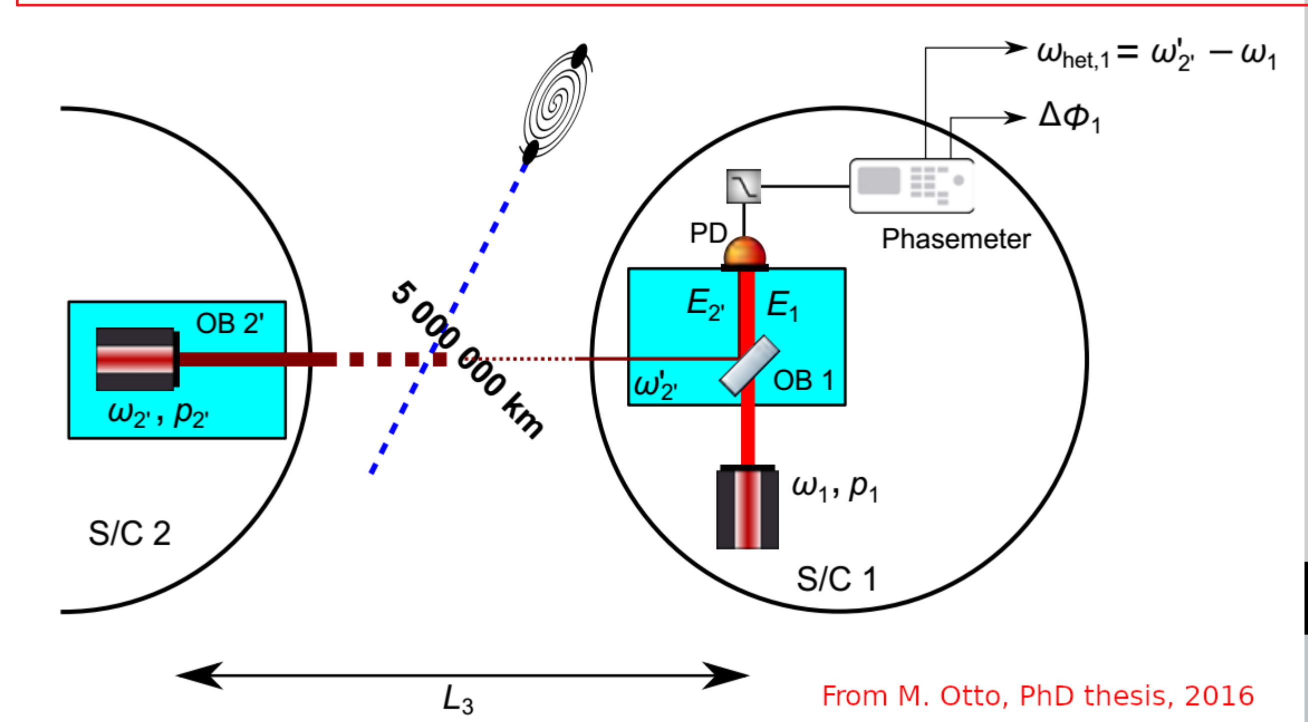
- This model is conducted by the collaboration of LDPG-WG6 INREP with supports from LIG, LISA PerfWG

$$
\begin{aligned}
b_{sc,2'\rightarrow 1} &= \mathbf{D}_3 \left[p_{2'} + N^{\mathrm{op}}_{TX,2'} - k_{2'}\hat{\mathbf{n}}_3 . \frac{[K\ \vec{\delta}_{2'} + \vec{\boldsymbol{\Delta}}_{2'}]}{1+K} + \frac{K}{1+K} N^{sens}_{2'} \right] \\
&\quad + ds_3 + gw_3 - k_{2'}\hat{\mathbf{n}}_{3'} . \frac{[K\ \vec{\delta}_1 + \vec{\boldsymbol{\Delta}}_1]}{1+K} + \frac{K}{1+K} N^{sens}_1 + N^{\mathrm{op}}_{RX,1} \\
b_{TM,1'\rightarrow 1} &= p_{1'} + \mu_{1'\rightarrow 1} + N^{\mathrm{op}}_{TM,1} \\
b_{ref,1'\rightarrow 1} &= p_{1'} + \mu_{1'\rightarrow 1} + N^{\mathrm{op}}_{ref,1} \\
b_{sc,1\rightarrow 1} &= p_1 + N^{\mathrm{op}}_{loc\rightarrow sc,1} \\
b_{TM,1\rightarrow 1} &= p_1 + 2k_1\hat{\mathbf{n}}_{3'} . \frac{(\vec{\boldsymbol{\Delta}}_1 - \vec{\delta}_1)}{1+K} + N^{\mathrm{op}}_{loc\rightarrow TM,1} \\
b_{ref,1\rightarrow 1} &= p_1 + N^{\mathrm{op}}_{loc\rightarrow ref,1}
\end{aligned}
$$

Beam model

- This model is conducted by the collaboration of LDPG-WG6 INREP with supports from LIG, LISA PerfWG

$$
\begin{aligned}
b_{sc,2'\to 1} \quad = \quad & \mathbf{D}_3 \left[p_{2'} + N^{\mathrm{op}}_{TX,2'} - k_{2'} \hat{\mathbf{n}}_3 . \frac{[K\ \vec{\delta}_{2'} + \vec{\mathbf{\Delta}}_{2'}]}{1+K} + \frac{K}{1+K} N^{sens}_{2'} \right] \\
& + ds_3 + gw_3 - k_{2'} \hat{\mathbf{n}}_{3'} . \frac{[K\ \vec{\delta}_1 + \vec{\mathbf{\Delta}}_1]}{1+K} + \frac{K}{1+K} N^{sens}_1 + N^{\mathrm{op}}_{RX,1}
\end{aligned}
$$



From M. Otto, PhD thesis, 2016

Beam model

- This model is conducted by the collaboration of LDPG-WG6 INREP with supports from LIG, LISA PerfWG

$$b_{sc,2'\to 1} = \mathbf{D}_3 \Big[p_{2'} + N^{op}_{TX,2'} - k_{2'}$$

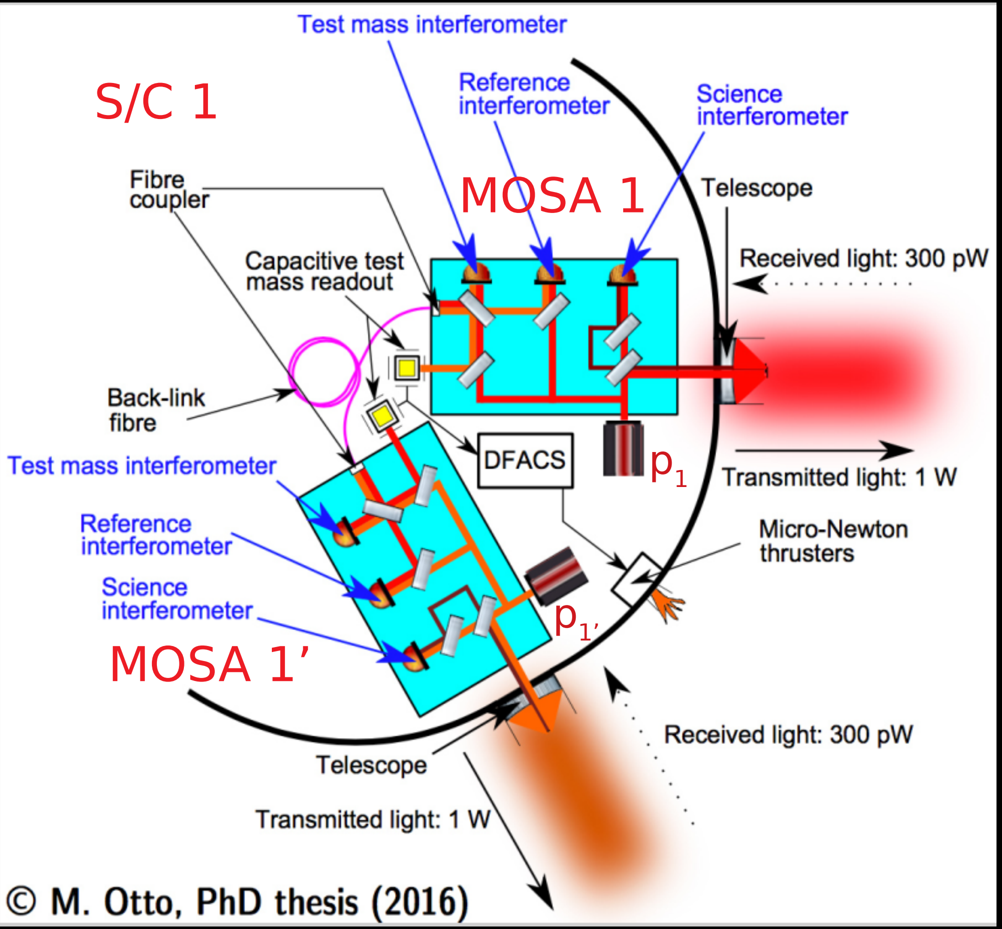
$$+ds_3 + gw_3 - k_{2'} \hat{\mathbf{n}}_{3'}.$$

$$\boxed{\begin{aligned} b_{TM,1'\to 1} &= p_{1'} + \mu_{1'\to 1} + N^{op}_{TM,1} \\ b_{ref,1'\to 1} &= p_{1'} + \mu_{1'\to 1} + N^{op}_{ref,1} \end{aligned}}$$

$$b_{sc,1\to 1} = p_1 + N^{op}_{loc\to sc,1}$$

$$b_{TM,1\to 1} = p_1 + 2k_1 \hat{\mathbf{n}}_{3'}.\frac{(\vec{\mathbf{\Delta}}_1 - \vec{\delta}_1)}{1+K}$$

$$b_{ref,1\to 1} = p_1 + N^{op}_{loc\to ref,1}$$

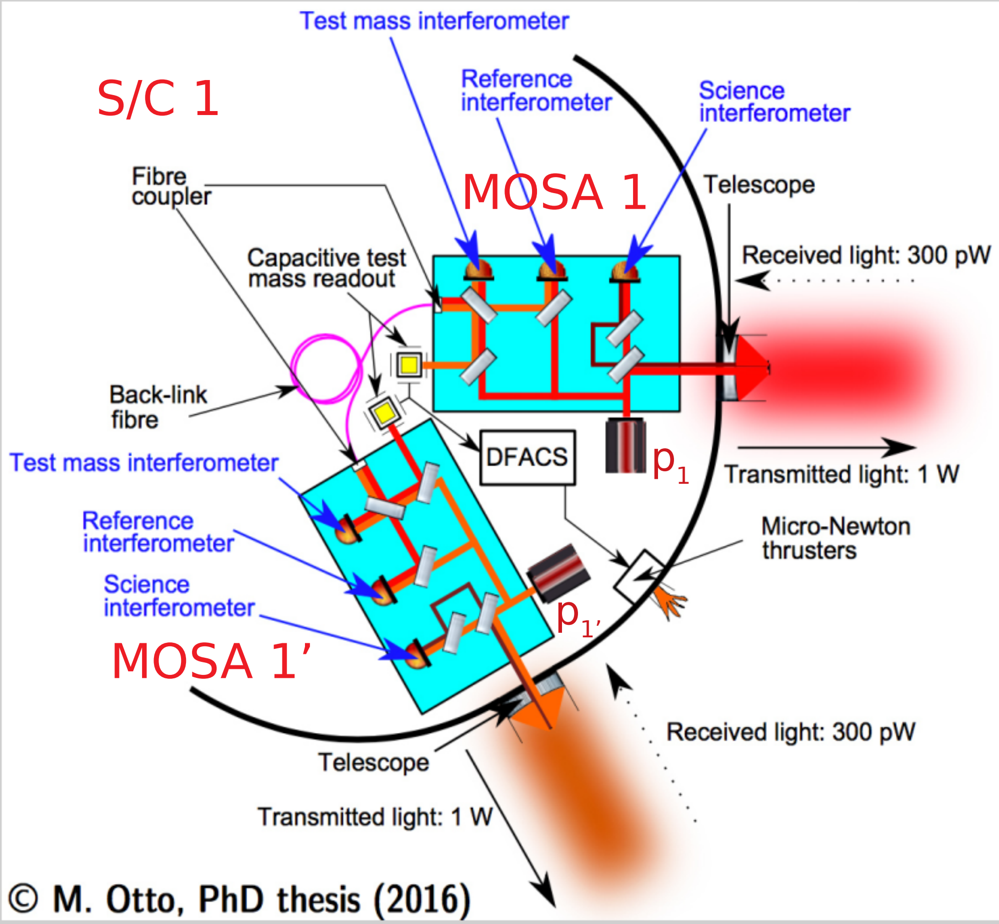


Beam model

- This model is conducted by the collabor
  supports from LIG, LISA PerfWG

$$
\begin{aligned}
b_{sc,2'\rightarrow 1} &= \mathbf{D}_3 \Big[p_{2'} + N^{\mathrm{op}}_{TX,2'} - k \\
&\quad + ds_3 + gw_3 - k_{2'} \hat{\mathbf{n}}_{3'}. \\
b_{TM,1'\rightarrow 1} &= p_{1'} + \mu_{1'\rightarrow 1} + N^{\mathrm{op}}_{TM,1} \\
b_{ref,1'\rightarrow 1} &= p_{1'} + \mu_{1'\rightarrow 1} + N^{\mathrm{op}}_{ref,1}
\end{aligned}
$$

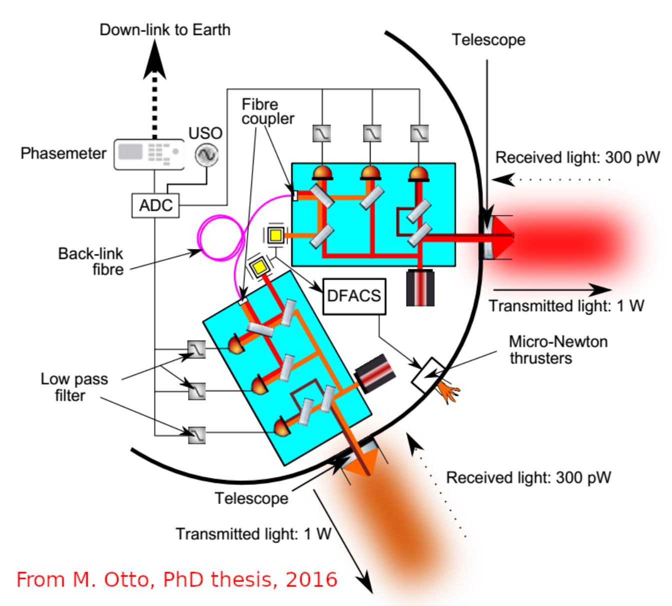
$$
\boxed{
\begin{aligned}
b_{sc,1\rightarrow 1} &= p_1 + N^{\mathrm{op}}_{loc\rightarrow sc,1} \\
b_{TM,1\rightarrow 1} &= p_1 + 2k_1 \hat{\mathbf{n}}_{3'}.\frac{(\vec{\mathbf{\Delta}}_1 - \vec{\delta}_1)}{1+K} + N^{\mathrm{op}}_{loc\rightarrow TM,1} \\
b_{ref,1\rightarrow 1} &= p_1 + N^{\mathrm{op}}_{loc\rightarrow ref,1}
\end{aligned}
}
$$



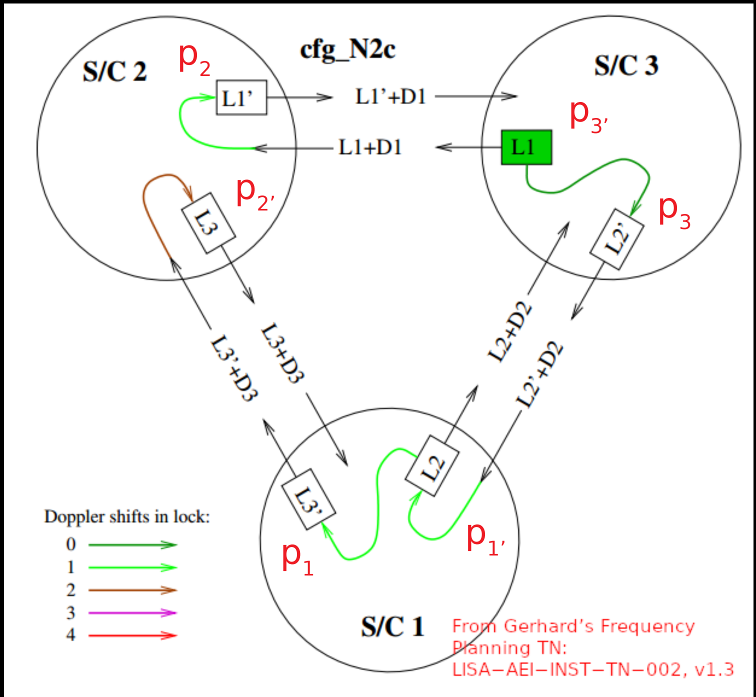
IFO measurements and laser locking

- IFO signal measurements are obtained from the interference of the beams

$$
\begin{aligned}
s_1^c &= \mathcal{F}\left[\theta_1^{2'}\left(b_{sc,2'\to 1} - b_{sc,1\to 1}\right) + N_{sc,1}^{ro}\right] \\
\epsilon_1 &= \mathcal{F}\left[\theta_1^{1'}\left(b_{TM,1'\to 1} - b_{TM,1\to 1}\right) + N_{\epsilon,1}^{ro}\right] \\
\tau_1 &= \mathcal{F}\left[\theta_1^{1'}\left(b_{ref,1'\to 1} - b_{ref,1\to 1}\right) + N_{\tau,1}^{ro}\right]
\end{aligned}
$$



- For the locking scheme cfg_N2c,

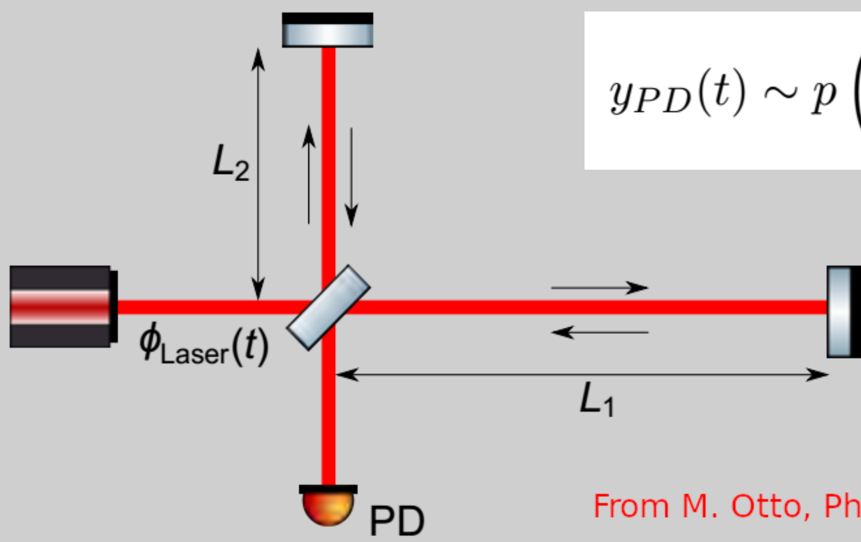


$$
\begin{aligned}
p_2 &= \theta_2^{3'}(N_{s,2}^{ro} - os_{3'\to 2}) + b_{sc,3'\to 2} - N_{loc\to sc,2}^{op} \\
p_3 &= \theta_3^{3'}(N_{\tau,3}^{ro} - os_{3'\to 3}) + b_{ref,3'\to 3} - N_{loc\to ref,3}^{op} \\
p_{1'} &= \theta_{1'}^{3}(N_{s,1'}^{ro} - os_{3\to 1'}) + b_{sc,3\to 1'} - N_{loc\to sc,1'}^{op} \\
p_1 &= \theta_1^{1'}(N_{\tau,1}^{ro} - os_{1'\to 1}) + b_{ref,1'\to 1} - N_{loc\to ref,1}^{op} \\
p_{2'} &= \theta_{2'}^{1}(N_{s,2'}^{ro} - os_{1\to 2'}) + b_{sc,1\to 2'} - N_{loc\to sc,2'}^{op}
\end{aligned}
$$

Introduction to TDI

- The laser frequency noise comes from the unequal armlengths in the Michelson interferometry.
- This is the most dominant noise source in LISA case, about 8 orders of magnitude to the strain sensitivity spectral density at ~ 30 mHz

$$y_{PD}(t) \sim p\left(t - \frac{2L_1}{c}\right) - p\left(t - \frac{2L_2}{c}\right)$$



From M. Otto, PhD Thesis, 2016

Figure 3.1: An unequal arm Michelson interferometer with armlengths L_1 and L_2 and physical roundtrip times $\frac{2L_1}{c}$ and $\frac{2L_2}{c}$. The laser comprises light with phase $\phi_{\text{Laser}}(t) = \omega t + p(t)$ propagated through the unequal arms. The PD at the output port detects the differential phase of the two light rays.

Introduction to TDI

- A post-processing algorithm to suppress the dominant laser frequency noise to be below the noise requirement, but still preserves the gravitational wave signals.
- In the case of LISA, GW signal is contained in the long-arm (science) signals. They represent 6 one-way link measurements, between one local and one remote laser beam.

$$s_1 = \mathbf{D}_3 p_2 - p_1, \quad s_{1'} = \mathbf{D}_{2'} p_3 - p_1\,,$$
$$s_2 = \mathbf{D}_1 p_3 - p_2, \quad s_{2'} = \mathbf{D}_{3'} p_1 - p_2\,,$$
$$s_3 = \mathbf{D}_2 p_1 - p_3, \quad s_{3'} = \mathbf{D}_{1'} p_2 - p_3\,.$$

?

$$\mathcal{F}\{s_i, \mathbf{D}_i\} = \sum_k \mathcal{P}_k\{\mathbf{D}_i\} p_k = 0$$

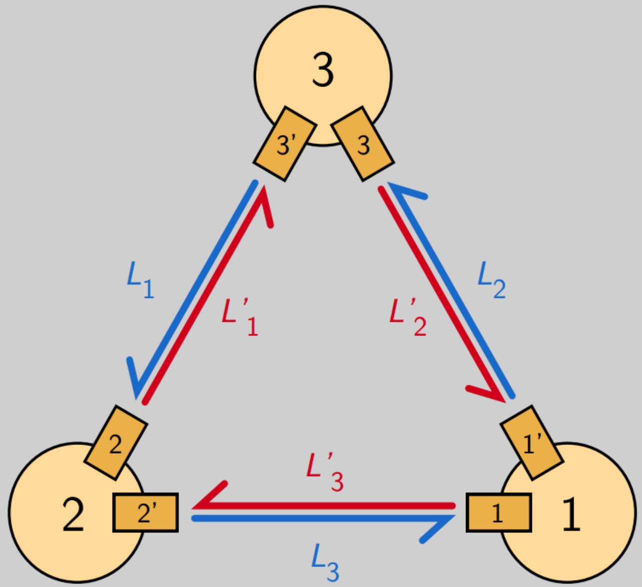


Figure 3.4: Conventions for labeling spacecraft, MOSAs, lasers, optical benches and arm-lengths. Primed indices are used for arms pointing clockwise, and for MOSAs and optical benches receiving light clockwise. From J-B. Bayle, PhD thesis, 2019

Introduction to TDI

- The basic idea of Time Delay Interferometry: combine the one-way link measurements with suitable time shift (delay) to have a pair of equal virtual photon paths.

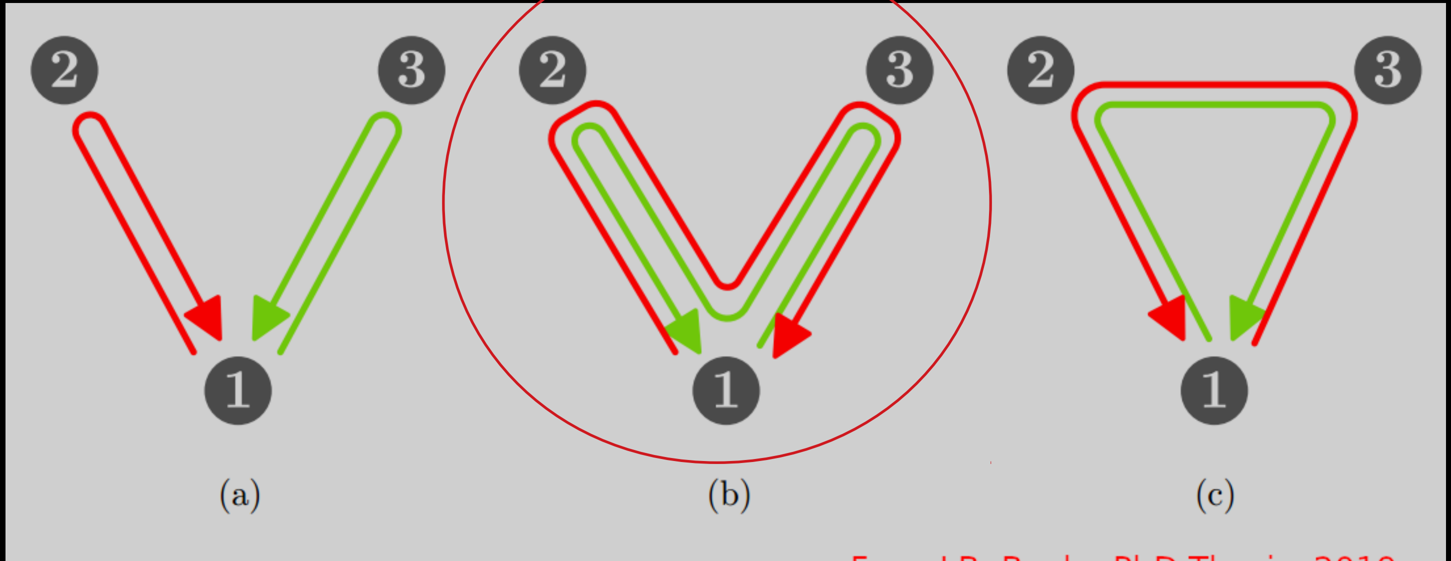
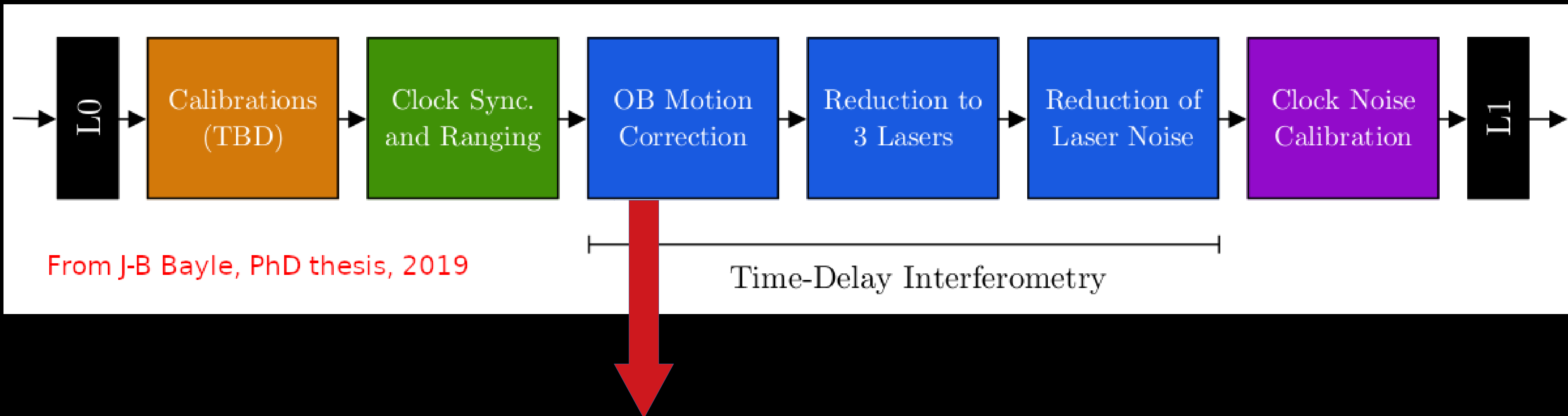


Figure 4.5: Examples of laser noise-calling virtual interferences. Panel (a) cancel laser noise under the assumption that armlengths are equal and constant. The first-generation Michelson X_1 and Sagnac α_1 are represented, respectively, in panels (b) and (c).

- Depend on the constellation configuration => generations of TDI.

TDI formulation

- Calibration chain involving TDI steps

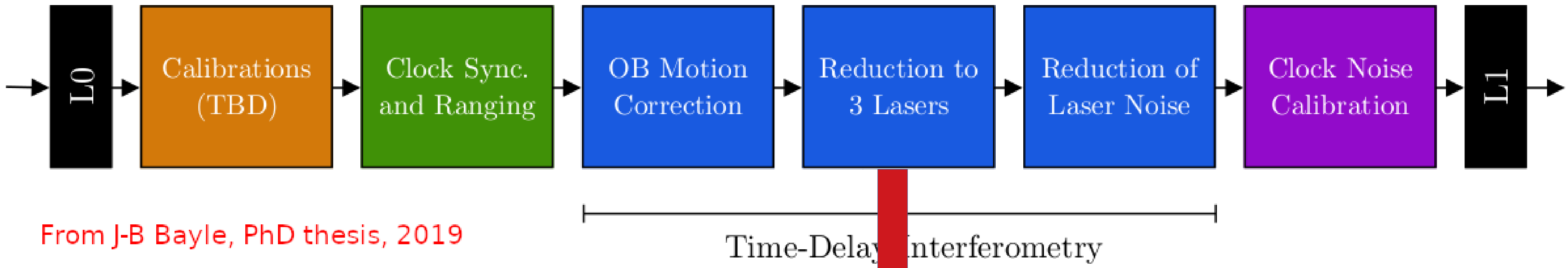


From J-B Bayle, PhD thesis, 2019

$$
\begin{aligned}
\xi_1 &= s_1^c - \theta_1^{2'} \theta_1^{1'} \frac{\lambda_1}{\lambda_{2'}} \frac{\epsilon_1(t) - \tau_1(t)}{2} - \theta_1^{2'} \theta_{2'}^{2} \frac{\mathcal{D}_3 \left[\epsilon_{2'}(t) - \tau_{2'}(t)\right]}{2} \\
\xi_{1'} &= s_{1'}^c - \theta_{1'}^{3} \theta_{1'}^{1} \frac{\lambda_{1'}}{\lambda_3} \frac{\epsilon_{1'}(t) - \tau_{1'}(t)}{2} - \theta_{1'}^{3} \theta_{3}^{3'} \frac{\mathcal{D}_{2'} \left[\epsilon_3(t) - \tau_3(t)\right]}{2}
\end{aligned}
$$

TDI formulation

- Calibration chain involving TDI steps

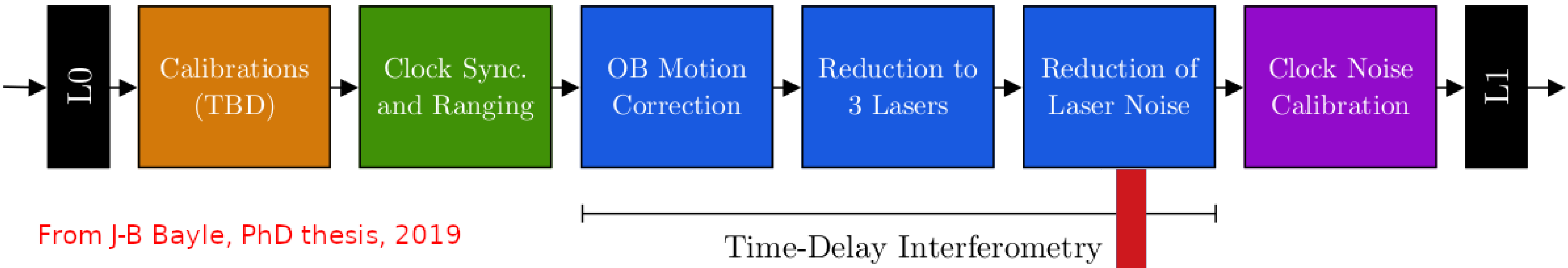


From J-B Bayle, PhD thesis, 2019

$$
\begin{aligned}
\eta_1(t) &= \theta_1^{2'} \xi_1(t) + \frac{\theta_{2'}^{2} \mathcal{D}_3 \tau_{2'}(t) - \theta_2^{2'} \mathcal{D}_3 \tau_2(t)}{2} \\
\eta_{1'}(t) &= \theta_{1'}^{3} \xi_{1'}(t) - \frac{\theta_{1'}^{1} \tau_{1'}(t) - \theta_1^{1'} \tau_1(t)}{2}
\end{aligned}
$$

TDI formulation

- Calibration chain involving TDI steps



From J-B Bayle, PhD thesis, 2019

$$X_{1.5} = (1 - \mathcal{D}_{33'})(\eta_{1'} + \mathcal{D}_{2'}\eta_3) - (1 - \mathcal{D}_{2'2})(\eta_1 + \mathcal{D}_3\eta_{2'}) \,.$$

$$\begin{aligned} X_{2.0} = {} & (1 - \mathcal{D}_{33'2'2})\left[(\eta_{1'} + \mathcal{D}_{2'}\eta_3) + \mathcal{D}_{2'2}(\eta_1 + \mathcal{D}_3\eta_{2'})\right] \\ & - (1 - \mathcal{D}_{22'3'3})\left[(\eta_1 + \mathcal{D}_3\eta_{2'}) + \mathcal{D}_{33'}(\eta_{1'} + \mathcal{D}_{2'}\eta_3)\right] . \end{aligned}$$

- Attention: there are many possible TDI combination!

Assumptions and approximations

- The main assumption in first part of the study: all the noises are uncorrelated.

- Another assumption is due to the fact that all non-suppressed noises are secondary noises, so we could neglect the varying arm-lengths affect → approx. A1 . Consequently,

$$X_{2.0} \approx (1 - \mathcal{D}_{2'233'})\, X_{1.5}$$

- Some approximations we may use to simplify the results:

  + all arm-lengths are equal → approx. A2

  + all noises of the same type have the same input PSD → approx. A3

  + all nominal laser frequencies are equal, so are their wave-lengths → approx. A4

- All signals are in units of fractional frequency deviation.

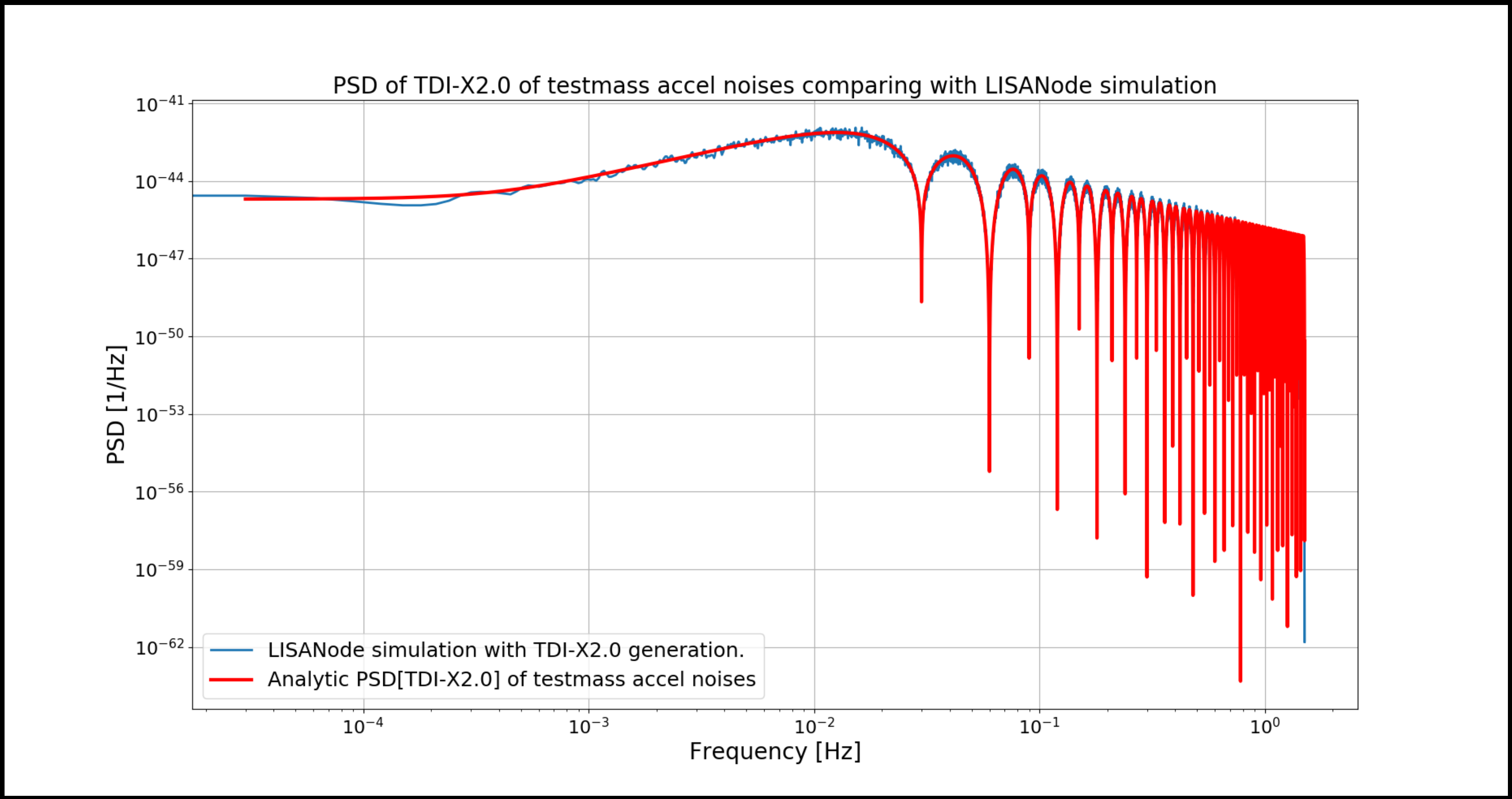
Some highlights of the result

- The demonstration of the computation procedure and the result of transfer function for non-suppressed noises are presented in the backups
- Laser locking has no significant impact to the propagation of non-suppressed noises if we assume that the arm-lengths are constant and the nominal laser frequencies are equal.
- With all previous approximations, only PSD TDI-XX and CSD-XY are needed to calculate to build the transfer function matrix
- Comparison between analytic PSD of some noises and the simulated data (using LISANode simulator): good match.

Result for TM acceleration noise

$$PSD\left[X_{1.5}^{TMaccel}\right] = \frac{16}{c^2}\sin^2(\omega L)\left[3+\cos(2\omega L)\right]S_\delta$$
$$PSD\left[X_{2.0}^{TMaccel}\right] = \frac{64}{c^2}\sin^2(\omega L)\sin^2(2\omega L)\left[3+\cos(2\omega L)\right]S_\delta$$

$$CSD\left[XY_{1.5}^{TMaccel}\right] = -\frac{16}{c^2}\sin(\omega L)\sin(2\omega L)\,S_\delta$$
$$CSD\left[XY_{2.0}^{TMaccel}\right] = -\frac{64}{c^2}\sin(\omega L)\sin^3(2\omega L)\,S_\delta$$



Conclusion

- In summary,
  - LISA is prospective GWs detector to “hear” the Universe.
  - TDI algorithm is necessary to suppress laser noise and other noises.
  - The computation of transfer function of non-suppressed noises through TDI algorithm with 1.5 and 2.0 generations has been presented and could be used widely for other noises.
  - No impact of laser locking to the propagation of non-suppressed noises, in the limitations of fixing arm-lengths and equal nominal laser frequencies.
  - The analytic PSDs of some noises are matched with the data simulation.
- For the next steps,
  - Consider the noise correlation in the study (WIP)
  - Examine the propagation of the suppressed noises (laser, clock, S/C jitter noises) with/without laser locking (partly done by Olaf Hartwig et al.)
  - The DFACs control loop should be studied and implemented more precise with the help from LISA Dynamics team (with Henri Inchaupsé and other colleagues)

=> Our results will be used by ESA and its prime companies to flow down the requirements from science to instrument subsystem.

Thank you for your attention!

Backups

Strategy of the study

Analytic study

- Use beam model w/wo laser locking scheme to construct IFO measurements
- Propagate IFO signals through TDI formulation
- Compute the spectral density of the noises in TDI variables: PSD and CSD

Simulation study

- Implement non-suppressed noises in LISANode
- Run the LISANode for each noise to get TDI variables
- Compare the PSDs of TDI variables between analytic function and simulation

→ validate the analytic results

Mathematica

- for supporting and double-checking the analytic results
- structure the study to be reproducible

Transfer function matrix of non-suppressed noises through TDI

Procedure for computing transfer function

- If considering laser locking, substitute the laser noises from the locking scheme into the beam model.
- Ignore all the noises in the beams except the one that is considered
- Compute the intermediary variables
- Compute the TDI Michelson combination, write them in the product of nested delay operator applied to each noise
- Compute the PSD/CSD of each term individually
  - use approx. A1 $\rightarrow$ commute delay operators
  - Fourier transform of nested delay operators
  - some patterns to look up
- Sum up components
- Use some assumptions/approximations to get simplified results

Propagation of tess-mass acceleration noise (1): IFO signals

- For simple demonstration, we ignore the laser locking scheme. One interested the computation with laser locking could refer to the LDPG-WG6 INREP technical note.

- Neglect all other noises except TM acceleration noise in beam formulas, and consider only the projection of TM displacement on the sensitive axis, $\delta_i = \vec{\delta}_i.\vec{\mathbf{n}}_{(i+2)'}$ and $\delta_{i'} = -\vec{\delta}_{i'}.\vec{\mathbf{n}}_{i+1}$

- The formulation of the IFO signals measurement:

$$\begin{cases} s_1^c &= \theta_1^{2'} k_{2'} \dfrac{K(\mathbf{D}_3\delta_{2'} - \delta_1)}{1+K} \\ \tau_1 &= 0 \\ \epsilon_1 &= 2\theta_1^{1'} k_1 \dfrac{\delta_1}{1+K} \end{cases} \qquad \begin{cases} s_{1'}^c &= -\theta_{1'}^3 k_3 \dfrac{K(\mathbf{D}_{2'}\delta_3 - \delta_{1'})}{1+K} \\ \tau_{1'} &= 0 \\ \epsilon_{1'} &= -2\theta_{1'}^1 k_{1'} \dfrac{\delta_{1'}}{1+K} \end{cases}$$

- Neglect the ranging error, so the TDI delay operator is perfectly the same with the propagation delay operator $\mathbf{D}_i \approx \mathcal{D}_i$

- Work in units of fractional frequency deviation, so $k_i = \dfrac{1}{c}$

Propagation of tess-mass acceleration noise (2): Intermediary variables

- The next step is to compute the intermediary variables, for the S/C jitter removal and the reduction of free-running laser number.

$$
\begin{aligned}
\xi_1 &= s_1^c - \theta_1^{2'}\theta_1^{1'}\frac{\lambda_1}{\lambda_{2'}}\frac{\epsilon_1(t)-\tau_1(t)}{2} - \theta_1^{2'}\theta_{2'}^{2}\frac{\mathcal{D}_3\epsilon_{2'}(t)-\mathcal{D}_3\tau_{2'}(t)}{2} \\
&= \frac{1}{c}\theta_1^{2'}\left(\mathcal{D}_3\delta_{2'} - \frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\delta_1\right) \\
\xi_{1'} &= s_{1'}^c - \theta_{1'}^{3}\theta_{1'}^{1}\frac{\lambda_{1'}}{\lambda_3}\frac{\epsilon_{1'}(t)-\tau_{1'}(t)}{2} - \theta_{1'}^{3}\theta_{3}^{3'}\frac{\mathcal{D}_{2'}\left[\epsilon_3(t)-\tau_3(t)\right]}{2} \\
&= -\frac{1}{c}\theta_{1'}^{3}\left(\mathcal{D}_{2'}\delta_3 - \frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\delta_{1'}\right)
\end{aligned}
$$

$$
\begin{aligned}
\eta_1 &= \theta_1^{2'}\xi_1(t) + \frac{\theta_{2'}^{2}\mathcal{D}_3\tau_{2'}(t) - \theta_2^{2'}\mathcal{D}_3\tau_2(t)}{2} \\
&= \frac{1}{c}\left(\mathcal{D}_3\delta_{2'} - \frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\delta_1\right) \\
\eta_{1'} &= \theta_{1'}^{3}\xi_{1'}(t) - \frac{\theta_{1'}^{1}\tau_{1'}(t) - \theta_1^{1'}\tau_1(t)}{2} \\
&= -\frac{1}{c}\left(\mathcal{D}_{2'}\delta_3 - \frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\delta_{1'}\right)
\end{aligned}
$$

Propagation of tess-mass acceleration noise (3): TDI formulation

- With the fixing arm-lengths (approx. A1), the TDI-X in both generations are

$$X_{1.5}^{TMaccel} = \frac{1}{c}\left[(1-\mathcal{D}_{2'2})\left(\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}+\mathcal{D}_{33'}\right)\delta_1 + (1-\mathcal{D}_{33'})\left(\frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}+\mathcal{D}_{2'2}\right)\delta_{1'} \right.$$
$$\left. - \left(1+\frac{K+\frac{\lambda_{2'}}{\lambda_1}}{K+1}\right)(1-\mathcal{D}_{2'2})\mathcal{D}_3\delta_{2'} - \left(1+\frac{K+\frac{\lambda_3}{\lambda_{1'}}}{K+1}\right)(1-\mathcal{D}_{33'})\mathcal{D}_{2'}\delta_3\right].$$

$$X_{2.0} \approx (1-\mathcal{D}_{2'233'})\,X_{1.5}$$

- The TDI-Y 1.5 and 2.0 are deduced from the TDI-X by indices permutation

$$Y_{1.5}^{TMaccel} = \frac{1}{c}\left[(1-\mathcal{D}_{3'3})\left(\frac{K+\frac{\lambda_2}{\lambda_{3'}}}{K+1}+\mathcal{D}_{11'}\right)\delta_2 + (1-\mathcal{D}_{11'})\left(\frac{K+\frac{\lambda_{2'}}{\lambda_1}}{K+1}+\mathcal{D}_{3'3}\right)\delta_{2'} \right.$$
$$\left. - \left(1+\frac{K+\frac{\lambda_{3'}}{\lambda_2}}{K+1}\right)(1-\mathcal{D}_{3'3})\mathcal{D}_1\delta_{3'} - \left(1+\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\right)(1-\mathcal{D}_{11'})\mathcal{D}_{3'}\delta_1\right].$$

$$Y_{2.0} \approx (1-\mathcal{D}_{3'311'})\,Y_{1.5}$$

Propagation of tess-mass acceleration noise (4): Some patterns for PSD/CSD computation

- to be more generalized, we don't present the specific PSD and CSD computation of the TM accel noise here, but show some patterns that one could use for deriving PSD and CSD. The other patterns exist in the PSD and CSD could be computed as the same process as following

$$\begin{aligned} PSD\left[\pm\left(1-\mathcal{D}_{ii'}\right)x(t)\right](\omega) &= \left\langle \left[\widetilde{\left(1-\mathcal{D}_{ii'}\right)x(t)}\right](\omega)\times\left[\widetilde{\left(1-\mathcal{D}_{ii'}\right)x(t)}\right]^{*}(\omega)\right\rangle \\ &= \left\langle\left(1-e^{-j\omega(L_i+L_{i'})}\right)\left(1-e^{j\omega(L_i+L_{i'})}\right)\widetilde{x}(\omega)\widetilde{x}^{*}(\omega)\right\rangle \\ &= \left\langle\left(2-e^{-j\omega(L_i+L_{i'})}-e^{j\omega(L_i+L_{i'})}\right)\widetilde{x}(\omega)\widetilde{x}^{*}(\omega)\right\rangle \\ &= 2\left(1-\cos\left[\omega(L_i+L_{i'})\right]\right)\left\langle\widetilde{x}(\omega)\widetilde{x}^{*}(\omega)\right\rangle \\ &= 4\sin^2\left(\omega\bar{L}_i\right)S_x, \end{aligned}$$

$$\begin{aligned} PSD[(a\pm b\mathcal{D}_{ii'})x(t)] &= \left\langle\left(a\pm be^{-j\omega(L_i+L_{i'})}\right)\left(a\pm be^{j\omega(L_i+L_{i'})}\right)\tilde{x}(\omega)\tilde{x}^{*}(\omega)\right\rangle \\ &= \left[a^2+b^2\pm 2ab\cos\omega(L_i+L_{i'})\right]S_x \end{aligned}$$

Propagation of tess-mass acceleration noise (4): Some patterns for PSD/CSD computation

$$
\begin{aligned}
PSD\left[\pm\left(1+\mathcal{D}_{ii'}\right)\left(1-\mathcal{D}_{kk'}\right)x(t)\right](\omega) &= \Big\langle\left(1+e^{-j\omega(L_i+L_{i'})}\right)\left(1+e^{j\omega(L_i+L_{i'})}\right) \\
&\quad\times\left(1-e^{-j\omega(L_k+L_{k'})}\right)\left(1-e^{j\omega(L_k+L_{k'})}\right)\widetilde{x}(\omega)\widetilde{x}^*(\omega)\Big\rangle \\
&= 2\left(1+\cos\left[\omega(L_i+L_{i'})\right]\right) \\
&\qquad \times 2\left(1-\cos\left[\omega(L_k+L_{k'})\right]\right)\langle\widetilde{x}(\omega)\widetilde{x}^*(\omega)\rangle \\
&= 16\cos^2\left(\omega\frac{L_i+L_{i'}}{2}\right)\sin^2\left(\omega\frac{L_k+L_{k'}}{2}\right)S_x, \qquad (85)
\end{aligned}
$$

$$
\begin{aligned}
CSD\left[XY\right] &= \Big\langle\left[\widetilde{(a+b\mathcal{D}_{ii'})\,x(t)}\right](\omega)\times\left[\widetilde{\left(1-\mathcal{D}_{jj'}\right)\mathcal{D}_{j_1j_2\ldots j_n}x(t)}\right]^*(\omega)\Big\rangle \\
&= \Big\langle\left(a+be^{-j\omega(L_i+L_{i'})}\right)\left(1-e^{j\omega(L_j+L_{j'})}\right)e^{j\omega(L_{j_1}+L_{j_2}+\ldots+L_{j_n})}\widetilde{x}(\omega)\widetilde{x}^*(\omega)\Big\rangle \\
&= e^{j\omega(L_{j_1}+L_{j_2}+\ldots+L_{j_n}-\bar{L}_i+\bar{L}_j)}\left(ae^{j\omega\bar{L}_i}+be^{-j\omega\bar{L}_i}\right)\left(e^{-j\omega\bar{L}_j}-e^{j\omega\bar{L}_j}\right)\langle\widetilde{x}(\omega)\widetilde{x}^*(\omega)\rangle \\
&= -2j\sin\left(\omega\bar{L}_j\right)e^{j\omega(L_{j_1}+L_{j_2}+\ldots+L_{j_n}-\bar{L}_i+\bar{L}_j)}\left(ae^{j\omega\bar{L}_i}+be^{-j\omega\bar{L}_i}\right)S_x \qquad (90)
\end{aligned}
$$

Propagation of tess-mass acceleration noise (5): PSD of TDI-XX

Use previous patterns, we could compute the PSD/CSD for each of noise terms. Then we sum up to get the final result

$$
\begin{aligned}
PSD\left[X_{1.5}^{TMaccel}\right] \quad = \quad & \frac{4}{c^2}\sin^2\frac{\omega(L_2+L_{2'})}{2}\left\{\left(1+\frac{K+\frac{\lambda_{2'}}{\lambda_1}}{K+1}\right)^2 S_{\delta_{2'}}\right. \\
& \left. +\left[\left(\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\right)^2+1+2\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\cos\omega(L_3+L_{3'})\right]S_{\delta_1}\right\} \\
& +\frac{4}{c^2}\sin^2\frac{\omega(L_3+L_{3'})}{2}\left\{\left(1+\frac{K+\frac{\lambda_3}{\lambda_{1'}}}{K+1}\right)^2 S_{\delta_3}\right. \\
& \left. +\left[\left(\frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\right)^2+1+2\frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\cos\omega(L_2+L_{2'})\right]S_{\delta_{1'}}\right\}
\end{aligned}
$$

$$
PSD[X_{2.0}] \quad \approx \quad 4\sin^2\frac{\omega(L_{2'}+L_2+L_3+L_{3'})}{2}PSD[X_{1.5}]
$$

Propagation of tess-mass acceleration noise (6): CSD of TDI-XY

$$
\begin{aligned}
CSD\left[XY_{1.5}^{TMaccel}\right] \quad = \quad & -\frac{4}{c^2}\sin\frac{\omega(L_1+L_{1'})}{2}\sin\frac{\omega(L_2+L_{2'})}{2}e^{\frac{j\omega(L_1+L_{1'}-L_2-L_{2'}-L_3+L_{3'})}{2}}\left(1+2K+\frac{\lambda_1}{\lambda_{2'}}\right) \\
& \times\left[\left(1+2K+\frac{\lambda_1}{\lambda_{2'}}\right)\cos\frac{\omega(L_3+L_{3'})}{2}+j\left(\frac{\lambda_1}{\lambda_{2'}}-1\right)\sin\frac{\omega(L_3+L_{3'})}{2}\right]S_{\delta_1} \\
& -\frac{4}{c^2}\sin\frac{\omega(L_1+L_{1'})}{2}\sin\frac{\omega(L_2+L_{2'})}{2}e^{\frac{j\omega(L_1+L_{1'}-L_2-L_{2'}-L_3+L_{3'})}{2}}\left(1+2K+\frac{\lambda_{2'}}{\lambda_1}\right) \\
& \times\left[\left(1+2K+\frac{\lambda_{2'}}{\lambda_1}\right)\cos\frac{\omega(L_3+L_{3'})}{2}+j\left(1-\frac{\lambda_{2'}}{\lambda_1}\right)\sin\frac{\omega(L_3+L_{3'})}{2}\right]S_{\delta_{2'}}
\end{aligned}
$$

With the approximation of fixing arm-lengths, the CSD of TDI-XY 2.0 is related to the CSD of TDI-XY 1.5 via following formula

$$
\begin{aligned}
CSD[XY_{2.0}] \quad \approx \quad & CSD[XY_{1.5}] \\
& \times\left[1-e^{-j\omega(L_2+L_{2'}+L_3+L_{3'})}-e^{j\omega(L_3+L_{3'}+L_1+L_{1'})}+e^{j\omega(L_1+L_{1'}-L_2-L_{2'})}\right]
\end{aligned}
$$

Propagation of tess-mass acceleration noise (7): Simplified results

Use the approximations $S_{\delta_i} = S_{\delta_{i'}} = S_\delta$ $L_i = L_{i'} = L.$ $\lambda_i = \lambda.$

$$
\begin{aligned}
PSD\left[X_{1.5}^{TMaccel}\right] &= \frac{16}{c^2}\sin^2(\omega L)\left[3+\cos(2\omega L)\right]S_\delta \\
PSD\left[X_{2.0}^{TMaccel}\right] &= \frac{64}{c^2}\sin^2(\omega L)\sin^2(2\omega L)\left[3+\cos(2\omega L)\right]S_\delta
\end{aligned}
$$

$$
\begin{aligned}
CSD\left[XY_{1.5}^{TMaccel}\right] &= -\frac{16}{c^2}\sin(\omega L)\sin(2\omega L)\,S_\delta \\
CSD\left[XY_{2.0}^{TMaccel}\right] &= -\frac{64}{c^2}\sin(\omega L)\sin^3(2\omega L)\,S_\delta
\end{aligned}
$$

Test-mass acceleration noise: PSD

$$
\begin{aligned}
PSD\left[X_{1.5}^{TMaccel}\right] &= \frac{4}{c^2}\sin^2\frac{\omega(L_2+L_{2'})}{2}\left\{\left(1+\frac{K+\frac{\lambda_{2'}}{\lambda_1}}{K+1}\right)^2 S_{\delta_{2'}}\right. \\
&\left.+\left[\left(\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\right)^2+1+2\frac{K+\frac{\lambda_1}{\lambda_{2'}}}{K+1}\cos\omega(L_3+L_{3'})\right]S_{\delta_1}\right\} \\
&+\frac{4}{c^2}\sin^2\frac{\omega(L_3+L_{3'})}{2}\left\{\left(1+\frac{K+\frac{\lambda_3}{\lambda_{1'}}}{K+1}\right)^2 S_{\delta_3}\right. \\
&\left.+\left[\left(\frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\right)^2+1+2\frac{K+\frac{\lambda_{1'}}{\lambda_3}}{K+1}\cos\omega(L_2+L_{2'})\right]S_{\delta_{1'}}\right\}
\end{aligned}
$$

- The PSD of TDI-X 2.0 was computed in the same way. The result equals to the PSD of TDI-X 1.5 multiplied by a factor of

$$4\sin^2\frac{\omega(L_{2'}+L_2+L_3+L_{3'})}{2}$$

- Use the approximations A2, A3, A4

$$
\begin{aligned}
PSD\left[X_{1.5}^{TMaccel}\right] &= \frac{16}{c^2}\sin^2(\omega L)\left[3+\cos(2\omega L)\right]S_\delta \\
PSD\left[X_{2.0}^{TMaccel}\right] &= \frac{64}{c^2}\sin^2(\omega L)\sin^2(2\omega L)\left[3+\cos(2\omega L)\right]S_\delta
\end{aligned}
$$

Test-mass acceleration noise: CSD

$$
\begin{aligned}
CSD\left[XY_{1.5}^{TMaccel}\right] &= -\frac{4}{c^2}\sin\frac{\omega(L_1+L_{1'})}{2}\sin\frac{\omega(L_2+L_{2'})}{2}e^{\frac{j\omega(L_1+L_{1'}-L_2-L_{2'}-L_3+L_{3'})}{2}}\left(1+2K+\frac{\lambda_1}{\lambda_{2'}}\right)\\
&\times\left[\left(1+2K+\frac{\lambda_1}{\lambda_{2'}}\right)\cos\frac{\omega(L_3+L_{3'})}{2}+j\left(\frac{\lambda_1}{\lambda_{2'}}-1\right)\sin\frac{\omega(L_3+L_{3'})}{2}\right]S_{\delta_1}\\
&-\frac{4}{c^2}\sin\frac{\omega(L_1+L_{1'})}{2}\sin\frac{\omega(L_2+L_{2'})}{2}e^{\frac{j\omega(L_1+L_{1'}-L_2-L_{2'}-L_3+L_{3'})}{2}}\left(1+2K+\frac{\lambda_{2'}}{\lambda_1}\right)\\
&\times\left[\left(1+2K+\frac{\lambda_{2'}}{\lambda_1}\right)\cos\frac{\omega(L_3+L_{3'})}{2}+j\left(1-\frac{\lambda_{2'}}{\lambda_1}\right)\sin\frac{\omega(L_3+L_{3'})}{2}\right]S_{\delta_{2'}}
\end{aligned}
$$

$$
\begin{aligned}
CSD[XY_{2.0}] &\approx CSD[XY_{1.5}]\\
&\times\left[1-e^{-j\omega(L_2+L_{2'}+L_3+L_{3'})}-e^{j\omega(L_3+L_{3'}+L_1+L_{1'})}+e^{j\omega(L_1+L_{1'}-L_2-L_{2'})}\right]
\end{aligned}
$$

- Use the approximations A2, A3, A4

$$
\begin{aligned}
CSD\left[XY_{1.5}^{TMaccel}\right] &= -\frac{16}{c^2}\sin(\omega L)\sin(2\omega L)\,S_\delta\\
CSD\left[XY_{2.0}^{TMaccel}\right] &= -\frac{64}{c^2}\sin(\omega L)\sin^3(2\omega L)\,S_\delta
\end{aligned}
$$

Backlink noise

$$PSD\left[X_{1.5}^{fibre}\right] = 4\sin^2\frac{\omega(L_3+L_{3'})}{2}\sin^2\frac{\omega(L_2+L_{2'})}{2}\left(S_{\mu_{1\to 1'}}+S_{\mu_{1'\to 1}}\right)$$

$$\begin{aligned} PSD\left[X_{2.0}^{fibre}\right] &= 16\sin^2\frac{\omega(L_2+L_{2'}+L_3+L_{3'})}{2} \\ &\times\sin^2\frac{\omega(L_3+L_{3'})}{2}\sin^2\frac{\omega(L_2+L_{2'})}{2}\left(S_{\mu_{1\to 1'}}+S_{\mu_{1'\to 1}}\right) \end{aligned}$$

- Use the approximations A2, A3, A4

$$\begin{aligned} PSD\left[X_{1.5}^{fibre}\right] &\approx 8\sin^4(\omega L)S_\mu \\ PSD\left[X_{2.0}^{fibre}\right] &\approx 32\sin^4(\omega L)\sin^2(2\omega L)S_\mu \end{aligned}$$

- the CSD of TDI-XY

$$\begin{aligned} CSD\left[XY_{1.5}^{fibre}\right] &= 0 \\ CSD\left[XY_{2.0}^{fibre}\right] &= 0 \end{aligned}$$

Readout noise: PSD

- Use the approximations A2, A3, A4

$$
\begin{aligned}
PSD\left[X_{1.5}^{ro}\right] &= 4\sin^2\left(\omega L\right) \\
&\times \left\{4S_{N_s^{ro}} + 4\frac{1+K^2-2K\cos^2(\omega L)}{(1+K)^2}S_{N_\tau^{ro}} + \frac{(1-K)^2}{(1+K)^2}[3+\cos(2\omega L)]S_{N_\epsilon^{ro}}\right\} \qquad (284) \\
PSD\left[X_{2.0}^{ro}\right] &= 16\sin^2\left(\omega L\right)\sin^2\left(2\omega L\right) \\
&\times \left\{4S_{N_s^{ro}} + 4\frac{1+K^2-2K\cos^2(\omega L)}{(1+K)^2}S_{N_\tau^{ro}} + \frac{(1-K)^2}{(1+K)^2}[3+\cos(2\omega L)]S_{N_\epsilon^{ro}}\right\}
\end{aligned}
$$

- in the extreme cases, K goes to 0 (neglect control loop) or infinity (perfect control loop)

$$
\begin{aligned}
PSD\left[X_{1.5}^{ro}\right] &= 4\sin^2\left(\omega L\right)\left[4S_{N_s^{ro}} + 4S_{N_\tau^{ro}} + \left[3+\cos(2\omega L)\right]S_{N_\epsilon^{ro}}\right] \\
PSD\left[X_{2.0}^{ro}\right] &= 16\sin^2\left(\omega L\right)\sin^2\left(2\omega L\right)\left[4S_{N_s^{ro}} + 4S_{N_\tau^{ro}} + \left[3+\cos(2\omega L)\right]S_{N_\epsilon^{ro}}\right]
\end{aligned}
$$

Readout noise: CSD

- Use the approximations A2, A3, A4

$$
\begin{aligned}
CSD\left[XY^{ro}_{1.5}\right] &= -4\sin(\omega L)\left[S_{N_s^{ro}} + 4\frac{(1-K)^2}{(1+K)^2}\sin(2\omega L)\left(S_{N_\tau^{ro}} + S_{N_\epsilon^{ro}}\right)\right] \\
CSD\left[XY^{ro}_{2.0}\right] &= -16\sin(\omega L)\sin^2(2\omega L)\left[S_{N_s^{ro}} + 4\frac{(1-K)^2}{(1+K)^2}\sin(2\omega L)\left(S_{N_\tau^{ro}} + S_{N_\epsilon^{ro}}\right)\right]
\end{aligned}
$$

- in the extreme cases, K goes to 0 (neglect control loop) or infinity (perfect control loop)

$$
\begin{aligned}
CSD\left[XY^{ro}_{1.5}\right] &= -4\sin(\omega L)\left[S_{N_s^{ro}} + 4\sin(2\omega L)\left(S_{N_\tau^{ro}} + S_{N_\epsilon^{ro}}\right)\right] \\
CSD\left[XY^{ro}_{2.0}\right] &= -16\sin(\omega L)\sin^2(2\omega L)\left[S_{N_s^{ro}} + 4\sin(2\omega L)\left(S_{N_\tau^{ro}} + S_{N_\epsilon^{ro}}\right)\right]
\end{aligned}
$$

Optical path noises: PSD

- Use the approximations A2, A3, A4

$$
\begin{aligned}
PSD\left[X^{op}_{1.5}\right] &= 4\sin^2(\omega L)\Bigg[4\left(S_{N^{op}_{RX}} + S_{N^{op}_{TX}} + S_{N^{op}_{loc\to sc}}\right) \\
&+ \frac{(1-K)^2}{(1+K)^2}\left[3+\cos(2\omega L)\right]\left(S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}}\right) \\
&+ 4\frac{1+K^2-2K\cos(2\omega L)}{(1+K)^2}\left(S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}}\right)\Bigg] \\
PSD\left[X^{op}_{2.0}\right] &= 16\sin^2(\omega L)\sin^2(2\omega L)\Bigg[4\left(S_{N^{op}_{RX}} + S_{N^{op}_{TX}} + S_{N^{op}_{loc\to sc}}\right) \\
&+ \frac{(1-K)^2}{(1+K)^2}\left[3+\cos(2\omega L)\right]\left(S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}}\right) \\
&+ 4\frac{1+K^2-2K\cos(2\omega L)}{(1+K)^2}\left(S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}}\right)\Bigg]
\end{aligned}
$$

Optical path noises: PSD

- in the extreme cases, K goes to 0 (neglect control loop) or infinity (perfect control loop)

$$
\begin{aligned}
PSD\left[X_{1.5}^{op}\right] &= 16\sin^2(\omega L)\left(S_{N_{RX}^{op}} + S_{N_{TX}^{op}} + S_{N_{loc\to sc}^{op}} + S_{N_{\tau}^{op}} + S_{N_{loc\to ref}^{op}}\right) \\
&\quad +4\sin^2(\omega L)\left[3+\cos(2\omega L)\right]\left(S_{N_{\epsilon}^{op}} + S_{N_{loc\to tm}^{op}}\right) \\
PSD\left[X_{2.0}^{op}\right] &= 64\sin^2\omega L\sin^2(2\omega L)\left(S_{N_{RX}^{op}} + S_{N_{TX}^{op}} + S_{N_{loc\to sc}^{op}} + S_{N_{\tau}^{op}} + S_{N_{loc\to ref}^{op}}\right) \\
&\quad +16\sin^2(\omega L)\sin^2(2\omega L)\left[3+\cos(2\omega L)\right]\left(S_{N_{\epsilon}^{op}} + S_{N_{loc\to tm}^{op}}\right)
\end{aligned}
$$

Optical path noises: CSD

- Use the approximations A2, A3, A4

$$
\begin{aligned}
CSD[XY^{op}_{1.5}] &= -4\sin(\omega L)\sin(2\omega L)\left(S_{N^{op}_{TX}} + S_{N^{op}_{RX}} + S_{N^{op}_{loc\to sc}}\right)\\
&\quad -4\frac{(1-K)^2}{(1+K)^2}\sin(\omega L)\sin(2\omega L)\left(S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}} + S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}}\right)\\
CSD[XY^{op}_{2.0}] &= -16\sin(\omega L)\sin^3(2\omega L)\left(S_{N^{op}_{TX}} + S_{N^{op}_{RX}} + S_{N^{op}_{loc\to sc}}\right)\\
&\quad -16\frac{(1-K)^2}{(1+K)^2}\sin(\omega L)\sin^3(2\omega L)\left(S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}} + S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}}\right)
\end{aligned}
$$

- in the extreme cases, K goes to 0 (neglect control loop) or infinity (perfect control loop)

$$
\begin{aligned}
CSD\left[XY^{op}_{1.5}\right] &= -4\sin(\omega L)\sin(2\omega L)\Big(S_{N^{op}_{RX}} + S_{N^{op}_{TX}} + S_{N^{op}_{loc\to sc}}\\
&\qquad + S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}} + S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}}\Big)\\
CSD\left[XY^{op}_{2.0}\right] &= -16\sin(\omega L)\sin^3(2\omega L)\Big(S_{N^{op}_{RX}} + S_{N^{op}_{TX}} + S_{N^{op}_{loc\to sc}}\\
&\qquad + S_{N^{op}_{\epsilon}} + S_{N^{op}_{loc\to tm}} + S_{N^{op}_{\tau}} + S_{N^{op}_{loc\to ref}}\Big)
\end{aligned}
$$

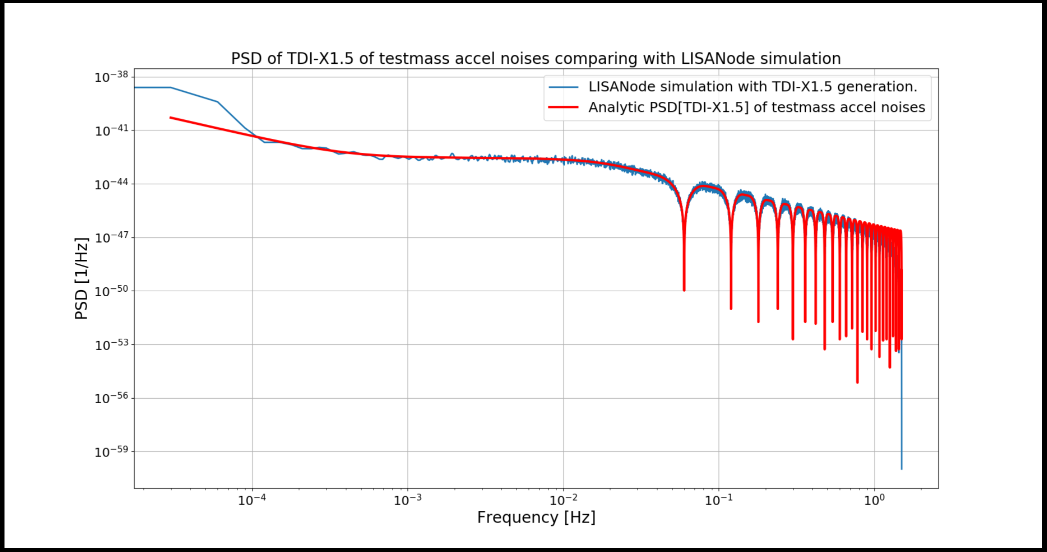
Noise implementation in LISANode

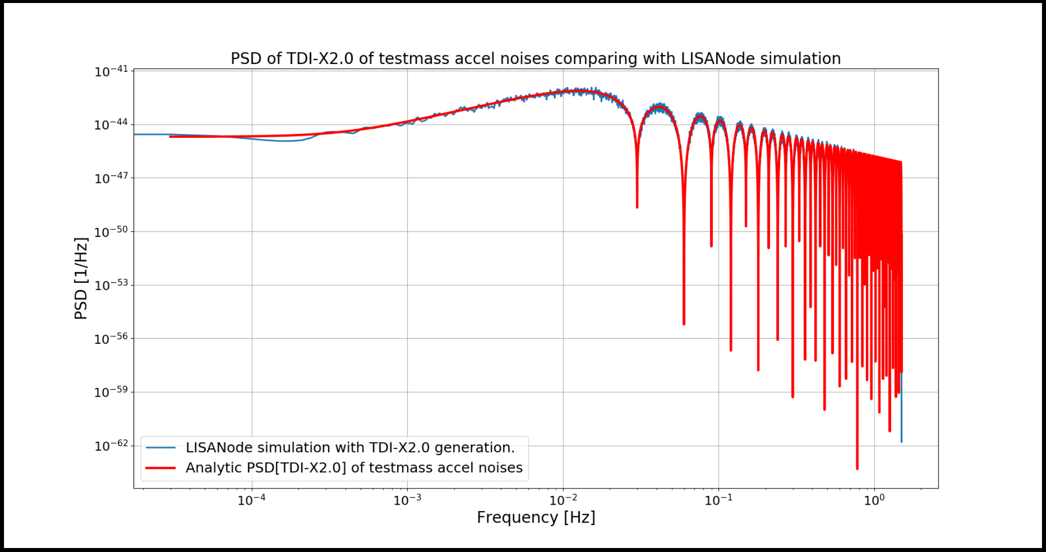
- In LISANode, noises are generated through some atomic nodes that produce white noises over simulation time.
  - We could indicate the PSD of the noise source and adjust the shape of the noise spectrum through some filter nodes.
  - The noises then are connected into the dedicated beams
- For the non-suppressed noises, we have
  - added the backlink noise node
  - modified the test-mass acceleration noise with the DFACs control loop transfer function K(f)
  - redistributed the optical path noises
- The laser locking implementation has been developing by Olaf Hartwig and other colleagues from AEI, Hannover.

Comparison results for some noises (1): Test-mass acceleration noise

- choose the input PSD of noise source as

$$
\begin{aligned}
S_\delta^{1/2} &= 2.4 \times 10^{-15} \frac{\mathrm{m/s^2}}{\sqrt{\mathrm{Hz}}} \sqrt{1 + \left(\frac{0.4\mathrm{mHz}}{f}\right)^2} \\
\rightarrow S_\delta &= \left(\frac{2.4 \times 10^{-15}}{2\pi f c}\right)^2 \frac{1}{\mathrm{Hz}} \left[1 + \left(\frac{0.4\mathrm{mHz}}{f}\right)^2\right]
\end{aligned}
$$





Comparison results for some noises (2): Readout noise

- choose input PSDs of the noise sources as

$$S^{1/2}_{N^{ro}_s} = S^{1/2}_{N^{ro}_\tau} = S^{1/2}_{N^{ro}_\epsilon} = 10^{-12}\frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}}$$

$$\rightarrow S_{N^{ro}_s} = S_{N^{ro}_\tau} = S_{N^{ro}_\epsilon} = \left(\frac{2\pi f}{c}10^{-12}\right)^2 \frac{1}{\mathrm{Hz}}$$